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Fractional Derivatives and Smoothing in Nonlinear Conservation Laws

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Let $\epsilon > 0$ be an arbitrary (small) positive number so that $\sigma(u)(t_*+\epsilon,\underline{x})$ is absolutely continuous and (1) holds for $t = t_* + \epsilon$ and almost every x. Invoke the estimates (34) and (35) to obtain

$$\sigma(u)_x(t_* + \epsilon, x) = -\int_{[0,t_*+\epsilon]} k(t-s)u_t(\mathrm{d}s, x) \leq -\delta k\big(\tau(x_* - x) + \epsilon\big), \quad x \in (0, x_*),$$

and so for $x \in (0, x_*)$,

$$\sigma(u)(t_* + \epsilon, x_*) - \sigma(u)(t_* + \epsilon, x) \leq -\delta \int_0^{x_*-x} k\big(\tau(y) + \epsilon\big)\,\mathrm{d}y.$$

Finally, let $\epsilon \downarrow 0$ and apply the monotone convergence theorem together with (29). The result is

$$\lim_{\epsilon\downarrow 0}\Big(\sigma(u)(t_* + \epsilon, x_*) - \sigma(u)(t_* + \epsilon, x)\Big) = -\infty,$$

which is impossible.

This contradiction shows that u cannot have any discontinuities and the proof of Theorem 5 is complete. □

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The second case is the one where

$$\limsup_{t\downarrow 0} \frac{k(t)}{p(t)} \le \frac{2}{3}. \tag{32}$$

In this case, there is a number $T \in (0,1)$ such that $k(t)/p(t) \le \frac{3}{4}$ when $t \in (0,T)$ and we have

$$\frac{k(t)(-p'(t))}{p(t)^2} = \frac{k(t)\big(p(t)-k(t)\big)}{tp(t)^2} \ge \frac{1}{4}\frac{k(t)}{\int_0^t k(s)\,ds}, \quad t \in (0,T).$$

Since $\lim_{t\downarrow 0}\int_0^t k(s)\,ds = 0$ we see again that (30) holds.

If neither (31) nor (32) holds, then there are infinitely many nonoverlapping intervals (s_j,t_j), $j=1,2,\dots$ contained in $(0,1)$ such that

$$\frac{k(t_j)}{p(t_j)} = \frac{2}{3} \ge \frac{k(t)}{p(t)} \ge \frac{k(s_j)}{p(s_j)} = \frac{1}{3}, \quad t \in (s_j,t_j), \quad j=1,2,\dots.$$

Since k is nonincreasing, it follows that we must have

$$\frac{p(s_j)}{p(t_j)} \ge 2, \quad j=1,2,\dots,$$

and therefore we get

$$\begin{aligned}\int_0^1 \frac{k(t)(-p'(t))}{p(t)^2}\,dt &\ge \frac{1}{3}\sum_{j=1}^{\infty}\int_{s_j}^{t_j}\frac{-p'(t)}{p(t)}\,dt \\ &= \frac{1}{3}\sum_{j=1}^{\infty}\ln\left(\frac{p(s_j)}{p(t_j)}\right) \ge \frac{1}{3}\sum_{j=1}^{\infty}\ln(2) = +\infty.\end{aligned}$$

This completes the proof. □

We now apply Lemma 7 with $\alpha = \sqrt{C}/2$. By (21) and by (28) we have

$$\frac{1}{\tau(\Delta x)}\int_0^{\tau(\Delta x)} k(s)\,ds = \frac{\sqrt{C}}{2\Delta x} \le \frac{1}{2\Delta t}\int_0^{2\Delta t} k(s)\,ds,$$

which implies that $\tau(\Delta x) \ge 2\Delta t$. In view of the monotonicity of u and the definition of Δt, we therefore have

$$u(t, x_* - \Delta x) \le u_+ - \delta, \quad t < t_* - \tau(\Delta x). \tag{33}$$

Now recall that $\Delta x \in (0, x_*)$ was arbitrary. Thus we may in fact write (33) as

$$u(t,x) \le u_+ - \delta, \quad 0 \le t \le \tau(x_* - x), \quad x \in (0,x_*). \tag{34}$$

On the other hand, from the monotonicity of u it follows that

$$u(t,x) \ge u_+, \quad t > t_* \quad x \in (0,x_*). \tag{35}$$

On the other hand we have, because u takes its values in $[0,1]$ and the support of $\rho_{\Delta t}$ is contained in $[0,\Delta t]$ that

$$
\begin{aligned}
&\sum_{j=0}^{m} f(t_0, x_0 + j\Delta x) \\
&\qquad = \int_{t_0}^{t_0+\Delta t} \Big((\sigma(u) * \rho_{\Delta t})(s, x_0 + (m+1)\Delta x) - (\sigma(u) * \rho_{\Delta t})(s, x_0) \Big)\,\mathrm{d}s \\
&\qquad \geq -\sigma(1)\rho_{\Delta t}([0,\Delta t])\Delta t.
\end{aligned}
$$

This gives the desired contradiction and we have established (20).

In order to exploit the fact, expressed by (21) that the level curves of u run almost parallel to the x-axis in the vicinity of (t_*, x_*) we need the following simple result.

Lemma 7. *Let $k \in L^1_{\mathrm{loc}}(\mathbb{R}^+;\mathbb{R})$ be nonnegative and nonincreasing on $(0,\infty)$ with $\lim_{t\downarrow 0} k(t) = +\infty$. Let $\alpha > 0$ be some constant and define the function $\tau : (0, \frac{\alpha}{k(\infty)}) \to (0,\infty)$ by*

$$
\frac{1}{\tau(\underline{x})} \int_0^{\tau(\underline{x})} k(s)\,\mathrm{d}s = \frac{\alpha}{\underline{x}}. \tag{28}
$$

Then

$$
\int_0^y k\big(\tau(x)\big)\,\mathrm{d}x = +\infty, \tag{29}
$$

when $y < \frac{\alpha}{k(\infty)}$.

Proof. Define the (strictly decreasing) function p by

$$
p(\underline{t}) = \frac{1}{\underline{t}} \int_0^{\underline{t}} k(s)\,\mathrm{d}s.
$$

In the integral in (29) we change variables by taking $t = \tau(x)$, or equivalently, $p(t) = \alpha/x$, so that $\mathrm{d}x = (-\alpha p'(t)/p(t)^2)\,\mathrm{d}t$, and we conclude that (29) holds if and only if we have

$$
\int_0^1 \frac{k(t)(-p'(t))}{p(t)^2}\,\mathrm{d}t = +\infty. \tag{30}
$$

Depending on the behavior of $k(t)/p(t)$ as t approaches 0, we distinguish three different cases. Suppose first that

$$
\liminf_{t\downarrow 0} \frac{k(t)}{p(t)} \geq \frac{1}{3}. \tag{31}
$$

In this case, there is a number $T \in (0,1)$ such that $k(t)/p(t) \geq \frac{1}{4}$ when $t \in (0,T)$ and we have

$$
\int_0^1 \frac{k(t)(-p'(t))}{p(t)^2}\,\mathrm{d}t \geq \int_0^T \frac{1}{4}\frac{-p'(t)}{p(t)}\,\mathrm{d}t = \frac{1}{4}\Big(\lim_{t\downarrow 0} \ln\big(p(t)\big) - \ln\big(p(T)\big)\Big) = +\infty.
$$

by (ii). Use (24) on the right hand side of (23) to obtain

$$f_x(t,x) = \int_x^{x+\Delta x} \Big(v(t+\Delta t, y) - v(t,y)\Big)\,dy, \quad t>0, \quad x>0, \tag{25}$$

where $v \stackrel{\text{def}}{=} k * \rho_{\Delta t} * h(k * u_t)$. From our assumptions on k it follows that $k * \rho_{\Delta t}$ is nonincreasing (on $(0,\infty)$), and that $k*\rho_{\Delta t}*k$ is nondecreasing on $(0,\Delta t]$. Therefore, by the fact that $u(\underline{t},x)$ is nondecreasing for each fixed x and satisfies $u \le 1$, we may apply Lemma 6 to get

$$f_x \le \frac{2\Delta x}{c_\sigma}(k * \rho_{\Delta t} * k)(\Delta t), \quad t>0, \quad x>0.$$

Hence, for $t>0$ and $0 \le x_1 \le x_2$,

$$f(t,x_2) - f(t,x_1) \le \frac{2\Delta x}{c_\sigma}(k * \rho_{\Delta t} * k)(\Delta t)(x_2 - x_1). \tag{26}$$

In order to estimate f, note that because u is monotone in both of its variables and continuous from the right in the first, we have

$$u(s,x_*) \le u_-, \quad t_0 < s < t_*,$$

and, using the definition of Δt,

$$u(s,x_0) \ge u_+ - \delta, \quad t_0 - \Delta t < s < t_*.$$

Therefore,

$$(\sigma(u) * \rho_{\Delta t})(s,x_*) - (\sigma(u) * \rho_{\Delta t})(s,x_0) \le \delta_\sigma \rho_{\Delta t}([0,\Delta t]), \quad t_0 < s < t_*,$$

where $\delta_\sigma \stackrel{\text{def}}{=} \sigma(u_+ - \delta) - \sigma(u_-)$. So, because $t_* = t_0 + \Delta t$ and $x_* = x_0 + \Delta x$, we get by (22)

$$f(t_0,x_0) \le -\delta_\sigma \rho_{\Delta t}([0,\Delta t])\Delta t. \tag{27}$$

From the relations (26) and (27) we arrive at (20) in the following manner. Assume that (20) does not hold and define the integer m by

$$m \stackrel{\text{def}}{=} \left\lfloor \frac{2\sigma(1)}{\delta_\sigma} \right\rfloor.$$

It follows from (26) (with $x_1 = x_0$ and $x_2 = x_0 + j\Delta x$) and from (27) that we have

$$f(t_0, x_0 + j\Delta x) \le -\tfrac{1}{2}\delta_\sigma \rho_{\Delta t}([0,\Delta t])\Delta t, \quad j = 0,1,2,\dots,m.$$

Add these inequalities to obtain

$$\sum_{j=0}^{m} f(t_0, x_0 + j\Delta x) < -\sigma(1)\Delta t \rho_{\Delta t}([0,\Delta t]).$$

that is

$$\rho_{\Delta t}([0,\Delta t]) \geq \frac{\Delta t}{\int_0^{\Delta t} k(s)\,\mathrm{d}s}.$$

Moreover, by the definition of $\rho_{\Delta t}$ we have $(k * k * \rho_{\Delta t})(\Delta t) = \int_0^{\Delta t} k(s)\,\mathrm{d}s$ and therefore inequality (20) says that

$$\frac{1}{\Delta t}\int_0^{\Delta t} k(s)\,\mathrm{d}s \geq \frac{\sqrt{C}}{\Delta x}. \tag{21}$$

The conclusion is thus that (20)—once established—will give us an upper bound for Δt in terms of Δx. In particular, observe that for unbounded kernels k we have that Δt is small compared to Δx, i.e., the level curves of u are close to being parallell to the x-axis near (t_*, x_*).

To establish (20), we define

$$f(t,x) \stackrel{\text{def}}{=} \int_t^{t+\Delta t} \Big((\sigma(u) * \rho_{\Delta t})(s, x+\Delta x) - (\sigma(u) * \rho_{\Delta t})(s,x) \Big)\,\mathrm{d}s, \quad t > 0, \quad x \geq 0. \tag{22}$$

(Note that if in (22) we have $\rho_{\Delta t} = \rho$, then $f(t,x) = -\int_t^{t+\Delta t}(u(s,x+\Delta x) - u(s,x))\,\mathrm{d}s$. However, for technical reasons we are forced to work with the cut-off resolvent $\rho_{\Delta t}$.)

By Theorem 4.(a), f is differentiable with respect to the second variable. This gives the first equality in the equation below. The second follows by Theorem 4.(c). To obtain the third, note that Theorem 4.(a) and the growth condition on σ in (ii) imply that for almost every t the function $u(t,\underline{x})$ is absolutely continuous.

$$\begin{aligned} f_x(t,x) &= \int_t^{t+\Delta t} \Big((\sigma(u)_x * \rho_{\Delta t})(s,x+\Delta x) - (\sigma(u)_x * \rho_{\Delta t})(s,x) \Big)\,\mathrm{d}s \\ &= -\int_t^{t+\Delta t} \Big(\frac{\partial}{\partial s}(k * \rho_{\Delta t} * u)(s,x+\Delta x) - \frac{\partial}{\partial s}(k * \rho_{\Delta t} * u)(s,x) \Big)\,\mathrm{d}s \\ &= -\int_x^{x+\Delta x} \Big((k * \rho_{\Delta t} * u_x)(t+\Delta t, y) - (k * \rho_{\Delta t} * u_x)(t,y) \Big)\,\mathrm{d}y. \end{aligned} \tag{23}$$

By (1) we have

$$u_x(t,x) = -h(t,x)\int_{[0,t]} k(t-s) u_t(\mathrm{d}s, x), \quad t > 0, \quad x > 0, \tag{24}$$

where

$$h(t,x) = \begin{cases} \dfrac{u_x(t,x)}{\sigma(u)_x(t,x)}, & \text{if } u_x(t,x) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that

$$\|h\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)} \leq \frac{1}{c_\sigma},$$

because we assumed that k_2 is nonincreasing. Since k_1 is nonincreasing the second term is nonpositive and the proof is completed. □

Proof of Theorem 5. First note that the solution u that exists according to Theorem 4 is such that $u(t,x) = 1$ when $t > 0$ and $x < 0$, and that since $u(t,\underline{x})$ belongs to $D(A)$ for almost every $t > 0$ it follows from the monotonicity properties that u is continuous in $\mathbb{R}^+ \times \mathbb{R}^- \setminus \{(0,0)\}$. Thus we may, without loss of generality, take u to be continuous from the right in the second variable, that is, $u(t,x) = \lim_{h\downarrow 0} u(t,x+h)$.

Assume now that u is discontinuous at some point (t_*,x_*) where $t_* > 0$ and $x_* > 0$. Let

$$u_+ \stackrel{\text{def}}{=} \limsup_{(t,x)\to(t_*,x_*)} u(t,x),$$
$$u_- \stackrel{\text{def}}{=} \liminf_{(t,x)\to(t_*,x_*)} u(t,x),$$
$$\delta \stackrel{\text{def}}{=} \frac{u_+ - u_-}{2}.$$

For every $\Delta x \in (0,x_*)$ we define the numbers x_0, t_0, and Δt by

$$\begin{aligned} x_0 &= x_* - \Delta x, \\ \Delta t &= \tfrac{1}{2}\Big(t_* - \sup\{\, s \in [0,t_*] \mid u(s,x_0) < u_+ - \delta \,\}\Big), \\ t_0 &= t_* - \Delta t. \end{aligned}$$

Observe that $t_* - 2\Delta t > 0$, because $x_0 > 0$ and $\lim_{t\downarrow 0} u(t,x_0) = 0$ by Theorem 4.(e).

Now let ρ be the resolvent of first kind of k, that is ρ satisfies the equation

$$\int_{[0,t]} k(t-s)\rho(\mathrm{d}s) = 1, \quad t > 0, \tag{19}$$

(which is the same as (6) when $\gamma = 0$) and define the restriction $\rho_{\Delta t}$ to $[0,\Delta t]$ by

$$\rho_{\Delta t}(E) = \rho(E \cap [0,\Delta t]),$$

for each Borel set $E \subset \mathbb{R}^+$. Recall that ρ, hence $\rho_{\Delta t}$, is a nonnegative measure.

We shall prove that

$$(\Delta x)^2 > C\frac{\rho_{\Delta t}([0,\Delta t])\Delta t}{(k * k * \rho_{\Delta t})(\Delta t)}, \tag{20}$$

where

$$C = \frac{c_\sigma\big(\sigma(u_+ - \delta) - \sigma(u_-)\big)^2}{8\sigma(1)},$$

and c_σ is the constant defined in (ii). To throw some light on (20), we observe that this inequality can be used to yield a more transparent estimate. From (19) we get after an integration that

$$\Delta t \le \int_0^{\Delta t} k(s)\,\mathrm{d}s\,\rho([0,\Delta t]),$$

Lemma 6. *Let $h > 0$, suppose $\psi \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathbb{R})$ is nonnegative, and let μ be a nonnegative finite measure on $\mathbb{R}^+$. Assume that k_1 and $k_2 \in L^1_{\mathrm{loc}}(\mathbb{R}^+;\mathbb{R})$ are nonnegative and nonincreasing on $(0,\infty)$ and such that the convolution $k_1 * k_2$ is nondecreasing on $(0,h]$. If*

$$v(t) \stackrel{\text{def}}{=} \int_0^t k_1(t-s)\psi(s)\int_{[0,s]} k_2(s-r)\mu(\mathrm{d}r)\,\mathrm{d}s, \quad t \geq 0,$$

then

$$v(t) - v(t-h) \leq 2\mu([0,t])\|\psi\|_{L^\infty([0,t])}(k_1 * k_2)(h), \quad 0 < h < t.$$

Proof. Let

$$A(t,r) \stackrel{\text{def}}{=} \int_r^t k_1(t-s)\psi(s)k_2(s-r)\,\mathrm{d}s, \quad 0 \leq r \leq t,$$

so that we have

$$v(t) = \int_{[0,t]} A(t,r)\mu(\mathrm{d}r), \quad t \geq 0,$$

and therefore

$$\begin{aligned} v(t) - v(t-h) &\leq \int_{(t-h,t]} A(t,r)\mu(\mathrm{d}r) + \int_{[0,t-h]} \big(A(t,r) - A(t-h,r)\big)\mu(\mathrm{d}r) \\ &\leq \mu([0,t])\Big(\sup_{r\in[t-h,t]} A(t,r) + \sup_{r\in[0,t-h]} \big(A(t,r) - A(t-h,r)\big)\Big), \quad 0 < h < t. \end{aligned}$$

Now it is clear that we have

$$\begin{aligned} \sup_{r\in[t-h,t]} A(t,r) &\leq \|\psi\|_{L^\infty([0,t])} \sup_{r\in[t-h,t]} \int_r^t k_1(t-s)k_2(s-r)\,\mathrm{d}s \\ &\leq \|\psi\|_{L^\infty([0,t])}(k_1 * k_2)(h), \quad 0 < h < t, \end{aligned}$$

since we assumed that $k_1 * k_2$ was nondecreasing on $[0,h]$. For the second term we use the decomposition

$$\begin{aligned} A(t,r) - A(t-h,r) &= \int_{t-h}^t k_1(t-s)\psi(s)k_2(s-r)\,\mathrm{d}s \\ &\quad + \int_r^{t-h} \big(k_1(t-s) - k_1(t-h-s)\big)\psi(s)k_2(s-r)\,\mathrm{d}s. \end{aligned}$$

The first term is easy to handle and we get

$$\begin{aligned} &\sup_{r\in[0,t-h]} \int_{t-h}^t k_1(t-s)\psi(s)k_2(s-r)\,\mathrm{d}s \\ &\qquad \leq \sup_{r\in[0,t-h]} \|\psi\|_{L^\infty([0,t])} \int_0^h k_1(h-s)k_2(t-r-h+s)\,\mathrm{d}s \\ &\qquad \leq \|\psi\|_{L^\infty([0,t])}(k_1 * k_2)(h), \end{aligned}$$

where μ_λ is a nonnegative Borel measure such that $\mu_\lambda(\mathbb{R}^+) \le 1$ and such that when we define k_λ by

$$k_\lambda(\underline{t}) = \frac{1}{\lambda}(1 - \mu_\lambda([0,\underline{t}])),$$

then we have $k_\lambda \to k$ in $L^1_{\mathrm{loc}}(\mathbb{R}^+; R)$ as $\lambda \downarrow 0$. (In [5], μ_λ was chosen so that the operator $\frac{\mathrm{d}}{\mathrm{d}t} k_\lambda *$ is the Yosida approximation of the operator $\frac{\mathrm{d}}{\mathrm{d}t} k*$, but it is easy, in this case where $\gamma = 0$, to see that it is sufficient that the properties mentioned above are satisfied.)

If the function $v(\underline{t}, \underline{x})$ is nonnegative, nondecreasing in its first and nonincreasing in its second variable, and $v(\underline{t}, 0) \le 1$, then the same holds true for the function

$$\int_{[0,\underline{t}]} v(\underline{t} - s, \underline{x}) \mu_\lambda(\mathrm{d}s).$$

Since equation (17) can be solved by iteration, it follows from Lemma 3 that the solution u_λ satisfies (d).

To obtain uniqueness for this extended solution we observe that if (1) is written in the form (7) (with r replaced by the measure resolvent ρ if necessary) we can conclude that the solutions we have to consider satisfy $u - \chi_{\mathbb{R}^-} \in L^1_{\mathrm{loc}}(\mathbb{R}^+; L^1(\mathbb{R}; \mathbb{R}))$. Let u and v be two such solutions and let $w = u - v$. By the same argument that was used in the proof of Lemma 3 we get

$$\int_{\mathbb{R}} \big(\sigma(u)_x(t,x) - \sigma(v)_x(t,x)\big) \operatorname{sign}\big(u(t,x) - v(t,x)\big)\, \mathrm{d}x = 0, \tag{18}$$

for almost every $t > 0$. On the other hand, since $w(0, \underline{x}) = 0$ we have for almost every $t > 0$

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{\partial}{\partial t} \int_0^t k(t-s) w(s,x)\, \mathrm{d}s \operatorname{sign}(w(t,x))\, \mathrm{d}x \\
&\quad = \int_{\mathbb{R}} k(t) w(t,x) \operatorname{sign}(w(t,x))\, \mathrm{d}x \\
&\qquad + \int_{\mathbb{R}} \int_{(0,t]} \big(w(t-s,x) - w(t,x)\big) k'(\mathrm{d}s) \operatorname{sign}(w(t,x))\, \mathrm{d}x \\
&\quad \ge k(t) \int_{\mathbb{R}} |w(t,x)|\, \mathrm{d}x + \int_{(0,t]} \Big(\int_{\mathbb{R}} |w(t-s,x)|\, \mathrm{d}x - \int_{\mathbb{R}} \big|w(t,x)\big|\, \mathrm{d}x \Big) k'(\mathrm{d}s) \\
&\quad = \frac{\partial}{\partial t} \int_0^t k(t-s) \int_{\mathbb{R}} \big|w(s,x)\big|\, \mathrm{d}x\, \mathrm{d}s.
\end{aligned}$$

We conclude from (1) and (18) that

$$\int_0^t k(t-s) \int_{\mathbb{R}} \big|u(s,x) - v(s,x)\big|\, \mathrm{d}x\, \mathrm{d}s \le 0, \quad t > 0.$$

It follows that we must have $u = v$ and we have established the uniqueness of the solution. $\square$

For the proof of Theorem 5 we need the following technical lemma.

Suppose next that $f \in L^1(\mathbb{R}^+;\mathbb{R})$ is such that $\sigma(f)$ has bounded variation. For technical reasons we assume for the moment that f is continuous and $f(0) = 1$. Let $\lambda > 0$ and write again $u = J_\lambda(f)$ so that we have $\sigma(u)' = A_\lambda(f)$ and (13).

Now the set $\{ x \in \mathbb{R}^+ \mid \sigma(u)'(x) \neq 0 \}$ can be written as a union $\cup_{j=1}^{J}(a_j, b_j)$ of nonoverlapping intervals where $J \le \infty$ and for all j we have that $\sigma(u)'$ is strictly positive or strictly negative on (a_j, b_j) and $\sigma(u)'(a_j) = \sigma(u)'(b_j) = 0$. (Here we use the assumption that $f(0) = 1$ so that $\sigma(u)'(0) = 0$.) If (a_j, b_j) is one such subinterval of $(0,\infty)$, then it follows by (13) that $f(a_j) = u(a_j)$ and $f(b_j) = u(b_j)$, or equivalently,

$$\int_{a_j}^{b_j} |\sigma(u)'(x)|\,\mathrm{d}x = \big|\sigma(u(b_j)) - \sigma(u(a_j)\big| = \big|\sigma(f(b_j)) - \sigma(f(a_j))\big|. \tag{15}$$

(Note that this result holds in the case where $b_j = \infty$ as well, because $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} u(x) = 0$.) Since the intervals are nonoverlapping, it follows from (15) that

$$\begin{aligned} \|A_\lambda(f)\|_{L^1(\mathbb{R}^+)} &= \sum_{j=1}^{J} \int_{a_j}^{b_j} |\sigma(u)'(x)|\,\mathrm{d}x \\ &= \sum_{j=1}^{J} \big|\sigma(f(b)) - \sigma(f(a))\big| \le \mathrm{Var}(\sigma(f);\mathbb{R}^+), \quad \lambda > 0. \end{aligned} \tag{16}$$

Every $f \in L^1(\mathbb{R}^+;\mathbb{R})$ such that $\sigma(f)$ has bounded variation can be approximated in $L^1(\mathbb{R}^+;\mathbb{R})$ by functions f_n that are continuous, satisfy $f_n(0) = 1$, and are such that

$$\mathrm{Var}(\sigma(f_n);\mathbb{R}^+) \le \big|\sigma(1) - \sigma(f(0))\big| + \mathrm{Var}(\sigma(f);\mathbb{R}^+).$$

Because A_λ is Lipschitz continuous we can therefore remove the extra regularity assumption and from (16) we conclude that

$$\|A_\lambda(f)\|_{L^1(\mathbb{R}^+)} \le \big|\sigma(1) - \sigma(f(0))\big| + \mathrm{Var}(\sigma(f);\mathbb{R}^+), \quad \lambda > 0.$$

This concludes the proof. □

Proof of Theorem 4. We see directly that $0 \in \hat{D}(A)$ and therefore it follows from [5, Thm. 2] that (9) with $\gamma = 0$ and $v = 0$ has a strong solution u, when A is defined as in Lemma 3. We can extend u as 1 on $\mathbb{R}^+ \times \mathbb{R}^-$ and then we immediately get (a) and (c). Once we have established (d) we get (b) as well.

If we can show that u is nondecreasing in its first and nonincreasing in its second variable, then the claim (e) follows from [5, Cor. 1].

The strong solution u is by the results in [5] obtained as the limit in $L^1_{\mathrm{loc}}(\mathbb{R}^+; L^1(\mathbb{R}^+;\mathbb{R}))$ as $\lambda \downarrow 0$ of a sequence of functions $u_\lambda : \mathbb{R}^+ \to L^1(\mathbb{R}^+;\mathbb{R})$ satisfying the equation

$$u_\lambda(t) = J_\lambda\Big(\int_{[0,t]} u_\lambda(t-s)\mu_\lambda(\mathrm{d}s)\Big), \quad t \ge 0, \tag{17}$$

where $J \le \infty$, $|c_j| = 1$ for all j, and where we choose the intervals (a_j, b_j) so that $u - v$ does not change sign on (a_j, b_j), and $u(a_j) - v(a_j) = u(b_j) - v(b_j) = 0$. Thus

$$\|u - v\|_{L^1(\mathbb{R}^+)} \le \|u - v + \lambda\big(A(u) - A(v)\big)\|_{L^1(\mathbb{R}^+)},$$

and so A is accretive.

It remains to prove that $R(I+\lambda A) = L^1(\mathbb{R}^+;\mathbb{R})$ for all $\lambda > 0$. Let $f \in C_c(\mathbb{R}^+;\mathbb{R})$ be a function with compact support and let $\lambda > 0$. Then there exists a function y that is a solution of the equation

$$\lambda y'(x) + \psi\big(y(x)\big) = f(x), \quad x \ge 0, \quad y(0) = \sigma(1),$$

and for large values of the argument y is a monotone function converging to 0. Thus it follows that y' and $\psi(y) \in L^1(\mathbb{R}^+;\mathbb{R})$ and if we let $u = \psi(y)$ we see that we have a solution of the equation

$$u + \lambda A(u) = f. \tag{12}$$

If now f is an arbitrary function in $L^1(\mathbb{R}^+;\mathbb{R})$ we can approximate it by functions $f_n \in C_c(\mathbb{R}^+;\mathbb{R})$ and we can find solutions u_n of the equation $u_n + \lambda A(u_n) = f_n$. The accretivity implies that

$$\|u_n - u_m\|_{L^1(\mathbb{R}^+;\mathbb{R})} \le \|f_n - f_m\|_{L^1(\mathbb{R}^+;\mathbb{R})},$$

and we see that there is a function $u \in L^1(\mathbb{R}^+;\mathbb{R})$ such that $u_n \to u$. But then the functions $A(u_n)$ converge as well, and since A is closed we conclude that u satisfies (12). Thus A is m-accretive.

Next let f and $g \in L^1(\mathbb{R}^+;\mathbb{R})$ be such that $f(\underline{x}) \overset{\text{a.e.}}{\le} g(\underline{x})$ and denote, for every $\lambda > 0$, $J_\lambda(f)$ and $J_\lambda(g)$ by u and v, respectively. Thus we know that

$$u(x) + \lambda\sigma(u)'(x) = f(x), \quad x \ge 0 \quad u(0) = 1, \tag{13}$$

and

$$v(x) + \lambda\sigma(v)'(x) = g(x), \quad x \ge 0 \quad v(0) = 1. \tag{14}$$

The functions u and v are continuous. Therefore, if $u(x) \le v(x)$ does not hold for all $x \in \mathbb{R}^+$, then there is a nonempty interval $(a, b) \subset (0, \infty)$ such that $u(a) = v(a)$ and $u(x) > v(x)$ for all $x \in (a, b)$. But by (13) and (14) this implies that $\sigma(u)'(x) - \sigma(v)'(x) < 0$, for almost every $x \in (a, b)$, and $\sigma(u)(a) = \sigma(v)(a)$. It follows that $\sigma(u)(x) - \sigma(v)(x) < 0$, for all $x \in (a, b)$ which cannot possibly hold because σ is increasing. From this contradiction we conclude that $J_\lambda(f)(\underline{x}) \le J_\lambda(g)(\underline{x})$ for every $\lambda > 0$.

Let $f \in L^1(\mathbb{R}^+;\mathbb{R})$ be nonnegative and nonincreasing with $f(0+) \le 1$. Let $\lambda > 0$ be arbitrary. Denoting again $J_\lambda(f)$ by u, we obtain equation (13). Using the same kind of argument as above, we easily conclude that we must have $u(\underline{x}) \ge f(\underline{x})$ and it follows from the equation that $\sigma(u)$ and hence u must be nonnegative and nonincreasing.

Then the solution of (1) *with* $u_0 = \chi_{\mathbb{R}^-}$ *that exists according to Theorem 4 is continuous in* $\mathbb{R}^+ \times \mathbb{R} \setminus \{(0,0)\}$.

Our proof of Theorem 5 relies on the monotonicity properties of u stated in (d) of Theorem 4. These properties are obviously not satisfied by solutions corresponding to arbitrary initial functions u_0. Thus an extension of Theorem 5 to include arbitrary u_0 (of bounded variation) is a nontrivial task. Moreover, we have not—sofar—established the absolute continuity of u. Consequently, the existence of acceleration waves is unsettled.

The formulation (1) may give a new way to approach the "entropy" solution of (2). For this approach to be of more general use, we clearly need to extend Theorem 5 to allow for any $u_0 \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$ and to examine whether the resulting solution satisfies some modified entropy inequality.

Let us finally remark that one may view the left side of (1) as the sum of two maximal monotone mappings, see [3]. However, for our purposes this approach does not appear productive.

3. Proofs.

Proof of Lemma 3. The definitions of $D(A)$ and A do not change if we replace $\sigma(\underline{u})$ by $\sigma(\underline{u}) - \sigma(0)$. Thus we may assume that $\sigma(0) = 0$.

Denote the inverse function of σ by ψ. Then it follows that

$$u(\underline{x}) = \psi\Big(\sigma(1) + \int_0^{\underline{x}} A(u)(s)\,\mathrm{d}s\Big), \quad u \in D(A), \tag{11}$$

and we claim on the basis of this equality that A is a closed operator. Suppose that $\sigma(\mathbb{R}) = (a,b)$ where $-\infty \le a < b \le \infty$ and that $u_n \to u$ and $A(u_n) \to w$ in $L^1(\mathbb{R}^+)$ as $n \to \infty$. For each $x \ge 0$ we know that the sequence $\sigma(1) + \int_0^{\underline{x}} A(u_n)(s)\,\mathrm{d}s$ converges and we observe that at those points x where the limit is a or b we must have $\lim_{n\to\infty}|u_n(x)| = \infty$. Thus the measure of these points is 0. At all other points we can invoke the continuity of ψ on (a,b) and conclude that

$$u(x) = \psi\Big(\sigma(1) + \int_0^{\underline{x}} w(s)\,\mathrm{d}s\Big).$$

But this means that we have

$$\int_0^{\underline{x}} w(s)\,\mathrm{d}s \overset{\text{a.e.}}{=} \sigma\big(u(\underline{x})\big) - \sigma(1),$$

and this implies that (when we modify u on a set with measure 0) $u \in D(A)$ and $A(u) = w$. Thus A is closed.

If u and $v \in D(A)$, then $\sigma(u)$ and $\sigma(v)$ are continuous and converge toward 0 at infinity. If we define $\operatorname{sign}(s) = 1$ when $s > 0$, $\operatorname{sign}(s) = -1$ when $s < 0$ and $\operatorname{sign}(0) = 0$, then we can write

$$\int_0^\infty (\sigma(u)'(x) - \sigma(v)'(x))\operatorname{sign}\big(u(x) - v(x)\big)\,\mathrm{d}x$$
$$= \sum_{j=1}^{J} \int_{a_j}^{b_j} (\sigma(u)'(x) - \sigma(v)'(x))(-1)^{c_j}\,\mathrm{d}x = 0,$$

Lemma 3. *Assume that $\sigma \in C(\mathbb{R};\mathbb{R})$ is strictly increasing. Let*

$$D(A) = \{\, u \in L^1(\mathbb{R}^+;\mathbb{R}) \mid \sigma(u) \in AC(\mathbb{R}^+;\mathbb{R}),\ u(0) = 1,\ \sigma(u)' \in L^1(\mathbb{R}^+;\mathbb{R}) \,\},$$

and

$$A(u) = \sigma(u)', \quad u \in D(A).$$

Then A is a closed, m-accretive operator in $L^1(\mathbb{R}^+;\mathbb{R})$.
Moreover,

(a) *if f and $g \in L^1(\mathbb{R}^+;\mathbb{R})$ and $f(\underline{x}) \overset{\text{a.e.}}{\le} g(\underline{x})$ then $J_\lambda(f)(\underline{x}) \le J_\lambda(g)(\underline{x})$ for every $\lambda > 0$,*
(b) *if $f \in L^1(\mathbb{R}^+;\mathbb{R})$ is nonnegative and nonincreasing with $f(0+) \le 1$, then $J_\lambda(f)$ is nonnegative and nonincreasing, for every $\lambda > 0$,*
(c) *every $f \in L^1(\mathbb{R}^+;\mathbb{R})$ with the property that $\sigma(f) \in BV(\mathbb{R}^+;\mathbb{R})$, belongs to $\hat{D}(A)$.*

This is essentially the same result as [1, Thm. 3.1], but the fact that we consider integrable functions on $\mathbb{R}^+$, and not on $\mathbb{R}$, forces us to assume that σ is strictly increasing and not merely strictly monotone. For completeness, we give a proof below.

With Lemma 3 at hand we may apply [5, Thm. 2] (with $X = L^1(\mathbb{R}^+;\mathbb{R})$, A as in Lemma 3, $\gamma = 0$, and $v = 0$) to (9). This gives the existence of a unique (see [5, Thm. 1]) strong solution of (10). It is possible to show that this solution has some desirable monotonicity properties and that one can extend it to a solution of (1). Thus we get the following result.

Theorem 4. *Assume that*

(i) *$k \in L^1_{\text{loc}}(\mathbb{R}^+;\mathbb{R})$ is nonnegative and nonincreasing with $k(0+) = \infty$,*
(ii) *$\sigma \in C(\mathbb{R};\mathbb{R})$ is strictly increasing.*

Then there exists a solution u of (1) with $u_0 = \chi_{\mathbb{R}^-}$ such that

(a) *$\underline{x} \mapsto \sigma(u)(t,\underline{x})$ is absolutely continuous in $\mathbb{R}$ for almost every $t > 0$,*
(b) *$\underline{t} \mapsto \int_0^t k(\underline{t}-s)(u(s,x)-\chi_{\mathbb{R}^-}(x))\,ds$ is absolutely continuous in $\mathbb{R}^+$ for every $x \in \mathbb{R}$,*
(c) *equation (1) holds almost everywhere in $\mathbb{R}^+ \times \mathbb{R}$,*
(d) *$\underline{t} \mapsto u(\underline{t},x)$ is nondecreasing for each $x \in \mathbb{R}$ and $\underline{x} \mapsto u(t,\underline{x})$ is nonincreasing for each $t > 0$,*
(e) *$\lim_{t\downarrow 0} u(t,x) = 0$ for every $x > 0$ and $u(t,x) = 1$ for every $t \ge 0$ and $x \le 0$.*

Moreover, the solution is unique among all solutions that are such that $\sigma(u)_x \in L^1_{\text{loc}}(\mathbb{R}^+;L^1(\mathbb{R};\mathbb{R}))$ and $u - \chi_{\mathbb{R}^-} \in BV_{\text{loc}}(\mathbb{R}^+;L^1(\mathbb{R};\mathbb{R}))$.

In the final step, we show that the solution given by Theorem 4 is continuous. For this result we are forced to assume that the rate of increase of σ is bounded away from 0. The proof of this step requires some rather detailed estimates.

Theorem 5. *Assume that*

(i) *$k \in L^1_{\text{loc}}(\mathbb{R}^+;\mathbb{R})$ is nonnegative and nonincreasing with $k(0+) = \infty$,*
(ii) *$\sigma \in C(\mathbb{R};\mathbb{R})$ and there is a constant $c_\sigma > 0$ such that $\sigma(v)-\sigma(w) \ge c_\sigma(v-w)$ when $v \ge w$.*

Theorem 2. *Let $k \in L^1_{\mathrm{loc}}(\mathbb{R}^+;\mathbb{R})$ be nonnegative and nonincreasing on $(0,\infty)$ with $k(0+)=\infty$. Then the solution u of the problem*

$$\frac{\partial}{\partial t}\int_0^t k(t-s)\big(u(s,x)-\chi_{\mathbb{R}^-}(x)\big)\,\mathrm{d}s + u_x(t,x) = 0, \quad t \ge 0, \quad x \in \mathbb{R}, \tag{8}$$

has the following properties.

(a) *u is continuous on $\mathbb{R}^+ \times \mathbb{R} \setminus \{(0,0)\}$,*
(b) *$\underline{t} \mapsto u(\underline{t},x)$ is absolutely continuous in $\mathbb{R}^+$ for each $x \in \mathbb{R}$,*
(c) *$\underline{x} \mapsto u(t,\underline{x})$ is absolutely continuous in $\mathbb{R}$ for each $t > 0$,*
(d) *$\underline{t} \mapsto u(\underline{t},x)$ is nondecreasing for each $x \in \mathbb{R}$ and $\underline{x} \mapsto u(t,\underline{x})$ is nonincreasing for each $t > 0$,*
(e) *$u(t,x) > 0$ when $t > 0$ and $x \in \mathbb{R}$, that is, the speed of propagation is infinite.*

In [9] the equation studied is formulated as a linear version of (7) and the initial condition on $\mathbb{R}^-$ is replaced by a boundary condition on the line $x = 0$, but in our opinion, it is more natural to work with (1) than with (7), both in the linear and the nonlinear case.

Equation (1) (and (5)) may be viewed as a particular case of the (abstract) equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma(u(t)-v) + \int_0^t k(t-s)\big(u(s)-v\big)\,\mathrm{d}s\right) + A(u(t)) = 0, \quad t \ge 0, \tag{9}$$

considered in [5]. This is the approach taken below in our first result. In (9), the constant γ is nonnegative, $u : \mathbb{R}^+ \to X$ with X a real Banach space, A is an m-accretive operator in X, and v is the initial value.

2. Statement of results.

It turns out that if $u_0 = \chi_{\mathbb{R}^-}$ and σ is increasing, then one can replace (1) where $x \in \mathbb{R}$ by the equation

$$\begin{aligned}&\frac{\partial}{\partial t}\int_0^t k(t-s)\big(u(s,x)-u_0(x)\big)\,\mathrm{d}s + \sigma(u)_x(t,x) = 0, \quad t \ge 0, \quad x > 0,\\ &u(t,0) = 1, \quad t > 0, \qquad u(0,x) = 0, \quad x > 0.\end{aligned} \tag{10}$$

To analyze (10) we first establish that $u \mapsto \sigma(u)_x$, with a domain taking into account the boundary condition in (10) defines an m-accretive operator in $L^1(\mathbb{R}^+;\mathbb{R})$.

Recall that A is an m-accretive operator in a Banach space X, with norm $\|\bullet\|$, provided

$$\|u-v\| \le \|u-v+\lambda\big(A(u)-A(v)\big)\|, \quad \lambda > 0,$$

for all u, $v \in D(A)$ and provided $R(I+\lambda A) = X$ for $\lambda > 0$. (In general, A can be a multivalued operator and the definition has to be interpreted in the way that $A(u)$ stands for an arbitrary element of $A(u)$.) The Yosida approximation is then defined to be $A_\lambda = \frac{1}{\lambda}(I - J_\lambda)$ where $J_\lambda = (I+\lambda A)^{-1}$. The operator A_λ is a Lipschitz continuous operator. We denote by $\hat{D}(A)$ the set $\{\, u \in X \mid \sup_{\lambda>0}\|A_\lambda(u)\| < \infty \,\}$.

one does have locally existing smooth solutions and globally existing weak solutions. Moreover, it is proved that under appropriate conditions on the initial data, the time derivative of a smooth solution blows up in finite time. The structure of possible shocks is analyzed and the existence of shocks for the case where $a(\underline{t})$ is a decaying exponential is demonstrated. Moreover, it is shown that the integral term causes the shock strength to decay exponentially to zero in time.

In [8], the more general (different nonlinearities) equation

$$u_t + \sigma(u)_x + a * \psi(u)_x = 0, \tag{4}$$

is examined. It is shown that first derivatives of locally existing smooth solutions may blow up in finite time.

In [2], the existence of global weak L^∞-solutions of (4) is proved. This is done by the method of compensated compactness. In both [8] and [2], $a \in C^1(\mathbb{R}^+)$.

In general the effort in these papers goes toward demonstrating that solutions of (3) and (4) retain the features of solutions of (2). This is contrary to our philosophy as regards the analysis of (1).

To see how (1) is related to equations (3) and (4), consider the equation

$$\gamma u_t + k * u_t + \sigma(u)_x = 0, \tag{5}$$

where $\gamma \geq 0$, together with the initial condition $u(0, \underline{x}) = u_0(\underline{x})$. If $\gamma = 0$, then this is the same equation as (1). We claim that (5) with $\gamma = 1$ corresponds to (3), or to (4) provided $\sigma = \psi$. To see that this claim holds, let r be the resolvent of the first kind of $\gamma\delta_0 + k$, i.e., the solution of the equation

$$\gamma r(t) + \int_0^t k(t-s)r(s)\,\mathrm{d}s = 1, \quad t > 0. \tag{6}$$

If $\gamma > 0$, then it is clear that such a solution exists. If $\gamma = 0$ and k is nonnegative and nonincreasing, then one can use the following lemma, [6, Thm. 5.5.5], to conclude that the resolvent exists.

Lemma 1. *Let $k \in L^1_{\mathrm{loc}}(\mathbb{R}^+; \mathbb{R})$ be nonnegative and nonincreasing on $(0, \infty)$ and not identically 0. Then k has a nonnegative (measure) resolvent of the first kind. This resolvent has no discrete part iff $k(0+) = \infty$.*

Note that even if $k(0+) = \infty$ the resolvent ρ given by this lemma need not be induced by a locally integrable function, see [4]. Thus, if $\gamma = 0$ then (6) in general reads $\int_{[0,t]} k(t-s)\rho(\mathrm{d}s) = 1$, for $t > 0$. Also observe that in our model case, where $\gamma = 0$ and $k(\underline{t}) = \underline{t}^{-\alpha}$, for some $\alpha \in (0,1)$, one has $r(\underline{t}) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\underline{t}^{\alpha-1}$.

A combination of (5) and (6) yields

$$u + r * \sigma(u)_x = u_0, \tag{7}$$

which is thus equivalent to (5) for any $\gamma \geq 0$. But equation (7) is nothing but (4) with $\sigma = \psi$, provided $r(0) = 1$ and $r' = a$. On the other hand, $r(0) = 1/\gamma$ and so the claim follows.

The Riemann problem for the linear version of (7) has been examined in detail in [9]. In particular, the following theorem, which corresponds to the case $\gamma = 0$ considered in this paper, may be extracted from this work, see [9, Thm. 2, p. 324].

FRACTIONAL DERIVATIVES AND SMOOTHING IN NONLINEAR CONSERVATION LAWS

GUSTAF GRIPENBERG AND STIG-OLOF LONDEN

ABSTRACT. It is shown that the solution of the Riemann problem

$$\frac{\partial}{\partial t}\int_0^t k(t-s)(u(s,x)-u_0(x))\,\mathrm{d}s+(\sigma(u))_x(t,x)=0,$$

where $u_0=\chi_{\mathbb{R}^-}$, is continuous when $t>0$. Here k is locally integrable, nonnegative, and nonincreasing on $\mathbb{R}^+$ with $k(0+)=\infty$.

1. INTRODUCTION.

This work treats the Riemann problem for the nonlinear scalar equation

$$\frac{\partial}{\partial t}\int_0^t k(t-s)\big(u(s,x)-u_0(x)\big)\,\mathrm{d}s+\sigma(u)_x(t,x)=0,\quad t>0,\quad x\in\mathbb{R}. \tag{1}$$

We take $k\in L^1_{\mathrm{loc}}(\mathbb{R}^+;\mathbb{R})$ to be nonnegative and nonincreasing on $(0,\infty)$ with $k(0+)=\infty$. In particular, we include the case where $\frac{\partial}{\partial t}(k*u)$ represents the fractional derivative of u, i.e., $k(\underline{t})=\underline{t}^{-\alpha}$ where $0<\alpha<1$. (Here $*$ denotes convolution with respect to the first variable.) Note that the formulation of (1) includes the initial condition $u(0,\underline{x})=u_0(\underline{x})$. We show, under weak assumptions on σ, that the solution u of (1) is continuous.

If $k(\underline{t})\,\mathrm{d}\underline{t}$ is replaced by the unit point mass $\delta_0(\mathrm{d}\underline{t})$ at the origin, then (1) reduces to the nonlinear conservation law

$$u_t+\sigma(u)_x=0. \tag{2}$$

It is well known that solutions u of (2) in general are discontinuous, i.e., they exhibit shocks. On the other hand, (2) does have weak solutions existing globally. Our result shows that if, in (2), u_t is replaced by $\frac{\partial^\alpha u}{\partial t^\alpha}$, where $0<\alpha<1$, then the solutions of the Riemann problem are continuous.

Several papers analyze (2) with an integral term added. In [7], the equation

$$u_t+\sigma(u)_x+a*\sigma(u)_x=0, \tag{3}$$

with $u(t,x)$ for $t\le 0$ given (and the convolution taken over $\mathbb{R}$ with $a(t)=0$ when $t<0$), is considered under the assumption that $a\in C^1(\mathbb{R}^+;\mathbb{R})$. It is shown that

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