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REMARKS ON SOME INTERPOLATION SPACES

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Dedicated to Enrico MAGENES

Abstract. One purpose of this paper is to introduce a new idea of associating microlocal measures to bounded functions of bounded variation, and even to functions in the interpolation space $\big(H^1(\Omega), L^2(\Omega)\big)_{1/2,\infty}$, in order to replace and improve some methods from geometric measure theory. Some properties of similar interpolation spaces are reviewed first.

I do not remember when I met Enrico MAGENES for the first time. My first visit to Italy was for a CIME course in Varenna in 1970, and he may have come there, or I may have met him a few months after at ICM70 in Nice, but I definitely met him three years after on my second visit to Italy, when I stopped for a few days in Pavia on my way to Trieste. He kindly guided my visit to the famous Certosa di Pavia, and insisted that I should come back again as my visit could not be complete without seeing the facade; it was not that some work was being done with scaffoldings all around as it unfortunately happens sometimes when one visits a famous monument, but it was in November and the fog limited our visibility to a few meters. I did visit Pavia a few other times, not quite as often as I would have liked, and in that warm atmosphere that emanated from E. MAGENES I always felt as if there was now an Italian branch on my family tree.

I cannot remember if my thesis advisor, Jacques-Louis LIONS, had told me to read any of the three volumes that he had written with Enrico MAGENES, but I did read the first one entirely and some parts of the others two and it was then that I first learned about the theory of interpolation in the case of HILBERT spaces [L1&M]. J.-L. LIONS did give me his article with Jack PEETRE to read [L2&P], which improved my understanding of the case of BANACH spaces, and he mentioned some improvements by J. PEETRE, some of which I later found useful when I wrote my thesis: the first part [T1] answered a question that J.-L. LIONS had asked me in order to generalize one of his results [L3]; the second part also answered another question of J.-L. LIONS but never appeared as the corrections proposed by the editor never reached me due to some defect in the transatlantic mail system, and as my personal difficulties about writing (and also reading) were genuine, I could not put my mind on questions of publication for quite a long time after this incident. Many of my results remained then unpublished, sometimes mentioned by J.-L. LIONS as "to appear" instead of "personal communication" which would have been more appropriate, as I could not have initiated the too arduous process for me of preparing a manuscript for publication without someone insisting that a particular result of mine was worth publishing.

Progressively, I moved away from functional analysis and set my new goal of understanding more about Continuum Mechanics and Physics, so I could not find enough motivation to write down some of these technical results which I had obtained before.

Today, however, I realize that even for questions of Continuum Mechanics, some technical results from interpolation theory might be more important than I had thought previously. I did mention orally some remarks in this sense in two meetings during the Summer of 1992, the first one in Oberwolfach and the second one in Trento, where I had the pleasure to meet again Enrico MAGENES, and it is therefore with great pleasure that I dedicate to him some of these old remarks, mixed with new ideas.

I. A problem in singular perturbations and boundary layers.

In the early 70s, J.-L. LIONS was working on questions of singular perturbations, some of them involving questions of boundary layers; as he often did, he was teaching the subject and writing a book on it at the same time [L4]. One of the many questions that he was considering was to describe the boundary layer correction in a variational elliptic problem of the type

$$-\varepsilon^2 \Delta u_\varepsilon + u_\varepsilon = f \text{ in } \Omega, \quad u_\varepsilon = 0 \text{ on } \partial\Omega, \tag{I.1}$$

Ω being a bounded open set of R^N. For $\varepsilon > 0$, this problem has a natural variational framework in $V = H_0^1(\Omega)$ when f is given in $V' = H^{-1}(\Omega)$ or $H = L^2(\Omega)$, and one obtains the natural bounds

$$||u_\varepsilon||_V \leq \frac{C}{\varepsilon^2}||f||_{V'}, \text{ and } ||u_\varepsilon||_H \leq \frac{C}{\varepsilon}||f||_{V'}, \tag{I.2}$$

which can be improved in the case $f \in L^2(\Omega)$, to give

$$||u_\varepsilon||_V \leq \frac{C}{\varepsilon}||f||_H, \text{ and } ||u_\varepsilon||_H \leq C||f||_H, \tag{I.3}$$

where, as usual, C denotes various constants, not necessarily all equal. One then turns to the question of describing the behaviour of u_ε as ε tends to 0, assuming at least that $f \in L^2(\Omega)$, and the first result is that

$$u_\varepsilon \to f \text{ in } L^2(\Omega) \text{ strong}, \tag{I.4}$$

but without any uniform estimate, while if $f \in H_0^1(\Omega)$ one can write an equation for $u_\varepsilon - f$ and deduce that

$$||u_\varepsilon - f||_H \leq C\varepsilon||f||_V. \tag{I.5}$$

One arrives then at the interesting question of studying the boundary layer, i.e. of describing what occurs in the case where f is smooth but not 0 on the boundary $\partial\Omega$ of Ω. In that case, the standard technique consists in performing a local change of variable, using ε as a characteristic length for rescaling in the normal direction, and one then discovers that one needs a correction in $e^{-\frac{d(x)}{\varepsilon}}$, with $d(x)$ denoting the distance to $\partial\Omega$; after a few computations, this gives the error estimate

$$||u_\varepsilon - f||_H \leq C\sqrt{\varepsilon} \text{ if } f \text{ and } \partial\Omega \text{ are smooth enough}. \tag{I.6}$$

The question which puzzled me was that this classical method is not well adapted to discovering what is the minimum regularity that one should ask for f and for the boundary $\partial\Omega$ in order to have an estimate like (I.6). With Bernard NIVELET, who was working on this kind of problem at the time, I did some quite technical computations in order to check that the estimate is still true in the case where f is smooth and Ω is a square, even though $\Delta(e^{-\frac{d(x)}{\varepsilon}})$ has singularities on the diagonals, but I then found a method that could give an estimate like (I.6) without describing the correction at all. My method has a few advantages for what concerns regularity hypotheses: for what concerns the regularity of f, it does not even ask that f belong to $H^1(\Omega)$, as $f \in H^s(\Omega)$ is enough for $s > \frac{1}{2}$, and even a little less, as we will see when I will describe the precise interpolation spaces which I used. For what concerns the regularity of the coefficients, it applies if $-\Delta$ is replaced by a second order operator with bounded measurable coefficients $-\sum_{i,j=1}^N \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j})$ assuming of course that it is uniformly elliptic, i.e. there exists $\alpha > 0$ such that $\sum_{i,j=1}^N a_{ij}\xi_i\xi_j \geq \alpha|\xi|^2$ for all $\xi \in R^N$ and almost every $x \in \Omega$. For what concerns the regularity of the boundary, it applies to strongly LIPSCHITZ domains; actually, if the goal is only to prove an estimate like (I.6), I conjecture that the regularity hypothesis for $\partial\Omega$ can be weakened by a careful use of nonlinear interpolation as I did in my thesis for what concerns the regularity of coefficients needed for regularity theorems obtained by interpolation.

For simplicity, I describe the method for $-\Delta$. My result was mentioned in [L4] for $f \in H^1(\Omega)$, but I had shown it for $f \in H^s(\Omega)$ with $s > \frac{1}{2}$, and this result was rediscovered later by Yoshio KONISHI [K]. The discussion will center on two families of interpolation spaces and I first recall what these interpolation spaces are, using the K-method of J. PEETRE (which, together with its dual J-method, is a simplification of the methods introduced by J.-L. LIONS & J. PEETRE [L2&P], which already generalized a few earlier ideas).

Definition 1: If E_0 and E_1 are two BANACH spaces embedded in a common topological space so that their sum $E_0 + E_1$ is defined, then for $t > 0$ and $a \in E_0 + E_1$ one denotes $K(t,a) = \inf(||a_0||_0 + t||a_1||_1)$, where the infimum is taken over all the decompositions $a = a_0 + a_1$, with $a_i \in E_i$, and where $||.||_i$ denotes the norm on E_i. Then for $0 < \theta < 1$ and $1 \leq p \leq \infty$, $(E_0, E_1)_{\theta,p}$ denotes the Banach space of $a \in E_0 + E_1$ such that $t^{-\theta}K(t,a) \in L^p(0,\infty;\frac{dt}{t})$, the norm on this space being the corresponding norm of $t^{-\theta}K(t,a)$ in $L^p(0,\infty;\frac{dt}{t})$. For convenience, we will write $E(\theta,p)$ instead of $(E_0, E_1)_{\theta,p}$.

Remark 2: Describing what these spaces are in particular circumstances is not always easy, but one always has $E(\theta,p) \in E(\theta,q)$ if $1 \le p \le q \le \infty$. These spaces satisfy the interpolation property, namely if a linear operator T from $E_0 + E_1$ into $F_0 + F_1$ is continuous from E_0 into F_0 and continuous from E_1 into F_1, then it is continuous from $E(\theta,p)$ into $F(\theta,p)$ with norm $\le C||T||_0^{1-\theta}||T||_1^{\theta}$, where $||T||_i$ denotes the norm of T from E_i into F_i.

Notation 3: For $1 \le p \le \infty$, $X(p,\Omega) = \left(H_0^1(\Omega), L^2(\Omega)\right)_{1/2,p}$ and $Y(p,\Omega) = \left(H^1(\Omega), L^2(\Omega)\right)_{1/2,p}$.

Theorem 4: If $f \in X(\infty,\Omega)$, then the estimate (I.6) holds, namely

$$||u_\varepsilon - f||_H \le C\sqrt{\varepsilon}||f||_{X(\infty,\Omega)}. \tag{I.7}$$

Theorem 5: One has $X(\infty,\Omega) \subset Y(\infty,\Omega)$. If Ω is a strongly Lipschitz bounded open subset of R^N, then $Y(1,\Omega) \subset X(\infty,\Omega)$.

Remark 6: As for $s > \frac{1}{2}$ one has $H^s(\Omega) = \left(H^1(\Omega), L^2(\Omega)\right)_{1-s,2} \subset Y(1,\Omega)$ one deduces that (I.7) holds if the norm in $X(\infty,\Omega)$ is replaced by a norm in $H^s(\Omega)$ with $s > \frac{1}{2}$.

Theorem 4 has not much content as it is almost the definition of the interpolation space $X(\infty,\Omega)$ mentioned in the statement, but Theorem 5 and its corollary stated in Remark 6, is a more interesting statement.

Proof of Theorem 4: The linear mapping T defined by $Tf = u_\varepsilon - f$, with u_ε defined by equation (I.1), maps $H_0^1(\Omega)$ into $L^2(\Omega)$ with a norm $\le C\varepsilon$ by (I.5) and maps $L^2(\Omega)$ into $L^2(\Omega)$ with a norm $\le C$ by (I.3); therefore, by the interpolation property, it maps $X(\infty,\Omega) = \left(H_0^1(\Omega), L^2(\Omega)\right)_{1/2,\infty}$ into $L^2(\Omega)$ with a norm $\le C\sqrt{\varepsilon}$, which gives (I.7).

Proof of Theorem 5: As $H_0^1(\Omega) \subset H^1(\Omega)$, the interpolation property implies that $X(\infty,\Omega) \subset Y(\infty,\Omega)$.

In order to prove the second inclusion, one first uses a LIPSCHITZ partition of unity to localize what happens around various points of Ω or the (compact) boundary $\partial\Omega$; of course for $u \in Y(1,\Omega)$ one has $u = \sum_k \theta_k u$ with all θ_k LIPSCHITZ and the interpolation property implies that each $\theta_k u$ belongs to $Y(1,\Omega)$. As Ω is a strongly LIPSCHITZ domain, around a given point of the boundary, one can consider that Ω is replaced by $\Omega' = \{x \in R^N : x_N > F(x_1,\ldots,x_{N-1})\}$ where F is a LIPSCHITZ map, and then by $R_+^N = \{x \in R^N : x_N > 0\}$ by considering $x_N - F(x_1,\ldots,x_{N-1})$; of course the interpolation property is again used to see that spaces like $Y(1,\Omega)$ behave nicely by LIPSCHITZ changes of variable. It is enough then to show that $Y(1,R_+^N) \subset X(\infty,R_+^N)$, and then put all the pieces back together by using once more the interpolation property to deal with spaces $X(\infty)$ for various open sets.

For $u \in Y(1,R_+^N)$ we want to show that $u \in X(\infty,R_+^N)$. For this we consider that u is extended by 0 for $x_N < 0$, take a C^∞ function ρ having compact support in $x_N > 1$ and having integral 1, and for $0 < \varepsilon \le 1$ we define u_ε by the convolution product

$$u_\varepsilon(x) = \int_{R_+^N} \varepsilon^{-N}\rho\left(\frac{y}{\varepsilon}\right)u(x-y)dy \tag{I.8}$$

so that u_ε has its support in $x_N \ge \varepsilon$ and therefore belongs to $H_0^1(R_+^N)$. Using now $H = L^2(R_+^N)$ and $V = H^1(R_+^N)$, we want to show that

$$||u - u_\varepsilon||_H \le C\sqrt{\varepsilon}, \text{ and } ||\text{grad}(u_\varepsilon)||_H \le \frac{C}{\sqrt{\varepsilon}}, \tag{I.9}$$

which will imply that $u \in X(\infty,R_+^N)$ because it shows that $K(t,u) \le C\left(\frac{1}{\sqrt{\varepsilon}} + t\sqrt{\varepsilon}\right)$ and therefore by taking $\varepsilon = \frac{1}{t}$ that $K(t,u) \le C\sqrt{t}$ for $t \ge 1$ (as $H_0^1(R_+^N) \subset L^2(R_+^N)$, one has $K(t,u) \le Ct$ by writing $u = 0 + u$, and there is no need to check that $K(t,u) \le C\sqrt{t}$ for $t \le 1$). All the required inequalities will follow from the fact that $u \in Y(1,R_+^N)$ implies

$$\|u(.-y)-u(.)\|_H \le C\sqrt{|y|}\|u\|_{Y(1,R^N_+)} \text{ for } y \in R^N. \tag{I.10}$$

Let us first admit (I.10). Then, because ρ has integral 1, one has

$$u_\varepsilon(x)-u(x)=\int_{R^N_+}\varepsilon^{-N}\rho\Big(\frac{y}{\varepsilon}\Big)\Big(u(x-y)-u(x)\Big)dy \tag{I.11}$$

and therefore taking the L^2 norm gives

$$\|u_\varepsilon-u\|_H \le \int_{R^N_+}\varepsilon^{-N}\Big|\rho\Big(\frac{y}{\varepsilon}\Big)\Big|.\sqrt{|y|}dy \le C\sqrt{\varepsilon}. \tag{I.12}$$

On the other hand as u_ε is a convolution product, one has

$$\partial_j u_\varepsilon(x)=\int_{R^N_+}\varepsilon^{-(N+1)}\partial_j\rho\Big(\frac{y}{\varepsilon}\Big)u(x-y)dy=\int_{R^N_+}\varepsilon^{-(N+1)}\partial_j\rho\Big(\frac{y}{\varepsilon}\Big)\Big(u(x-y)-u(x)\Big)dy, \tag{I.13}$$

as the integral of $\partial_j\rho$ is 0, and this gives

$$\|\partial_j u_\varepsilon\|_H \le \int_{R^N_+}\varepsilon^{-(N+1)}\Big|\partial_j\rho\Big(\frac{y}{\varepsilon}\Big)\Big|.\sqrt{|y|}dy \le \frac{C}{\sqrt{\varepsilon}}. \tag{I.14}$$

It remains to prove (I.10). For $y_N=0$, inequality (I.10) holds even for $u\in Y(\infty,R^N_+)$ because one has $\|u(.-y)-u(.)\|_H \le C|y|.\|u\|_V$ and $\|u(.-y)-u(.)\|_H \le 2\|u\|_H$ and the interpolation property gives (I.10), but this argument does not work if $y_N\ne 0$, but it does apply for the case of R^N, as we will use later. Let us consider now the case where $y_N\ne 0$ and $y_j=0$ for $j\ne N$, and denote by Pu the extension of u which is even in x_N, i.e.

$$Pu(x_1,\dots,x_{N-1},x_N)=u(x_1,\dots,x_{N-1},x_N) \text{ if } x_N>0 \text{ or } u(x_1,\dots,x_{N-1},-x_N) \text{ if } x_N<0. \tag{I.15}$$

The interpolation property shows that $u\in Y(1,R^N_+)$ implies $Pu\in Y(1,R^N)$ and therefore by applying the preceding argument to Pu on R^N, one deduces that

$$\|Pu(.-y)-Pu(.)\|_H \le C\sqrt{|y|}\|u\|_{Y(1,R^N)} \text{ for } y\in R^N, \tag{I.16}$$

and this will give (I.10) if we show the inequality

$$\|u\|_{L^2(0<x_N<h)} \le C\sqrt{h}\|u\|_{Y(1,R^N_+)}. \tag{I.17}$$

Indeed for a function φ of a single variable one has

$$|\varphi(t_0)| \le C\|\varphi\|^{\frac12}_{H^1(R)}\|\varphi\|^{\frac12}_{L^2(R)}, \tag{I.18}$$

where C is independent of t_0, and therefore

$$\|\varphi\|_{L^2(0<t<h)} \le C\sqrt{h}\|\varphi\|^{\frac12}_{H^1(R)}\|\varphi\|^{\frac12}_{L^2(R)}, \tag{I.19}$$

which after integration in $x_1,\dots,x_{N-1}$ gives

$$\|u\|_{L^2(0<x_N<h)} \le C\sqrt{h}\|u\|^{\frac12}_{H^1(R)}\|u\|^{\frac12}_{L^2(R)}. \tag{I.20}$$

This last inequality implies (I.17) by a classical argument of LIONS & PEETRE [L2&P] (which uses the J-method), namely if a linear map S satisfies

$$\|Su\|_Z \le C\|u\|^{1-\theta}_{E_0}\|u\|^{\theta}_{E_1} \text{ for all } u\in E_0\cap E_1, \tag{I.21}$$

then it satisfies

$$\|Su\|_Z \le C\|u\|_{E(\theta,1)}. \tag{I.22}$$

Remark 7: By interpolation, if $u \in X(\infty,\Omega)$, then the extension of u by 0 outside Ω gives a function in $X(\infty, R^N)$, and therefore an inequality of type (I.10) holds, giving then an inequality of type (I.16), i.e. by denoting $d(x)$ the distance of x to $\partial\Omega$, $u \in X(\infty,\Omega)$ implies

$$\|u\|_{L^2(0<d(x)<h)} \le C\sqrt{h} \text{ for } 0 < h \le 1. \tag{I.23}$$

The preceding proof actually shows that $u \in X(\infty,\Omega)$ if and only if $u \in Y(\infty,\Omega)$ and (I.23) holds for u, as this inequality is conserved by localization and LIPSCHITZ change of variable.

II. An interpolation space on Ω with traces exactly in $L^2(\partial\Omega)$.

If inequality (I.17) was true with $Y(1,R^N)$ replaced by $Y(p,R^N)$ with $p > 1$, then by taking squares, dividing by h and letting h tend to 0, it would imply that functions in $Y(p,R^N)$ have a trace on $x_N = 0$ belonging to $L^2(R^N)$, and this is not true. However, the space $Y(1,R^N)$ does have $L^2(R^{N-1})$ traces on $x_N = 0$ (as follows from (I.17)), a fact which I learned in 1975 in a talk of Shmuel AAGMON, who was describing some work he had done with Lars HÖRMANDER concerning questions of spectral theory [A&H]. The trace result obviously extends to LIPSCHITZ hypersurfaces (S. AGMON & L. HÖRMANDER were actually interested in traces on spheres, and working with the FOURIER transform of $Y(1,R^N)$ instead); S. AAGMON & L. HÖRMANDER also proved that any $L^2(R^{N-1})$ function could be a such a trace, not by an explicit construction but by an argument using the transposed operator.

Of course, one cannot obtain information on the trace by interpolation as one is exactly considering a limiting case beyond which traces are not defined. Around 1985, in collaboration with Michel ARTOLA, we constructed an explicit lifting of the trace from $L^2(\partial\Omega)$ into $Y(1,\Omega)$ by applying a method which I had derived earlier for proving the classical result of Emilio GAGLIARDO [Ga] that every function in $L^1(\partial\Omega)$ is the trace of a function in $W^{1,1}(\Omega)$; my method is quite similar to E. GAGLIARDO's original method. For simplicity, I consider the case of R^2, and present both results for completeness and in order to show the similarities of the two questions.

For $f \in L^1(R)$ let us construct $u \in W = W^{1,1}(R^2)$ such that $u(x,0) = f(x)$ almost everywhere. Continuous functions with compact support are dense in $L^1(R)$, and can be approximated uniformly by continuous piecewise affine functions by refining the mesh size, so there exists a mesh size h and a continuous function g affine on every interval $(ih,(i+1)h)$, such that $\|f-g\|_{L^1(R)} \le \alpha\|f\|_{L^1(R)}$ with $\alpha < 1$. The interest of this class of functions g is that there exists a constant C independent of g and h such that $\|\frac{dg}{dx}\|_{L^1(R)} \le \frac{C}{h}\|g\|_{L^1(R)}$. A function $G \in W = W^{1,1}(R^2)$ with trace g is then given explicitly by

$$G(x,y) = g(x)e^{-\frac{|y|}{h}}, \tag{II.1}$$

and one immediately checks that it satisfies

$$\|G\|_W \le C\|g\|_{L^1(R)}. \tag{II.2}$$

Then one repeats the procedure with f replaced by $f-g$ and one creates then a series of functions G (with smaller and smaller mesh sizes) whose sum is the desired function.

For $f \in L^2(R)$ let us construct $u \in Y(1,R^2)$ such that $u(x,0) = f(x)$ almost everywhere. By the same argument, there exists a mesh size h and a continuous function g affine on every interval $(ih,(i+1)h)$, such that $\|f-g\|_{L^2(R)} \le \alpha\|f\|_{L^2(R)}$ with $\alpha < 1$. We make the same observation that $\|\frac{dg}{dx}\|_{L^2(R)} \le \frac{C}{h}\|g\|_{L^2(R)}$ and use a similar formula

$$G(x,y) = g(x)e^{-\frac{|y|}{h}}, \tag{II.3}$$

in order to generate a function with trace g (notice the different power of h in the exponential). It remains to estimate the norm of G in $Y(1,R^2)$. One has $||G||_{L^2(R^2)} \leq C\sqrt{h}||g||_{L^2(R)}$ and $||G||_{L^2(R^2)} + ||\frac{\partial G}{\partial x}||_{L^2(R^2)} + ||\frac{\partial G}{\partial y}||_{L^2(R^2)} \leq \frac{C}{\sqrt{h}}||g||_{L^2(R)}$, and therefore

$$||G||_{Y(1,R^2)} \leq C||G||^{\frac{1}{2}}_{L^2(R^2)}||G||^{\frac{1}{2}}_{H^1(R^2)} \leq C||g||_{L^2(R)}. \tag{II.4}$$

Then one repeats the procedure with f replaced by $f-g$ and this creates then a series of function G (with smaller and smaller mesh sizes) whose sum is the desired function.

III. Regularity for evolution variational inequalities.

Just after having found the role played by $X(\infty,\Omega)$ and $Y(\infty,\Omega)$ in the singular perturbation problem, I discovered another situation where a space like $Y(\infty)$ arose naturally, this time in relation with a question of regularity in time of solutions of evolution variational inequalities (of parabolic type).

In the usual Hilbert setting $V \subset H = H' \subset V'$, V being dense in H and the respective norms being denoted by $||.||$, $|.|$ and $||.||_*$, let $A \in \mathcal{L}(V,V')$ be such that there exists $\alpha > 0$ and $\beta \in R$ such that

$$(Au,u) \geq \alpha||u||^2 - \beta|u|^2 \text{ for all } u \in V, \tag{III.1}$$

and then let $f \in L^2(0,T;V')$ and $u_0 \in H$. Then there exists a unique solution $u \in L^2(0,T;V) \cap C([0,T];H)$ of

$$\frac{du}{dt} + Au = f \text{ in } (0,T); \quad u(0) = u_0, \tag{III.2}$$

and it obviously satisfies $\frac{du}{dt} \in L^2(0,T;V')$, because both f and Au belong to this space. The evolution variational inequality consists in having for each t a nonempty closed convex set $K(t)$ in V and ask instead of (III.2) that u satisfy

$$\Big(\frac{du}{dt} + Au, v-u\Big) \geq (f,v-u) \text{ a.e. } t \in (0,T) \text{ for all } v \in K(t) \text{ a.e. } t \in (0,T),$$

$$u(t) \in K(t) \text{ for } t \in [0,T]; \quad u(0) = u_0. \tag{III.3}$$

Of course, one adds that u_0 belongs to the closure of $K(0)$ in H, and one requires some reasonable dependence for $K(t)$ with respect to t, but the problem is that if such a solution u exists one cannot deduce where $\frac{du}{dt}$ will be. One must then use a weaker notion of solution, corresponding to a formulation where $\frac{du}{dt}$ is not used, and then one must ask about the regularity of that solution. In my argument, I was using a particular hypothesis for $K(t)$, namely that

$$K(t) \text{ is nondecreasing in } t, \tag{III.4}$$

and what I obtained was not an estimate on the derivative of u in t, but on a derivative of order $\frac{1}{2}$, involving a space like $Y(\infty)$, but with values in H. Let us consider first the case of smooth solutions.

Theorem 8: Under assumptions (III.1) and (III.4), there is a constant C such that if u is a strong solution of (III.3), i.e. such that $u \in L^2(0,T;V) \cap C([0,T];H)$ and $\frac{du}{dt} \in L^2(0,T;V')$, then it satisfies

$$\int_0^T |u(t)-u(t-h)|^2 dt \leq Ch\Big(\int_0^T ||f(t)||_*^2 dt + |u_0|^2 + 1\Big) \text{ for } 0 < h < T, \tag{III.5}$$

where u had been extended by 0 for $t < 0$.

Proof. By choosing a constant $v \in K(0)$, which is admissible because of (III.4), the inequality (III.3) gives

$$\frac{d\big(|u-v|^2\big)}{dt} + 2\alpha||u-v||^2 - 2\beta|u-v|^2 \leq 2||f-Av||_*||u|| \leq \alpha||u-v||^2 + \frac{1}{\alpha}||f-Av||_*^2, \tag{III.6}$$

which implies the first elementary estimate

$$\int_0^T \|u(t)-v\|^2 dt + \sup_{t\in[0,T]} |u(t)-v|^2 \leq C\Big(\int_0^T \|f(t)-Av\|_*^2 dt + |u_0 - v|^2\Big), \tag{III.7}$$

where C depends upon α, β, T and the best constant γ such that $\|v\| \geq \gamma|v|$ for every $v \in V$. In order to obtain the less elementary estimate (III.5), I made a translation in order to have $0 \in K(0)$, and chose the test function v to be

$$v(t) = \frac{1}{h}\int_{t-h}^{t} u(s)ds,\ t\in(0,T) \tag{III.8}$$

with $h > 0$ arbitrary and u being extended by 0 for $t < 0$. Again, this is an admissible test function because of (III.4) and the extension of u by 0 for $t < 0$ gives $v(0) = 0$, which is admissible because $0 \in K(0)$. We then integrate (III.3) from 0 to T with this choice of v. One has

$$\int_0^T \Big(\frac{du}{dt}, v-u\Big)dt = -\frac{1}{2}|u(T)|^2 + \frac{1}{2}|u_0|^2 + \Big(u(T), v(T)\Big) - \frac{1}{h}\int_0^T \Big(u(t), u(t)-u(t-h)\Big)dt, \tag{III.9}$$

so that

$$\int_0^T \Big(\frac{du}{dt}, v-u\Big)dt \leq -\frac{1}{2h}\int_0^T |u(t)-u(t-h)|^2 dt + \frac{1}{2}|u_0|^2 + \frac{1}{2}|v(T)|^2. \tag{III.10}$$

As all the other terms are bounded by constants independent of h because of the first estimate (III.7), one obtains a bound independent of h for $\frac{1}{h}\int_0^T |u(t)-u(t-h)|^2 dt$, i.e. the estimate (III.5).

Remark 9: For weak solutions, one must be a little more careful, but the idea is the same. One uses a time discretization, choosing a mesh size $k = \Delta t = \frac{T}{N}$, approximating f by $f_n = \frac{1}{k}\int_{(n-1)k}^{nk} f(t)dt$ for $n = 1,\ldots,N$, and defining u_n, expected to be an approximation of $u(n\Delta t)$, by an implicit scheme: solving for $n = 1,\ldots,N$, the variational inequalities

$$\frac{1}{k}(u_n - u_{n-1}, v-u_n) + (Au_n, v-u_n) \geq (f_n, v-u_n) \text{ for all } v\in K(nk), \text{ with } u_n \in K(nk). \tag{III.11}$$

Assuming that we have made a translation so that $0 \in K(0) \in K(nk)$, we can take $v = 0$ in (III.11) and obtain the discrete analog of (III.7), i.e.

$$k\sum_{n=1}^{N} \|u_n\|^2 + \sup_n |u_n|^2 \leq C\Big(\int_0^T \|f(t)\|_*^2 dt + |u_0|^2\Big), \tag{III.12}$$

as well as

$$\sum_{n=1}^{N} |u_n - u_{n-1}|^2 \leq C, \tag{III.13}$$

with C independent of N. The analog of (III.5) is

$$\sum_{n=1}^{N} |u_n - u_{n-m}|^2 \leq Cm, \tag{III.14}$$

with C independent of m and N, and it is obtained by choosing $v = v_n = \frac{1}{m}(u_{n-1} + \ldots + u_{n-m})$, and summing in n after multiplying by k. Indeed the analog of the left hand side of (III.9) is

$$X = \sum_{n=1}^{N} \Big(u_n - u_{n-1}, \frac{1}{m}(u_{n-1} + \ldots + u_{n-m}) - u_n \Big), \tag{III.15}$$

which transforms into a few more terms than the right hand side of (III.9), as $-\frac{1}{2}|u(T)|^2 + \frac{1}{2}|u_0|^2$ is replaced by $-\frac{1}{2}|u_N|^2 + \frac{1}{2}|u_0|^2 - \frac{1}{2}\sum_{n=1}^{N}|u_n - u_{n-1}|^2$; then the other terms are $(u_N.v_N)$ replacing $\big(u(T), v(T)\big)$, and $-\frac{1}{m}\sum_{n=1}^{N-1}(u_n, u_n - u_{n-m})$ replacing $-\frac{1}{h}\int_0^T (u(t), u(t) - u(t-h))dt$. This gives for the analog of (III.10) the inequality

$$X \leq -\frac{1}{2m}\sum_{n=1}^{N-1}|u_n - u_{n-m}|^2 + \frac{1}{2}|u_0|^2 + \frac{1}{2}|v_N|^2, \tag{III.16}$$

which with (III.11) implies (III.14) as all the other terms appearing are bounded by constants independent of m by (III.12). Having solved (III.11) for all n, one creates a function by interpolation, and any reasonable method will give a function satisfying (III.5). Passing to the limit in (III.11) is not difficult if one uses a test function such that $v(t) \in K(t)$ for every $t \in [0,T]$ and which is smooth enough, for example such that $\frac{dv}{dt} \in L^\infty(0,T;H)$, so that the derivative in t can be used for the test function v instead of the solution u.

I remember applying the same idea to other nonlinear problems. A first one was an abstract pseudo-NAVIER-STOKES setting, $\frac{du}{dt} + Au + B(u,u) = f$, for which I was considering variational inequalities if my memory is correct, as I have misplaced my personal notes on that question long ago, but J.-L. LIONS did mention it in a joint book with Roland GLOWINSKI and Raymond TRÉMOLIÈRES [G&L&T] (although they wrongly attributed me a co-author for that result). A second one, answering a question of Haïm BRÉZIS at a meeting in Pau in February 1972, was an equation of the type $\frac{\partial u}{\partial t} + \partial\varphi(u) \in f$ with φ convex.

IV. What functional spaces could one use for hyperbolic equations?

At a meeting in Oxford in the Summer of 1982, I did mention functional spaces like $Y(\infty)$ as a possible answer to a question of Andrew MAJDA. I had heard mentioned before that the space of BV functions, although quite useful for studying a scalar quasi-linear hyperbolic equation, did not seem adequate for treating the case of systems of conservation laws; A. MAJDA was adding the remark that for linear hyperbolic systems in more than one space variable the L^2 setting seemed almost necessary and that the study of small perturbations of a nonlinear system added the constraint of using a functional space adapted to linearized problems, so that a space like BV had to be excluded. A space like $Y(\infty, R^N)$ seemed to me a good substitute, as it was using a L^2 setting, while at the same time it contained many functions having discontinuities, as piecewise smooth functions with discontinuities along smooth hypersurfaces do belong to $Y(\infty, R^N)$.

Unfortunately, my only result in that direction concerns a scalar equation in only one space variable, and it uses different interpolation spaces than $Y(\infty, R)$, so that I still do not know if the space $Y(\infty, R^N)$ is useful for hyperbolic equations. The framework is that of entropy solutions of

$$\frac{\partial u}{\partial t} + \frac{\partial\big(f(u)\big)}{\partial x} = 0 \text{ in the sense of distributions in } R \times (0,T); \quad u(x,0) = v(x) \text{ a.e. in } R, \tag{IV.1}$$

obtained, for example, by the vanishing viscosity method. We assume for simplicity that f is a C^1 function. For $\varepsilon > 0$, the regularized equation

$$\frac{\partial u^\varepsilon}{\partial t} + \frac{\partial\big(f(u^\varepsilon)\big)}{\partial x} - \varepsilon\frac{\partial^2 u^\varepsilon}{\partial x^2} = 0 \text{ a.e. in } R \times (0,T); \quad u^\varepsilon(x,0) = v(x) \text{ a.e. in } R, \tag{IV.2}$$

defines a (nonlinear) semigroup $S^\varepsilon(t)$, which satisfies

$$||S^\varepsilon(t)v||_{L^p(R)} \leq ||v||_{L^p(R)} \text{ for } 1 \leq p \leq \infty. \tag{IV.3}$$

$$\|S^\epsilon(t)v - S^\epsilon(t)w\|_{L^1(R)} \le \|v - w\|_{L^1(R)}. \tag{IV.4}$$

As ε tends to 0, u^ϵ converges to the particular solution of (IV.1) which satisfies the following entropy condition

$$\frac{\partial\big(\varphi(u)\big)}{\partial t} + \frac{\partial\big(\psi(u)\big)}{\partial x} \le 0 \text{ in the sense of distributions in } R \times (0,T), \tag{IV.5}$$

for every convex function φ, called an entropy by Peter LAX, the entropy flux ψ being defined by $\psi' = f'\varphi'$; the entropy solution corresponds to a semigroup $S(t)$ which satisfies the same inequalities (IV.3)-(IV.4) as $S^\epsilon(t)$ does. As $S^\epsilon(t)$ and $S(t)$ commute with translation, one can deduce immediately that

$$\|\tau_h S(t)v - S(t)v\|_{L^1(R)} \le \|\tau_h v - v\|_{L^1(R)}, \tag{IV.6}$$

where τ_h denotes the translation by h, i.e. $\tau_h u(x,t) = u(x-h,t)$. From (IV.6), and the analog of (IV.3) for $p = 1$, one can deduce that

$$\|S(t)v\|_{BV(R)} \le C\|v\|_{BV(R)}, \tag{IV.7}$$

and by (nonlinear) interpolation that

$$\|S(t)v\|_W \le C\|v\|_W, \tag{IV.8}$$

where W is any interpolation space between $BV(R)$ or $W^{1,1}(R)$ and $L^1(R)$, like $W^{s,1}(R)$ for $0 < s < 1$. It could seem natural to deduce existence results in interpolation spaces between $BV(R)$ and $L^\infty(R)$, but the known (nonlinear) interpolation theorems do not apply, as they require a LIPSCHITZ or HÖLDER hypothesis for the mapping on some of the spaces, and this does not hold for either $BV(R)$ or $L^\infty(R)$. However, I was able to derive such an interpolation result, but by using a linear interpolation argument as follows.

Theorem 10: The semigroups $S^\epsilon(t)$ and $S(t)$ satisfy the estimate

$$\|S^\epsilon(t)v\|_{Z(\theta,p)} + \|S(t)v\|_{Z(\theta,p)} \le C\|v\|_{Z(\theta,p)}, \tag{IV.9}$$

where $Z(\theta,p) = \big(BV(R), L^\infty(R)\big)_{\theta,p}$, with $0 < \theta < 1$ and $1 \le p \le \infty$, and C is independent of ε.
Proof: For $v \in L^\infty(R)$, the solution of u_ϵ of (IV.2) is uniformly bounded, by an application of the maximum principle, and satisfies

$$\|u^\epsilon(\cdot,t)\|_{L^\infty(R)} \le \|v\|_{L^\infty(R)}. \tag{IV.10}$$

We define $a^\epsilon(x,t) = f'\big(u^\epsilon(x,t)\big)$, and consider the linear equation

$$\frac{\partial w}{\partial t} + a^\epsilon\frac{\partial w}{\partial x} - \varepsilon\frac{\partial^2 w}{\partial x^2} = 0 \text{ a.e. in } R \times (0,T); \quad w(x,0) = w_0(x) \text{ a.e. in } R, \tag{IV.11}$$

so that, because of (IV.2) and the fact that u^ϵ is smooth enough to have $\frac{\partial(f(u^\epsilon))}{\partial x} = f'(u^\epsilon)\frac{\partial u^\epsilon}{\partial x}$ the solution of (IV.10) corresponding to the initial data v is u^ϵ. By the maximum principle, the solution w of (IV.10) satisfies

$$\|w(\cdot,t)\|_{L^\infty(R)} \le \|w_0\|_{L^\infty(R)}. \tag{IV.12}$$

On the other hand the function $z = \frac{\partial w}{\partial x}$ satisfies

$$\frac{\partial z}{\partial t} + \frac{\partial(a^\epsilon z)}{\partial x} - \varepsilon\frac{\partial^2 z}{\partial x^2} = 0 \text{ a.e. in } R \times (0,T); \quad w(x,0) = w_0(x) \text{ a.e. in } R, \tag{IV.13}$$

so that, by the maximum principle applied to the dual problem, one has

$$\left\|\frac{\partial w}{\partial x}(\cdot,t)\right\|_{L^1(R)} \leq \left\|\frac{\partial w_0}{\partial x}\right\|_{L^1(R)}. \tag{IV.14}$$

The interesting fact is that the inequalities (IV.12) and (IV.14) are independent of the regularity of a^ϵ, as long as a^ϵ is smooth enough, which is indeed the case because $\epsilon > 0$ and u^ϵ is reasonably smooth. As $BV(R)$ is included in $L^\infty(R)$, a result which does not extend to more than one space variable, one may put on $BV(R)$ the norm

$$\|\varphi\|_{BV(R)} = \|\varphi\|_{L^\infty(R)} + \left\|\frac{\partial \varphi}{\partial x}\right\|_{\mathcal{M}(\mathcal{R})}, \tag{IV.15}$$

where$\mathcal{M}(\mathcal{R})$ is the space of Radon measure with finite total mass, and easily deduce from (IV.12) and (IV.14) that one has

$$\|w(\cdot,t)\|_{BV(R)} \leq \|w_0\|_{BV(R)}. \tag{IV.16}$$

Then one can use linear interpolation theory to deduce from (IV.12) and (IV.16) that one also has

$$\|w(\cdot,t)\|_{Z(\theta,p)} \leq C\|w_0\|_{Z(\theta,p)}. \tag{IV.17}$$

One then applies the result to $w_0 = v$, which gives $w = u^\epsilon$ and the desired estimate for $S^\epsilon(t)v$ in (IV.11), and then letting ϵ tend to 0, one obtains the same estimate for $S(t)v$, proving (IV.11).

Remark 11: The result for u^ϵ is actually true with $BV(R)$ replaced by $W^{1,1}(R)$, but of course, the limit u may have discontinuities and take values in $BV(R)$ but not in $W^{1,1}(R)$.

Remark 12: It is not difficult to check that

$$Y(p,R) = \left\{u \in H = L^2(R) : \frac{1}{\sqrt{h}}\|\tau_h u - u\|_H \in L^p\left(0,\infty;\frac{dh}{h}\right)\right\}, \tag{IV.18}$$

but because $BV(R)$ is not embedded in $L^2(R)$, we also consider the spaces

$$Z(p,R) = \left\{u \in L^\infty(R) : \frac{1}{\sqrt{h}}\|\tau_h u - u\|_H \in L^p\left(0,\infty;\frac{dh}{h}\right)\right\}. \tag{IV.19}$$

With this notation, one has $Z(\frac{1}{2},2) \subset Z(\infty,R)$. Indeed one has

$$\|\tau_h v - v\|_{L^1(R)} \leq C|h|.\|v\|_{BV(R)} \text{ and } \|\tau_h v - v\|_{L^\infty(R)} \leq 2\|v\|_{L^\infty(R)}, \tag{IV.20}$$

so that by interpolation one obtains

$$\|\tau_h v - v\|_{L^2(R)} \leq C\sqrt{h}\|v\|_{Z(1/2,2)}, \tag{IV.21}$$

and as this is true for all h, and as $Z(\theta,p) \subset L^\infty(R)$, this implies

$$\|v\|_{Y(\infty,R)} \leq C\|v\|_{Z(1/2,2)}. \tag{IV.22}$$

The space $Z(\frac{1}{2},2)$ is not included in $Z(p,R)$ for $p < \infty$, as there are functions in $BV(R)$ which do not belong to $Y(p,R)$ or $Z(p,R)$ for $p < \infty$: let $\varphi \in D(0,\infty)$ have integral 1, and define u by $u(x) = 0$ for $x \leq 0$ and $u(x) = 1 - \int_0^x \varphi(y)dy$ for $x \geq 0$, so that $u \in BV(R)$ and has its FOURIER transform which decays like $\frac{c}{\xi}$ with $c \neq 0$, and therefore belongs to $Y(\infty,R)$ but not to $Y(p,R)$ for $p < \infty$.

The preceding construction showing that some discontinuities are not allowed in $Y(p,R)$ or $Z(p,R)$ for $p < \infty$ (although not all functions of $Y(p,R)$ or $Z(p,R)$ are continuous for $p > 1$) shows that in the family of functional spaces $Y(p,R^N)$ or $Z(p,R^N)$ with $1 \leq p \leq \infty$, only $Y(\infty,R^N)$ or better $Z(p,R^N)$ could be adapted to conservation laws.

Remark 13: In the other direction, the spaces $Y(p,R)$ are not included in $Z(\frac{1}{2},2)$ for $p>1$, as $Z(\frac{1}{2},2)$ is included in $L^\infty(R)$, while $Y(p,R)$ is not for $p>1$. One can construct an unbounded function belonging to $Y(p,R)$ in the following way: let φ be a real even nonnegative C^∞ function having support in $1\le|x|\le 2$ and integral 1, and let ψ be its FOURIER transform, which is a real even C^∞ function, belonging to $S(R)$ and such that $\psi(0)=1$; then the function f defined by $f(x)=\sum_{n\in Z}a_n\psi(2^n x)$ has FOURIER transform $\mathbf{F}f(\xi)=\sum_{n\in Z}a_n 2^{-n}\varphi(2^{-n}\xi)$ and belongs to $Y(p,R)$ if and only if the sequence a_n belongs to l^p, and f will be infinite at 0 by choosing the coefficients a_n to be nonnegative numbers such that the sequence a_n does not belong to l^1.

As $Y(1,R)$ only contains continuous functions, one has $Y(1,R)\subset Z(1,R)$ and one can expect that $Y(1,R)$ is continuously embedded in $Z(\frac{1}{2},2)$, but I do not know if that is true.

$Y(1,R)$ is actually embedded in the larger space $Z(\frac{1}{2},\infty)$. In order to show that it is continuously embedded in any space $Z(\frac{1}{2},p)$, it is enough (and therefore equivalent as $H^1(R)\subset Y(1,R)$) to prove that $H^1(R)$ is continuously embedded in $Z(\frac{1}{2},p)$, i.e. that there exists a constant C such that

$$||u||_{Z(1/2,p)}\le C||u||_{H^1(R)} \text{ for every } u\in H^1(R). \tag{IV.23}$$

Then for $\lambda>0$, one applies (IV.23) to the rescaled function u_λ defined by $u_\lambda(x)=u(\lambda x)$ for $x\in R$; as $||u_\lambda||_{BV(R)}=||u||_{BV(R)}$ and $||u_\lambda||_{L^\infty(R)}=||u||_{L^\infty(R)}$ for all $\lambda>0$, one deduces from the definition of interpolation spaces that $||u_\lambda||_{Z(1/2,p)}=||u||_{Z(1/2,p)}$ for all $\lambda>0$; on the other hand the $L^2(R)$ norm of u_λ varies with λ in a different manner than the norm of $\frac{du_\lambda}{dx}$, so that (IV.23) applied to u_λ gives

$$||u||_{Z(1/2,p)}\le \frac{C}{\sqrt{\lambda}}||u||_{L^2(R)}+C\sqrt{\lambda}\Big|\Big|\frac{du}{dx}\Big|\Big|_{L^2(R)} \text{ for every } u\in H^1(R). \tag{IV.24}$$

Choosing the best λ gives

$$||u||_{Z(1/2,p)}\le C||u||^{\frac{1}{2}}_{L^2(R)}\Big|\Big|\frac{du}{dx}\Big|\Big|^{\frac{1}{2}}_{L^2(R)}, \tag{IV.25}$$

which implies that there exists a constant C such that

$$||u||_{Z(1/2,p)}\le C||u||_{Y(1,R)}. \tag{IV.26}$$

I do not know if (IV.23) holds for $p=2$, but it does hold for $p=\infty$. For proving this, it is enough to prove (IV.23) for $p=\infty$ and for nonnegative u, as for a general $u\in H^1(R)$ both u_- and u_+ belong to $H^1(R)$. For a nonnegative $u\in H^1(R)$ and $\alpha>0$, let v_α be defined by

$$v_\alpha(x)=(u(x)-\alpha)_+, \tag{IV.27}$$

so that

$$||u-v_\alpha||_{L^\infty(R)}\le\alpha, \tag{IV.28}$$

and $v_\alpha\in BV(R)$ as $v_\alpha\in H^1(R)$ and is 0 outside a set of finite measure $f(\alpha)$, with $f(\alpha)$ satisfying $f(\alpha)\alpha^2\le\int_R|u(x)|^2dx$, and therefore one obtains

$$||v_\alpha||_{BV(R)}\le\frac{1}{\alpha}\Big|\Big|\frac{du}{dx}\Big|\Big|_{L^2(R)}||u||_{L^2(R)}, \tag{IV.29}$$

and (IV.28)-(IV.29) give directly (IV.23) for $p=\infty$.

Remark 14: The preceding arguments apply to slightly larger spaces, modeled on the Lorentz space $L^{2,\infty}(R)$ instead of $L^2(R)$. Let

$$V_1=L^{2,\infty}(R) \text{ and } V_0=\Big\{u\in V_1,\frac{du}{dx}\in V_1\Big\}. \tag{IV.30}$$

Then one has

$$V(\frac{1}{2},1) \subset Z(\frac{1}{2},\infty) \subset V(\frac{1}{2},\infty). \tag{IV.31}$$

As $Y(\infty, R)$ is not embedded in $L^\infty(R)$, neither is the larger space $V(\frac{1}{2},\infty)$, but one has

$$V(\frac{1}{2},\infty) \subset BMO(R). \tag{IV.32}$$

On the other hand $V(\frac{1}{2},1)$ is larger than $Y(1,R)$ but still contains only continuous functions.

Remark 15: Unfortunately, the argument used in Theorem 10 does not extend to the case of more than one space variable. Indeed, one should then look at equations of the form

$$\frac{\partial w}{\partial t} + \sum_{j=1}^{N} a_j \frac{\partial w}{\partial x_j} - \varepsilon \Delta w = 0 \text{ a.e. in } R \times (0,T); \quad w(x,0) = w_0(x) \text{ a.e. in } R, \tag{IV.33}$$

for which the maximum principle does apply so that (IV.12) is satisfied, but the difficulty comes from the fact that the derivatives $\frac{\partial w}{\partial x_k}$ do not satisfy simple equations with good L^1 bounds.

So my search for discovering new functional spaces for hyperbolic problems, eventually defined as interpolation spaces using more classical ones, is not much advanced as the only result that I have obtained concerns a scalar equation in one space variable, while the main idea behind this search for new spaces was that one must think of systems, and in a realistic multidimensional setting.

V. Associating microlocal measures to bounded BV functions.

My more recent contribution to this list of remarks about interpolation spaces, was to use them in order to associate microlocal measures to BV functions.

There are many problems in Continuum Mechanics where interfaces occur, and often they do move. In conservation laws, for example, shocks can be created and it is not a completely understood question to decide which discontinuities are physically relevant; it is believed that only the value of the solution on both sides of the shock are relevant, and that an eventual internal structure of shock layers only helps understanding what is the right admissibility criterion for shocks. However, in some other problems, the geometry of an interface may play a role, for example when it moves in its normal direction with a velocity depending upon the mean curvature of the interface, or a more general function of its curvature tensor, as appears in some models in material science. Interfaces with small wrinkles may also appear in physical problems, corresponding to the concept of generalized surfaces that Laurence C. YOUNG introduced a long time ago, but instead of following his approach using a probability measure for the normal, I want to use tools which are more adapted to partial differential equations, like microlocal measures.

In approximating (IV.1) by the vanishing viscosity method (IV.2), one obtains an estimate

$$\sqrt{\varepsilon}\text{grad}(u^\varepsilon) \text{ bounded in } L^2\Big(0,T;L^2(R)\Big), \tag{V.1}$$

and

$$\varepsilon|\text{grad}(u^\varepsilon)|^2 \rightharpoonup \mu \text{ in the sense of distributions in } R \times (0,T), \tag{V.2}$$

and the nonnegative measure μ appears in the entropy condition, as the entropy solution u must satisfy

$$\frac{1}{2}\frac{\partial(u^2)}{\partial t} + \frac{\partial\Big(g(u)\Big)}{\partial x} + \mu = 0 \text{ in the sense of distributions in } R \times (0,T), \tag{V.3}$$

where $g(u) = \int_0^u z f'(z)dz$. The measure μ lives on the shock set of the solution u, but it does not know directly what is the velocity of any shock curve it lives on. If instead of a measure in (x,t) one constructs a microlocal measure in $\Big((x,t),(\xi,\tau)\Big)$, it might contain some geometrical information about its support. Let us consider for example a sequence of the form

$$u^\varepsilon(x,t) = U\Big(\frac{x-st}{\varepsilon}\Big), \tag{V.4}$$

where U is a traveling wave joining the constant states a and b – with $f(b) - f(a) = s(a-b)$ of course – satisfying $U' \in L^2(R)$. Then the measure μ is supported by the line of discontinuity $x = st$, and its constant strength corresponds to $\int_R |U'(\sigma)|^2 d\sigma$; on the other hand, the H-measure corresponding to $\sqrt{\varepsilon}\mathrm{grad}(u^\varepsilon)$ contains a little more information as it lives on $\tau + s\xi = 0$, so that it can read directly what the speed of propagation is.

Instead of following the approach of geometric measure theory, which starts from a function of bounded variation and then analyzes it in order to find the surfaces of discontinuity and their normals, the new idea consists in creating some microlocal measures associated to u, which would contain in a decoded way the geometrical information needed. Of course, the microlocal measures would be either an object without characteristic length like the H-measures which I have introduced [T2], identical to the microlocal defect measures introduced independently by Patrick GÉRARD [Gé1], or an object with a characteristic length like the semi-classical measures of P. GÉRARD [Gé2], identical to the Wigner measures of Pierre-Louis LIONS & Thierry PAUL [L'&Pa]; it could as well be another object that still has to be discovered.

My purpose here is to see how the space $Z(\infty, R^N)$, and the particular approach of our first subject of singular perturbations, plays a role in a similar question: associate some microlocal measures to any bounded BV function. What I do is actually to associate some H-measures to the space of functions $Z(\infty, R^N)$; with the same arguments used for one dimension in Remark 12, one has

$$||\tau_h v - v||_{L^1(R^N)} \le |h|.||\mathrm{grad}(v)||_{\mathcal{M}(\mathcal{R}^N)}; \quad ||\tau_h v - v||_{L^\infty(R^N)} \le 2||v||_{L^\infty(R^N)}, \tag{V.5}$$

and therefore one deduces that $Z(\frac{1}{2}, 2) \subset Z(\infty, R^N)$, i.e.

$$||v||_{Z(\infty,R^N)} \le C||v||_{Z(1/2,2)}. \tag{V.6}$$

For $u \in Z(\infty, R^N)$, we consider the equation

$$-\varepsilon^2 \Delta u_\varepsilon + u_\varepsilon = u \text{ in } R^N, \tag{V.7}$$

for which have the bounds

$$\varepsilon||\mathrm{grad}(u_\varepsilon)||_{L^2(R^N)} \le C||u||_{L^2(R^N)}, \tag{V.8}$$

$$||\mathrm{grad}(u_\varepsilon)||_{L^2(R^N)} \le C||\mathrm{grad}(u)||_{L^2(R^N)}, \tag{V.9}$$

from which one can deduce

$$\sqrt{\varepsilon}||\mathrm{grad}(u_\varepsilon)||_{L^2(R^N)} \le C||u||_{Z(\infty,R^N)}. \tag{V.10}$$

The preceding inequality can be proved by interpolation, based on the following decomposition: for $v \in Z(\infty, R^N)$ so that $||\tau_h v - v||_{L^2(R^N)} \le C\sqrt{|h|}$, and for $\rho \in D(R^N)$ with integral 1, one defines

$$v_\alpha(x) = \frac{1}{\alpha^N}\int_{R^N} \rho\Big(\frac{y}{\alpha}\Big) v(x-y)\,dy, \tag{V.11}$$

for $\alpha > 0$ so that

$$||v_\alpha - v||_{L^2(R^N)} \le C\sqrt{\alpha}||\frac{\partial v_\alpha}{\partial x_k}||_{L^2(R^N)} \le \frac{C}{\sqrt{\alpha}}. \tag{V.12}$$

If one defines the FOURIER transform $\mathbf{F}$ by

$$\mathbf{F}f(\xi) = \int_{R^N} f(x) e^{-2i\pi(x.\xi)}\,dx, \tag{V.13}$$

so that $\mathbf{F}$ induces an isometry on $L^2(R^N)$ whose inverse is obtained by changing i into $-i$, then the preceding decomposition shows that for $u \in Y(\infty, R^N)$ or $u \in Z(\infty, R^N)$, $Fu(\xi)$ belongs to an interpolation space and that there exists a constant C such that

$$\int_{r<|\xi|<2r} |\xi|.|\mathbf{F}u(\xi)|^2 d\xi \leq C, \text{ for every } r > 0. \tag{V.14}$$

The spaces $Y(\infty, R^N)$ or $Z(\infty, R^N)$ are then characterized as the spaces of functions in $L^2(R^N)$ or $L^\infty(R^N)$ satisfying (V.14) (notice that if $u \in Z(\infty, R^N)$ then $\mathbf{F}u$ may contain a Dirac mass at 0).

For $u \in Z(\infty, R^N)$, the estimate (V.10) implies that

$$\sqrt{\varepsilon}\,\mathrm{grad}(u_\varepsilon) \rightharpoonup 0 \text{ in } L^2(R^N) \text{ weak}, \tag{V.15}$$

and therefore, after eventual extraction of a subsequence, $\sqrt{\varepsilon}\,\mathrm{grad}(u_\varepsilon)$ defines a H-measure μ, which is a $N \times N$ hermitian nonnegative matrix of Radon measures on $R^N \times S^{N-1}$. Because the curl of $\sqrt{\varepsilon}\,\mathrm{grad}(u_\varepsilon)$ is 0, the localization principle implies that $\mu = (\xi \otimes \xi)\nu$ with ν being a nonnegative Radon measure on $R^N \times S^{N-1}$. Equation (V.7) gives $\mathbf{F}u_\varepsilon(\xi) = \frac{1}{1+4\varepsilon^2\pi^2|\xi|^2}\mathbf{F}u(\xi)$, so that

$$\mathbf{F}\Big(\sqrt{\varepsilon}\,\mathrm{grad}(u_\varepsilon)\Big)(x) = \sqrt{\varepsilon}\frac{2i\pi\xi}{1+4\varepsilon^2\pi^2|\xi|^2}\mathbf{F}u(\xi), \tag{V.16}$$

i.e. $\mathbf{F}(\sqrt{\varepsilon}\,\mathrm{grad}(u_\varepsilon))(\xi)$ has the form $f(\varepsilon\xi)\sqrt{|\xi|}\mathbf{F}u(\xi)$ with $f(z) = \frac{2i\pi\sqrt{z}}{1+4\pi^2|z|^2}$ so that in particular f is continuous in R^N, $O(\sqrt{|z|})$ near 0 and $O(\frac{1}{|z|\sqrt{|z|}})$ near infinity; the information (V.10) and (V.15) is then easily obtained as a consequence of (V.14) and the fact that f tends to 0 at 0 and at infinity. One can then derive a formula for the value of the measure ν on a test function $|\varphi(x)|^2\psi(x)$, where φ is a LIPSCHITZ function with compact support and $\psi \in C(S^{N-1})$: let $\eta \to 0$ be the subsequence extracted for defining the H-measure μ, then

$$\langle \nu, |\varphi|^2\psi\rangle = \lim_{\eta\to 0}\int_{R^N} \psi\Big(\frac{\xi}{|\xi|}\Big)4\pi^2\frac{\eta|\xi|}{(1+4\eta^2\pi^2|\xi|^2)^2}.|\mathbf{F}(\varphi u)(\xi)|^2 d\xi. \tag{V.17}$$

One can deduce from (V.17) something about the support of ν: if for a Lipschitz function φ with compact support, one has

$$\lim_{r\to\infty}\int_{r<|\xi|<2r} |\xi|.|\mathbf{F}(\varphi u)(\xi)|^2 d\xi = 0, \tag{V.18}$$

then $\langle \nu, |\varphi|^2\psi\rangle = 0$ for all $\psi \in C(S^{N-1})$ and therefore $\nu = 0$ in $\{x : \varphi(x) \neq 0\} \times S^{N-1}$. In particular, if $u \in BV(R^N)$ and is continuous on an open set ω, then $\nu = 0$ on $\omega \times S^{N-1}$; indeed let φ be a Lipschitz function with compact support in ω, then $\varphi u \in BV(R^N)$ and is uniformly continuous, so it satisfies

$$||\tau_h\varphi u - \varphi u||_{L^1(R^N)} \leq C|h|, \text{ and } ||\tau_h\varphi u - \varphi u||_{L^\infty(R^N)} \leq o(1), \tag{V.19}$$

and therefore

$$||\tau_h\varphi u - \varphi u||_{L^2(R^N)} \leq o(\sqrt{|h|}), \tag{V.20}$$

which does imply (V.18).

If u is the characteristic function of an open set ω with finite perimeter, and a subsequence corresponds to a H-measure μ characterized by a scalar nonnegative measure ν, then the support of ν is included in $\partial\omega \times S^{N-1}$. Of course, one can ask the same question for general sets of finite perimeter, or even for sets with infinite perimeter but such that u belongs to $Z(\infty, R^N)$. Another natural question is to decide under what hypothesis the measure ν will only see points of the form $(x, \pm n(x))$ with $x \in \partial\omega$ and $n(x)$ some kind of a normal to $\partial\omega$ at x. Obviously if the boundary is piecewise smooth, this will be the case because u will satisfy equations of the type

$$\sum_{j=1}^{N} a_j(x)\frac{\partial u}{\partial x_j} = 0, \text{ written as } \sum_{j=1}^{N} \frac{\partial(a_j(x)u)}{\partial x_j} - \Big(\mathrm{div}(a)\Big)u = 0, \tag{V.21}$$

with some functions a_j smooth enough for applying the localization principle and deduce

$$\Big(\sum_{j=1}^{N} a_j(x)\xi_j\Big)\nu = 0, \tag{V.22}$$

i.e. a_j continuous and $\mathrm{div}(a) \in L^\infty(R^N)$; if the functions a_j are C^1, then one can use the transport theorem to obtain more information. For example, if ω is the square $(0,1) \times (0,1)$, and one looks for information near the open edge $(0,1) \times \{0\}$ then the localization principle implies that the restriction π of ν to that edge is such that there exist nonnegative functions α, $\beta \in L^1(R^N)$ such that for every continuous function Φ one has

$$\langle \pi, \Phi\Big((x,y),(\xi,\eta)\Big)\rangle = \int_0^1 \Big[\alpha(x)\Phi\Big((x,0),(0,1)\Big) + \beta(x)\Phi\Big((x,0),(0,-1)\Big)\Big]\,dx, \tag{V.23}$$

while the transport theorem implies that α and β are constant; a more careful analysis of a one dimensional problem shows that u_ε looks like $\frac{1}{2} - \mathrm{sign}(x)e^{-\frac{|y|}{\varepsilon}}$ near the edge and one can deduce that $\alpha = \beta = \frac{1}{8}$. The localization principle or transport theorem for H-measures give no information about ν near the vertices, and one must look again at the precise behaviour of u_ε for determining what ν looks like near a vertex, and one finds that ν does not charge any of the vertices.

In general, I do not know if one can avoid the choice of a subsequence in the definition of ν. As the integral of ν with respect to ξ is obtained by taking $\varphi = 1$ in formula (V.17), extracting a subsequence might be necessary, but I do not know if that happens for $u \in Z(\infty, R^N)$ or $u \in BV(R^N) \cap L^\infty(R^N)$, as these spaces are not simply characterized through their FOURIER transform.

One advantage of the preceding procedure of attaching microlocal measures to functions in $BV(R^N) \cap L^\infty(R^N)$ is that it can be done for the larger space $Z(\infty, R^N)$, and another advantage is that one can do a similar analysis for sequences. If a sequence v_ε is bounded in $Z(\infty,\, R^N)$, then the sequence u_ε defined by

$$-\varepsilon^2 \Delta u_\varepsilon + u_\varepsilon = v_\varepsilon \text{ in } R^N, \tag{V.24}$$

for which one has the bounds

$$\sqrt{\varepsilon}\|\mathrm{grad}(u_\varepsilon)\|_{L^2(R^N)} \le C\|v_\varepsilon\|_{Z(\infty,R^N)} \le C, \tag{V.25}$$

and

$$\sqrt{\varepsilon}\,\mathrm{grad}(u_\varepsilon) \rightharpoonup 0 \text{ in } L^2(R^N) \text{ weak}, \tag{V.26}$$

so that one can extract a subsequence such that $\sqrt{\varepsilon}\,\mathrm{grad}(u_\varepsilon)$ defines a H-measure μ, which must be of the form $(\xi \otimes \xi)\nu$ by the localization principle. If the functions v_ε are initial data for an evolution problem where the characteristic length ε is used, the H-measure associated to the initial data or the solution of the evolution problem could then become interesting objects for understanding the evolution of the oscillations that one could put in the initial data.

Although I have only used the language of H-measures, the preceding analysis has made use of a characteristic length, and therefore one could also use the language of semi-classical/WIGNER measures. Certainly, the approach seems in defect for problems where many different scales occur, or problems where one does not really know what are the characteristic scales involved. More research should be done, of course, in order to decide if these ideas will help developing a new approach to problems usually attacked with the technical methods of geometric measure theory, for example for existence and regularity of solutions of some variational problems.

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