NAMT
93- 030

Relaxation of Bulk and Interfacial Energies

Ana Cristina Barosso
Irene Fonseca
Carnegie Mellon University
Guy Bouchitte
Universite de Toulon et du Var
Giuseppe Buttazzo
Universita di Pisa

Research Report No. 93-NA-030

September 1993

Sponsors

U.S. Army Research Office
Research Triangle Park
NC 27709

National Science Foundation
1800 G Street, N.W.
Washington, DC 20550

Relaxation of Bulk and Interfacial Energies

Ana Cristina Barroso
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

Guy Bouchitté
Départment de Mathématiques
Université de Toulon et du Var
BP132, 83957 La Garde Cedex, France

Giuseppe Buttazzo
Dipartimento di Matematica
Università di Pisa
Via Buonarroti, 2
56127, Pisa, Italy

Irene Fonseca
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

March 1993

Abstract

In this paper we obtain an integral representation for the relaxation in $BV(\Omega;\mathbf{R}^p)$ of the functional

$$u \mapsto \int_{\Omega} f(x, \nabla u(x))dx + \int_{\Sigma(u)} \varphi(x, [u](x), \nu(x))dH_{N-1}(x)$$

with respect to the BV weak topology.

Contents

1 Introduction

In this paper we study the relaxation $\mathcal{F}(\cdot)$ with respect to the BV weak topology of the functional defined in $SBV(\Omega;\mathbf{R}^p)$ by

$$u \mapsto E(u) := \int_\Omega f(x,\nabla u(x))dx + \int_{\Sigma(u)} \varphi(x,[u](x),\nu(x))dH_{N-1}(x) \tag{1.1}$$

where $[u](x)$ denotes the jump of u at x, $\Sigma(u)$ is the jump set of u and the distributional derivative Du is represented by $Du = \nabla u dx + [u]\otimes\nu dH_{N-1}\lfloor\Sigma(u) + C(u)$. Assuming that $f(x,\cdot)$ is quasiconvex and has linear growth and that φ grows at most linearly in the second argument, under some technical continuity conditions (see Section 2) we obtain the integral representation

$$\mathcal{F}(u) = \int_\Omega g(x,\nabla u(x))dx + \int_{\Sigma(u)} h(x,[u](x),\nu(x))dH_{N-1}(x) + \int_\Omega g^\infty(x,dC(u)) \tag{1.2}$$

where g is the quasiconvexification of the inf-convolution of f and φ_0 given by

$$f\nabla\varphi_0(x,A) := \inf\{f(x,A-a\otimes b)+\varphi_0(x,a,b) : a\in\mathbf{R}^p, b\in\mathbf{R}^N\}$$

and h is given by

$$h(x_0,\xi,\nu) := \inf\left\{\int_{Q_\nu} f^\infty(x_0,\nabla u(x))dx + \int_{\Sigma(u)\cap Q_\nu} \varphi(x_0,[u](x),\nu(x))dH_{N-1}(x) : u\in\mathcal{A}(\xi,\nu)\right\},$$

$$\begin{aligned}\mathcal{A}(\xi,\nu) := \Big\{ & v\in SBV_{loc}(S_\nu;\mathbf{R}^p) : v(y)=0 \text{ if } y\cdot\nu = -\frac{1}{2},\ v(y)=\xi \text{ if } y\cdot\nu=\frac{1}{2} \\ & \text{and } v \text{ is periodic with period one in the directions of } \nu_1,\dots,\nu_{N-1}\Big\},\end{aligned}$$

$\{\nu_1,\dots,\nu_{N-1},\nu\}$ forms an orthonormal basis of $\mathbf{R}^N$ and S_ν is the strip

$$S_\nu = \left\{y\in\mathbf{R}^N : |y\cdot\nu| < \frac{1}{2}\right\}.$$

In the above f^∞ (resp. g^∞) denotes the recession function of f (resp. g) given by

$$f^\infty(x,A) := \limsup_{t\to+\infty}\frac{f(x,tA)}{t}$$

and φ_0 is the positively homogeneous of degree one function defined by

$$\varphi_0(x,\xi,\nu) = \lim_{t\to 0^+}\frac{\varphi(x,t\xi,\nu)}{t}.$$

The lack of a coercivity hypotheses on f forces us to work within the framework of the BV weak topology. However, if one assumes that there exist constants C, C_1 such that

$$C\|A\| - C_1 \le f(x,A)$$

then it is possible to relax with respect to the L^1 topology.

Functionals of the form (1.1) model many problems in Mathematical Physics, for example variational problems for phase transitions, where the function spaces involved should allow discontinuous, vector-valued functions u. Minima of the functional $E(\cdot)$, when $E(\cdot)$ is lower semicontinuous, are obtained via the Direct Methods of the Calculus of Variations. However, for nonconvex problems $E(\cdot)$ is not lower semicontinuous and so, to obtain the effective energy of the system, one studies the relaxed functional

$$\bar{F}(u) = \sup\{G(u) : G \text{ is l.s.c., } G \leq E\}.$$

Since the BV weak topology is not metrizable it turns out (see [12]) that

$$\bar{F}(u) = \inf\{\liminf_{i\in I} E(u_i) : (u_i)_{i\in I} \text{ is a net converging to } u \text{ in the } BV \text{ weak topology}\}.$$

We will restrict ourselves to finding an integral representation for

$$\mathcal{F}(u) = \inf\{\liminf_{n\to+\infty} E(u_n) : u_n \to u \text{ in } BV \text{ weak}\}$$

since the effective energy can always be obtained by using minimizing sequences.

When no surface energy term is present and assuming an explicit dependence of f on u, Fonseca and Müller [19] (see also [5]) obtained the following representation for the relaxation $\mathcal{F}(\cdot)$ in BV of the functional

$$u \mapsto \int_\Omega f(x, u(x), \nabla u(x))dx \tag{1.3}$$

with respect to the L^1 topology:

$$\begin{aligned}\mathcal{F}(u) &= \int_\Omega f(x,u(x),\nabla u(x))dx + \int_{\Sigma(u)} K_f(x,u^-(x),u^+(x),\nu(x))dH_{N-1}(x) + \\ &+ \int_\Omega f^\infty(x,u(x),dC(u)).\end{aligned}$$

Their surface energy density $K_f : \Omega \times \mathbf{R}^p \times \mathbf{R}^p \times S^{N-1} \to [0,+\infty)$ is given by

$$K_f(x,a,b,\nu) = \inf\left\{\int_{Q_\nu} f^\infty(x,v(y),\nabla v(y))dy : v \in \mathcal{A}(a,b,\nu)\right\}$$

where

$$\begin{aligned}\mathcal{A}(a,b,\nu) := \Big\{&v \in SBV_{loc}(S_\nu;\mathbf{R}^p) : v(y) = a \text{ if } y\cdot\nu = -\frac{1}{2},\ v(y) = b \text{ if } y\cdot\nu = \frac{1}{2} \\ &\text{and } v \text{ is periodic with period one in the directions of } \nu_1,\ldots,\nu_{N-1}\Big\}\end{aligned}$$

and if f does not depend explicitly on u it turns out that $K_f(x,a,b,\nu) = f^\infty(x,(b-a)\otimes\nu)$ (see [5] and [19], Remark 2.17).

Ambrosio and Braides [3], [4] obtained lower semicontinuity and relaxation results for the functional (1.1) in the scalar case and assuming that $\varphi \geq \alpha > 0$ and that f has superlinear growth at infinity. A similar problem was studied by Bouchitté, Braides and Buttazzo [10] in the isotropic (radial), scalar case.The lower semicontinuity of the functional (1.1) in the space $SBV(\Omega;\mathbf{R}^p)$, under the above assumptions, was generalized by Ambrosio [2] to the vector-valued case.

We recently became aware of a result of Braides and Coscia [11] providing an integral representation of the relaxation with respect to the L^1 topology of the functional

$$u \mapsto \int_\Omega f(\nabla u(x))dx + \int_{\Sigma(u)} \varphi([u](x) \otimes \nu(x))dH_{N-1}(x).$$

No growth condition on f is assumed, however φ is required to be positively homogeneous of degree one and locally bounded in N independent directions of $\mathbf{R}^N$.

We organize the paper as follows; in Section 2 we mention some results on functions of bounded variation and we state our main results (see Theorems 2.13 and 2.14). In Section 3 we prove a slicing lemma that allows us to modify a sequence near the boundary without increasing its total energy, as well as some properties of the density functions g and h which will be of later use in the paper. In Section 4, using the blow-up method introduced by Fonseca and Müller in [18], we obtain a lower bound for the relaxation $\mathcal{F}(u)$ and in Section 5 an upper bound is obtained in the case where $u \in SBV(\Omega;\mathbf{R}^p)$. Assuming that φ is positively homogeneous of degree one, in Section 6 we extend our previous results to arbitrary BV functions and we find an explicit formula for the function h, namely

$$h(x,\xi,\nu) = g^\infty(x,\xi \otimes \nu).$$

Finally in Section 7 we complete the proof of Theorem 2.13 by removing the requirement that φ be positively homogeneous of degree one.

We remark that when $\varphi = \varphi_0$, i.e. in the homogeneous case, the proof of Theorem 2.13 is simplified since we do not need the blow-up of the Cantor part to relax in BV.

2 Preliminaries. Statement of the Theorem

In what follows $\Omega \subset \mathbf{R}^N$ is an open, bounded set, $p, N \geq 1, \{e_1, \ldots, e_N\}$ is the standard orthonormal basis of $\mathbf{R}^N$ and $M^{p\times N}$ is the vector space of all $p \times N$ real matrices endowed with the norm $||A|| := (tr(A^T A))^{\frac{1}{2}}$.

Given $\nu \in S^{N-1} := \{x \in \mathbf{R}^N : ||x|| = 1\}$ we denote by Q_ν an open unit cube centered at the origin with two of its faces normal to ν, i.e.

$$Q_\nu := \left\{x \in \mathbf{R}^N : |x \cdot \nu_i| < \frac{1}{2}, |x \cdot \nu| < \frac{1}{2}, i = 1, \ldots, N-1\right\}$$

for some orthonormal basis of $\mathbf{R}^N$ $\{\nu_1, \nu_2, \ldots, \nu_{N-1}, \nu\}$.

We briefly recall some facts on functions of bounded variation which will be of later use in this paper. For more details we refer the reader to Ambrosio, Mortola and Tortorelli [7], Evans and Gariepy [15], Federer [16], Giusti [21] and Ziemer [25].

Definition 2.1 *A function $u \in L^1(\Omega;\mathbf{R}^p)$ is said to be of* bounded variation, $u \in BV(\Omega;\mathbf{R}^p)$, *if for all $i \in \{1, \ldots, p\}$, $j \in \{1, \ldots, N\}$ there exists a Radon measure μ_{ij} such that*

$$\int_\Omega u_i(x) \frac{\partial \phi}{\partial x_j}(x)\, dx = -\int_\Omega \phi(x)\, d\mu_{ij}$$

for every $\phi \in C_0^1(\Omega)$. The distributional derivative Du is the matrix-valued measure with components μ_{ij}.

Definition 2.2 *A set $A \subset \Omega$ is said to be of* finite perimeter *in Ω if $\chi_A \in BV(\Omega)$, where χ_A denotes the characteristic function of A. The* perimeter *of A in Ω is defined by*

$$Per_\Omega(A) := \sup\left\{\int_A div\phi(x)dx : \phi \in C_0^1(\Omega;\mathbf{R}^N),\ ||\phi||_\infty \le 1\right\}.$$

For $u \in BV(\Omega;\mathbf{R}^p)$ the *approximate upper and lower limit* of each component u_i, for all $i \in \{1,\dots,p\}$, are given by

$$u_i^+(x) := inf\left\{t \in \mathbf{R} : \lim_{\epsilon\to0^+} \frac{1}{\epsilon^N}\mathcal{L}_N\left[\{u_i > t\} \cap B(x,\epsilon)\right] = 0\right\}$$

and

$$u_i^-(x) := sup\left\{t \in \mathbf{R} : \lim_{\epsilon\to0^+} \frac{1}{\epsilon^N}\mathcal{L}_N\left[\{u_i < t\} \cap B(x,\epsilon)\right] = 0\right\}$$

where $B(x,\epsilon)$ is the open ball centered at x and with radius ϵ. The set $\Sigma(u)$ is called the *singular set of u* or *jump set* and is defined by

$$\Sigma(u) = \bigcup_{i=1}^{p}\left\{x \in \Omega : u_i^-(x) < u_i^+(x)\right\}.$$

It is well known that $\Sigma(u)$ is $N-1$ rectifiable, i.e.

$$\Sigma(u) = \bigcup_{n=1}^{\infty} K_n \cup E$$

where $H_{N-1}(E) = 0$ and K_n is a compact subset of a C^1 hypersurface.

Theorem 2.3 *If $u \in BV(\Omega;\mathbf{R}^p)$ then*

i) for $\mathcal{L}_N$ a.e. $x \in \Omega$

$$\lim_{\epsilon\to0^+} \frac{1}{\epsilon}\left\{\int_{B(x,\epsilon)} |u(y) - u(x) - \nabla u(x)\cdot(y-x)|^{\frac{N}{N-1}}dy\right\}^{\frac{N-1}{N}} = 0;$$

ii) for H_{N-1} a.e. $x \in \Sigma(u)$ there exists a unit vector $\nu(x) \in S^{N-1}$, normal to $\Sigma(u)$ at x, and there exist vectors $u^-(x), u^+(x) \in \mathbf{R}^p$ such that

$$\lim_{\epsilon\to0^+} \frac{1}{\epsilon^N}\int_{\{y\in B(x,\epsilon):(y-x)\cdot\nu(x)>0\}} |u(y) - u^+(x)|^{\frac{N}{N-1}}dy = 0,$$

$$\lim_{\epsilon\to0^+} \frac{1}{\epsilon^N}\int_{\{y\in B(x,\epsilon):(y-x)\cdot\nu(x)<0\}} |u(y) - u^-(x)|^{\frac{N}{N-1}}dy = 0;$$

iii) for H_{N-1} a.e. $x_0 \in \Omega\setminus\Sigma(u)$

$$\lim_{\epsilon\to0^+} \frac{1}{meas(B(x_0,\epsilon))}\int_{B(x_0,\epsilon)} |u(x) - u(x_0)|dx = 0$$

and for H_{N-1} a.e. $x_0 \in \Sigma(u)$

$$\lim_{\epsilon\to0^+} \frac{1}{meas(B(x_0,\epsilon))}\int_{B(x_0,\epsilon)} u(x)dx = \frac{u^+(x_0) + u^-(x_0)}{2}.$$

We remark that in general $(u_i)^\pm \neq (u^\pm)_i$. In the following we shall denote by $[u](x)$ the jump of u at x defined by

$$[u](x) = u^+(x) - u^-(x).$$

If $u \in BV(\Omega;\mathbf{R}^p)$ then Du may be represented as

$$Du = \nabla u\, dx + (u^+ - u^-) \otimes \nu\, dH_{N-1} \lfloor \Sigma(u) + C(u) \tag{2.1}$$

where ∇u is the density of the absolutely continuous part of Du with respect to the N-dimensional Lebesgue measure $\mathcal{L}_N$, H_{N-1} is the $N-1$ dimensional Hausdorff measure and $C(u)$ is the so-called Cantor part. The three measures in (2.1) are mutually singular; if $H_{N-1}(B) < +\infty$ then $|C(u)|(B) = 0$ and there exists a Borel set E such that $\mathcal{L}_N(E) = 0$ and $|C(u)|(X) = |C(u)|(X \cap E)$ for all Borel sets $X \subset \Omega$, where $|\mu|$ denotes the total variation measure of μ. When $C(u) = 0$ we say that u is a special BV function and we write $u \in SBV(\Omega;\mathbf{R}^p)$. This space was introduced by Ambrosio and De Giorgi in [6]. The following lower semi-continuity result holds : if $u_n \in BV(\Omega;\mathbf{R}^p)$ converges to u in $L^1(\Omega;\mathbf{R}^p)$ then

$$|Du|(\Omega) \le \liminf_{n\to+\infty} |Du_n|(\Omega).$$

Lemma 2.4 *Let $u \in BV(\Omega;\mathbf{R}^p)$ and let $\rho \in C_0^\infty(\mathbf{R}^N)$ be a nonnegative function such that*

$$\int_{\mathbf{R}^N} \rho(x)\, dx = 1, supp\, \rho = \overline{B}(0,1),\ \rho(x) = \rho(-x) \text{ for every } x \in \mathbf{R}^N.$$

Let $\rho_n(x) := n^N \rho(nx)$ and

$$u_n(x) := (u * \rho_n)(x) = \int_\Omega u(y)\rho_n(x-y)dy.$$

Then

i)

$$\int_{B(x_0,\epsilon)} h(x)|\nabla u_n(x)|dx \le \int_{B(x_0,\epsilon+\frac{1}{n})} (h * \rho_n)(x)|Du(x)|$$

whenever $dist(x_0,\partial\Omega) > \epsilon + \frac{1}{n}$ and h is a nonnegative Borel function;

ii)

$$\lim_{n\to+\infty} \int_{B(x_0,\epsilon)} \theta(\nabla u_n(x))\, dx = \int_{B(x_0,\epsilon)} \theta(Du(x))$$

for every function θ positively homogeneous of degree one and for every $\epsilon \in (0, dist(x_0,\partial\Omega))$ such that $|Du|(\partial B(x_0,\epsilon)) = 0$;

iii) if, in addition, $u \in L^\infty(\Omega;\mathbf{R}^p)$ then for every $x_0 \in \Omega \setminus \Sigma(u)$

$$u_n(x_0) \to u(x_0) \text{ and } (|u_n - u| * \rho_n)(x_0) \to 0$$

as $n \to +\infty$.

The proof of this lemma can be found in [7], Lemma 4.5. The next result is proved in [19], Lemma 2.6.

Lemma 2.5 *For H_{N-1} a.e. $x_0 \in \Sigma(u)$*

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon^{N-1}} \int_{\Sigma(u)\cap(x_0+\epsilon Q_{\nu(x_0)})} |u^+(x) - u^-(x)| dH_{N-1}(x) = |u^+(x_0) - u^-(x_0)|.$$

The following version of the Besicovitch Differentiation Theorem was proven by Ambrosio and Dal Maso, [5] Proposition 2.2.

Theorem 2.6 *If λ and μ are Radon measures in Ω, $\mu \geq 0$, then there exists a Borel set $E \subset \Omega$ such that $\mu(E) = 0$ and for every $x \in \operatorname{supp}\mu \setminus E$*

$$\frac{d\lambda}{d\mu}(x) := \lim_{\epsilon \to 0^+} \frac{\lambda(x+\epsilon C)}{\mu(x+\epsilon C)}$$

exists and is finite whenever C is a bounded, convex, open set containing the origin.

We remark that in the above result the exceptional set E does not depend on C. An immediate consequence is given below.

Theorem 2.7 *If μ is a nonnegative Radon measure and if $f \in L^1_{loc}(\mathbf{R}^N, \mu)$ then*

$$\lim_{\epsilon \to 0^+} \frac{1}{\mu(x+\epsilon C)} \int_{x+\epsilon C} |f(y) - f(x)| d\mu(y) = 0$$

for μ a.e. $x \in \mathbf{R}^N$ and for every bounded, convex, open set C containing the origin.

Recently, Alberti [1] showed that the density of the Cantor part C(u) is a rank-one matrix (see also [5]). Taking into consideration Theorem 2.6 we have the following property:

Theorem 2.8 *If $u \in BV(\Omega; \mathbf{R}^p)$ then for $|C(u)|$ a.e. $x \in \Omega$*

$$A(x) := \lim_{\epsilon \to 0^+} \frac{D(u)(x+\epsilon X)}{|D(u)|(x+\epsilon X)} = \lim_{\epsilon \to 0^+} \frac{C(u)(x+\epsilon X)}{|C(u)|(x+\epsilon X)}$$

exists and is a rank-one matrix of norm one, for every convex, open set X containing the origin.

The following two results can be found in [19], Lemma 2.13 and Proposition A.1. They will be used in Section 4 when we treat the density of $\mathcal{F}(\cdot)$ with respect to the Cantor part of the derivative Du.

Lemma 2.9 *Let μ be a nonnegative Radon measure on $\mathbf{R}^N$. For μ a.e. $x_0 \in \mathbf{R}^N$ and for every $0 < t < 1$ one has*

$$\limsup_{\epsilon \to 0^+} \frac{\mu(B(x_0, t\epsilon))}{\mu(B(x_0, \epsilon))} \geq t^N.$$

Proposition 2.10 *Let $\{\mu_k\}$ be a sequence of $\mathbf{R}^p$-valued Radon measures on Ω such that $|\mu_k|(\Omega) \to 1$ and $\mu_k(\Omega) \to a$ where $|a| = 1$. Then*

$$|\mu_k - (\mu_k \cdot a)a|(\Omega) \to 0.$$

In Section 5 we will need to approximate a set of finite perimeter by polyhedral sets and thus we will need the following theorem which is proved in [8], Lemma 3.1.

Theorem 2.11 *Let A be a subset of Ω such that $Per_\Omega(A) < +\infty$. There exists a sequence of polyhedral sets $\{A_k\}$ (i.e. A_k are bounded, strongly Lipschitz domains with $\partial A_k = H_1 \cup H_2 \cup \ldots \cup H_p$ where each H_i is a closed subset of a hyperplane of the type $\{x \in \mathbf{R}^N : x \cdot \nu_i = \alpha_i\}$) satisfying the following properties:*

i) $\mathcal{L}_N[((A_k \cap \Omega) \setminus A) \cup (A \setminus (A_k \cap \Omega))] \to 0$ *as* $k \to +\infty$;

ii) $Per_\Omega(A_k) \to Per_\Omega(A)$ *as* $k \to +\infty$;

iii) $H_{N-1}(\partial A_k \cap \partial\Omega) = 0$;

iv) $\mathcal{L}_N(A_k) = \mathcal{L}_N(A)$.

Let $f : \Omega \times M^{p\times N} \to [0, +\infty)$ and $\varphi : \Omega \times \mathbf{R}^p \times S^{N-1} \to [0, +\infty)$ be continuous functions satisfying the following hypotheses:

- (H0) $f(x, \cdot)$ is quasiconvex for all $x \in \Omega$;
- (H1) there exists a constant $C > 0$ such that

$$0 \le f(x, A) \le C(1 + ||A||)$$

  for all $(x, A) \in \Omega \times M^{p\times N}$;
- (H2) for every $x_0 \in \Omega$ and for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f(x_0, A) - f(x, A)| \le \epsilon C(1 + ||A||)$$

  for all $(x, A) \in \Omega \times M^{p\times N}$.

We recall that the *recession function of f* is defined by

$$f^\infty(A) := \limsup_{t\to+\infty} \frac{f(tA)}{t}.$$

If $f(x, \cdot)$ is quasiconvex and has linear growth (see (H1) above) $f^\infty(x, \cdot)$ is a quasiconvex, positively homogeneous of degree one function (see [19]). We assume further that

- (H3) there exist constants $c, L > 0$, $0 < m < 1$ such that

$$\left| f^\infty(x, A) - \frac{f(x, tA)}{t} \right| \le c\frac{1}{t^m}$$

  for every $(x, A) \in \Omega \times M^{p\times N}$ with $||A|| = 1$ and for all $t > 0$ such that $t > L$;

- (H4) there exists a constant $C_1 > 0$ such that

$$0 \le \varphi(x,\xi,\nu) \le C_1|\xi|$$

for all $(x,\xi,\nu) \in \Omega \times \mathbf{R}^p \times S^{N-1}$;

- (H5) for every $x_0 \in \Omega$ and for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |\varphi(x_0,\xi,\nu) - \varphi(x,\xi,\nu)| \le \epsilon C_1|\xi|$$

for all $(x,\xi,\nu) \in \Omega \times \mathbf{R}^p \times S^{N-1}$;

- (H6) φ is subadditive i.e.

$$\varphi(x,\xi_1+\xi_2,\nu) \le \varphi(x,\xi_1,\nu) + \varphi(x,\xi_2,\nu)$$

$\forall(x,\nu) \in \Omega \times S^{N-1}$ and $\forall \xi_1,\xi_2 \in \mathbf{R}^p$.

We define the positively homogeneous of degree one function

$$\varphi_0(x,\xi,\nu) = \lim_{t\to 0^+} \frac{\varphi(x,t\xi,\nu)}{t}.$$

Under hypothesis (H6) it turns out that (see [10] and [9])

$$\varphi_0(x,\xi,\nu) = \sup_{t>0} \frac{\varphi(x,t\xi,\nu)}{t}.$$

We will also need the following:

- (H7) there exist constants $C, l, \alpha > 0$ such that

$$\left|\varphi_0(x,\xi,\nu) - \frac{\varphi(x,t\xi,\nu)}{t}\right| \le Ct^\alpha$$

for every $(x,\xi,\nu) \in \Omega \times \mathbf{R}^p \times S^{N-1}$ with $|\xi| = 1$ and for all t such that $t < l$.

We remark that quasiconvexity of f (H0) and subadditivity of φ (H6) are necessary conditions for lower semi-continuity of the functional

$$\int_\Omega f(x,\nabla u(x))dx + \int_{\Sigma(u)} \varphi(x,[u](x),\nu(x))dH_{N-1}(x)$$

(see Morrey [22], [23] and Ambrosio and Braides [3], [4]). Under assumptions (H1) and (H3) it turns out that the lim sup in the definition of f^∞ is actually a limit. It is an easy consequence of the definition of the recession function that

Lemma 2.12 *Under hypotheses (H1) and (H2) it follows that*

i) for all $(x,A) \in \Omega \times M^{p\times N}$

$$0 \le f^\infty(x,A) \le C||A||;$$

ii) for every $x_0 \in \Omega$ and for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f^\infty(x_0, A) - f^\infty(x, A)| \le \epsilon C||A||$$

for all $(x, A) \in \Omega \times M^{p\times N}$.

Our goal in this paper is to find an integral representation for the relaxation $\mathcal{F}(\cdot)$ in $BV(\Omega;\mathbf{R}^p)$ of the functional defined on $SBV(\Omega;\mathbf{R}^p)$ by

$$u \mapsto E(u) := \int_\Omega f(x, \nabla u(x))dx + \int_{\Sigma(u)} \varphi(x, [u](x), \nu(x))dH_{N-1}(x)$$

with respect to the BV weak topology, namely

$$\mathcal{F}(u) := \inf_{\{u_n\}} \left\{ \liminf_{n\to+\infty} E(u_n) : u_n \in SBV(\Omega;\mathbf{R}^p), u_n \to u \text{ in } L^1(\Omega;\mathbf{R}^p), \sup_n |Du_n|(\Omega) < +\infty \right\}.$$

Notice that we do not allow $u_n \in BV(\Omega;\mathbf{R}^p)$ because any $u \in BV(\Omega;\mathbf{R}^p)$ can be approached by Cantor-Vitali functions v_n for which $E(v_n) = 0$ and this would imply that $\mathcal{F}(u) \equiv 0$.

In what follows, if g is a positively homogeneous function of degree one and if μ is an $\mathbf{R}^m$*-valued* measure, we use the notation

$$\int_\Omega g(d\mu) = \int_\Omega g\left(\frac{d\mu}{d|\mu|}(x)\right) d|\mu|(x).$$

Let $(\xi, \nu) \in \mathbf{R}^p \times S^{N-1}$, let $\{\nu_1, \dots, \nu_{N-1}, \nu\}$ form an orthonormal basis of $\mathbf{R}^N$ and define the class of admissible functions

$$\mathcal{A}(\xi,\nu) := \Big\{ v \in SBV_{loc}(S_\nu;\mathbf{R}^p) : v(y) = 0 \text{ if } y\cdot\nu = -\frac{1}{2},\ v(y) = \xi \text{ if } y\cdot\nu = \frac{1}{2}$$
$$\text{and } v \text{ is periodic with period one in the directions of } \nu_1, \dots, \nu_{N-1}\Big\},$$

where S_ν is the strip

$$S_\nu = \left\{ y \in \mathbf{R}^N : |y\cdot\nu| < \frac{1}{2} \right\}$$

and where the boundary values of v are understood in the sense of traces. A function v is said to be *periodic with period one in the direction of ν_i* if

$$v(y) = v(y + k\nu_i)$$

for all $k \in \mathbf{Z}, y \in S_\nu$.

The main result of this paper is the following

Theorem 2.13 *Under hypotheses (H0)-(H7), if $u \in BV(\Omega;\mathbf{R}^p)$ then*

$$\mathcal{F}(u) = \int_\Omega g(x, \nabla u(x))dx + \int_{\Sigma(u)} h(x, [u](x), \nu(x))dH_{N-1}(x) + \int_\Omega g^\infty(x, dC(u)) =: \mathcal{F}^*(u)$$

where

$$g(x_0, A) := \inf\Big\{ \int_Q f(x_0, \nabla u(x))dx + \int_{\Sigma(u)\cap Q} \varphi_0(x_0, [u](x), \nu(x))dH_{N-1}(x) :$$
$$u \in SBV(Q;\mathbf{R}^p), u|_{\partial Q}(x) = Ax\Big\}$$

and

$$h(x_0,\xi,\nu) := \inf\left\{\int_{Q_\nu} f^\infty(x_0,\nabla u(x))dx + \int_{\Sigma(u)\cap Q_\nu} \varphi(x_0,[u](x),\nu(x))dH_{N-1}(x) : u \in \mathcal{A}(\xi,\nu)\right\}.$$

In Proposition 6.3 we obtain an explicit formula for the function g involving the inf-convolution of f and φ_0 defined by

$$f\nabla\varphi_0(x_0,A) := \inf\{f(x_0,A-a\otimes b)+\varphi_0(x_0,a,b) : a\in\mathbf{R}^p, b\in\mathbf{R}^N\}.$$

Assuming further that φ is positively homogeneous of degree one i.e.

- (H8) $\varphi(x,\lambda\xi,\nu)=\lambda\varphi(x,\xi,\nu)$ for all $\lambda>0$ and all $(x,\xi,\nu)\in\Omega\times\mathbf{R}^p\times S^{N-1}$

we show that (cf. Theorem 6.7 and Proposition 7.2)

Theorem 2.14 *If f and φ satisfy (H0)-(H8) then, for any $u\in BV(\Omega;\mathbf{R}^p)$,*

$$\begin{aligned}\mathcal{F}(u) = &\int_\Omega Q(f\nabla\varphi)(x,\nabla u(x))dx + \int_{\Sigma(u)} Q^\infty(f\nabla\varphi)(x,[u](x)\otimes\nu(x))dH_{N-1}(x) + \\ &+\int_\Omega Q^\infty(f\nabla\varphi)\left(x,\frac{dC(u)}{d|C(u)|}(x)\right)d|C(u)|(x)\end{aligned}$$

where $Q(f\nabla\varphi)$ denotes the quasiconvexification of the inf-convolution of f and φ.

Remark 2.15 *i) Notice that if $\varphi=0$ and taking into account (H0), $\mathcal{F}(u)$ reduces to the expression that was obtained by Fonseca and Müller in [19] (see also Ambrosio and Dal Maso [5]), where it was proven that the relaxation in $BV(\Omega;\mathbf{R}^p)$ of the functional defined in $W^{1,1}(\Omega;\mathbf{R}^p)$ by*

$$u\mapsto\int_\Omega f(x,\nabla u(x))dx$$

is given by

$$\begin{aligned}\overline{\mathcal{F}}(u) &:= \int_\Omega f(x,\nabla u(x))dx+\int_{\Sigma(u)} f^\infty(x,[u](x)\otimes\nu(x))dH_{N-1}(x)+ \\ &\quad+\int_\Omega f^\infty(x,dC(u)).\end{aligned}$$

Since one can approach any $u\in BV(\Omega;\mathbf{R}^p)$ by a sequence $u_n\in C^\infty(\Omega;\mathbf{R}^p)\cap BV(\Omega;\mathbf{R}^p)$ such that $u_n\to u\in L^1(\Omega;\mathbf{R}^p)$ and $|Du_n|(\Omega)\to|Du|(\Omega)$ it follows that

$$\mathcal{F}(u)\le\overline{\mathcal{F}}(u).$$

If $\varphi\ge f^\infty$, which we do not expect in general, using the lower semi-continuity of $\overline{\mathcal{F}}(\cdot)$ in $BV(\Omega;\mathbf{R}^p)$ we conclude that the above inequality is actually an equality i.e.

$$\mathcal{F}(u)=\overline{\mathcal{F}}(u).$$

ii) We remark also that when $f = f(x,u,\nabla u)$ then the surface energy density of $\mathcal{F}$, denoted by $K_f(x,u^-,u^+,\nu)$ (see [19]), may not be translation invariant i.e. it is not clear that

$$K_f(x,u^-,u^+,\nu) = \hat{K}_f(x,[u],\nu).$$

In this case, the surface energy density of $E(\cdot)$ may be of the form

$$\psi(x,u^-,u^+,\nu) \neq \hat{\psi}(x,[u],\nu).$$

The following example shows that the relaxation of an energy which includes both bulk and interfacial energy terms is not equal to the sum of the relaxations of each term separatly. In particular, as we will see, the relaxation part corresponding to the bulk term may be altered by the initial surface energy.

Example 2.16 *Let $f(x,A) = \|A\|$, $\varphi(x,\xi,\nu) = \alpha|\xi|$. Then if $u \in BV(\Omega;\mathbf{R}^p)$ we have*

i) for $\alpha \geq 1$

$$\mathcal{F}(u) = \int_\Omega \|\nabla u(x)\| dx + \int_{\Sigma(u)} |[u](x)| dH_{N-1}(x) + |C(u)|(\Omega);$$

ii) for $0 < \alpha < 1$

$$\mathcal{F}(u) = \alpha \left[\int_\Omega \|\nabla u(x)\| dx + \int_{\Sigma(u)} |[u](x)| dH_{N-1}(x) + |C(u)|(\Omega) \right].$$

Proof. Here $f^\infty(x,\xi\otimes\nu) = |\xi|$.

i) By lower semi-continuity of the functional

$$u \mapsto |Du|(\Omega) = \int_\Omega \|\nabla u(x)\| dx + \int_{\Sigma(u)} |[u](x)| dH_{N-1}(x) + |C(u)|(\Omega),$$

given $u_n \in SBV(\Omega;\mathbf{R}^p)$ such that $u_n \to u$ in $L^1(\Omega;\mathbf{R}^p)$ and if $\alpha \geq 1$ one has

$$\begin{aligned} &\int_\Omega \|\nabla u(x)\| dx + \int_{\Sigma(u)} |[u](x)| dH_{N-1}(x) + |C(u)|(\Omega) \leq \\ &\leq \liminf_{n\to+\infty} \left[\int_\Omega \|\nabla u_n(x)\| dx + \int_{\Sigma(u_n)} |[u_n](x)| dH_{N-1}(x) \right] \leq \liminf_{n\to+\infty} E(u_n). \end{aligned}$$

Taking the infimum over all such u_n we get

$$\int_\Omega \|\nabla u(x)\| dx + \int_{\Sigma(u)} |[u](x)| dH_{N-1}(x) + |C(u)|(\Omega) \leq \mathcal{F}(u).$$

Conversely, given $u \in BV(\Omega;\mathbf{R}^p)$ let $u_n \in W^{1,1}(\Omega;\mathbf{R}^p)$ be such that $u_n \to u$ in $L^1(\Omega;\mathbf{R}^p)$ and

$$\begin{aligned} E(u_n) &= |Du_n|(\Omega) \to |Du|(\Omega) = \\ &= \int_\Omega \|\nabla u(x)\| dx + \int_{\Sigma(u)} |[u](x)| dH_{N-1}(x) + |C(u)|(\Omega). \end{aligned}$$

Then,

$$\mathcal{F}(u) \leq \liminf_{n\to+\infty} E(u_n) = \int_\Omega ||\nabla u(x)||dx + \int_{\Sigma(u)} |[u](x)|dH_{N-1}(x) + |C(u)|(\Omega).$$

ii) Given $u_n \in SBV(\Omega;\mathbf{R}^p)$ such that $u_n \to u$ in $L^1(\Omega;\mathbf{R}^p)$, by lower semi-continuity of the functional

$$u \mapsto \alpha|Du|(\Omega) = \alpha\left[\int_\Omega ||\nabla u(x)||dx + \int_{\Sigma(u)} |[u](x)|dH_{N-1}(x) + |C(u)|(\Omega)\right]$$

and if $\alpha < 1$ one has

$$\alpha\left[\int_\Omega ||\nabla u(x)||dx + \int_{\Sigma(u)} |[u](x)|dH_{N-1}(x) + |C(u)|(\Omega)\right] \leq$$

$$\leq \alpha \liminf_{n\to+\infty}\left[\int_\Omega ||\nabla u_n(x)||dx + \int_{\Sigma(u_n)} |[u_n](x)|dH_{N-1}(x)\right] \leq \liminf_{n\to+\infty} E(u_n).$$

Taking the infimum over all such u_n we get

$$\alpha\left[\int_\Omega ||\nabla u(x)||dx + \int_{\Sigma(u)} |[u](x)|dH_{N-1}(x) + |C(u)|(\Omega)\right] \leq \mathcal{F}(u).$$

Conversely, given $u \in BV(\Omega;\mathbf{R}^p)$ let u_n be a sequence of piecewise constant functions such that $u_n \to u$ in $L^1(\Omega;\mathbf{R}^p)$ and $|Du_n|(\Omega) \to |Du|(\Omega)$. Then,

$$\begin{aligned}\mathcal{F}(u) &\leq \liminf_{n\to+\infty} E(u_n) = \liminf_{n\to+\infty} \alpha \int_{\Sigma(u_n)} |[u_n](x)|dH_{N-1}(x) = \\ &= \alpha \liminf_{n\to+\infty} |Du_n|(\Omega) = \alpha|Du|(\Omega) = \\ &= \alpha\left[\int_\Omega ||\nabla u(x)||dx + \int_{\Sigma(u)} |[u](x)|dH_{N-1}(x) + |C(u)|(\Omega)\right].\end{aligned}$$

■

Remark 2.17 *For simplicity we start by proving Theorems 2.13 and 2.14 under the additional coercivity hypothesis*

- *(H9) there exists $\alpha > 0$ such that*

$$\inf_{t>0} \frac{f(x,tA)}{t} \geq \alpha||A|| \; \forall (x,A) \in \Omega \times M^{p\times N};$$

$$\inf_{t>0} \frac{\varphi(x,t\xi,\nu)}{t} \geq \alpha|\xi| \; \forall (x,\xi,\nu) \in \Omega \times \mathbf{R}^p \times S^{N-1}.$$

We shall see in Proposition 7.2 that this hypothesis can be removed.

We divide the proof of Theorem 2.13 into several parts. In the first one, proved in Section 4, we show that

$$\mathcal{F}(u) \;\geq\; \int_\Omega g(x,\nabla u(x))dx + \int_{\Sigma(u)} h(x,[u](x),\nu(x))dH_{N-1}(x) + \int_\Omega g^\infty(x,dC(u))$$

and in Section 5 we show the reverse inequality for SBV functions. In Section 6 we extend the result to arbitrary BV functions in the case where φ is positively homogeneous of degree one. Assuming (H8) we prove that $h(x,\xi,\nu) = g^\infty(x,\xi\otimes\nu)$ and we show that $g(x,A) = Q(f\nabla\varphi_0)(x,A)$ holds even in the inhomogeneous case. The proof of Theorems 2.13 and 2.14 are obtained in Section 7 and follow from the previous results.

3 Some Properties of the Density Functions

In this section we prove some properties of the density functions g and h which will be of use in Sections 4 and 5.

In the following lemma we use the slicing method introduced by Fonseca and Rybka [20] to modify a sequence near the boundary without increasing its total energy. This lemma will be used in Section 4 to treat the density of $\mathcal{F}(\cdot)$ with respect to the absolutely continuous and jump parts of the derivative Du.

Lemma 3.1 *Given $u \in BV(Q;\mathbf{R}^p)$ let $\mathcal{G}_n(u,\cdot)$ be a sequence of measures such that*

$$0 \leq \mathcal{G}_n(u,A) \leq C(|Du|(A) + \mathcal{L}_N(A)) \tag{3.1}$$

for every Borel set $A \subset Q$ and for all n. Let $u_n, v_n \in SBV(Q;\mathbf{R}^p)$ be such that $\lim_{n\to+\infty} ||u_n - v_n||_{L^1(Q;\mathbf{R}^p)} = 0$ and $\sup_n |Du_n|(Q) < +\infty$. Then, for every $0 < \delta < 1$ there exists a subsequence $\{v_{n_k}\}$ and a sequence $\{w_k\} \subset SBV(Q;\mathbf{R}^p)$ such that $w_k = v_{n_k}$ on ∂Q,

$$||w_k - v_{n_k}||_{L^1(Q;\mathbf{R}^p)} \leq ||u_{n_k} - v_{n_k}||_{L^1(Q;\mathbf{R}^p)}$$

and

$$\begin{aligned}\liminf_{n\to+\infty} \mathcal{G}_n(u_n,Q) \geq & \limsup_{k\to+\infty} \mathcal{G}_{n_k}(w_k,Q) - C meas(Q\setminus(1-\delta)Q) - \\ & -C\sup_m |Dv_m|(Q\setminus(1-\delta)Q).\end{aligned}$$

Proof. We follow an argument used in [17]. If necessary by extracting a subsequence, assume without loss of generality that

$$\liminf_{n\to+\infty} \mathcal{G}_n(u_n,Q) = \lim_{n\to+\infty} \mathcal{G}_n(u_n,Q) < +\infty. \tag{3.2}$$

Define $M := \sup_n |Du_n|(Q)$. Let $Q_k := Q \setminus \frac{k+1-\delta}{k+1}Q$, let P_k be an integer such that

$$\frac{P_k}{k} > M$$

and divide Q_k into P_k slices of equal width

$$Q_k = \bigcup_{i=1}^{P_k} S_i^{(k)}$$

where $S_i^{(k)}$ are of the form $S_i^{(k)} = \lambda_{i+1}^{(k)} Q \setminus \lambda_i^{(k)} Q$ with $0 < \lambda_i^{(k)} < 1$, $\lambda_1^{(k)} = \frac{k+1-\delta}{k+1}$, $\lambda_{P_k+1}^{(k)} = 1$. We claim that for every k, n there exists a slice $S_i^{(k)}$ such that

$$\int_{S_i^{(k)}} \|\nabla u_n(x)\| dx + \int_{S_i^{(k)} \cap \Sigma(u_n)} |[u_n](x)| dH_{N-1}(x) < \frac{1}{k}. \tag{3.3}$$

Indeed, if this were false there would exist k, n such that for all $i \in \{1, \ldots, P_k\}$

$$\int_{S_i^{(k)}} \|\nabla u_n(x)\| dx + \int_{S_i^{(k)} \cap \Sigma(u_n)} |[u_n](x)| dH_{N-1}(x) \geq \frac{1}{k}$$

and therefore we would obtain

$$\begin{aligned} M &\geq \int_Q \|\nabla u_n(x)\| dx \quad + \quad \int_{Q \cap \Sigma(u_n)} |[u_n](x)| dH_{N-1}(x) \geq \\ &\geq \sum_{i=1}^{P_k} \left[\int_{S_i^{(k)}} \|\nabla u_n(x)\| dx \quad + \quad \int_{S_i^{(k)} \cap \Sigma(u_n)} |[u_n](x)| dH_{N-1}(x) \right] \\ &\qquad \geq \frac{P_k}{k} > M \end{aligned}$$

which is a contradiction. Since we have finitely many slices and infinitely many indices, it is possible to choose a slice $S_{(1)}$ belonging to $\{S_i^{(1)} : i = 1, \ldots, P_1\}$ such that (3.3) with k = 1 is satisfied by a subsequence $\{u_n^{(1)}\}$ of $\{u_n\}$. Since $\lim_{n \to +\infty} \|u_n - v_n\|_{L^1(Q;\mathbf{R}^p)} = 0$ we may always assume that the subsequence $\{u_n^{(1)}\}$ satisfies

$$\frac{1}{meas(S_{(1)})} \int_{S_{(1)}} |u_n^{(1)}(x) - v_n^{(1)}(x)| dx < 1$$

for the corresponding subsequence $\{v_n^{(1)}\}$ of $\{v_n\}$ and for sufficiently large n. Next we choose a slice $S_{(2)}$ in $\{S_i^{(2)} : i = 1, \ldots, P_2\}$ such that (3.3) with k = 2 is satisfied by a subsequence $\{u_n^{(2)}\}$ of $\{u_n^{(1)}\}$ and so that

$$\frac{1}{meas(S_{(2)})} \int_{S_{(2)}} |u_n^{(2)}(x) - v_n^{(2)}(x)| dx < \frac{1}{2}$$

for the corresponding subsequence $\{v_n^{(2)}\}$ of $\{v_n^{(1)}\}$. By induction, let $S_{(m)}$ be a slice in $\{S_i^{(m)} : i = 1, \ldots, P_m\}$ such that (3.3) with k = m is satisfied by a subsequence $\{u_n^{(m)}\}$ of $\{u_n^{(m-1)}\}$ and so that

$$\frac{1}{meas(S_{(m)})} \int_{S_{(m)}} |u_n^{(m)}(x) - v_n^{(m)}(x)| dx < \frac{1}{m} \tag{3.4}$$

for the corresponding subsequence $\{v_n^{(m)}\}$ of $\{v_n^{(m-1)}\}$. We write $S_{(k)} = [-\alpha_k, \alpha_k]^N \setminus [-\beta_k, \beta_k]^N$ where $0 < \beta_k < \alpha_k$, $\alpha_k, \beta_k \nearrow \frac{1}{2}$. Let $\{\phi_k\}$ be a family of smooth cut-off functions such that

$$\begin{aligned} &0 \leq \phi_k \leq 1 \\ &\phi_k = 0 \text{ in } Q \setminus [-\alpha_k, \alpha_k]^N \\ &\phi_k = 1 \text{ in } [-\beta_k, \beta_k]^N \\ &\| \nabla \phi_k \|_\infty = O\left(\frac{1}{meas(S_{(k)})} \right) \end{aligned}$$

and define

$$w_k(x) := (1-\phi_k(x))v_k^{(k)}(x) + \phi_k(x)u_k^{(k)}(x).$$

Notice that $w_k \in SBV(Q;\mathbf{R}^p)$ and

$$\begin{aligned} w_k &= v_k^{(k)} \text{ in } Q \setminus [-\alpha_k,\alpha_k]^N \\ w_k &= u_k^{(k)} \text{ in } [-\beta_k,\beta_k]^N \end{aligned} \tag{3.5}$$

and in $S_{(k)}$

$$\nabla w_k(x) = \nabla v_k^{(k)}(x) + \phi_k(x)(\nabla u_k^{(k)}(x) - \nabla v_k^{(k)}(x)) + (u_k^{(k)}(x) - v_k^{(k)}(x)) \otimes \nabla\phi_k(x). \tag{3.6}$$

By (3.3)–(3.6) we have

$$\begin{aligned} &\int_{S_{(k)}\cap\Sigma(w_k)} |[w_k](x)|dH_{N-1}(x) \le \\ &\le \int_{S_{(k)}\cap\Sigma(u_k^{(k)})} |[u_k^{(k)}(x)|dH_{N-1}(x) + \int_{S_{(k)}\cap\Sigma(v_k^{(k)})} |[v_k^{(k)}(x)|dH_{N-1}(x) < \\ &< \frac{1}{k} + \sup_m |Dv_m|(Q\setminus(1-\delta)Q)\,, \end{aligned} \tag{3.7}$$

$$\int_{S_{(k)}} ||\nabla u_k^{(k)}(x)||dx < \frac{1}{k} \tag{3.8}$$

and

$$\frac{1}{meas(S_{(k)})}\int_{S_{(k)}} |u_k^{(k)}(x) - v_k^{(k)}(x)|dx < \frac{1}{k}. \tag{3.9}$$

Hence we may write

$$\begin{aligned} \mathcal{G}_k(w_k,Q) &= \mathcal{G}_k\left(u_k^{(k)},(-\beta_k,\beta_k)^N\right) + \mathcal{G}_k\left(v_k^{(k)},Q\setminus[-\alpha_k,\alpha_k]^N\right) + \mathcal{G}_k\left(w_k,S_{(k)}\right) \le \\ &\le \mathcal{G}_k\left(u_k^{(k)},Q\right) + \mathcal{G}_k\left(v_k^{(k)},Q\setminus[-\alpha_k,\alpha_k]^N\right) + \mathcal{G}_k\left(w_k,S_{(k)}\right) \end{aligned}$$

where by (3.1)

$$\begin{aligned} &\mathcal{G}_k\left(v_k^{(k)},Q\setminus[-\alpha_k,\alpha_k]^N\right) \le C|Dv_k^{(k)}|\left(Q\setminus[-\alpha_k,\alpha_k]^N\right) + C meas\left(Q\setminus[-\alpha_k,\alpha_k]^N\right) \le \\ &\le C\sup_m |Dv_m|(Q\setminus(1-\delta)Q) + C meas(Q\setminus(1-\delta)Q) \end{aligned}$$

and by (3.1) and (3.6)–(3.9)

$$\begin{aligned} &\mathcal{G}_k\left(w_k,S_{(k)}\right) \le C|Dw_k|\left(S_{(k)}\right) + C meas\left(S_{(k)}\right) \le \\ &\le C\int_{S_{(k)}} ||\nabla w_k(x)||dx + C\int_{S_{(k)}\cap\Sigma(w_k)} |[w_k](x)|dH_{N-1}(x) + C meas\left(S_{(k)}\right) \le \\ &\le C\int_{S_{(k)}} ||\nabla v_k^{(k)}(x)||dx + C\int_{S_{(k)}} ||\nabla u_k^{(k)}(x)||dx + C meas\left(S_{(k)}\right) + \\ &+ \frac{C}{meas\left(S_{(k)}\right)}\int_{S_{(k)}} |u_k^{(k)}(x) - v_k^{(k)}(x)|dx + C\int_{S_{(k)}\cap\Sigma(w_k)} |[w_k](x)|dH_{N-1}(x) \le \\ &\le C\sup_m |Dv_m|(Q\setminus(1-\delta)Q) + \frac{C}{k} + C meas(Q\setminus(1-\delta)Q)\,. \end{aligned}$$

∎

Thus the conclusion follows.

Remark 3.2 *Under the conditions of Lemma 3.1, if $v_n \equiv u_0$ i.e. if $\|u_n - u_0\|_{L^1(\Omega;\mathbf{R}^p)} \to 0$ and if $|Du_0|(Q \setminus (1-\delta)Q) \to 0$ as $\delta \to 0^+$ then considering $\delta = \frac{1}{j}$ and after extracting a diagonal subsequence it follows from the above lemma that there exists a sequence $\{w_k\}$ such that $w_k = u_0$ on ∂Q, $\|w_k - u_0\|_{L^1(\Omega;\mathbf{R}^p)} \to 0$ and*

$$\liminf_{n\to+\infty} \mathcal{G}_n(u_n;Q) \geq \limsup_{k\to+\infty} \mathcal{G}_{n_k}(w_k;Q).$$

Consider the density function g as in Theorem 2.13 i.e.

$$\begin{aligned} g(x_0,A) \; := \; \inf \Big\{ \int_Q f(x_0,\nabla u(x))dx + \int_{\Sigma(u)\cap Q} \varphi_0(x_0,[u](x),\nu(x))dH_{N-1}(x) : \\ u \in SBV(Q;\mathbf{R}^p), u|_{\partial Q}(x) = Ax \Big\}. \end{aligned}$$

The following proposition establishes some properties of g.

Proposition 3.3 *Assume that (H0)-(H7) and (H9) hold. Then*

i) assuming that Theorem 2.13 holds for SBV functions, $g(x,\cdot)$ is quasiconvex, for all $x \in \Omega$;

ii)

$$\begin{aligned} g(x_0,A) \; = \; & \inf_{\{u_n\}} \Big\{ \liminf_{n\to+\infty} \Big[\int_Q f(x_0,\nabla u_n(x))dx + \\ & + \int_{Q\cap\Sigma(u_n)} \varphi_0(x_0,[u_n](x),\nu_n(x))dH_{N-1}(x) \Big] : \\ & u_n \in SBV(Q;\mathbf{R}^p), u_n \to Ax \text{ in } L^1(Q;\mathbf{R}^p) \Big\} =: g^*(x_0,A). \end{aligned}$$

Proof. i) We show that if u_n, $u \in W^{1,1}(\Omega;\mathbf{R}^p)$ are such that $u_n \rightharpoonup u$ in $W^{1,1}(\Omega;\mathbf{R}^p)$ then

$$\int_\Omega g(x,\nabla u(x))dx \leq \liminf_{n\to+\infty} \int_\Omega g(x,\nabla u_n(x))dx.$$

Since quasiconvexity is a necessary condition for lower semi-continuity this proves that $g(x,\cdot)$ is quasiconvex. We remark that quasiconvexity of g is not needed to prove Theorem 2.13 for SBV functions, indeed it will only be used in Section 6 when we extend our relaxation result to arbitrary BV functions (cf. Lemma 6.5 and Propositions 6.3 and 6.4). Thus, as $u_n \in W^{1,1}(\Omega;\mathbf{R}^p)$ we may apply Theorem 2.13 to u_n to obtain $v_n \in SBV(\Omega;\mathbf{R}^p)$ such that

$$\|v_n - u_n\|_{L^1(\Omega;\mathbf{R}^p)} < \frac{1}{n}$$

and

$$|\mathcal{F}(u_n) - E(v_n)| < \frac{1}{n}.$$

Then $v_n \to u$ in $L^1(\Omega;\mathbf{R}^p)$, by (H9) $\sup_n |Dv_n|(\Omega) < +\infty$ and so

$$\begin{aligned} \int_\Omega g(x,\nabla u(x))dx = \mathcal{F}(u) \leq \liminf_{n\to+\infty} E(v_n) = \\ = \liminf_{n\to+\infty} \mathcal{F}(u_n) = \liminf_{n\to+\infty} \int_\Omega g(x,\nabla u_n(x))dx. \end{aligned}$$

ii) For any Borel set $B \subset Q$ and $x_0 \in \Omega$ let

$$\mathcal{G}_{x_0}(u;B) = \int_B f(x_0, \nabla u(x))dx + \int_{B\cap\Sigma(u)} \varphi_0(x_0,[u](x),\nu(x))dH_{N-1}(x).$$

Given any sequence $u_n \in SBV(Q;\mathbf{R}^p)$ such that $u_n \to Ax$ in $L^1(Q;\mathbf{R}^p)$ we may use the previous lemma and Remark 3.2 to modify it in order to obtain a sequence $w_n \in SBV(Q;\mathbf{R}^p)$ such that $w_n|_{\partial Q}(x) = Ax$ and

$$\limsup_{n\to+\infty}\left[\int_Q f(x_0,\nabla w_n(x))dx + \int_{Q\cap\Sigma(w_n)} \varphi_0(x_0,[w_n](x),\mu_n(x))dH_{N-1}(x)\right] \leq$$
$$\leq \liminf_{n\to+\infty}\left[\int_Q f(x_0,\nabla u_n(x))dx + \int_{Q\cap\Sigma(u_n)} \varphi_0(x_0,[u_n](x),\nu_n(x))dH_{N-1}(x)\right].$$

It follows that

$$g(x_0,A) \leq \liminf_{n\to+\infty}\left[\int_Q f(x_0,\nabla u_n(x))dx + \int_{Q\cap\Sigma(u_n)} \varphi_0(x_0,[u_n](x),\nu_n(x))dH_{N-1}(x)\right]$$

for any such u_n and so

$$g(x_0,A) \leq g^*(x_0,A).$$

Conversely, assume that

$$g(x_0,A) = \lim_{n\to+\infty}\left[\int_Q f(x_0,\nabla u_n(x))dx + \int_{Q\cap\Sigma(u_n)} \varphi_0(x_0,[u_n](x),\nu_n(x))dH_{N-1}(x)\right]$$

where $u_n \in SBV(Q;\mathbf{R}^p)$ and $u_n|_{\partial Q}(x) = Ax$. Then we may write $u_n(x) = Ax + \phi_n(x)$ where $\phi_n|_{\partial Q} = 0$. Extend ϕ_n periodically to $\mathbf{R}^N$ with period one and define

$$u_{n,m}(x) := Ax + \frac{1}{m}\phi_n(mx).$$

Then

$$\lim_n \lim_m \|u_{n,m} - Ax\|_{L^1(Q;\mathbf{R}^p)} = 0 \qquad (3.10)$$

since, for n fixed, by periodicity of ϕ_n and the Riemann-Lebesgue Lemma

$$\lim_m \int_Q |\phi_n(mx)|dx = \int_Q |\phi_n(x)|dx.$$

On the other hand, since $\Sigma(u_{n,m}) = \frac{\Sigma(u_n)}{m}$, by the periodicity of ϕ_n and since φ_0 is positively homogeneous of degree one we have

$$\int_Q f(x_0,\nabla u_{n,m}(x))dx = \int_Q f(x_0, A + \nabla\phi_n(mx))dx =$$
$$= \frac{1}{m^N}\int_{mQ} f(x_0,A+\nabla\phi_n(x))dx = \int_Q f(x_0,\nabla u_n(x))dx$$

and

$$\int_{Q\cap\Sigma(u_{n,m})} \varphi_0(x_0,[u_{n,m}](x),\nu_{n,m}(x))dH_{N-1}(x) =$$
$$= \int_{Q\cap\frac{\Sigma(u_n)}{m}} \varphi_0(x_0,\frac{1}{m}[\phi_n](mx),\nu_n(mx))dH_{N-1}(x) =$$
$$= \frac{1}{m^N}\int_{mQ\cap\Sigma(u_n)} \varphi_0(x_0,[\phi_n](x),\nu_n(x))dH_{N-1}(x) =$$
$$= \int_{Q\cap\Sigma(u_n)} \varphi_0(x_0,[u_n](x),\nu_n(x))dH_{N-1}(x).$$

Hence, choosing a diagonalizing sequence in (3.10), we obtain a sequence $\{v_k\}$ such that $v_k \to Ax$ in $L^1(Q;\mathbf{R}^p)$ and

$$g^*(x_0,A) \le \lim_{k\to+\infty}\left[\int_Q f(x_0,\nabla v_k(x))dx + \int_{Q\cap\Sigma(v_k)} \varphi_0(x_0,[v_k](x),\mu_k(x))dH_{N-1}(x)\right] =$$

$$\lim_{n\to+\infty}\left[\int_Q f(x_0,\nabla u_n(x))dx + \int_{Q\cap\Sigma(u_n)} \varphi_0(x_0,[u_n](x),\nu_n(x))dH_{N-1}(x)\right] = g(x_0,A).$$

■

Proposition 3.4 *Under hypotheses (H0)-(H7) and (H9) it follows that*

$$\begin{aligned} g^\infty(x_0,A) &= \inf\left\{\int_Q f^\infty(x_0,\nabla u(x))dx + \int_{Q\cap\Sigma(u)} \varphi_0(x_0,[u](x),\nu(x))dH_{N-1}(x) :\right. \\ &\qquad u \in SBV(Q;\mathbf{R}^p),\ u|_{\partial Q}(x) = Ax\} =: \Psi(x_0,A) \end{aligned}$$

Proof. Let $w_n \in SBV(Q;\mathbf{R}^p)$ be such that $w_n(x) = Ax$ on ∂Q and

$$\Psi(x_0,A) = \lim_{n\to+\infty}\left[\int_Q f^\infty(x_0,\nabla w_n(x))dx + \int_{Q\cap\Sigma(w_n)} \varphi_0(x_0,[w_n](x),\nu_n(x))dH_{N-1}(x)\right].$$

Assume that

$$g^\infty(x_0,A) = \lim_{n\to+\infty}\frac{g(x_0,t_nA)}{t_n}$$

where $t_n \to +\infty$. Then, as $t_nw_n|_{\partial Q}(x) = t_nAx$, we have by positive homogeneity of φ_0

$$\begin{aligned} &g^\infty(x_0,A) \le \\ &\le \limsup_{n\to+\infty}\frac{1}{t_n}\left[\int_Q f(x_0,t_n\nabla w_n(x))dx + \int_{Q\cap\Sigma(w_n)} \varphi_0(x_0,t_n[w_n](x),\nu_n(x))dH_{N-1}(x)\right] \le \\ &\le \limsup_{n\to+\infty}\left[\int_Q f^\infty(x_0,\nabla w_n(x))dx + \int_{Q\cap\Sigma(w_n)} \varphi_0(x_0,[w_n](x),\nu_n(x))dH_{N-1}(x)\right] + \\ &+ \limsup_{n\to+\infty}\left[\int_Q \frac{1}{t_n}f(x_0,t_n\nabla w_n(x)) - f^\infty(x_0,\nabla w_n(x))dx\right]. \end{aligned}$$

By (H3) and since $t_n \to +\infty$ it follows that

$$\limsup_{n\to+\infty}\left[\int_Q \frac{1}{t_n} f(x_0, t_n\nabla w_n(x)) - f^\infty(x_0, \nabla w_n(x))dx\right] = 0$$

so we conclude that

$$g^\infty(x_0, A) \le \Psi(x_0, A).$$

Conversely, given $t > 0$ let $v \in SBV(Q;\mathbf{R}^p)$ be such that $v|_{\partial Q}(x) = tAx$ and

$$g(x_0, tA) \ge \int_Q f(x_0, \nabla v(x))dx + \int_{Q\cap\Sigma(v)} \varphi_0(x_0, [v](x), \nu(x))dH_{N-1}(x) - \epsilon t. \tag{3.11}$$

Then, using the fact that φ_0 is positively homogeneous of degree one

$$\frac{g(x_0,tA)}{t} \ge \int_Q f^\infty(x_0, \frac{1}{t}\nabla v(x))dx + \int_{Q\cap\Sigma(v)} \varphi_0(x_0, \frac{1}{t}[v](x), \nu(x))dH_{N-1}(x) - \epsilon -$$
$$- \int_Q \left|\frac{1}{t} f(x_0, \nabla v(x)) - f^\infty(x_0, \frac{1}{t}\nabla v(x))\right| dx$$

where by (H3), (H1) and Hölder's inequality

$$\int_Q \left|\frac{1}{t} f(x_0, \nabla v(x)) - f^\infty(x_0, \frac{1}{t}\nabla v(x))\right| dx \le$$
$$\le \int_{Q\cap\{||\nabla v(x)||>L\}} \frac{C}{t}||\nabla v(x)||^{1-m}dx + \int_{Q\cap\{||\nabla v(x)||\le L\}} \frac{C}{t}(1 + ||\nabla v(x)||)dx \le$$
$$\le \frac{C}{t}\left(\int_Q ||\nabla v(x)||dx\right)^{1-m} + \frac{C}{t}.$$

However, by (H9) and (3.11)

$$\left(\int_Q ||\nabla v(x)||dx\right)^{1-m} \le C\,(g(x_0, tA) + \epsilon t)^{1-m}$$

and so

$$\lim_{t\to+\infty} \frac{C}{t}\left(\int_Q ||\nabla v(x)||dx\right)^{1-m} = 0.$$

Therefore, since $\frac{v}{t}|_{\partial Q}(x) = Ax$,

$$g^\infty(x_0, A) = \limsup_{t\to+\infty} \frac{g(x_0,tA)}{t} \ge$$
$$\ge \limsup_{t\to+\infty}\left[\int_Q f^\infty(x_0, \frac{1}{t}\nabla v(x))dx + \int_{Q\cap\Sigma(v)} \varphi_0(x_0, \frac{1}{t}[v](x), \nu(x))dH_{N-1}(x)\right] - \epsilon \ge$$
$$\ge \Psi(x_0, A) - \epsilon.$$

■

The result now follows by letting $\epsilon \to 0^+$.

Proposition 3.5 *Under hypotheses (H0)-(H5) and (H9) the following hold:*

i) $h(x,\xi,\nu) \le g^\infty(x,\xi\otimes\nu)$, *for all* $(x,\xi,\nu)\in\Omega\times\mathbf{R}^p\times S^{N-1}$;

ii) let u_0 *be given by*

$$u_0(x) := \begin{cases} \xi & \text{if } x\cdot\nu > 0 \\ 0 & \text{if } x\cdot\nu < 0, \end{cases}$$

then

$$\begin{aligned} h(x_0,\xi,\nu) &= \inf_{\{u_n\}}\Bigg\{\liminf_{n\to+\infty}\Bigg[\int_{Q_\nu} f^\infty(x_0,\nabla u_n(x))dx + \\ &+ \int_{Q_\nu\cap\Sigma(u_n)} \varphi(x_0,[u_n](x),\nu_n(x))dH_{N-1}(x)\Bigg] : \\ &u_n\in SBV(Q_\nu;\mathbf{R}^p),\ u_n\to u_0 \text{ in } L^1(Q_\nu;\mathbf{R}^p)\Bigg\} =: h^*(x_0,\xi,\nu). \end{aligned}$$

Proof. i) It is well known that $\mathcal{F}(\cdot)$ is lower semi-continuous in BV for the L^1 topology. Indeed, if $v_n, v\in BV(\Omega;\mathbf{R}^p)$ are such that $v_n\to v$ in $L^1(\Omega;\mathbf{R}^p)$ then for every n there exists $\bar{u}_n\in SBV(\Omega;\mathbf{R}^p)$ such that

$$||\bar{u}_n - v_n||_{L^1(\Omega;\mathbf{R}^p)} < \frac{1}{n}$$

and

$$|\mathcal{F}(v_n) - E(\bar{u}_n)| < \frac{1}{n}.$$

Then $\bar{u}_n\to v$ in $L^1(\Omega;\mathbf{R}^p)$ and so

$$\mathcal{F}(v) \le \liminf_{n\to+\infty} E(\bar{u}_n) = \liminf_{n\to+\infty}\mathcal{F}(v_n).$$

Now consider an arbitrary $u\in BV(\Omega;\mathbf{R}^p)$. By Fonseca and Müller's result (see [19] and also [5]) let $u_n\in W^{1,1}(\Omega;\mathbf{R}^p)$ be such that $u_n\to u$ in $L^1(\Omega;\mathbf{R}^p)$ and

$$\begin{aligned} &\int_\Omega g(x,\nabla u(x))dx + \int_{\Sigma(u)} g^\infty(x,[u](x)\otimes\nu(x))dH_{N-1}(x) + \int_\Omega g^\infty(x,dC(u)) = \\ &= \lim_{n\to+\infty}\int_\Omega g(x,\nabla u_n(x))dx = \lim_{n\to+\infty}\mathcal{F}(u_n). \end{aligned}$$

Then, by lower semi-continuity of $\mathcal{F}(\cdot)$, we have

$$\begin{aligned} &\mathcal{F}(u) = \int_\Omega g(x,\nabla u(x))dx + \int_{\Sigma(u)} h(x,[u](x),\nu(x))dH_{N-1}(x) + \int_\Omega g^\infty(x,dC(u)) \le \\ &\liminf_{n\to+\infty}\mathcal{F}(u_n) = \int_\Omega g(x,\nabla u(x))dx + \int_{\Sigma(u)} g^\infty(x,[u](x)\otimes\nu(x))dH_{N-1}(x) + \int_\Omega g^\infty(x,dC(u)). \end{aligned}$$

Given the arbitrariness of u we conclude that

$$h(x,\xi,\nu) \le g^\infty(x,\xi\otimes\nu).$$

ii) For any Borel set $B\subset Q_\nu$ and $x_0\in\Omega$ let

$$\mathcal{G}_{x_0}(u;B) = \int_B f^\infty(x_0,\nabla u(x))dx + \int_{B\cap\Sigma(u)} \varphi(x_0,[u](x),\nu(x))dH_{N-1}(x).$$

Then, given any sequence $u_n \in SBV(Q_\nu;\mathbf{R}^p)$ such that $u_n \to u_0$ in $L^1(Q_\nu;\mathbf{R}^p)$ we may use Lemma 3.1 and Remark 3.2 to modify it in order to obtain a sequence $w_n \in \mathcal{A}(\xi,\nu)$ such that

$$\limsup_{n\to+\infty}\left[\int_{Q_\nu} f^\infty(x_0,\nabla w_n(x))dx + \int_{Q_\nu\cap\Sigma(w_n)}\varphi(x_0,[w_n](x),\mu_n(x))dH_{N-1}(x)\right] \le$$

$$\le \liminf_{n\to+\infty}\left[\int_{Q_\nu} f^\infty(x_0,\nabla u_n(x))dx + \int_{Q_\nu\cap\Sigma(u_n)}\varphi(x_0,[u_n](x),\nu_n(x))dH_{N-1}(x)\right].$$

It follows that

$$h(x_0,\xi,\nu) \le$$
$$\le \liminf_{n\to+\infty}\left[\int_{Q_\nu} f^\infty(x_0,\nabla u_n(x))dx + \int_{Q_\nu\cap\Sigma(u_n)}\varphi(x_0,[u_n](x),\nu_n(x))dH_{N-1}(x)\right]$$

for any such u_n and so

$$h(x_0,\xi,\nu) \le h^*(x_0,\xi,\nu).$$

Conversely, assume that

$$h(x_0,\xi,\nu) =$$
$$= \lim_{n\to+\infty}\left[\int_{Q_\nu} f^\infty(x_0,\nabla u_n(x))dx + \int_{Q_\nu\cap\Sigma(u_n)}\varphi(x_0,[u_n](x),\nu_n(x))dH_{N-1}(x)\right]$$

where $u_n \in \mathcal{A}(\xi,\nu)$. For simplicity of notation we set $\nu = e_N$ and denote Q_ν by Q. Since u_n is periodic in the directions of $e_1,\dots,e_{N-1}$ we may extend it periodically with period one to the strip S_{e_N} (see Section 2) and define

$$u_{n,m}(x) := \begin{cases} \xi & \text{if } x_N > \frac{1}{2m} \\ u_n(mx) & \text{if } |x_N| < \frac{1}{2m} \\ 0 & \text{if } x_N < -\frac{1}{2m}. \end{cases}$$

Then, with $Q' := \{x \in Q : x_N = 0\}$, by periodicity of $u_n(\cdot,x_N)$ and the Riemann-Lebesgue Lemma

$$\lim_n \lim_m \|u_{n,m} - u_0\|_{L^1(Q;\mathbf{R}^p)} = 0$$

since

$$\int_{-\frac{1}{2m}}^{\frac{1}{2m}}\int_{Q'} |u_n(mx) - u_0(x)|dx =$$
$$= \frac{1}{m}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{Q'} |u_n(mx',t) - u_0(x)|dx'dt \xrightarrow[m\to+\infty]{} 0.$$

On the other hand,

$$\int_Q f^\infty(x_0,\nabla u_{n,m}(x))dx = \int_{-\frac{1}{2m}}^{\frac{1}{2m}}\int_{Q'} f^\infty(x_0,m\nabla u_n(mx))dx =$$
$$= \int_Q f^\infty(x_0,\nabla u_n(mx',t))dx'dt \xrightarrow[m\to+\infty]{} \int_Q f^\infty(x_0,\nabla u_n(x))dx$$

and

$$\int_{Q\cap\Sigma(u_{n,m})}\varphi(x_0,[u_{n,m}](x),\nu_{n,m}(x))dH_{N-1}(x)=$$
$$=\int_{\{x\in Q:|x_N|<\frac{1}{2m}\}\cap\frac{\Sigma(u_n)}{m}}\varphi(x_0,[u_n](mx),\nu_n(mx))dH_{N-1}(x)=$$
$$=\frac{1}{m^{N-1}}\int_{mQ\cap\{|x_N|<\frac{1}{2}\}\cap\Sigma(u_n)}\varphi(x_0,[u_n](x),\nu_n(x))dH_{N-1}(x)=$$
$$=\int_{Q\cap\Sigma(u_n)}\varphi(x_0,[u_n](x),\nu_n(x))dH_{N-1}(x)$$

where we have used the periodicity of u_n in the first $N-1$ variables. Thus, by a diagonalizing procedure, we obtain a sequence $w_n = u_{n,m(n)}$ such that $w_n \to u_0$ in $L^1(Q;\mathbf{R}^p)$ and

$$h^*(x_0,\xi,\nu)\le\liminf_{n\to+\infty}\left[\int_Q f^\infty(x_0,\nabla w_n(x))dx+\int_{Q\cap\Sigma(w_n)}\varphi(x_0,[w_n](x),\mu_n(x))dH_{N-1}(x)\right]=$$

$$=\lim_{n\to+\infty}\left[\int_Q f^\infty(x_0,\nabla u_n(x))dx+\int_{Q\cap\Sigma(u_n)}\varphi(x_0,[u_n](x),\nu_n(x))dH_{N-1}(x)\right]=h(x_0,\xi,\nu).$$

∎

The following proposition will be used in Section 5 to obtain an upper bound for the density of $\mathcal{F}(\cdot)$ with respect to the jump part of the derivative Du.

Proposition 3.6 *Under hypotheses (H0)-(H6) and (H9) the following hold:*

i) $h(x,\xi,\nu)\le C|\xi|$, *for all* $(x,\xi,\nu)\in\Omega\times\mathbf{R}^p\times S^{N-1}$;

ii) for every $x_0\in\Omega$ *and every* $\epsilon>0$ *there exists a* $\delta>0$ *such that*

$$|x-x_0|<\delta\Rightarrow|h(x_0,\xi,\nu)-h(x,\xi,\nu)|\le\epsilon C(1+|\xi|)$$

for all $(x,\xi,\nu)\in\Omega\times\mathbf{R}^p\times S^{N-1}$;

iii) for all (x,ξ,ν), $(x,\xi',\nu)\in\Omega\times\mathbf{R}^p\times S^{N-1}$

$$|h(x,\xi,\nu)-h(x,\xi',\nu)|\le C|\xi-\xi'|;$$

iv) h is upper semi-continuous in $\Omega\times\mathbf{R}^p\times S^{N-1}$.

Proof. i) Let u_0 be defined as

$$u_0(x):=\begin{cases}\xi & \text{if } x\cdot\nu>0\\ 0 & \text{if } x\cdot\nu<0.\end{cases}$$

Then $u_0\in\mathcal{A}(\xi,\nu)$, $\nabla u_0=0$ a.e. and $\Sigma(u_0)=\{x\in Q_\nu : x\cdot\nu=0\}$. Thus, by (H4)

$$h(x,\xi,\nu)\le\varphi(x,\xi,\nu)H_{N-1}(Q_\nu\cap\Sigma(u_0))\le C|\xi|.$$

ii) Fix $x_0 \in \Omega$ and $\epsilon > 0$. By Lemma 2.12 ii) and (H5) choose $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$|f^\infty(x_0, A) - f^\infty(x, A)| \leq \epsilon C\|A\| \tag{3.12}$$

and

$$|\varphi(x_0, \xi, \nu) - \varphi(x, \xi, \nu)| \leq \epsilon C|\xi|. \tag{3.13}$$

For all $n \in \mathbf{N}$ choose $u_n \in \mathcal{A}(\xi, \nu)$ such that

$$\int_{Q_\nu} f^\infty(x_0, \nabla u_n(x))dx + \int_{Q_\nu \cap \Sigma(u_n)} \varphi(x_0, [u_n](x), \nu_n(x))dH_{N-1}(x) \leq$$
$$\leq h(x_0, \xi, \nu) + \frac{1}{n}.$$

By (H9) and part i) it follows that

$$\int_{Q_\nu} \|\nabla u_n(x)\|dx \leq \frac{h(x_0, \xi, \nu) + 1}{C} \leq C\,(1 + |\xi|)$$

and

$$\int_{Q_\nu \cap \Sigma(u_n)} |[u_n](x)|dH_{N-1}(x) \leq \frac{h(x_0, \xi, \nu) + 1}{C} \leq C\,(1 + |\xi|).$$

Hence, if $|x - x_0| < \delta$, by (3.12) and (3.13) we have

$$h(x, \xi, \nu) - h(x_0, \xi, \nu) \leq$$
$$\leq \int_{Q_\nu} |f^\infty(x, \nabla u_n(y)) - f^\infty(x_0, \nabla u_n(y))|\,dy +$$
$$+ \int_{Q_\nu \cap \Sigma(u_n)} |\varphi(x, [u_n](y), \nu_n(y)) - \varphi(x_0, [u_n](y), \nu_n(y))|\,dH_{N-1}(y) + \frac{1}{n}$$
$$\leq \int_{Q_\nu} \epsilon C\|\nabla u_n(y)\|dy + \int_{Q_\nu \cap \Sigma(u_n)} \epsilon C|[u_n](y)|dH_{N-1}(y) + \frac{1}{n} \leq$$
$$\leq \epsilon C\,(1 + |\xi|) + \frac{1}{n}.$$

Letting $n \to +\infty$ we obtain

$$h(x, \xi, \nu) - h(x_0, \xi, \nu) \leq \epsilon C\,(1 + |\xi|).$$

In a similar way we get

$$h(x_0, \xi, \nu) - h(x, \xi, \nu) \leq \epsilon C\,(1 + |\xi|).$$

iii) Let $u \in \mathcal{A}(\xi, \nu)$, let θ be a smooth cut-off function such that $0 \leq \theta \leq 1$, $\theta(t) = 0$ if $t \geq \frac{1}{2}$, $\theta(t) = 1$ if $t \leq \frac{1}{4}$ and define

$$u^*(x) := \begin{cases} u(2x) & \text{if } |x \cdot \nu| < \frac{1}{4} \\ \theta(x \cdot \nu)\xi + (1 - \theta(x \cdot \nu))\xi' & \text{if } \frac{1}{4} < x \cdot \nu < \frac{1}{2} \\ 0 & \text{if } -\frac{1}{2} < x \cdot \nu < -\frac{1}{4}. \end{cases}$$

Then $u^* \in \mathcal{A}(\xi', \nu)$ and so

$$h(x,\xi',\nu) \le \int_{Q_\nu} f^\infty(x, \nabla u^*(y))dy + \int_{Q_\nu \cap \Sigma(u^*)} \varphi(x, [u^*](y), \nu^*(y))dH_{N-1}(y)$$

where, by Lemma 2.12 i) and periodicity of u

$$\begin{aligned}
\int_{Q_\nu} f^\infty(x, \nabla u^*(y))dy &= \int_{\{|y\cdot\nu|<\frac{1}{4}\}} \int_{Q'_\nu} f^\infty(x, 2\nabla u(2y))dy + \\
&+ \int_{\{\frac{1}{4}<y\cdot\nu<\frac{1}{2}\}} \int_{Q'_\nu} f^\infty(x, (\xi-\xi')\otimes\theta'(y\cdot\nu)\nu)dy + \\
&+ \int_{\{-\frac{1}{2}<y\cdot\nu<\frac{1}{4}\}} \int_{Q'_\nu} f^\infty(x, 0)dy \le \\
&\le \frac{2}{2^N} \int_{\{|z\cdot\nu|<\frac{1}{2}\}} \int_{2Q'_\nu} f^\infty(x, \nabla u(z))dz + C|\xi-\xi'| = \\
&= \int_{Q_\nu} f^\infty(x, \nabla u(z))dz + C|\xi-\xi'|
\end{aligned}$$

and

$$\begin{aligned}
&\int_{Q_\nu \cap \Sigma(u^*)} \varphi(x, [u^*](y), \nu^*(y))dH_{N-1}(y) = \\
&= \int_{\{x\in Q_\nu : |x\cdot\nu|<\frac{1}{4}\}\cap\frac{1}{2}\Sigma(u)} \varphi(x, [u](2y), \nu(2y))dH_{N-1}(y) = \\
&= \frac{1}{2^{N-1}} \int_{\{x\in 2Q_\nu : |x\cdot\nu|<\frac{1}{2}\}\cap\Sigma(u)} \varphi(x, [u](y), \nu(y))dH_{N-1}(y) = \\
&= \int_{Q_\nu \cap \Sigma(u)} \varphi(x, [u](y), \nu(y))dH_{N-1}(y)
\end{aligned}$$

so that

$$\begin{aligned}
&h(x,\xi',\nu) \le \\
&\le \int_{Q_\nu} f^\infty(x, \nabla u(y))dy + \int_{Q_\nu \cap \Sigma(u)} \varphi(x, [u](y), \nu(y))dH_{N-1}(y) + C|\xi-\xi'|.
\end{aligned}$$

Taking the infimum over all such $u \in \mathcal{A}(\xi, \nu)$ we conclude that

$$h(x,\xi',\nu) \le h(x,\xi,\nu) + C|\xi-\xi'|$$

and in a similar way we can show that

$$h(x,\xi,\nu) \le h(x,\xi',\nu) + C|\xi-\xi'|.$$

iv) By iii) it suffices to show that $(x,\nu) \mapsto h(x,\xi,\nu)$ is upper semi-continuous, for every $\xi \in \mathbf{R}^p$. It is clear that

$$\begin{aligned}
&h(x,\xi,\nu) = \\
&= \inf\left\{ \int_Q f^\infty(x, \nabla u(y)R^T)dy + \int_{Q\cap\Sigma(u)} \varphi(x, [u](y), \mu(y))dH_{N-1}(y) : \right. \\
&\left. R \text{ is a rotation, } Re_N = \nu,\ u \in \mathcal{A}(\xi, e_N) \right\}.
\end{aligned}$$

Let $(x_n,\nu_n) \to (x,\nu)$ and choose a rotation R such that $Re_N = \nu$. Given $\epsilon > 0$ let $u_\epsilon \in \mathcal{A}(\xi, e_N)$ be such that

$$\left| h(x,\xi,\nu) - \int_Q f^\infty(x, \nabla u_\epsilon(y)R^T)dy - \int_{Q\cap\Sigma(u_\epsilon)} \varphi(x,[u_\epsilon](y),\mu_\epsilon(y))dH_{N-1}(y)\right| < \epsilon. \tag{3.14}$$

As $f^\infty(x,\cdot)$ is Lipschitz (by quasiconvexity and growth condition) we have

$$|f^\infty(x,A) - f^\infty(x,B)| \le C||A-B||. \tag{3.15}$$

Let K be a compact subset of Ω containing a neighbourhood of x and for fixed $\epsilon > 0$, choose $\delta > 0$ such that Lemma 2.12 ii) and (H5) are satisfied uniformly in K i.e.

$$y, y' \in K,\ |y-y'| < \delta \Rightarrow |f^\infty(y,A) - f^\infty(y',A)| \le \epsilon C||A||,\ \forall A \in M^{p\times N} \tag{3.16}$$

and for all $(\xi,\mu) \in \mathbb{R}^p \times S^{N-1}$,

$$y, y' \in K,\ |y-y'| < \delta \Rightarrow |\varphi(y,\xi,\mu) - \varphi(y',\xi,\mu)| \le \epsilon C|\xi|. \tag{3.17}$$

Choosing rotations R_n such that $R_n e_N = \nu_n$, by (3.14)–(3.17), by (H9) and for n large enough, it follows that

$$\left| \int_Q f^\infty(x,\nabla u_\epsilon(y)R^T)dy + \int_{Q\cap\Sigma(u_\epsilon)} \varphi(x,[u_\epsilon](y),\mu_\epsilon(y))dH_{N-1}(y) - \right.$$
$$\left. - \int_Q f^\infty(x_n,\nabla u_\epsilon(y)R_n^T)dy - \int_{Q\cap\Sigma(u_\epsilon)} \varphi(x_n,[u_\epsilon](y),\mu_\epsilon(y))dH_{N-1}(y)\right| < C\epsilon|Du_\epsilon|(Q) \le C\epsilon.$$

Hence,

$$\begin{aligned}
&h(x_n,\xi,\nu_n) \le \\
&\le \int_Q f^\infty(x_n,\nabla u_\epsilon(y)R_n^T)dy + \int_{Q\cap\Sigma(u_\epsilon)} \varphi(x_n,[u_\epsilon](y),\mu_\epsilon(y))dH_{N-1}(y) \le \\
&\le C\epsilon + \int_Q f^\infty(x,\nabla u_\epsilon(y)R^T)dy + \int_{Q\cap\Sigma(u_\epsilon)} \varphi(x,[u_\epsilon](y),\mu_\epsilon(y))dH_{N-1}(y) \le \\
&\le C\epsilon + h(x,\xi,\nu)
\end{aligned}$$

and so, letting $\epsilon \to 0^+$, we conclude that

$$\limsup_{n\to+\infty} h(x_n,\xi,\nu_n) \le h(x,\xi,\nu).$$

■

The following truncation lemma will be used in the next two sections to control the error terms that appear in estimates involving φ and φ_0. This result was obtained in collaboration with G. Alberti and L. Ambrosio during the meeting on "Calculus of Variations and Nonlinear Elasticity" held at Cortona on May 24–28, 1993, and its proof relies on a type of averaging slicing method that was introduced by De Giorgi. The idea of truncating on the range rather than on the domain is also due to De Giorgi and this argument has been used recently by P. Celada and G. Dal Maso (see [13]).

We recall that if $u \in BV(\Omega;\mathbf{R}^p)\cap L^\infty(\Omega;\mathbf{R}^p)$ then $\Sigma(u)$ is the complement of the set of Lebesgue points of u i.e. $x_0 \notin \Sigma(u)$ if and only if there exists $y_0 \in \mathbf{R}^p$ such that

$$\lim_{\delta\to 0^+} \fint_{B(x_0,\delta)} |u(x) - y_0|dx = 0.$$

As it turns out, y_0 is unique and we set

$$\tilde{u}(x_0) := y_0,$$

the *approximate limit* of u at x_0. We recall also that (see Vol'pert [24]) if $u \in BV(\Omega;\mathbf{R}^p)$ (respectively $u \in SBV(\Omega;\mathbf{R}^p)$) and $\varphi \in C_0^\infty(\mathbf{R}^p;\mathbf{R}^m)$ then $\varphi o u \in BV(\Omega;\mathbf{R}^m)$ (resp. $\varphi o u \in SBV(\Omega;\mathbf{R}^m)$, $D(\varphi\circ u) = D\varphi(\tilde{u})Du$ on $\Omega\setminus\Sigma(u)$, $\Sigma(\varphi\circ u)\subset\Sigma(u)$ and $(\varphi\circ u)^\pm(x) = \varphi(u^\pm(x))$. This result was generalized by Ambrosio and Dal Maso to the case where φ is Lipschitz.

In the following $\mathcal{B}(\Omega)$ denotes the set of all Borel subsets of Ω.

Lemma 3.7 *Let $\mathcal{G} : BV(\Omega;\mathbf{R}^p)\times\mathcal{B}(\Omega)\to[0,+\infty]$ (resp. $\mathcal{G} : SBV(\Omega;\mathbf{R}^p)\times\mathcal{B}(\Omega)\to[0,+\infty]$) satisfy*

i) $\mathcal{G}(u,\cdot)$ is a Borel measure for every $u\in BV(\Omega;\mathbf{R}^p)$ (resp. $u\in SBV(\Omega;\mathbf{R}^p)$);

ii) $\mathcal{G}(u,B) = \mathcal{G}(v,B)$ for every $B\in\mathcal{B}(\Omega)$ and every $u,v\in BV(\Omega;\mathbf{R}^p)$ (resp. $u,v\in SBV(\Omega;\mathbf{R}^p)$) such that $Du\lfloor B = Dv\lfloor B$;

iii) $\mathcal{G}(u,B)\le C_0(meas(B) + |Du|(B))$ for every $B\in\mathcal{B}(\Omega)$ and every $u\in BV(\Omega;\mathbf{R}^p)$ (resp. $u\in SBV(\Omega;\mathbf{R}^p)$).

Let $u_0\in BV(\Omega;\mathbf{R}^p)\cap L^\infty(\Omega;\mathbf{R}^p)$. Then for every $\epsilon>0$ and every $R>0$ there exists $C = C(\epsilon,R,C_0,||u_0||_\infty)$ such that for every $u\in BV(\Omega;\mathbf{R}^p)$ (resp. $u\in SBV(\Omega;\mathbf{R}^p)$) with $||u||_{BV}\le R$ there exists $u_\epsilon\in BV(\Omega;\mathbf{R}^p)\cap L^\infty(\Omega;\mathbf{R}^p)$ satisfying

a) $||u_\epsilon||_\infty\le C$;

b) $\mathcal{G}(u_\epsilon,\Omega)\le\mathcal{G}(u,\Omega)+\epsilon$;

c) $||u_\epsilon - u_0||_{L^1(\Omega;\mathbf{R}^p)}\le||u-u_0||_{L^1(\Omega;\mathbf{R}^p)}$;

d) $|Du_\epsilon|(\Omega)\le|Du|(\Omega)$.

Proof. Fix $k\in\mathbf{N}$ and let $i\in\{\lambda,\dots,k\}$ where $\lambda\in\mathbf{N}$ is given by $\lambda = [\ln||u_0||_\infty]+1$ ($[t]$ denotes the integer part of t). Let $\varphi_i\in C_0^\infty(\mathbf{R}^p;\mathbf{R}^p)$ satisfy

$$\varphi_i(x) = \begin{cases} x, & |x| < e^i \\ 0, & |x|\ge e^{i+1}, \end{cases}$$

$||\nabla\varphi_i||_\infty\le 1$ and $|\varphi_i(x)|\le\min\{e^i,|x|\}$. We define $u_i\in BV(\Omega;\mathbf{R}^p)\cap L^\infty(\Omega;\mathbf{R}^p)$ (resp. $u_i\in SBV(\Omega;\mathbf{R}^p)\cap L^\infty(\Omega;\mathbf{R}^p)$) by

$$u_i(x) := \varphi_i(u(x)).$$

Clearly $\|u_i\|_\infty \le e^i$ and as $\|\nabla\varphi_i\|_\infty \le 1$ we conclude that

$$|Du_i|(\Omega) \le |Du|(\Omega).$$

On the other hand, by choice of λ it follows that $|u_0(x)| \le \|u_0\|_\infty < e^i$ and so $u_0(x) = \varphi_i(u_0(x))$. Therefore,

$$\|u_i - u_0\|_{L^1(\Omega;\mathbf{R}^p)} = \int_{\{|\tilde{u}|<e^i\}} |u(x) - u_0(x)|dx + \int_{\{|\tilde{u}|\ge e^i\}} |\varphi_i(u(x)) - \varphi_i(u_0(x))|dx \le$$
$$\le \|u - u_0\|_{L^1(\Omega;\mathbf{R}^p)}$$

where we have used the fact that φ_i has Lipschitz constant less than or equal to 1. Let

$$\Omega_i := \{x \in \Omega : |u^-(x)|,\ |u^+(x)| < e^i\}.$$

Because $\mathcal{G}(u_i,\cdot)$ is a measure we have

$$\mathcal{G}(u_i,\Omega) \le \mathcal{G}(u_i,\Omega_i) + \mathcal{G}(u_i,\Omega \setminus \Omega_i)$$

where, by locality,

$$\mathcal{G}(u_i,\Omega_i \setminus \Sigma(u_i)) = \mathcal{G}(u,\Omega_i \setminus \Sigma(u_i))$$

since $Du_i\lfloor(\Omega_i \setminus \Sigma(u_i)) = D\varphi_i(\tilde{u}(x))Du\lfloor(\Omega_i \setminus \Sigma(u_i)) = Du\lfloor(\Omega_i \setminus \Sigma(u_i))$. Also,

$$\mathcal{G}(u_i,\Omega_i \cap \Sigma(u_i)) = \mathcal{G}(u,\Omega_i \cap \Sigma(u_i))$$

because if $x \in \Omega_i \cap \Sigma(u_i)$ then $x \in \Omega_i \cap \Sigma(u)$ and so $\varphi_i(u^\pm(x)) = u^\pm(x)$. Hence,

$$\mathcal{G}(u_i,\Omega) \le \mathcal{G}(u,\Omega_i) + \mathcal{G}(u_i,\Omega \setminus \Omega_i) \le \mathcal{G}(u,\Omega) + \mathcal{G}(u_i,\Omega \setminus \Omega_i). \tag{3.18}$$

On the other hand

$$\mathcal{G}(u_i,\Omega \setminus \Omega_i) \le C_0\left(meas(\Omega \setminus \Omega_i) + |Du_i|(\Omega \setminus \Omega_i)\right) \tag{3.19}$$

where, by Chebyshev's inequality and since $\mathcal{L}_N(\Sigma(u)) = 0$,

$$meas(\Omega \setminus \Omega_i) = meas\left[(\Omega \setminus \Omega_i) \setminus \Sigma(u)\right] = meas\left(\{x \in \Omega : |\tilde{u}(x)| \ge e^i\}\right) \le \frac{\|u\|_{L^1(\Omega;\mathbf{R}^p)}}{e^i} \le \frac{R}{e^i} \tag{3.20}$$

and

$$\sum_{i=\lambda}^{k} |Du_i|(\Omega \setminus \Omega_i) \le \sum_{i=\lambda}^{k} |Du_i|(\{e^i \le |\tilde{u}(x)| < e^{i+1}\} \setminus \Sigma(u)) +$$
$$+ \sum_{i=\lambda}^{k} \int_{\{x\in\Sigma(u):\max|u^\pm(x)|\ge e^i\}\cap\Sigma(u_i)} |u_i^+(x) - u_i^-(x)|dH_{N-1}(x) \le$$
$$\le |Du|(\Omega \setminus \Sigma(u)) + \sum_{i=\lambda}^{k} \int_{\{x\in\Sigma(u):\max|u^\pm(x)|\ge e^i\}\cap\Sigma(u_i)} |u_i^+(x) - u_i^-(x)|dH_{N-1}(x). \tag{3.21}$$

But,

$$\sum_{i=\lambda}^{k}\int_{\{x\in\Sigma(u):\max|u^{\pm}(x)|\geq e^{i}\}\cap\Sigma(u_i)}|u_i^+(x)-u_i^-(x)|dH_{N-1}(x)=$$

$$=\sum_{i=\lambda}^{k}\int_{\{x\in\Sigma(u):e^i<|u^-(x)|<|u^+(x)|<e^{i+1}\}}|u_i^+(x)-u_i^-(x)|dH_{N-1}(x)+ \tag{3.22}$$

$$+\sum_{i=\lambda}^{k}\int_{\{x\in\Sigma(u):e^i<|u^+(x)|<|u^-(x)|<e^{i+1}\}}|u_i^+(x)-u_i^-(x)|dH_{N-1}(x)+ \tag{3.23}$$

$$+\sum_{i=\lambda}^{k}\int_{\{x\in\Sigma(u):e^i\leq|u^+(x)|=|u^-(x)|<e^{i+1}\}}|u_i^+(x)-u_i^-(x)|dH_{N-1}(x)+ \tag{3.24}$$

$$+\sum_{i=\lambda}^{k}\int_{\{x\in\Sigma(u):|u^-(x)|<e^i<e^{i+1}\leq|u^+(x)|\}}|u_i^+(x)-u_i^-(x)|dH_{N-1}(x)+ \tag{3.25}$$

$$+\sum_{i=\lambda}^{k}\int_{\{x\in\Sigma(u):|u^+(x)|<e^i<e^{i+1}\leq|u^-(x)|\}}|u_i^+(x)-u_i^-(x)|dH_{N-1}(x)+ \tag{3.26}$$

$$+\sum_{i=\lambda}^{k}\int_{\{x\in\Sigma(u):e^i\leq|u^-(x)|<e^{i+1}\leq|u^+(x)|\}}|u_i^+(x)-u_i^-(x)|dH_{N-1}(x)+ \tag{3.27}$$

$$+\sum_{i=\lambda}^{k}\int_{\{x\in\Sigma(u):e^i\leq|u^+(x)|<e^{i+1}<\leq|u^-(x)|\}}|u_i^+(x)-u_i^-(x)|dH_{N-1}(x)+ \tag{3.28}$$

$$+\sum_{i=\lambda}^{k}\int_{\{x\in\Sigma(u):|u^-(x)|\leq e^i<|u^+(x)|<e^{i+1}\}}|u_i^+(x)-u_i^-(x)|dH_{N-1}(x)+ \tag{3.29}$$

$$+\sum_{i=\lambda}^{k}\int_{\{x\in\Sigma(u):|u^+(x)|\leq e^i<|u^-(x)|<e^{i+1}\}}|u_i^+(x)-u_i^-(x)|dH_{N-1}(x). \tag{3.30}$$

where each $x\in\Sigma(u)$ is accounted for in only one of the above sums and all of these, except for (3.25) and (3.26), have only one term. In the case of sum (3.25) we must count how many $i\in\{\lambda,\dots,k\}$ satisfy $|u^-(x)|<e^i<e^{i+1}\leq|u^+(x)|$. Solving with respect to i we conclude that this sum has at most $\ln\frac{|u^+(x)|}{|u^-(x)|}$ terms and so, for these x, we have

$$\sum_{i=\lambda}^{k}|u_i^+(x)-u_i^-(x)|=\sum_{i=\lambda}^{k}|u^-(x)|\leq|u^-(x)|\ln\frac{|u^+(x)|}{|u^-(x)|}\leq$$
$$\leq|u^+(x)|-|u^-(x)|\leq|u^+(x)-u^-(x)|.$$

Likewise for (3.26), we conclude that there are at most $\ln\frac{|u^+(x)|}{|u^-(x)|}$ terms and for these x

$$\sum_{i=\lambda}^{k}|u_i^+(x)-u_i^-(x)|=\sum_{i=\lambda}^{k}|u^+(x)|\leq|u^+(x)-u^-(x)|.$$

Therefore, using the fact that φ_i has Lipschitz constant less than or equal to 1, we obtain

$$\sum_{i=\lambda}^{k} \int_{\{x\in\Sigma(u):\max|u^{\pm}(x)|\geq e^i\}\cap\Sigma(u_i)} |u_i^+(x) - u_i^-(x)|dH_{N-1}(x) \leq \int_{\Sigma(u)} |u^+(x) - u^-(x)|dH_{N-1}(x)$$

so from (3.18)–(3.21) it follows that

$$\sum_{i=\lambda}^{k} \mathcal{G}(u_i,\Omega) \leq (k-\lambda+1)\mathcal{G}(u,\Omega) +$$

$$+C_0\left(R\sum_{i=\lambda}^{k}\frac{1}{e^i} + |Du|(\Omega\setminus\Sigma(u)) + \int_{\Sigma(u)}|u^+(x)-u^-(x)|dH_{N-1}(x)\right) \leq$$

$$\leq (k-\lambda+1)\mathcal{G}(u,\Omega) + C_0R\left(1+\frac{1}{e^{\lambda-1}(e-1)}\right)$$

and thus, by choice of λ,

$$\frac{1}{k-\lambda+1}\sum_{i=\lambda}^{k}\mathcal{G}(u_i,\Omega) \leq \mathcal{G}(u,\Omega) + \frac{C_0R}{k-[\ln\|u_0\|_\infty]}\left(1+\frac{1}{\|u_0\|_\infty(e-1)}\right). \tag{3.31}$$

Choose k large enough so that

$$\frac{C_0R}{k-[\ln\|u_0\|_\infty]}\left(1+\frac{1}{\|u_0\|_\infty(e-1)}\right) < \frac{\epsilon}{2}. \tag{3.32}$$

It is clear that (3.31) implies that there exists $i \in \{\lambda,\dots,k\}$ such that

$$\mathcal{G}(u_i,\Omega) \leq \mathcal{G}(u,\Omega) + \epsilon$$

with $\|u_i\|_\infty \leq e^i \leq e^k$ where k is given by (3.32). ∎

4 A Lower Bound for $\mathcal{F}(u)$ when $u \in BV(\Omega;\mathbf{R}^p)$

In this section we prove the first part of Theorem 2.13 namely that

$$\mathcal{F}(u) \geq \int_\Omega g(x,\nabla u(x))dx + \int_{\Sigma(u)} h(x,[u](x),\nu(x))dH_{N-1}(x) + \int_\Omega g^\infty(x,dC(u))$$

where $u \in BV(\Omega;\mathbf{R}^p)$. It is clear that the above inequality is equivalent to proving

Theorem 4.1 *Let (H0)-(H7) and (H9) hold, let $u_n \in SBV(\Omega;\mathbf{R}^p)$, $u \in BV(\Omega;\mathbf{R}^p)$ and suppose that $u_n \rightharpoonup u$ in $BV(\Omega;\mathbf{R}^p)$. Then*

$$\liminf_{n\to+\infty}\left[\int_\Omega f(x,\nabla u_n(x))dx + \int_{\Sigma(u_n)}\varphi(x,[u_n](x),\nu_n(x))dH_{N-1}(x)\right] \geq$$

$$\geq \int_\Omega g(x,\nabla u(x))dx + \int_{\Sigma(u)} h(x,[u](x),\nu(x))dH_{N-1}(x) + \int_\Omega g^\infty(x,dC(u)).$$

The proof of Theorem 4.1 makes use of Lemma 3.7 therefore we must ensure that its hypotheses are satisfied. We start by proving that, defining

$$\begin{aligned}\mathcal{F}(u;A) &:= \inf_{\{u_n\}}\left\{\liminf_{n\to+\infty}\left[\int_A f(x,\nabla u_n(x))dx + \int_{A\cap\Sigma(u_n)}\varphi(x,[u_n](x),\nu_n(x))dH_{N-1}(x)\right] : \right.\\ &\left. : u_n \in SBV(A;\mathbf{R}^p),\ u_n \to u \text{ in } L^1(A;\mathbf{R}^p),\ \sup_n |Du_n|(A) < +\infty\right\}\end{aligned}$$

where $A \subset \Omega$ is an open set and $u \in BV(\Omega;\mathbf{R}^p)$, then $\mathcal{F}(u;A)$ is a *variational functional* with respect to the L^1 topology i.e.

Proposition 4.2 *Under hypotheses (H1) and (H4) the following hold:*

i) $\mathcal{F}(\cdot;A)$ *is local, i.e.* $\mathcal{F}(u;A) = \mathcal{F}(v;A)$ *for every* $u, v \in SBV(A;\mathbf{R}^p)$ *verifying* $Du\lfloor A = Dv\lfloor A$*;*

ii) if, in addition (H9) holds, then $\mathcal{F}(\cdot;A)$ *is sequentially lower semi-continuous i.e. if* u_n, $u \in SBV(A;\mathbf{R}^p)$ *are such that* $u_n \to u$ *in* $L^1(A;\mathbf{R}^p)$ *then* $\mathcal{F}(u;A) \le \liminf_{n\to+\infty}\mathcal{F}(u_n;A)$;

iii) $\mathcal{F}(\cdot;A)$ *is the trace on* $\{U \subset \Omega : U$ *is open* $\}$ *of a Borel measure on the set* $\mathcal{B}(\Omega)$ *of all Borel subsets of* Ω*;*

iv) $0 \le \mathcal{F}(u;A) \le C\,(meas(A) + |Du|(A))$.

Proof. We begin by showing that

$$\mathcal{F}(u;A) \le C\,(meas(A) + |Du|(A)). \tag{4.1}$$

Consider a sequence $u_n \in SBV(A;\mathbf{R}^p)$ such that $u_n \to u$ in $L^1(A;\mathbf{R}^p)$ and $|Du_n|(A) \to |Du|(A)$. Then, by (H1) and (H4), we have

$$\begin{aligned}\mathcal{F}(u;A) &\le \liminf_{n\to+\infty}\left[\int_A f(x,\nabla u_n(x))dx + \int_{A\cap\Sigma(u_n)}\varphi(x,[u_n](x),\nu_n(x))dH_{N-1}(x)\right] \le\\ &\le \liminf_{n\to+\infty}\left[\int_A C(1+\|\nabla u_n(x)\|)dx + \int_{A\cap\Sigma(u_n)} C|[u_n](x)|dH_{N-1}(x)\right] \le\\ &\le C meas(A) + \limsup_{n\to+\infty} C|Du_n|(A) = C\,(meas(A) + |Du|(A)).\end{aligned}$$

The locality property is clear from the definition of $\mathcal{F}(u;A)$. To prove sequential lower semi-continuity of $\mathcal{F}(\cdot;A)$ we use a standard diagonalization procedure. Let u_n, $u \in SBV(A;\mathbf{R}^p)$ be such that $u_n \to u$ in $L^1(A;\mathbf{R}^p)$, $\sup_n |Du_n|(A) < +\infty$ and assume that

$$\liminf_{n\to+\infty}\mathcal{F}(u_n;A) = \lim_{n\to+\infty}\mathcal{F}(u_n;A) < +\infty,$$

$$\mathcal{F}(u_n;A) = \lim_{k\to+\infty}\left[\int_A f(x,\nabla u_n^k(x))dx + \int_{A\cap\Sigma(u_n^k)}\varphi(x,[u_n^k](x),\nu_n^k(x))dH_{N-1}(x)\right]$$

where $u_n^k \to u_n$ in $L^1(A;\mathbf{R}^p)$ as $k \to +\infty$, $\sup_k |Du_n^k|(A) < +\infty$. For all n let $p(n)$ be such that for every $k \ge p(n)$

$$\|u_n^k - u_n\|_{L^1(A;\mathbf{R}^p)} \le \frac{1}{n}.$$

Choose $s_n \geq p(n)$ such that

$$\left|\mathcal{F}(u_n;A) - \int_A f(x,\nabla u_n^{s_n}(x))dx + \int_{A\cap\Sigma(u_n^{s_n})} \varphi(x,[u_n^{s_n}](x),\nu_n^{s_n}(x))dH_{N-1}(x)\right| \leq \frac{1}{n}.$$

Clearly $u_n^{s_n} \to u$ in $L^1(A;\mathbf{R}^p)$, by (H9) $\sup_n |Du_n^{s_n}|(A) < +\infty$ and so

$$\begin{aligned}\mathcal{F}(u;A) &\leq \liminf_{n\to+\infty}\left[\int_A f(x,\nabla u_n^{s_n}(x))dx + \int_{A\cap\Sigma(u_n^{s_n})} \varphi(x,[u_n^{s_n}](x),\nu_n^{s_n}(x))dH_{N-1}(x)\right] = \\ &= \liminf_{n\to+\infty} \mathcal{F}(u_n;A).\end{aligned}$$

To establish iii) we use De Giorgi and Letta's criterion (see [14]):
A set function $\alpha : X \to [0,+\infty]$ is the trace of a Borel measure if

a) $\alpha(B) \leq \alpha(A)$ for every $A,\ B \in X$ with $B \subset A$;

b) $\alpha(A\cup B) \geq \alpha(A) + \alpha(B)$ for every $A,\ B \in X$ such that $A\cap B = \emptyset$;

c) $\alpha(A\cup B) \leq \alpha(A) + \alpha(B)$ for every $A,\ B \in X$;

d) $\alpha(A) = \sup\{\alpha(B) : B \subset\subset A\}$ for every $A \in X$.

Parts a) and b) follow trivially from the definition of $\mathcal{F}(u;A)$. To show c) and d) we prove that if A, B, C are open subsets of Ω with $B \subset\subset C \subset\subset A$ then

$$\mathcal{F}(u;A) \leq \mathcal{F}(u;C) + \mathcal{F}(u;A\setminus\overline{B}). \tag{4.2}$$

Suppose that (4.2) holds. To show d) fix $\epsilon > 0$ and let $B \subset\subset A$ be such that

$$meas(A\setminus\overline{B}) + |Du|(A\setminus\overline{B}) < \frac{\epsilon}{C}.$$

By (4.1) it follows that $\mathcal{F}(u;A\setminus\overline{B}) < \epsilon$ and so, if C is such that $B \subset\subset C \subset\subset A$, by (4.2) we conclude that

$$\mathcal{F}(u;A) \leq \mathcal{F}(u;C) + \epsilon$$

thus proving d). In order to obtain c), for $t \in (0,1)$ we define the sets

$$\begin{aligned}A_t &:= \{x \in A\cup B : t\,dist(x,A\setminus B) < (1-t)\,dist(x,B\setminus A)\}, \\ B_t &:= \{x \in A\cup B : t\,dist(x,A\setminus B) > (1-t)\,dist(x,B\setminus A)\}, \\ S_t &:= \{x \in A\cup B : t\,dist(x,A\setminus B) = (1-t)\,dist(x,B\setminus A)\}.\end{aligned}$$

Since $\mathcal{L}_N(\cup_t S_t) + |Du|(\cup_t S_t) < +\infty$ and the sets $\{S_t\}$ are pairwise disjoint, there exists $t_0 \in (0,1)$ such that $(\mathcal{L}_N + |Du|)(S_{t_0}) = 0$. Given $\epsilon > 0$ by (4.1) choose $K_1 \subset\subset A_{t_0}$, $K_2 \subset\subset B_{t_0}$ such that $\mathcal{F}(u;(A\cup B)\setminus(\overline{K_1}\cup\overline{K_2})) < \epsilon$ and let $K_1 \subset\subset H_1 \subset\subset A_{t_0}$, $K_2 \subset\subset H_2 \subset\subset B_{t_0}$. By (4.2), a) and b) we deduce that

$$\mathcal{F}(u;A\cup B) \leq \mathcal{F}(u;H_1\cup H_2) + \mathcal{F}(u;(A\cup B)\setminus(\overline{K_1}\cup\overline{K_2})) \leq \mathcal{F}(u;A) + \mathcal{F}(u;B) + \epsilon.$$

We now prove (4.2). We let A, B and C be open subsets of Ω such that $B \subset\subset C \subset\subset A$ and we assume that

$$\mathcal{F}(u;A\setminus\overline{B}) = \lim_{k\to+\infty}\left[\int_{A\setminus\overline{B}} f(x,\nabla u_k(x))dx + \int_{(A\setminus\overline{B})\cap\Sigma(u_k)} \varphi(x,[u_k](x),\nu_k(x))dH_{N-1}(x)\right] \quad (4.3)$$

$$\mathcal{F}(u;C) = \lim_{k\to+\infty}\left[\int_C f(x,\nabla v_k(x))dx + \int_{C\cap\Sigma(v_k)} \varphi(x,[v_k](x),\mu_k(x))dH_{N-1}(x)\right] \quad (4.4)$$

where $u_k \to u$ in $L^1(A\setminus\overline{B};\mathbb{R}^p)$, $\sup_k |Du_k|(A\setminus\overline{B}) < +\infty$ by (H9) and $v_k \to u$ in $L^1(C;\mathbb{R}^p)$ and (H9) implies that $\sup_k |Dv_k|(C) < +\infty$. In order to obtain an admissible sequence in the whole of A we will use the slicing method to connect u_k to v_k across $C\setminus\overline{B}$. We partition $C\setminus\overline{B}$ into two layers $S^1_{(2)}$ and $S^2_{(2)}$ of equal measure of the type $S = \{x \in C\setminus\overline{B} : 0 < \alpha < dist(x,\partial B) < \beta\}$. Define

$$M := \sup_k\{|Du_k|(A\setminus\overline{B}) + |Dv_k|(C)\}.$$

We claim that for every k there exists a layer $S \in \{S^1_{(2)}, S^2_{(2)}\}$ such that

$$\begin{aligned}&\int_S ||\nabla u_k(x)||dx + \int_{S\cap\Sigma(u_k)} |[u_k](x)|dH_{N-1}(x) + \\ &+ \int_S ||\nabla v_k(x)||dx + \int_{S\cap\Sigma(v_k)} |[v_k](x)|dH_{N-1}(x) \le \frac{M}{2}.\end{aligned} \quad (4.5)$$

Indeed if (4.5) were false there would exist k such that for every $S \in \{S^1_{(2)}, S^2_{(2)}\}$

$$\begin{aligned}&\int_S ||\nabla u_k(x)||dx + \int_{S\cap\Sigma(u_k)} |[u_k](x)|dH_{N-1}(x) + \\ &+ \int_S ||\nabla v_k(x)||dx + \int_{S\cap\Sigma(v_k)} |[v_k](x)|dH_{N-1}(x) > \frac{M}{2}.\end{aligned}$$

Then

$$\begin{aligned} M &\ge \int_{A\setminus\overline{B}} ||\nabla u_k(x)||dx + \int_{(A\setminus\overline{B})\cap\Sigma(u_k)} |[u_k](x)|dH_{N-1}(x) + \\ &\quad + \int_{C\setminus\overline{B}} ||\nabla v_k(x)||dx + \int_{(C\setminus\overline{B})\cap\Sigma(v_k)} |[v_k](x)|dH_{N-1}(x) \ge \\ &\ge \sum_{i=1}^{2}\left[\int_{S^i_{(2)}} ||\nabla u_k(x)||dx + \int_{S^i_{(2)}\cap\Sigma(u_k)} |[u_k](x)|dH_{N-1}(x) + \right. \\ &\quad \left. + \int_{S^i_{(2)}} ||\nabla v_k(x)||dx + \int_{S^i_{(2)}\cap\Sigma(v_k)} |[v_k](x)|dH_{N-1}(x)\right] > M\end{aligned}$$

which is a contradiction. Since we have two layers and infinitely many indices, we conclude that one of the layers

$$S_1 = \{x \in C\setminus\overline{B} : \alpha_1 < dist(x,\partial B) < \beta_1\} \in \{S^1_{(2)}, S^2_{(2)}\}$$

verifies

$$\int_{S_1} \|\nabla u_k^{(1)}(x)\|dx + \int_{S_1\cap\Sigma(u_k^{(1)})} |[u_k^{(1)}](x)|dH_{N-1}(x) +$$
$$+ \int_{S_1} \|\nabla v_k^{(1)}(x)\|dx + \int_{S_1\cap\Sigma(v_k^{(1)})} |[v_k^{(1)}](x)|dH_{N-1}(x) \le \frac{M}{2}$$

for a subsequence $\{u_k^{(1)}, v_k^{(1)}\}$ of $\{u_k, v_k\}$. As $u_k - v_k \to 0$ in $L^1(C\setminus\overline{B};\mathbf{R}^p)$ the above layer can also be chosen so that it satisfies

$$\frac{1}{meas(S_1)}\int_{S_1} \|u_k^{(1)}(x) - v_k^{(1)}(x)\|dx < \frac{1}{2}.$$

Next we divide $C\setminus\overline{B}$ into three layers $S^1_{(3)}$, $S^2_{(3)}$ and $S^3_{(3)}$ of equal measure. By the same reasoning as before one of these

$$S_2 = \{x \in C\setminus\overline{B} : \alpha_2 < dist(x,\partial B) < \beta_2\} \in \{S^i_{(3)} : i = 1,2,3\}$$

verifies

$$\int_{S_2} \|\nabla u_k^{(2)}(x)\|dx + \int_{S_2\cap\Sigma(u_k^{(2)})} |[u_k^{(2)}](x)|dH_{N-1}(x) +$$
$$+ \int_{S_2} \|\nabla v_k^{(2)}(x)\|dx + \int_{S_2\cap\Sigma(v_k^{(2)})} |[v_k^{(2)}](x)|dH_{N-1}(x) \le \frac{M}{3}$$

for a subsequence $\{u_k^{(2)}, v_k^{(2)}\}$ of $\{u_k^{(1)}, v_k^{(1)}\}$ and

$$\frac{1}{meas(S_2)}\int_{S_2} \|u_k^{(2)}(x) - v_k^{(2)}(x)\|dx < \frac{1}{3}.$$

Recursively we obtain layers

$$S_j = \{x \in C\setminus\overline{B} : \alpha_j < dist(x,\partial B) < \beta_j\} \in \{S^i_{(j+1)} : i = 1,\ldots,j+1\}$$

such that

$$\int_{S_j} \|\nabla u_k^{(j)}(x)\|dx + \int_{S_j\cap\Sigma(u_k^{(j)})} |[u_k^{(j)}](x)|dH_{N-1}(x) +$$
$$+ \int_{S_j} \|\nabla v_k^{(j)}(x)\|dx + \int_{S_j\cap\Sigma(v_k^{(j)})} |[v_k^{(j)}](x)|dH_{N-1}(x) \le \frac{M}{j+1} \tag{4.6}$$

for a subsequence $\{u_k^{(j)}, v_k^{(j)}\}$ of $\{u_k^{(j-1)}, v_k^{(j-1)}\}$ and

$$\frac{1}{meas(S_j)}\int_{S_j} \|u_k^{(j)}(x) - v_k^{(j)}(x)\|dx < \frac{1}{j+1}. \tag{4.7}$$

Let $\{\eta_j\}$ be a family of smooth cut-off functions such that $0 \le \eta_j \le 1$, $\|\nabla\eta_j\|_\infty = O\left(\frac{1}{meas(S_j)}\right)$ and

$$\eta_j = 0 \text{ in } A\setminus\left[\left\{x \in C : dist(x,\partial B) < \beta_j\right\}\right]$$

$$\eta_j = 1 \text{ in } \{x \in C : dist(x, \partial B) < \alpha_j\} \cup B.$$

Define $w_j(x) := (1 - \eta_j(x))u_k^{(j)}(x) + \eta_j(x)v_k^{(j)}(x)$. Clearly $w_j \to u$ in $L^1(A; \mathbb{R}^p)$ and by (4.6) and (4.7), $\sup_j |Dw_j|(A) < +\infty$. Thus, by (H1) and (H4)

$$\begin{aligned}
&\mathcal{F}(u; A) \leq \liminf_{j \to +\infty} \left[\int_A f(x, \nabla w_j(x))dx + \int_{A \cap \Sigma(w_j)} \varphi(x, [w_j](x), \theta_j(x))dH_{N-1}(x) \right] \leq \\
&\leq \mathcal{F}(u; A \setminus \overline{B}) + \mathcal{F}(u; C) + \\
&+ \limsup_{j \to +\infty} \left[\int_{S_j} C(1 + ||\nabla w_j(x)||)\, dx + \int_{S_j \cap \Sigma(w_j)} C|[w_j](x)|dH_{N-1}(x) \right]. \qquad (4.8)
\end{aligned}$$

Now, $\Sigma(w_j) \subseteq \Sigma(u_k^{(j)}) \cup \Sigma(v_k^{(j)})$ and

$$\nabla w_j(x) = \nabla u_k^{(j)}(x) + \eta_j(x) \left(\nabla v_k^{(j)}(x) - \nabla u_k^{(j)}(x) \right) + \left(v_k^{(j)}(x) - u_k^{(j)}(x) \right) \otimes \nabla \eta_j(x)$$

so from (4.8) we obtain

$$\begin{aligned}
&\mathcal{F}(u; A) \leq \mathcal{F}(u; A \setminus \overline{B}) + \mathcal{F}(u; C) + \limsup_{j \to +\infty} \left[\int_{S_j} C \left(1 + ||\nabla u_k^{(j)}(x)|| + ||\nabla v_k^{(j)}(x)|| \right) dx + \right. \\
&\quad + \frac{C}{meas(S_j)} \int_{S_j} |v_k^{(j)}(x) - u_k^{(j)}(x)|dx + \int_{S_j \cap \Sigma(u_k^{(j)})} C|[u_k^{(j)}](x)|dH_{N-1}(x) + \\
&\quad \left. + \int_{S_j \cap \Sigma(v_k^{(j)})} C|[v_k^{(j)}](x)|dH_{N-1}(x) \right] \leq \\
&\leq \mathcal{F}(u; A \setminus \overline{B}) + \mathcal{F}(u; C) + \limsup_{j \to +\infty} \left[C meas(S_j) + \frac{C}{j+1} + \frac{CM}{j+1} \right] \leq \\
&\leq \mathcal{F}(u; A \setminus \overline{B}) + \mathcal{F}(u; C)
\end{aligned}$$

where we have used (4.6)–(4.7). ∎

We now proceed with the proof of Theorem 4.1.

Proof. Step 1. Assume, without loss of generality, that

$$\begin{aligned}
&\liminf_{n \to +\infty} \left[\int_\Omega f(x, \nabla u_n(x))dx + \int_{\Sigma(u_n)} \varphi(x, [u_n](x), \nu_n(x))dH_{N-1}(x) \right] = \\
&= \lim_{n \to +\infty} \left[\int_\Omega f(x, \nabla u_n(x))dx + \int_{\Sigma(u_n)} \varphi(x, [u_n](x), \nu_n(x))dH_{N-1}(x) \right] < +\infty \qquad (4.9)
\end{aligned}$$

and define the sequence μ_n of Radon measures by

$$< \mu_n, \psi > := \int_\Omega \psi(x) f(x, \nabla u_n(x))dx + \int_{\Sigma(u_n)} \psi(x) \varphi(x, [u_n](x), u_n^+(x), \nu_n(x))dH_{N-1}(x)$$

for every $\psi \in C_0(\Omega)$. By (4.9) we have $\sup_n |\mu_n|(\Omega) < +\infty$ and so there exists a subsequence (still denoted μ_n) and a Radon measure μ such that $\mu_n \overset{*}{\rightharpoonup} \mu$ in the sense of measures i.e. for every $\psi \in C_0(\Omega)$

$$\lim_{n\to+\infty}\left[\int_\Omega \psi(x)f(x,\nabla u_n(x))dx + \int_{\Sigma(u_n)}\psi(x)\varphi(x,[u_n](x),u_n^+(x),\nu_n(x))dH_{N-1}(x)\right] =$$
$$= \int_\Omega \psi(x)d\mu(x).$$

By the Radon-Nikodym Theorem we may decompose μ as the sum of four mutually singular nonnegative measures

$$\mu = \mu_a\mathcal{L}_N + \mu_J|u^+ - u^-|H_{N-1}\lfloor\Sigma(u) + \mu_c|C(u)| + \mu_s.$$

Using the blow-up method introduced by Fonseca and Müller [18], we reduce the problem to verifying the pointwise inequalities

$$\mu_a(x_0) \geq g(x_0,\nabla u(x_0)) \text{ for } \mathcal{L}_N \text{ a.e. } x_0 \in \Omega, \tag{4.10}$$

$$\mu_J(x_0)|u^+(x_0) - u^-(x_0)| \geq h(x_0,[u](x_0),\nu(x_0)) \tag{4.11}$$

for H_{N-1} a.e. $x_0 \in \Omega \cap \Sigma(u)$ and

$$\mu_c(x_0) \geq g^\infty\left(x_0,\frac{dC(u)}{d|C(u)|}(x_0)\right) \tag{4.12}$$

for $|C(u)|$ a.e. $x_0 \in \Omega$. Assuming (4.10)–(4.12) hold, consider an increasing sequence of smooth cut-off functions $\psi_k \in C_0(\Omega)$ with $0 \leq \psi_k \leq 1$ and $\sup_k \psi_k(x) = 1$ in Ω. Then

$$\lim_{n\to+\infty}\left[\int_\Omega f(x,\nabla u_n(x))dx + \int_{\Sigma(u_n)}\varphi(x,[u_n](x),\nu_n(x))dH_{N-1}(x)\right] \geq$$
$$\geq \lim_{n\to+\infty}\left[\int_\Omega \psi_k(x)f(x,\nabla u_n(x))dx + \int_{\Sigma(u_n)}\psi_k(x)\varphi(x,[u_n](x),\nu_n(x))dH_{N-1}(x)\right]$$
$$= \int_\Omega \psi_k(x)d\mu(x) \geq$$
$$\geq \int_\Omega \psi_k(x)\mu_a(x)dx + \int_{\Sigma(u)}\psi_k(x)\mu_J(x)|u^-(x) - u^+(x)|dH_{N-1}(x) +$$
$$+ \int_\Omega \psi_k(x)\mu_c(x)d|C(u)|(x) \geq$$
$$\geq \int_\Omega \psi_k(x)g(x,\nabla u(x))dx + \int_{\Sigma(u)}\psi_k(x)h(x,[u](x),\nu(x))dH_{N-1}(x) + \int_\Omega \psi_k(x)g^\infty(x,dC(u)).$$

Letting $k \to +\infty$ and using Lebesgue's Monotone Convergence Theorem we conclude the result.

Step 2. We prove (4.10). By Theorems 2.3 i) and 2.6 for $\mathcal{L}_N$ a.e. $x_0 \in \Omega$ the following hold:

$$\lim_{\epsilon\to0^+}\frac{1}{\epsilon}\fint_{B(x_0,\epsilon)}|u(x) - u(x_0) - \nabla u(x_0)\cdot(x - x_0)|dx = 0, \tag{4.13}$$

$$\mu_a(x_0) = \lim_{\epsilon \to 0^+} \frac{\mu(Q(x_0,\epsilon))}{\mathcal{L}_N(Q(x_0,\epsilon))} \text{ exists and is finite.} \tag{4.14}$$

Select a point $x_0 \in \Omega$ such that the above properties hold. Let $0 < \eta < 1$ and let $\psi \in C_0^\infty(Q)$ be such that $0 \leq \psi \leq 1$ and $\psi = 1$ on ηQ. Using (4.14) we have

$$\begin{aligned}
\mu_a(x_0) &= \lim_{\delta \to 0^+} \frac{\mu(Q(x_0,\delta))}{\delta^N} \geq \limsup_{\delta \to 0^+} \frac{1}{\delta^N} \int_{x_0+\delta Q} \psi\left(\frac{x-x_0}{\delta}\right) d\mu(x) = \\
&= \limsup_{\delta \to 0^+} \frac{1}{\delta^N} \lim_{n \to +\infty} \left[\int_{x_0+\delta Q} \psi\left(\frac{x-x_0}{\delta}\right) f(x, \nabla u_n(x)) dx + \right. \\
&\left. + \int_{(x_0+\delta Q)\cap \Sigma(u_n)} \psi\left(\frac{x-x_0}{\delta}\right) \varphi(x, [u_n](x), \nu_n(x)) dH_{N-1}(x) \right] \geq \\
&\geq \limsup_{\delta \to 0^+} \limsup_{n \to +\infty} \left[\int_{\eta Q} f(x_0+\delta y, \nabla u_n(x_0+\delta y)) dy + \right. \\
&\left. + \frac{1}{\delta} \int_{\eta Q \cap \frac{\Sigma(u_n)-x_0}{\delta}} \varphi(x_0+\delta y, [u_n](x_0+\delta y), \nu_n(x_0+\delta y)) \, dH_{N-1}(y) \right].
\end{aligned} \tag{4.15}$$

Defining

$$u_{n,\delta}(y) := \frac{u_n(x_0+\delta y) - u(x_0)}{\delta}$$

and

$$w_0(y) := \nabla u(x_0) y$$

it follows by (4.13) that

$$\begin{aligned}
&\lim_{\delta \to 0^+} \lim_{n \to +\infty} \|u_{n,\delta} - w_0\|_{L^1(B(0,1);\mathbb{R}^p)} = \lim_{\delta \to 0^+} \frac{1}{\delta} \int_{B(0,1)} |u(x_0+\delta y) - u(x_0) - \delta \nabla u(x_0) y| dy = \\
&= \lim_{\delta \to 0^+} \frac{1}{\delta^{N+1}} \int_{B(x_0,\delta)} |u(x) - u(x_0) - \nabla u(x_0)(x-x_0)| dx = 0.
\end{aligned}$$

Also, as $\nabla u_{n,\delta}(y) = \nabla u_n(x_0+\delta y)$ and $\frac{\Sigma(u_n)-x_0}{\delta} = \Sigma(u_{n,\delta})$, if we define $\nu_{n,\delta}(y) := \nu_n(x_0+\delta y)$ we obtain from (4.15)

$$\begin{aligned}
\mu_a(x_0) \;&\geq\; \limsup_{\delta \to 0^+} \limsup_{n \to +\infty} \left[\int_{\eta Q} f(x_0+\delta y, \nabla u_{n,\delta}(y)) dy + \right. \\
&+\; \left. \frac{1}{\delta} \int_{\eta Q \cap \Sigma(u_{n,\delta})} \varphi(x_0+\delta y, \delta[u_{n,\delta}](y), \nu_{n,\delta}(y)) dH_{N-1}(y) \right] \geq \\
&\geq\; \liminf_{\delta \to 0^+} \liminf_{n \to +\infty} \left[\int_{\eta Q} f(x_0, \nabla u_{n,\delta}(y)) dy + \right. \\
&+\; \left. \int_{\eta Q \cap \Sigma(u_{n,\delta})} \frac{1}{\delta} \varphi(x_0, \delta[u_{n,\delta}](y), \nu_{n,\delta}(y)) dH_{N-1}(y) \right]
\end{aligned} \tag{4.16}$$

$$+ \liminf_{\delta\to 0^+}\liminf_{n\to+\infty}\left[\int_{\eta Q} f(x_0+\delta y,\nabla u_{n,\delta}(y)) - f(x_0,\nabla u_{n,\delta}(y))dy\right] +$$

$$+ \liminf_{\delta\to 0^+}\liminf_{n\to+\infty}\left[\int_{\eta Q\cap\Sigma(u_{n,\delta})} \frac{1}{\delta}\varphi(x_0+\delta y,\delta[u_{n,\delta}](y),\nu_{n,\delta}(y)) - \right.$$

$$\left. - \frac{1}{\delta}\varphi(x_0,\delta[u_{n,\delta}](y),\nu_{n,\delta}(y))dH_{N-1}(y)\right].$$

Fix $\epsilon > 0$. Then by (H2), (H9), (4.16) and for $\delta > 0$ small enough

$$\int_{\eta Q}\left|f(x_0+\delta y,\nabla u_{n,\delta}(y)) - f(x_0,\nabla u_{n,\delta}(y))\right| dy \leq$$

$$\leq \int_{\eta Q}\epsilon C\left(1+\|\nabla u_{n,\delta}(y)\|\right) dy \leq$$

$$\leq \epsilon C + \int_{\eta Q}\epsilon C f(x_0+\delta y,\nabla u_{n,\delta}(y))dy = O(\epsilon)$$

and by (H5), (H9) and (4.16)

$$\int_{\eta Q\cap\Sigma(u_{n,\delta})} \frac{1}{\delta}\left|\varphi(x_0+\delta y,\delta[u_{n,\delta}](y),\nu_{n,\delta}(y)) - \varphi(x_0,\delta[u_{n,\delta}](y),\nu_{n,\delta}(y))\right| dH_{N-1}(y) \leq$$

$$\leq \int_{\eta Q\cap\Sigma(u_{n,\delta})} C\epsilon|[u_{n,\delta}](y)|dH_{N-1}(y) \leq$$

$$\leq \epsilon C\int_{\eta Q\cap\Sigma(u_{n,\delta})} \varphi(x_0+\delta y,[u_{n,\delta}](y),\nu_{n,\delta}(y))dH_{N-1}(y) = O(\epsilon).$$

Therefore we obtain

$$\mu_a(x_0) \geq \liminf_{\delta\to 0^+}\liminf_{n\to+\infty}\left[\int_{\eta Q} f(x_0,\nabla u_{n,\delta}(y))dy + \right.$$

$$\left. + \int_{\eta Q\cap\Sigma(u_{n,\delta})} \frac{1}{\delta}\varphi(x_0,\delta[u_{n,\delta}](y),\nu_{n,\delta}(y))dH_{N-1}(y)\right] + O(\epsilon).$$

We can choose a sequence $\delta_k \to 0^+$ such that

$$\liminf_{n\to+\infty}\left[\int_{\eta Q} f(x_0,\nabla u_{n,\delta_k}(y))dy + \right.$$

$$\left. + \int_{\eta Q\cap\Sigma(u_{n,\delta_k})} \frac{1}{\delta_k}\varphi(x_0,\delta_k[u_{n,\delta_k}](y),\nu_{n,\delta_k}(y))dH_{N-1}(y)\right] + O(\epsilon) \leq$$

$$\leq \liminf_{\delta\to 0^+}\liminf_{n\to+\infty}\left[\int_{\eta Q} f(x_0,\nabla u_{n,\delta}(y))dy + \right.$$

$$\left. + \int_{\eta Q\cap\Sigma(u_{n,\delta})} \frac{1}{\delta}\varphi(x_0,\delta[u_{n,\delta}](y),\nu_{n,\delta}(y))dH_{N-1}(y)\right] + O(\epsilon) + \frac{1}{k}.$$

Choose n_k large enough so that $\|u_{n_k,\delta_k} - w_0\|_{L^1(Q;\mathbf{R}^p)} \leq \frac{1}{k}$ and

$$\liminf_{n\to+\infty}\left[\int_{\eta Q} f(x_0,\nabla u_{n,\delta_k}(y))dy + \right.$$
$$\left. + \int_{\eta Q\cap\Sigma(u_{n,\delta_k})} \frac{1}{\delta_k}\varphi(x_0,\delta_k[u_{n,\delta_k}](y),\nu_{n,\delta_k}(y))dH_{N-1}(y)\right] \geq$$
$$\geq \left[\int_{\eta Q} f(x_0,\nabla u_{n_k,\delta_k}(y))dy + \right.$$
$$\left. + \int_{\eta Q\cap\Sigma(u_{n_k,\delta_k})} \frac{1}{\delta_k}\varphi(x_0,\delta_k[u_{n_k,\delta_k}](y),\nu_{n_k,\delta_k}(y))dH_{N-1}(y)\right] - \frac{1}{k}.$$

Thus, defining $v_k(y) := u_{n_k,\delta_k}(y)$ and $\nu_k(y) := \nu_{n_k,\delta_k}(y)$ it follows that for all $0 < \eta < 1$

$$\mu_a(x_0) \geq \lim_{k\to+\infty}\left[\int_{\eta Q} f(x_0,\nabla v_k(y))dy + \int_{\eta Q\cap\Sigma(v_k)} \frac{1}{\delta_k}\varphi(x_0,\delta_k[v_k](y),\nu_k(y))dH_{N-1}(y)\right] + O(\epsilon) \tag{4.17}$$

where $v_k \to \nabla u(x_0)y$ in $L^1(Q;\mathbf{R}^p)$. By (H9) and (4.17) it follows that $\sup_k |Dv_k|(\eta Q) < +\infty$ and so by Lemma 3.7 there exists a sequence $\bar{v}_k \in SBV(\eta Q;\mathbf{R}^p)$ such that $\bar{v}_k \to \nabla u(x_0)y$ in $L^1(\eta Q;\mathbf{R}^p)$, $\sup_k \|\bar{v}_k\|_{L^\infty(\eta Q;\mathbf{R}^p)} = C(\epsilon) < +\infty$ and

$$\mu_a(x_0) \geq \lim_{k\to+\infty}\left[\int_{\eta Q} f(x_0,\nabla\bar{v}_k(x))dx + \frac{1}{\delta_k}\int_{\eta Q\cap\Sigma(\bar{v}_k)} \varphi(x_0,\delta_k[\bar{v}_k](x),\bar{\nu}_k(x))dH_{N-1}(x)\right] + O(\epsilon). \tag{4.18}$$

For k large enough so that $\delta_k|[\bar{v}_k]| < l$ (H7) yields

$$\begin{aligned}\mu_a(x_0) \quad &\geq \quad \liminf_{k\to+\infty}\left[\int_{\eta Q} f(x_0,\nabla\bar{v}_k(x))dx + \int_{\eta Q\cap\Sigma(\bar{v}_k)} \varphi_0(x_0,[\bar{v}_k](x),\bar{\nu}_k(x))dH_{N-1}(x)\right] - \\ &- \quad \limsup_{k\to+\infty}\int_{\eta Q\cap\Sigma(\bar{v}_k)} \delta_k^\alpha|[\bar{v}_k](x)|^{\alpha+1}dH_{N-1}(x) + O(\epsilon)\end{aligned}$$

where by (H9) and (4.18)

$$\int_{\eta Q\cap\Sigma(\bar{v}_k)} \delta_k^\alpha|[\bar{v}_k](x)|^{\alpha+1}dH_{N-1}(x) \leq$$
$$\leq \delta_k^\alpha\left(\sup_k\|\bar{v}_k\|_\infty\right)^\alpha\int_{\eta Q\cap\Sigma(\bar{v}_k)} |[\bar{v}_k](x)|dH_{N-1}(x) \to 0 \text{ as } k\to+\infty.$$

Thus

$$\mu_a(x_0) \geq \lim_{k\to+\infty}\left[\int_{\eta Q} f(x_0,\nabla\bar{v}_k(y))dy + \int_{\eta Q\cap\Sigma(\bar{v}_k)} \varphi_0(x_0,[\bar{v}_k](y),\bar{\nu}_k(y))dH_{N-1}(y)\right] + O(\epsilon) \tag{4.19}$$

where $\bar{v}_k \to \nabla u(x_0)y$ in $L^1(\eta Q;\mathbf{R}^p)$. To compare $\mu_a(x_0)$ with $g(x_0,\nabla u(x_0))$ we must modify the sequence $\bar{v}_k$ so that it satisfies the boundary condition and in such a way that the total energy does not increase. This is achieved by using Lemma 3.1 and Remark 3.2 applied to ηQ instead of Q and to the functional

$$\mathcal{G}_{x_0}(u,A)=\int_A f(x_0,\nabla u(x))dx+\int_{A\cap\Sigma(u)}\varphi_0(x_0,[u](x),\nu(x))dH_{N-1}(x).$$

We obtain a sequence $\{\xi_i\}$ in $SBV(Q;\mathbf{R}^p)$ such that $\xi_i=\nabla u(x_0)y$ on $Q\setminus\eta Q$, $\xi_i\to\nabla u(x_0)y$ in $L^1(Q;\mathbf{R}^p)$ and

$$\begin{aligned}&\limsup_{i\to+\infty}\left[\int_{\eta Q}f(x_0,\nabla\xi_i(x))dx+\int_{\eta Q\cap\Sigma(\xi_i)}\varphi_0(x_0,[\xi_i](x),\theta_i(x))dH_{N-1}(x)\right]\le\\&\le\liminf_{k\to+\infty}\left[\int_{\eta Q}f(x_0,\nabla\bar v_k(x))dx+\int_{\eta Q\cap\Sigma(\bar v_k)}\varphi_0(x_0,[\bar v_k](x),\bar\nu_k(x))dH_{N-1}(x)\right]\end{aligned}\tag{4.20}$$

where $\theta_i(x)$ is the normal to $\Sigma(\xi_i)$ at $x\in\eta Q\cap\Sigma(\xi_i)$. From (4.19) and (4.20), using the fact that $Q\cap\Sigma(\xi_i)=\eta Q\cap\Sigma(\xi_i)$, it follows that

$$\begin{aligned}\mu_a(x_0)\ \ge\ &\liminf_{i\to+\infty}\left[\int_Q f(x_0,\nabla\xi_i(x))dx+\int_{Q\cap\Sigma(\xi_i)}\varphi_0(x_0,[\xi_i](x),\theta_i(x))dH_{N-1}(x)-\right.\\&\left.-\int_{Q\setminus\eta Q}f(x_0,\nabla u(x_0))dx\right]+O(\epsilon)\ge\\&\ge g(x_0,\nabla u(x_0))+O(1-\eta)+O(\epsilon).\end{aligned}$$

The result now follows if we let first $\epsilon\to0^+$ and then $\eta\to1^-$.

Step 3. We now prove (4.11). By Lemma 2.5 and Theorems 2.3 ii) and 2.6 we know that for a.e. $x_0\in\Sigma(u)$

$$\lim_{\delta\to0^+}\frac{1}{\delta^{N-1}}\int_{\Sigma(u)\cap(x_0+\delta Q_{\nu(x_0)})}|u^-(x)-u^+(x)|dH_{N-1}(x)=|u^-(x_0)-u^+(x_0)|,\tag{4.21}$$

$$\lim_{\delta\to0^+}\frac{1}{\delta^N}\int_{\{x\in B(x_0,\delta):(x-x_0)\cdot\nu(x_0)>0\}}|u(x)-u^+(x_0)|^{\frac{N}{N-1}}dx=0,\tag{4.22}$$

$$\lim_{\delta\to0^+}\frac{1}{\delta^N}\int_{\{x\in B(x_0,\delta):(x-x_0)\cdot\nu(x_0)<0\}}|u(x)-u^-(x_0)|^{\frac{N}{N-1}}dx=0,\tag{4.23}$$

$$\mu_J(x_0)=\lim_{\delta\to0^+}\frac{\mu(x_0+\delta Q_{\nu(x_0)})}{|u^+-u^-|H_{N-1}\lfloor\Sigma(u)(x_0+\delta Q_{\nu(x_0)})}\ \text{exists and is finite.}\tag{4.24}$$

Choose a point $x_0\in\Omega\cap\Sigma(u)$ such that the above properties hold. For simplicity of notation write $Q:=Q_{\nu(x_0)}$, $Q^*:=\frac{1}{1+\eta}Q$ with $0<\eta<1$ and let $\psi\in C_0^\infty(Q)$ be such that $0\le\psi\le1$, $\psi=1$ on Q^*. Then, by (4.24) and (4.21)

$$\begin{aligned}\mu_J(x_0)=&\lim_{\delta\to0^+}\frac{\mu(x_0+\delta Q)}{|u^+-u^-|H_{N-1}\lfloor\Sigma(u)(x_0+\delta Q)}=\\=&\frac{1}{|u^+(x_0)-u^-(x_0)|}\lim_{\delta\to0^+}\frac{1}{\delta^{N-1}}\int_{x_0+\delta Q}d\mu(x)\ge\end{aligned}$$

$$\geq \frac{1}{|u^+(x_0)-u^-(x_0)|}\limsup_{\delta\to 0^+}\frac{1}{\delta^{N-1}}\int_{x_0+\delta Q}\psi\left(\frac{x-x_0}{\delta}\right)d\mu(x) =$$
$$= \frac{1}{|u^+(x_0)-u^-(x_0)|}\limsup_{\delta\to 0^+}\lim_{n\to+\infty}\frac{1}{\delta^{N-1}}\left[\int_{x_0+\delta Q}\psi\left(\frac{x-x_0}{\delta}\right)f(x,\nabla u_n(x))dx+\right.$$
$$\left.+\int_{(x_0+\delta Q)\cap\Sigma(u_n)}\psi\left(\frac{x-x_0}{\delta}\right)\varphi(x,[u_n](x),\nu_n(x))dH_{N-1}(x)\right]\geq$$
$$\geq \frac{1}{|u^+(x_0)-u^-(x_0)|}\limsup_{\delta\to 0^+}\limsup_{n\to+\infty}\left[\delta\int_{Q^*}f(x_0+\delta y,\nabla u_n(x_0+\delta y))dy+\right.$$
$$\left.\int_{Q^*\cap\frac{\Sigma(u_n)-x_0}{\delta}}\varphi(x_0+\delta y,[u_n](x_0+\delta y),\nu_n(x_0+\delta y))dH_{N-1}(y)\right]. \tag{4.25}$$

Define $u_{n,\delta}(y) := u_n(x_0+\delta y)$, $\nu_{n,\delta}(y) := \nu_n(x_0+\delta y)$ and

$$u_0(y) := \begin{cases} u^+(x_0) & \text{if } y\cdot\nu(x_0)>0 \\ u^-(x_0) & \text{if } y\cdot\nu(x_0)<0. \end{cases}$$

As $u_n \to u$ in $L^1(\Omega;\mathbf{R}^p)$ by (4.22) and (4.23) we obtain

$$\lim_{\delta\to 0^+}\lim_{n\to+\infty}\int_Q |u_{n,\delta}(y)-u_0(y)|dy =$$
$$= \lim_{\delta\to 0^+}\frac{1}{\delta^N}\int_{\{x\in x_0+\delta Q:(x-x_0)\cdot\nu(x_0)>0\}}|u(x)-u^+(x_0)|dx +$$
$$+ \lim_{\delta\to 0^+}\frac{1}{\delta^N}\int_{\{x\in x_0+\delta Q:(x-x_0)\cdot\nu(x_0)<0\}}|u(x)-u^-(x_0)|dx = 0.$$

On the other hand from (4.25), taking into account that $\Sigma(u_{n,\delta}) = \frac{\Sigma(u_n)-x_0}{\delta}$, one gets

$$\mu_J(x_0) \geq \frac{1}{|u^+(x_0)-u^-(x_0)|}\liminf_{\delta\to 0^+}\liminf_{n\to+\infty}\left[\int_{Q^*}f^\infty(x_0+\delta y,\nabla u_{n,\delta}(y))dy+\right.$$
$$+\int_{Q^*\cap\Sigma(u_{n,\delta})}\varphi(x_0+\delta y,[u_{n,\delta}](y),\nu_{n,\delta}(y))dH_{N-1}(y)+$$
$$\left.+\int_{Q^*}\delta f(x_0+\delta y,\frac{1}{\delta}\nabla u_{n,\delta}(y))-f^\infty(x_0+\delta y,\nabla u_{n,\delta}(y))dy\right]. \tag{4.26}$$

Now,

$$\int_{Q^*}\left|\delta f(x_0+\delta y,\frac{1}{\delta}\nabla u_{n,\delta}(y))-f^\infty(x_0+\delta y,\nabla u_{n,\delta}(y))\right|dy =$$
$$= \int_{Q^*\cap\{\|\nabla u_{n,\delta}\|\leq\delta L\}}\left|\delta f(x_0+\delta y,\frac{1}{\delta}\nabla u_{n,\delta}(y))-f^\infty(x_0+\delta y,\nabla u_{n,\delta}(y))\right|dy +$$
$$+\int_{Q^*\cap\{\|\nabla u_{n,\delta}\|>\delta L\}}\left|\delta f(x_0+\delta y,\frac{1}{\delta}\nabla u_{n,\delta}(y))-f^\infty(x_0+\delta y,\nabla u_{n,\delta}(y))\right|dy =: I_1+I_2.$$

By (H1) and the growth condition on f^∞

$$I_1 \le \int_{Q^* \cap \{\|\nabla u_{n,\delta}\| \le \delta L\}} \delta C_1 + 2C_1 \|\nabla u_{n,\delta}(y)\| dy \le C\delta = O(\delta)$$

and by (H3) with $t = \frac{1}{\delta}$, Hölder's inequality and (H9) we have

$$I_2 \le \int_{Q^* \cap \{\|\nabla u_{n,\delta}\| > \delta L\}} C \|\nabla u_{n,\delta}(y)\|^{1-m} \delta^m dy \le C\delta^m \left[\int_{Q^*} \|\nabla u_{n,\delta}(y)\| dy\right]^{1-m} =$$

$$= C\delta^m \left[\int_{Q^*} \delta \|\nabla u_n(x_0 + \delta y)\| dy\right]^{1-m} \le$$

$$\le C\delta^m \left[\int_{Q^*} C\delta + C\delta f(x_0 + \delta y, \nabla u_n(x_0 + \delta y)) dy\right]^{1-m} = O(\delta) + O(\delta^m)$$

since by (4.25) $\left\{\int_{Q^*} \delta f(x_0 + \delta y, \nabla u_n(x_0 + \delta y)) dy\right\}$ remains bounded. Therefore (4.26) reduces to

$$\begin{aligned}
\mu_J(x_0) \ge \frac{1}{|u^+(x_0) - u^-(x_0)|} \liminf_{\delta \to 0^+} \liminf_{n \to +\infty} \Bigg[\int_{Q^*} f^\infty(x_0 + \delta y, \nabla u_{n,\delta}(y)) dy + \\
+ \int_{Q^* \cap \Sigma(u_{n,\delta})} \varphi(x_0 + \delta y, [u_{n,\delta}](y), \nu_{n,\delta}(y)) dH_{N-1}(y)\Bigg] \\
\ge \frac{1}{|u^+(x_0) - u^-(x_0)|} \liminf_{\delta \to 0^+} \liminf_{n \to +\infty} \Bigg[\int_{Q^*} f^\infty(x_0, \nabla u_{n,\delta}(y)) dy + \\
+ \int_{Q^* \cap \Sigma(u_{n,\delta})} \varphi(x_0, [u_{n,\delta}](y), \nu_{n,\delta}(y)) dH_{N-1}(y)\Bigg] + \\
+ \frac{1}{|u^+(x_0) - u^-(x_0)|} \liminf_{\delta \to 0^+} \liminf_{n \to +\infty} \left[\int_{Q^*} f^\infty(x_0 + \delta y, \nabla u_{n,\delta}(y)) - f^\infty(x_0, \nabla u_{n,\delta}(y)) dy\right] + \\
+ \frac{1}{|u^+(x_0) - u^-(x_0)|} \liminf_{\delta \to 0^+} \liminf_{n \to +\infty} \Bigg[\int_{Q^* \cap \Sigma(u_{n,\delta})} \varphi(x_0 + \delta y, [u_{n,\delta}](y), \nu_{n,\delta}(y)) - \\
- \varphi(x_0, [u_{n,\delta}](y), \nu_{n,\delta}(y)) dH_{N-1}(y)\Bigg].
\end{aligned} \tag{4.27}$$

Fix $\epsilon > 0$. It follows from (H2), (H9) and (4.27) that for δ small enough

$$\begin{aligned}
&\int_{Q^*} \left| f^\infty(x_0 + \delta y, \nabla u_{n,\delta}(y)) - f^\infty(x_0, \nabla u_{n,\delta}(y)) \right| dy \le \\
&\le \int_{Q^*} \epsilon C_1 \|\nabla u_{n,\delta}(y)\| dy \le \\
&\le \int_{Q^*} \epsilon C f^\infty(x_0 + \delta y, \nabla u_{n,\delta}(y)) dy = O(\epsilon).
\end{aligned}$$

Also by (H5), (H9) and (4.27) we have

$$\int_{Q^* \cap \Sigma(u_{n,\delta})} \left| \varphi(x_0 + \delta y, [u_{n,\delta}](y), \nu_{n,\delta}(y)) - \varphi(x_0, [u_{n,\delta}](y), \nu_{n,\delta}(y)) \right| dH_{N-1}(y)$$

$$\leq \int_{Q^*\cap\Sigma(u_{n,\delta})} \epsilon C \left|[u_{n,\delta}](y)\right| dH_{N-1}(y) \leq$$

$$\leq \int_{Q^*\cap\Sigma(u_{n,\delta})} \epsilon C \varphi(x_0 + \delta y, [u_{n,\delta}](y), \nu_{n,\delta}(y)) dH_{N-1}(y) = O(\epsilon).$$

Hence,

$$\mu_J(x_0) \geq \frac{1}{|u^+(x_0) - u^-(x_0)|} \liminf_{\delta\to 0^+} \liminf_{n\to+\infty} \left[\int_{Q^*} f^\infty(x_0, \nabla u_{n,\delta}(y))dy + \right.$$

$$\left. + \int_{Q^*\cap\Sigma(u_{n,\delta})} \varphi(x_0, [u_{n,\delta}](y), \nu_{n,\delta}(y)) dH_{N-1}(y)\right] + O(\epsilon)$$

where $u_{n,\delta} \to u_0$ in $L^1(Q;\mathbf{R}^p)$ as $n \to +\infty$, $\delta \to 0^+$ and so, by a standard diagonalization argument, as used in Step 2, we obtain

$$\mu_J(x_0) \geq \frac{1}{|u^+(x_0) - u^-(x_0)|} \lim_{k\to+\infty} \left[\int_{Q^*} f^\infty(x_0, \nabla v_k(y))dy + \right.$$

$$\left. + \int_{Q^*\cap\Sigma(v_k)} \varphi(x_0, [v_k](y), \mu_k(y)) dH_{N-1}(y)\right] + O(\epsilon)$$

where $v_k \to u_0$ in $L^1(Q;\mathbf{R}^p)$. Making the change of variables $y = \frac{x}{1+\eta}$, setting $w_k(x) := v_k\left(\frac{x}{1+\eta}\right)$ and using the invariance of u_0 under the above change of variables we have $w_k \to u_0$ in $L^1(Q;\mathbf{R}^p)$ and

$$\mu_J(x_0) \geq \frac{1}{|u^+(x_0) - u^-(x_0)|} \left(\frac{1}{1+\eta}\right)^{N-1} \lim_{k\to+\infty} \left[\int_Q f^\infty(x_0, \nabla w_k(x))dx + \right.$$

$$\left. + \int_{Q\cap\Sigma(w_k)} \varphi(x_0, [w_k](x), \eta_k(x)) dH_{N-1}(x)\right] + O(\epsilon) \tag{4.28}$$

where $\eta_k(x) = \mu_k\left(\frac{x}{1+\eta}\right)$. To compare $\mu_J(x_0)|u^+(x_0) - u^-(x_0)|$ with $h(x_0, [u](x_0), \nu(x_0))$ we must modify the sequence $\{w_k\}$ in such a way that it meets the boundary condition and the total energy does not increase. This is done by using Lemma 3.1 and Remark 3.2 applied to the functional

$$\mathcal{G}_{x_0}(u, A) = \int_A f^\infty(x_0, \nabla u(x))dx + \int_{A\cap\Sigma(u)} \varphi(x_0, [u](x), \nu(x)) dH_{N-1}(x).$$

We obtain a sequence $\{\xi_i\} \in SBV(Q;\mathbf{R}^p)$ such that $\xi_i \to u_0$ in $L^1(Q;\mathbf{R}^p)$, $\xi_i = u_0$ on ∂Q and

$$\limsup_{i\to+\infty} \left[\int_Q f^\infty(x_0, \nabla \xi_i(x))dx + \int_{Q\cap\Sigma(\xi_i)} \varphi(x_0, [\xi_i](x), \theta_i(x)) dH_{N-1}(x)\right] \leq$$

$$\leq \liminf_{k\to+\infty} \left[\int_Q f^\infty(x_0, \nabla w_k(x))dx + \int_{Q\cap\Sigma(w_k)} \varphi(x_0, [w_k](x), \eta_k(x)) dH_{N-1}(x)\right] \tag{4.29}$$

where $\theta_i(x)$ is the normal to $\Sigma(\xi_i)$ at $x \in Q \cap \Sigma(\xi_i)$. From (4.28) and (4.29) we get

$$\mu_J(x_0) \geq \frac{(1+\eta)^{1-N}}{|u^+(x_0) - u^-(x_0)|} \liminf_{i\to+\infty} \left[\int_Q f^\infty(x_0, \nabla \xi_i(x))dx + \right.$$

$$+\int_{Q\cap\Sigma(\xi_i)}\varphi(x_0,[\xi_i](x),\theta_i(x))dH_{N-1}(x)\Big]+O(\epsilon)\geq$$

$$\geq\frac{(1+\eta)^{1-N}}{|u^+(x_0)-u^-(x_0)|}h(x_0,[u](x_0),\nu(x_0))+O(\epsilon).$$

Letting $\epsilon\to 0^+$ and $\eta\to 0^+$ we conclude that

$$\mu_J(x_0)\geq\frac{1}{|u^+(x_0)-u^-(x_0)|}h(x_0,[u](x_0),\nu(x_0)).$$

Step 4. Finally, we obtain a lower bound for the density of the Cantor part, namely (4.12). Let $Q=\left(-\frac{1}{2},\frac{1}{2}\right)^N$ and fix $t\in(0,1)$ with t close to 1. By Proposition 2.8, Lemma 2.9 and Theorem 2.6 we know that for $|C(u)|$ a.e. $x_0\in\Omega$ the following hold:

$$\lim_{\epsilon\to 0^+}\frac{|Du|(x_0+\epsilon Q)}{|C(u)|(x_0+\epsilon Q)}=1; \tag{4.30}$$

$$\liminf_{\epsilon\to 0^+}\frac{|Du|\left((x_0+\epsilon Q)\setminus(x_0+t\epsilon Q)\right)}{|Du|(x_0+\epsilon Q)}=O(1-t); \tag{4.31}$$

$$\lim_{\epsilon\to 0^+}\frac{|Du|(x_0+\epsilon Q)}{\epsilon^{N-1}}=0,\ \lim_{\epsilon\to 0^+}\frac{|Du|(x_0+\epsilon Q)}{\epsilon^N}=+\infty; \tag{4.32}$$

$$A_0:=\lim_{\epsilon\to 0^+}\frac{Du(x_0+\epsilon Q)}{|Du|(x_0+\epsilon Q)}\text{ exists and }\|A_0\|=1, A_0=a\otimes\nu; \tag{4.33}$$

$$\mu_c(x_0)=\lim_{\epsilon\to 0^+}\frac{\mu(x_0+\epsilon Q)}{|C(u)|(x_0+\epsilon Q)}=\lim_{\epsilon\to 0^+}\frac{\mu(x_0+\epsilon Q)}{|Du|(x_0+\epsilon Q)}. \tag{4.34}$$

Choose a point $x_0\in\Omega$ such that (4.30)–(4.34) hold and without loss of generality assume that $\nu=e_N$, $A_0=a\otimes e_N$ with $|a|=1$. Let $\psi\in C_0(Q)$ be such that $0\leq\psi\leq 1$ and $\psi\equiv 1$ on tQ. Then,

$$\begin{aligned}
\mu_c(x_0) &= \lim_{\epsilon\to 0^+}\frac{\mu(x_0+\epsilon Q)}{|Du|(x_0+\epsilon Q)}=\lim_{\epsilon\to 0^+}\frac{1}{|Du|(x_0+\epsilon Q)}\int_{x_0+\epsilon Q}d\mu(x)\geq\\
&\geq \limsup_{\epsilon\to 0^+}\frac{1}{|Du|(x_0+\epsilon Q)}\int_{x_0+\epsilon Q}\psi\left(\frac{x-x_0}{\epsilon}\right)d\mu(x)=\\
&= \limsup_{\epsilon\to 0^+}\frac{1}{|Du|(x_0+\epsilon Q)}\lim_{n\to+\infty}\Big[\int_{x_0+\epsilon Q}\psi\left(\frac{x-x_0}{\epsilon}\right)f(x,\nabla u_n(x))dx+\\
&\qquad +\int_{(x_0+\epsilon Q)\cap\Sigma(u_n)}\psi\left(\frac{x-x_0}{\epsilon}\right)\varphi(x,[u_n](x),\nu_n(x))dH_{N-1}(x)\Big]\geq\\
&\geq \limsup_{\epsilon\to 0^+}\limsup_{n\to+\infty}\frac{1}{|Du|(x_0+\epsilon Q)}\Big[\int_{x_0+t\epsilon Q}f(x,\nabla u_n(x))dx+\\
&\qquad +\int_{(x_0+t\epsilon Q)\cap\Sigma(u_n)}\varphi(x,[u_n](x),\nu_n(x))dH_{N-1}(x)\Big].
\end{aligned} \tag{4.35}$$

Also,

$$\begin{aligned}
\lim_{\epsilon\to 0^+}\lim_{n\to+\infty}\frac{\epsilon^{N-1}}{|Du|(x_0+\epsilon Q)}\int_Q\Big|u_n(x_0+\epsilon x)-\frac{1}{\epsilon^N}\int_{x_0+\epsilon Q}u_n(y)dy-\\
-\left[u(x_0+\epsilon x)-\frac{1}{\epsilon^N}\int_{x_0+\epsilon Q}u(y)dy\right]\Big|dx=0.
\end{aligned} \tag{4.36}$$

By (4.31) choose a sequence $r_k \to 0^+$ such that

$$\frac{|Du|\left((x_0 + r_kQ) \setminus (x_0 + t^2 r_kQ)\right)}{|Du|(x_0 + r_kQ)} = O(1-t) \tag{4.37}$$

and by (4.35) and (4.36), using a standard diagonalization argument, choose a subsequence $\{u_k\}$ (not relabelled) such that

$$\mu_c(x_0) \geq \limsup_{k\to+\infty} \frac{1}{|Du|(x_0 + r_kQ)} \left[\int_{x_0+tr_kQ} f(x, \nabla u_k(x))dx + \right.$$
$$\left. + \int_{(x_0+tr_kQ)\cap\Sigma(u_k)} \varphi(x, [u_k](x), \nu_k(x))dH_{N-1}(x)\right] \tag{4.38}$$

and

$$\|\bar{u}_k - z_k\|_{L^1(Q;\mathbb{R}^p)} \to 0 \tag{4.39}$$

where

$$\bar{u}_k(x) := \frac{r_k^{N-1}}{|Du|(x_0 + r_kQ)} \left[u_k(x_0 + r_kx) - \frac{1}{r_k^N}\int_{x_0+r_kQ} u_k(y)dy\right],$$
$$z_k(x) := \frac{r_k^{N-1}}{|Du|(x_0 + r_kQ)} \left[u(x_0 + r_kx) - \frac{1}{r_k^N}\int_{x_0+r_kQ} u(y)dy\right].$$

By (H2), (H5), (H9), (4.32) and (4.38)

$$\frac{1}{|Du|(x_0 + r_kQ)} \int_{x_0+tr_kQ} (dx + |Du_k|)$$

is bounded and so we obtain

$$\mu_c(x_0) \geq \limsup_{k\to+\infty} \frac{1}{|Du|(x_0 + r_kQ)} \left[\int_{x_0+tr_kQ} f(x_0, \nabla u_k(x))dx \right.$$
$$\left. + \int_{(x_0+tr_kQ)\cap\Sigma(u_k)} \varphi(x_0, [u_k](x), \nu_k(x))dH_{N-1}(x)\right].$$

Changing variables and setting

$$t_k := \frac{|Du|(x_0 + r_kQ)}{r_k^N} \to +\infty,$$

$$\theta_k := \frac{|Du|(x_0 + r_kQ)}{r_k^{N-1}} \to 0$$

we conclude that

$$\mu_c(x_0) \geq \limsup_{k\to+\infty} \left[\frac{1}{t_k}\int_{tQ} f(x_0, t_k\nabla\bar{u}_k(x))dx + \frac{1}{\theta_k}\int_{tQ\cap\Sigma(\bar{u}_k)} \varphi(x_0, \theta_k[\bar{u}_k](x), \nu_k(x))dH_{N-1}(x)\right]. \tag{4.40}$$

Now,

$$\int_{tQ}\left|\frac{1}{t_k}f(x_0,t_k\nabla\bar{u}_k(x))-f^\infty(x_0,\nabla\bar{u}_k(x))\right|dx =$$
$$=\int_{tQ\cap\{\|t_k\nabla\bar{u}_k\|\le L\}}\left|\frac{1}{t_k}f(x_0,t_k\nabla\bar{u}_k(x))-f^\infty(x_0,\nabla\bar{u}_k(x))\right|dx +$$
$$+\int_{tQ\cap\{\|t_k\nabla\bar{u}_k\|> L\}}\left|\frac{1}{t_k}f(x_0,t_k\nabla\bar{u}_k(x))-f^\infty(x_0,\nabla\bar{u}_k(x))\right|dx =: I_1+I_2$$

where, by (H1) and Lemma 2.12 i)

$$I_1 \le \int_{tQ\cap\{\|t_k\nabla\bar{u}_k\|\le L\}}\frac{1}{t_k}C\left(1+2\|t_k\nabla\bar{u}_k(x)\|\right)dx \le \frac{1}{t_k}C(1+2L)\to 0$$

and, by (H3), Hölder's inequality, (H9) and (4.40)

$$I_2 \le \int_{tQ\cap\{\|t_k\nabla\bar{u}_k\|> L\}} C\|\nabla\bar{u}_k(x)\|^{1-m}t_k^{-m}dx \le \frac{C}{t_k^m}\left[\int_{tQ}\|\nabla\bar{u}_k(x)\|dx\right]^{1-m}\to 0.$$

Thus, from (4.40) we obtain

$$\mu_c(x_0)\ge \limsup_{k\to+\infty}\left[\int_{tQ}f^\infty(x_0,\nabla\bar{u}_k(x))dx+\frac{1}{\theta_k}\int_{tQ\cap\Sigma(\bar{u}_k)}\varphi(x_0,\theta_k[\bar{u}_k](x),\nu_k(x))dH_{N-1}(x)\right]. \tag{4.41}$$

Since

$$\int_Q z_k(x)dx=\int_Q \bar{u}_k(x)dx=0$$

and

$$|D\bar{u}_k|(Q)=|Dz_k|(Q)=1$$

by (4.39) there exist subsequences (not relabelled) $\{z_k\}$ $\{\bar{u}_k\}$ and there exists $u_0\in BV(Q;\mathbf{R}^p)$ such that

$$z_k,\ \bar{u}_k\to u_0 \text{ in } L^1(Q;\mathbf{R}^p). \tag{4.42}$$

Now,

$$Dz_k(Q)=\frac{Du(x_0+r_kQ)}{|Du|(x_0+r_kQ)}\to A_0=a\otimes e_N$$

where $|a|=1$ and $|Dz_k|(Q)=1$ so by Proposition 2.10 it follows that

$$|Dz_k-(Dz_k\cdot A_0)A_0|(Q)\to 0$$

from which we conclude that

$$|Dz_k\cdot e_i|(Q)\to 0 \text{ for } i=1,\dots,N-1.$$

Since

$$|Du_0\cdot e_i|(Q)\le\liminf_{k\to+\infty}|Dz_k\cdot e_i|(Q)=0$$

we obtain

$$u_0(x) = \hat{u}_0(x_N) \in BV((-\frac{1}{2},\frac{1}{2});\mathbf{R}^p) \cap L^\infty((-\frac{1}{2},\frac{1}{2});\mathbf{R}^p). \tag{4.43}$$

Consider the smooth mollifications of u_0, $\xi_k(x) = \rho_k * u_0(x) = \hat{\xi}_k(x_N)$. By (4.43) we have

$$||\xi_k||_{L^\infty(Q;\mathbf{R}^p)} \leq const.$$

and (4.42) implies that

$$||\bar{u}_k - \xi_k||_{L^1(Q;\mathbf{R}^p)} \to 0.$$

Thus, fixing $\epsilon > 0$, by Lemma 3.7 there exists a sequence $\bar{\bar{u}}_k \in BV(Q;\mathbf{R}^p)$ such that $||\bar{\bar{u}}_k||_{L^\infty(Q;\mathbf{R}^p)} \leq C(\epsilon)$, $||\bar{\bar{u}}_k - \xi_k||_{L^1(Q;\mathbf{R}^p)} \to 0$ and

$$\int_{tQ} f^\infty(x_0, \nabla\bar{\bar{u}}_k(x))dx + \frac{1}{\theta_k}\int_{tQ\cap\Sigma(\bar{\bar{u}}_k)} \varphi(x_0, \theta_k[\bar{\bar{u}}_k](x), \nu_k(x))dH_{N-1}(x) \leq$$
$$\leq \int_{tQ} f^\infty(x_0, \nabla\bar{u}_k(x))dx + \frac{1}{\theta_k}\int_{tQ\cap\Sigma(\bar{u}_k)} \varphi(x_0, \theta_k[\bar{u}_k](x), \nu_k(x))dH_{N-1}(x) + \epsilon.$$

Then, since for k large enough $\theta_k||[\bar{\bar{u}}_k]||_\infty < l$, (H7), (H9) and (4.41) yield

$$\int_{tQ\cap\Sigma(\bar{\bar{u}}_k)} \left|\frac{1}{\theta_k}\varphi(x_0, \theta_k[\bar{\bar{u}}_k](x), \nu_k(x)) - \varphi_0(x_0, [\bar{\bar{u}}_k](x), \nu_k(x))\right| dH_{N-1}(x) \leq$$
$$\leq \int_{tQ\cap\Sigma(\bar{\bar{u}}_k)} \theta_k^\alpha |[\bar{\bar{u}}_k](x)|^{\alpha+1} dH_{N-1}(x) \leq \theta_k^\alpha C(\epsilon) \int_{tQ\cap\Sigma(\bar{\bar{u}}_k)} |[\bar{\bar{u}}_k](x)| dH_{N-1}(x) \to 0$$

so from (4.41) we get

$$\mu_c(x_0) \geq \limsup_{k\to+\infty}\left[\int_{tQ} f^\infty(x_0, \nabla\bar{\bar{u}}_k(x))dx + \int_{tQ\cap\Sigma(\bar{\bar{u}}_k)} \varphi_0(x_0, [\bar{\bar{u}}_k](x), \nu_k(x))dH_{N-1}(x)\right] - \epsilon. \tag{4.44}$$

Since for a.e. $\tau \in (t^2, t)$, $|Du_0|(\partial\tau Q) = 0$ and $\nabla\xi_k(\tau Q) - Dz_k(\tau Q) \to 0$ choose one such τ and choose $\delta > 0$ such that $|Du_0|(\delta(1-\delta)\tau Q) = 0$ and $\tau(1-\delta) > t^2$. Then, by Lemma 2.4 and (4.37) we have

$$\lim_{k\to+\infty} |\nabla\xi_k|(\tau Q \setminus \tau(1-\delta)Q) \leq |Du_0|(\tau Q \setminus \tau(1-\delta)Q) \leq \lim_{k\to+\infty} |Dz_k|(\tau Q \setminus \tau(1-\delta)Q)$$
$$\leq \lim_{k\to+\infty} \frac{|Du|((x_0 + r_k Q) \setminus (x_0 + t^2 r_k Q))}{|Du|(x_0 + r_k Q)} = O(1-t). \tag{4.45}$$

On τQ we have $\xi_k(x) = A_k x + p_k(x)$ where $p_k(x) = \hat{p}_k(x_N)$ is smooth, $\hat{p}_k(t) = 0$ if $t = \pm\frac{\tau}{2}$ and

$$A_k = \frac{\hat{\xi}_k(\frac{\tau}{2}) - \hat{\xi}_k(-\frac{\tau}{2})}{\tau} \otimes e_N.$$

We claim that

$$||A_k - A_0|| = O(1-t). \tag{4.46}$$

Indeed,

$$\begin{aligned}
&\limsup_{k\to+\infty}\|A_k-A_0\|\le\limsup_{k\to+\infty}\|A_k-\frac{A_0}{\tau^N}\|+|1-\frac{1}{\tau^N}|=\\
&=\limsup_{k\to+\infty}\frac{1}{\tau^N}|\nabla\xi_k(\tau Q)-A_0|+O(1-t)=\\
&=\limsup_{k\to+\infty}\frac{1}{\tau^N}|Dz_k(\tau Q)-A_0|+O(1-t)\le\\
&\le\limsup_{k\to+\infty}\frac{1}{\tau^N}\left[|Dz_k(\tau Q)-Dz_k(Q)|+|Dz_k(Q)-A_0|\right]+O(1-t)=\\
&=\frac{1}{\tau^N}\limsup_{k\to+\infty}\frac{|Du(x_0+\tau r_kQ)-Du(x_0+r_kQ)|}{|Du|(x_0+r_kQ)}+O(1-t)=\\
&=\frac{1}{\tau^N}\limsup_{k\to+\infty}\frac{|Du|\left((x_0+r_kQ)\setminus(x_0+\tau r_kQ)\right)}{|Du|(x_0+r_kQ)}+O(1-t)=O(1-t)
\end{aligned}$$

where we have used (4.37) and the fact that $t^2<\tau$. Since $\lim_{k\to+\infty}\|\bar{u}_k-\xi_k\|_{L^1(Q;\mathbf{R}^p)}=0$ by Lemma 3.1 there exists a subsequence $\{\xi_{n_k}\}$ and a sequence $\{w_k\}\in SBV(\tau Q;\mathbf{R}^p)$ such that $w_k|_{\partial\tau Q}(x)=\xi_{n_k}(x)=A_{n_k}x+p_{n_k}(x)$, $\|w_k-\xi_{n_k}\|_{L^1(Q;\mathbf{R}^p)}\to 0$ and by (4.44) and (4.45)

$$\begin{aligned}
\mu_c(x_0)\ge&\liminf_{k\to+\infty}\left[\int_{\tau Q}f^\infty(x_0,\nabla w_k(x))dx+\int_{\tau Q\cap\Sigma(w_k)}\varphi_0(x_0,[w_k](x),\nu_k(x))dH_{N-1}(x)\right]-\\
&-C meas(\tau Q\setminus\tau(1-\delta)Q)-C\sup_m|\nabla\xi_m|(\tau Q\setminus\tau(1-\delta)Q)-\epsilon=\\
=&\liminf_{k\to+\infty}\left[\int_{\tau Q}f^\infty(x_0,\nabla w_k(x))dx+\int_{\tau Q\cap\Sigma(w_k)}\varphi_0(x_0,[w_k](x),\nu_k(x))dH_{N-1}(x)\right]-\\
&-\epsilon+O(1-t).
\end{aligned}\tag{4.47}$$

Write $w_k(x)=A_{n_k}x+p_{n_k}(x)+q_k(x)$ where $q_k|_{\partial\tau Q}(x)=0$. Since p_{n_k} is also τQ periodic we may extend p_{n_k} and q_k τQ-periodically to $\mathbf{R}^N$ and define

$$w_{k,\delta}(x):=A_{n_k}x+\delta^2\psi_\delta(x)\left[p_{n_k}\left(\frac{x}{\delta^2}\right)+q_k\left(\frac{x}{\delta^2}\right)\right]$$

where $\psi_\delta:Q\to[0,1]$ is a smooth cut-off function satisfying $supp\psi_\delta\subset\subset\tau Q$, $\psi_\delta(x)=1$ if $x\in\{y\in\tau Q: dist(y,\partial\tau Q)>\delta\}=:Q^*_\delta$ and $\|\nabla\psi_\delta\|\le\frac{C}{\delta}$. Then $w_{k,\delta}|_{\partial\tau Q}(x)=A_{n_k}x$ and for every k

$$\lim_{\delta\to 0^+}\|w_{k,\delta}-A_{n_k}x\|_{L^1(\tau Q;\mathbf{R}^p)}\le\lim_{\delta\to 0^+}\delta^2\int_{\tau Q}\left|(p_{n_k}+q_k)\left(\frac{x}{\delta^2}\right)\right|dx=0$$

by periodicity of p_{n_k} and q_k. On the other hand

$$\begin{aligned}
&\int_{\tau Q}f^\infty(x_0,\nabla w_{k,\delta}(x))dx+\int_{\tau Q\cap\Sigma(w_{k,\delta})}\varphi_0(x_0,[w_{k,\delta}](x),\nu_{k,\delta}(x))dH_{N-1}(x)\le\\
&\le\int_{\tau Q}f^\infty(x_0,\nabla w_k(x))dx+\int_{\tau Q\cap\Sigma(w_{k,\delta})}\varphi_0(x_0,[w_{k,\delta}](x),\nu_{k,\delta}(x))dH_{N-1}(x)+\\
&\qquad+\int_{\tau Q\setminus Q^*_\delta}f^\infty(x_0,\nabla w_k(x))dx.
\end{aligned}\tag{4.48}$$

Since

$$\nabla w_{k,\delta} = A_{n_k} + \psi_\delta \left[\nabla p_{n_k}\left(\frac{x}{\delta^2}\right) + \nabla q_k\left(\frac{x}{\delta^2}\right)\right] + \delta^2 \left[p_{n_k}\left(\frac{x}{\delta^2}\right) + q_k\left(\frac{x}{\delta^2}\right)\right] \otimes \nabla \psi_\delta$$

we have by Lemma 2.12 i)

$$\begin{aligned} &\int_{\tau Q \setminus Q^*_\delta} f^\infty(x_0, \nabla w_k(x))dx \leq \\ &\leq C meas(\tau Q \setminus Q^*_\delta) + \delta^2 \int_{\tau Q \setminus Q^*_\delta} \left|p_{n_k}\left(\frac{x}{\delta^2}\right) + q_k\left(\frac{x}{\delta^2}\right)\right| \|\nabla \psi_\delta(x)\| dx + \\ &\qquad + \int_{\tau Q \setminus Q^*_\delta} |\psi_\delta(x)| \left\|(\nabla p_{n_k} + \nabla q_k)\left(\frac{x}{\delta^2}\right)\right\| dx \end{aligned}$$

where

$$meas(\tau Q \setminus Q^*_\delta) \xrightarrow[\delta \to 0^+]{} 0,$$

$$\delta^2 \int_{\tau Q \setminus Q^*_\delta} \left|p_{n_k}\left(\frac{x}{\delta^2}\right) + q_k\left(\frac{x}{\delta^2}\right)\right| \|\nabla \psi_\delta(x)\| dx \leq C\delta \int_{\tau Q} \left|(p_{n_k} + q_k)\left(\frac{x}{\delta^2}\right)\right| dx \xrightarrow[\delta \to 0^+]{} 0$$

since, by periodicity,

$$\int_{\tau Q} \left|(p_{n_k} + q_k)\left(\frac{x}{\delta^2}\right)\right| dx \xrightarrow[\delta \to 0^+]{} \int_{\tau Q} |p_{n_k} + q_k|(y)dy,$$

and finally, due to the equi-integrability of the sequence $\{\nabla p_{n_k} + \nabla q_k\}$,

$$\int_{\tau Q \setminus Q^*_\delta} \|(\nabla p_{n_k} + \nabla q_k)\| \left(\frac{x}{\delta^2}\right) dx \xrightarrow[\delta \to 0^+]{} 0.$$

Hence we conclude that

$$\int_{\tau Q \setminus Q^*_\delta} f^\infty(x_0, \nabla w_{k,\delta}(x))dx \xrightarrow[\delta \to 0^+]{} 0. \tag{4.49}$$

Also, the periodicity of q_k yields

$$\begin{aligned} &\int_{\tau Q \cap \Sigma(w_{k,\delta})} \varphi_0(x_0, [w_{k,\delta}](x), \nu_{k,\delta}(x)) dH_{N-1}(x) = \\ &= \int_{\tau Q \cap \delta^2 \Sigma(q_k)} \varphi_0\left(x_0, \delta^2 \psi_\delta(x)[q_k]\left(\frac{x}{\delta^2}\right), \nu_{k,\delta}(x)\right) dH_{N-1}(x) = \\ &= \delta^2 \int_{\tau Q \cap \delta^2 \Sigma(q_k)} \psi_\delta(x) \varphi_0\left(x_0, [q_k]\left(\frac{x}{\delta^2}\right), \nu_{k,\delta}(x)\right) dH_{N-1}(x) \leq \\ &\leq \delta^{2N} \int_{\frac{\tau Q}{\delta^2} \cap \Sigma(q_k)} \varphi_0(x_0, [q_k](x), \nu_k(x)) dH_{N-1}(x) = \\ &= \int_{\tau Q \cap \Sigma(q_k)} \varphi_0(x_0, [q_k](x), \nu_k(x)) dH_{N-1}(x). \end{aligned}$$

Hence, for every k choose $\delta = \delta(k)$ so that $w_{k,\delta(k)}|_{\partial\tau Q}(x) = A_{n_k}x$,

$$||w_{k,\delta(k)} - A_{n_k}x||_{L^1(\tau Q;\mathbf{R}^p)} < \frac{1}{k}$$

and by (4.47)–(4.49)

$$\mu_c(x_0) \geq \limsup_{k\to+\infty} \left[\int_{\tau Q} f^\infty(x_0, \nabla w_{k,\delta(k)}(x))dx + \right.$$
$$\left. + \int_{\tau Q\cap\Sigma(w_{k,\delta(k)})} \varphi_0(x_0, [w_{k,\delta(k)}](x), \nu_{k,\delta(k)}(x))dH_{N-1}(x) - \frac{1}{k}\right] - \epsilon + O(1-t).$$

Extending $w_{k,\delta(k)}$ as A_0x on $Q \setminus \tau Q$ we obtain a sequence $\bar{w}_k$ such that $\bar{w}_k|_{\partial Q}(x) = A_0x$ and by (4.46)

$$\mu_c(x_0) \geq \liminf_{k\to+\infty} \left[\int_Q f^\infty(x_0, \nabla\bar{w}_k(x))dx + \int_{Q\cap\Sigma(\bar{w}_k)} \varphi_0(x_0, [\bar{w}_k](x), \nu_k(x))dH_{N-1}(x) - \right.$$
$$\left. - \int_{Q\setminus\tau Q} ||A_0||dx - C\int_{\partial\tau Q} |A_{n_k}x - A_0x|dH_{N-1}(x)\right] - \epsilon + O(1-t) \geq$$
$$\geq g^\infty(x_0, A_0) - CO(1-\tau) - \epsilon + O(1-t).$$

Now it suffices to let $\epsilon \to 0^+$ and $t \to 1^-$. ∎

5 An Upper Bound for $\mathcal{F}(u)$ when $u \in SBV(\Omega;\mathbf{R}^p)$

In this section we continue the proof of Theorem 2.13 by obtaining an upper bound for the relaxation $\mathcal{F}(u)$ when $u \in SBV(\Omega;\mathbf{R}^p)$, namely

Proposition 5.1 *Let $u \in SBV(\Omega;\mathbf{R}^p)$ be given and assume that hypotheses (H0)-(H7) and (H9) hold. Then*

$$\mathcal{F}(u) \leq \int_\Omega g(x, \nabla u(x))dx + \int_{\Sigma(u)} h(x, [u](x), \nu(x))dH_{N-1}(x).$$

We follow the ideas presented in Ambrosio, Mortola and Tortorelli's paper [7] (see also [10]).

Proof. Step 1. We claim that if $u \in SBV(\Omega;\mathbf{R}^p)$ then

$$\mathcal{F}(u;\Omega\setminus\Sigma(u)) \leq \int_{\Omega\setminus\Sigma(u)} g(x, \nabla u(x))dx. \tag{5.1}$$

By Proposition 4.2 $\mathcal{F}(u;\cdot)$ is a Radon measure, absolutely continuous with respect to $\mathcal{L}_N + |Du|$. Thus (5.1) holds if and only if for $\mathcal{L}_N$ a.e. $x_0 \in \Omega$

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{L}_N}(x_0) \leq g(x_0, \nabla u(x_0)). \tag{5.2}$$

Writting $Du = \nabla u dx + D_s u$, by Theorems 2.6 and 2.7 and for $\mathcal{L}_N$ a.e. $x_0 \in \Omega$ we have

$$\lim_{\delta \to 0^+} \frac{|D_s u|(B(x_0,\delta))}{\delta^N} = 0, \tag{5.3}$$

$$\lim_{\delta \to 0^+} \fint_{B(x_0,\delta)} |\nabla u(x) - \nabla u(x_0)| dx = 0, \tag{5.4}$$

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{L}_N}(x_0) \text{ exists and is finite.} \tag{5.5}$$

Choose a point $x_0 \in \Omega$ such that (5.3)–(5.5) hold. By Proposition 3.3 ii) let $u_n \in SBV(Q;\mathbf{R}^p)$ be such that $u_n \to \nabla u(x_0)x$ in $L^1(Q;\mathbf{R}^p)$ and

$$g(x_0,\nabla u(x_0)) = \lim_{n\to+\infty} \left[\int_Q f(x_0, \nabla u_n(x))dx + \int_{Q\cap\Sigma(u_n)} \varphi_0(x_0,[u_n](x),\nu_n(x))dH_{N-1}(x) \right].$$

This, together with (H9) implies that $\sup_n |Du_n|(Q) < +\infty$ so by Lemma 3.7, given $\epsilon > 0$ there exists a sequence $\bar{u}_n \in SBV(Q;\mathbf{R}^p)$ such that $\bar{u}_n \to \nabla u(x_0)x$ in $L^1(Q;\mathbf{R}^p)$, $\sup_n \|\bar{u}_n\|_\infty < +\infty$ and

$$\lim_{n\to+\infty} \left[\int_Q f(x_0, \nabla \bar{u}_n(x))dx + \int_{Q\cap\Sigma(\bar{u}_n)} \varphi_0(x_0,[\bar{u}_n](x),\bar{\nu}_n(x))dH_{N-1}(x) \right] \le g(x_0,\nabla u(x_0)) + \epsilon. \tag{5.6}$$

Choose a sequence of numbers $\delta \in (0, dist(x_0,\partial\Omega))$ and consider the sequence given by

$$w_{n,\delta}(x) := (\rho_n * u)(x) + \delta \left[\bar{u}_n\left(\frac{x-x_0}{\delta}\right) - \nabla u(x_0)\left(\frac{x-x_0}{\delta}\right)\right]$$

where $\rho \in C_0^\infty(\mathbf{R}^N)$ is a nonnegative function such that

$$\int_{\mathbf{R}^N} \rho(x)dx = 1,\ supp\,\rho = \overline{B}(0,1),\ \rho(x) = \rho(-x)\ \forall x \in \mathbf{R}^N$$

and $\rho_n(x) := n^N \rho(nx)$. For each fixed $\delta > 0$ it is clear that $w_{n,\delta} \xrightarrow[n\to+\infty]{} u$ in $L^1(Q(x_0,\delta);\mathbf{R}^p)$ and hence

$$\begin{aligned}
&\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{L}_N}(x_0) = \lim_{\delta\to0^+} \frac{\mathcal{F}(u;Q(x_0,\delta))}{\delta^N} \le \\
&\le \liminf_{\delta\to0^+} \liminf_{n\to+\infty} \frac{1}{\delta^N} \left[\int_{Q(x_0,\delta)} f(x,\nabla w_{n,\delta}(x))dx + \right. \\
&\left. + \int_{Q(x_0,\delta)\cap\Sigma(w_{n,\delta})} \varphi(x,[w_{n,\delta}](x),\nu_{n,\delta}(x)dH_{N-1}(x) \right] = \\
&= \liminf_{\delta\to0^+} \liminf_{n\to+\infty} \left[\int_Q f(x_0+\delta y, \nabla(\rho_n * u)(x_0+\delta y) + \nabla\bar{u}_n(y) - \nabla u(x_0))dy + \right. \\
&\left. + \int_{Q\cap\Sigma(u_n)} \frac{1}{\delta}\varphi(x_0+\delta y, \delta[\bar{u}_n](x), \bar{\nu}_n(x)dH_{N-1}(x) \right].
\end{aligned} \tag{5.7}$$

Since $f(x_0+\delta y,\cdot)$ is a Lipschitz function (by quasiconvexity and (H1)) we have

$$|f(x_0+\delta y,\nabla(\rho_n * u)(x_0+\delta y)+\nabla\bar{u}_n(y)-\nabla u(x_0))-f(x_0+\delta y,\nabla\bar{u}_n(y))|\leq$$
$$\leq C\,|\nabla(\rho_n * u)(x_0+\delta y)-\nabla u(x_0)|$$

and so from (5.7) we obtain

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{L}_N}(x_0)\leq \limsup_{\delta\to 0^+}\limsup_{n\to+\infty}\Big[\int_Q C\,|\nabla(\rho_n * u)(x_0+\delta y)-\nabla u(x_0)|\,dy+$$
$$+\int_Q f(x_0+\delta y,\nabla\bar{u}_n(y))dy\;+\int_{Q\cap\Sigma(u_n)}\frac{1}{\delta}\varphi(x_0+\delta y,\delta[\bar{u}_n](y),\bar{\nu}_n(y))dH_{N-1}(y)\Big]\leq$$
$$\leq \limsup_{\delta\to 0^+}\limsup_{n\to+\infty}\Big\{\Big[\int_Q f(x_0,\nabla\bar{u}_n(y))dy+\int_{Q\cap\Sigma(u_n)}\varphi_0(x_0,[\bar{u}_n](y),\bar{\nu}_n(y))dH_{N-1}(y)\Big]+$$
$$+\int_Q f(x_0+\delta y,\nabla\bar{u}_n(y))-f(x_0,\nabla\bar{u}_n(y))dy+\int_Q C\,|\nabla(\rho_n * u)(x_0+\delta y)-\nabla u(x_0)|\,dy+$$
$$+\int_{Q\cap\Sigma(u_n)}\frac{1}{\delta}\varphi(x_0+\delta y,\delta[\bar{u}_n](y),\bar{\nu}_n(y))-\frac{1}{\delta}\varphi(x_0,\delta[\bar{u}_n](y),\bar{\nu}_n(y))dH_{N-1}(y)+$$
$$+\int_{Q\cap\Sigma(\bar{u}_n)}\frac{1}{\delta}\varphi(x_0,\delta[\bar{u}_n](y),\bar{\nu}_n(y))-\varphi_0(x_0,[\bar{u}_n](y),\bar{\nu}_n(y))dH_{N-1}(y)\Big\}=:$$
$$=:\limsup_{\delta\to 0^+}\limsup_{n\to+\infty}[I_1+I_2+I_3+I_4+I_5].$$

By (5.6)

$$\lim_{n\to+\infty} I_1\leq g(x_0,\nabla u(x_0))+\epsilon$$

and by (H2), (H9) and (5.6), we have for δ small enough

$$|I_2|\leq\int_Q|f(x_0+\delta y,\nabla\bar{u}_n(y))-f(x_0,\nabla\bar{u}_n(y))|\,dy\leq$$
$$\leq\int_Q\epsilon C\,(1+\|\nabla\bar{u}_n(y)\|)\,dy\leq$$
$$\leq C\epsilon+C\epsilon\int_Q f(x_0,\nabla\bar{u}_n(y))dy=O(\epsilon).$$

Also by (H5), (H9) and (5.6)

$$|I_4|\leq\int_{Q\cap\Sigma(\bar{u}_n)}\frac{1}{\delta}\,|\varphi(x_0+\delta y,\delta[\bar{u}_n](y),\bar{\nu}_n(y))-\varphi(x_0,\delta[\bar{u}_n](y),\bar{\nu}_n(y))|\,dH_{N-1}(y)\leq$$
$$\leq\int_{Q\cap\Sigma(\bar{u}_n)}\epsilon C|[\bar{u}_n](y)|dH_{N-1}(y)\leq$$
$$\leq\epsilon C\int_{Q\cap\Sigma(\bar{u}_n)}\varphi_0(x_0,[\bar{u}_n](y),\bar{\nu}_n(y))dH_{N-1}(y)=O(\epsilon).$$

For δ small enough, so that $\delta|[\bar{u}_n](y)|<l$, (H7) yields

$$|I_5|\leq\int_{Q\cap\Sigma(\bar{u}_n)}\left|\frac{1}{\delta}\varphi(x_0,\delta[\bar{u}_n](y),\bar{\nu}_n(y))-\varphi_0(x_0,[\bar{u}_n](y),\bar{\nu}_n(y))\right|dH_{N-1}(y)\leq$$

$$\leq \int_{Q\cap\Sigma(\bar{u}_n)} C\delta^{\alpha}|[\bar{u}_n](y)|^{\alpha+1}dH_{N-1}(y) \leq$$

$$\leq C\delta^{\alpha}\left(\sup_n \|\bar{u}_n\|_{\infty}\right)^{\alpha}\int_{Q\cap\Sigma(\bar{u}_n)} |[\bar{u}_n](y)|dH_{N-1}(y) = C(\epsilon)\delta^{\alpha}$$

by (H9) and (5.6). Finally, by Lemma 2.4 i) and by (5.4), setting $w_0(x) = \nabla u(x_0)x$ we have

$$\limsup_{\delta\to 0^+}\limsup_{n\to+\infty} I_3 = \limsup_{\delta\to 0^+}\limsup_{n\to+\infty}\frac{C}{\delta^N}\int_{Q(x_0,\delta)}|\nabla\rho_n * (u-w_0)(x)|dx \leq$$

$$\leq \limsup_{\delta\to 0^+}\limsup_{n\to+\infty}\frac{C}{\delta^N}\int_{B(x_0,\delta+\frac{1}{n})}|D(u-w_0)|(x) \leq$$

$$\leq \limsup_{\delta\to 0^+} C \fint_{B(x_0,\delta)}|\nabla u(x)-\nabla u(x_0)|dx + ess\limsup_{\delta\to 0^+}\frac{C}{\delta^N}|D_s u|(B(x_0,\delta)) = 0$$

by (5.3) and since for a.e. δ $|D_s u|(\overline{B}(x_0,\delta)) = |D_s u|(B(x_0,\delta))$. Thus we conclude that

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{L}_N}(x_0) \leq g(x_0,\nabla u(x_0)) + O(\epsilon)$$

and the result now follows by letting $\epsilon\to 0^+$.

Step 2. We claim that for any $u \in SBV(\Omega;\mathbf{R}^p)$

$$\mathcal{F}(u;\Sigma(u)) \leq \int_{\Sigma(u)} h(x,[u](x),\nu(x))dH_{N-1}(x). \tag{5.8}$$

The proof of (5.8) will be done in several steps according to the limit function u:

1) $u(x) = \xi\chi_E(x)$ with $Per_{\Omega}(E) < +\infty$, $\xi\in\mathbf{R}^p$;

2) $u(x) = \sum_{i=1}^n a_i\chi_{E_i}(x)$ where $\{E_i\}_{i=1}^n$ forms a partition of Ω into sets of finite perimeter, $a_i\in\mathbf{R}^p$;

3) general case $u\in SBV(\Omega;\mathbf{R}^p)$.

1) Suppose first that $u(x) = \xi\chi_E(x)$ with $Per_{\Omega}(E) < +\infty$. We show that for every open set $\Omega^*\subset\Omega$ we have

$$\mathcal{F}(u;\Omega^*) \leq \int_{\Omega^*} f(x,0)dx + \int_{\Omega^*\cap\Sigma(u)} h(x,\xi,\nu(x))dH_{N-1}(x). \tag{5.9}$$

The proof is again divided into several cases, depending on the interface of u and on the set Ω^*.

a) Assume that we have no explicit dependence on x on either f or φ and that $\Omega^* = a_0 + \eta Q_{\nu}$ for some $a_0\in\mathbf{R}^N$, $\eta > 0$. We claim that for

$$u(x) = \begin{cases} \xi & \text{if } (x-a_0)\cdot\nu > 0 \\ 0 & \text{if } (x-a_0)\cdot\nu < 0\end{cases}$$

we have

$$\mathcal{F}(u;\Omega^*) \leq f(0)\eta^N + h(\xi,\nu)\eta^{N-1}. \tag{5.10}$$

Notice that (5.9) reduces to (5.10) under these assumptions. For simplicity we assume that $\nu = e_N$ and we denote Q_ν by Q. Let Q' be the projection of Q on $\mathbf{R}^{N-1}$

$$Q' := \{x \in Q : x_N = 0\}$$

and let

$$Q^+ := \{x \in Q : x_N \geq 0\},\ Q^- := \{x \in Q : x_N \leq 0\}.$$

Suppose first that $a_0 = 0$ and $\eta = 1$ so that $\Omega^* = Q$ and

$$u(x) = \begin{cases} \xi & \text{if } x_N > 0 \\ 0 & \text{if } x_N < 0. \end{cases}$$

Then by proposition 3.5 ii) there exists $u_n \in \mathcal{A}(\xi,\nu)$ such that $u_n \to u$ in $L^1(Q;\mathbf{R}^p)$ and

$$h(\xi,\nu) = \lim_{n\to+\infty}\left[\int_Q f^\infty(\nabla u_n(x))dx + \int_{Q\cap\Sigma(u_n)} \varphi([u_n](x),\nu_n(x))dH_{N-1}(x)\right]. \tag{5.11}$$

Define the sequence $v_{n,k}$ as follows

$$v_{n,k}(x) = \begin{cases} \xi & \text{if } x_N > \frac{1}{2(2k+1)} \\ u_n((2k+1)x) & \text{if } |x_N| < \frac{1}{2(2k+1)} \\ 0 & \text{if } x_N < -\frac{1}{2(2k+1)}. \end{cases}$$

Then, for n fixed

$$||v_{n,k} - u||_{L^1(Q;\mathbf{R}^p)} = \int_{-\frac{1}{2(2k+1)}}^{\frac{1}{2(2k+1)}} \int_{Q'} |u_n((2k+1)x) - u(x)|dx'dx_N =$$

$$= \frac{1}{2k+1}\int_{-\frac{1}{2}}^{0}\int_{Q'} |u_n((2k+1)x',t)|dx'dt + \frac{1}{2k+1}\int_0^{\frac{1}{2}}\int_{Q'} |u_n((2k+1)x',t) - \xi|dx'dt$$

where by periodicity of u_n in the first $N-1$ variables and the Riemann-Lebesgue Lemma

$$\int_{-\frac{1}{2}}^{0}\int_{Q'} |u_n((2k+1)x',t)|dx'dt \to \int_{-\frac{1}{2}}^{0}\int_{Q'} |u_n(x)|dx'dx_N \text{ as } k \to +\infty$$

and

$$\int_0^{\frac{1}{2}}\int_{Q'} |u_n((2k+1)x',t) - \xi|dx'dt \to \int_0^{\frac{1}{2}}\int_{Q'} |u_n(x) - \xi|dx'dx_N \text{ as } k \to +\infty$$

so that

$$\lim_{n\to+\infty}\lim_{k\to+\infty} ||v_{n,k} - u||_{L^1(Q;\mathbf{R}^p)} = 0.$$

Hence, by a standard diagonalizing argument

$$\mathcal{F}(u;Q) \leq \limsup_{n\to+\infty}\limsup_{k\to+\infty}\left[\int_Q f(\nabla v_{n,k}(x))dx + \int_{Q\cap\Sigma(v_{n,k})} \varphi([v_{n,k}](x),\nu_{n,k}(x))dH_{N-1}(x)\right].$$

Now

$$\int_Q f(\nabla v_{n,k}(x))dx = \int_{-\frac{1}{2}}^{-\frac{1}{2(2k+1)}} \int_{Q'} f(0)dx +$$

$$+ \int_{\frac{1}{2(2k+1)}}^{\frac{1}{2}} \int_{Q'} f(0)dx + \int_{-\frac{1}{2(2k+2)}}^{\frac{1}{2(2k+1)}} \int_{Q'} f((2k+1)\nabla u_n((2k+1)x)dx$$

where by periodicity of u_n

$$\int_{-\frac{1}{2(2k+2)}}^{\frac{1}{2(2k+1)}} \int_{Q'} f((2k+1)\nabla u_n((2k+1)x)dx =$$

$$= \left(\frac{1}{2k+1}\right)^N \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{(2k+1)Q'} f((2k+1)\nabla u_n(y))dy =$$

$$= \frac{1}{2k+1} \int_Q f((2k+1)\nabla u_n(y))dy =$$

$$= \int_Q f^\infty(\nabla u_n(y))dy + \int_Q \frac{1}{2k+1} f((2k+1)\nabla u_n(y)) - f^\infty(\nabla u_n(y))dy$$

so by (H3)

$$\limsup_{k\to+\infty} \int_Q f(\nabla v_{n,k}(x))dx = f(0) + \int_Q f^\infty(\nabla u_n(y))dy.$$

On the other hand, again using the periodicity of u_n

$$\int_{Q\cap\Sigma(v_{n,k})} \varphi([v_{n,k}](x), \nu_{n,k}(x))dH_{N-1}(x) =$$

$$= \int_{\{x\in Q:|x_N|<\frac{1}{2(2k+1)}\}\cap\frac{1}{2k+1}\Sigma(u_n)} \varphi([u_n]((2k+1)x), \nu_n((2k+1)x)dH_{N-1}(x) =$$

$$= \left(\frac{1}{2k+1}\right)^{N-1} \int_{\{x\in(2k+1)Q:|x_N|<\frac{1}{2}\}\cap\Sigma(u_n)} \varphi([u_n](y), \nu_n(y))dH_{N-1}(y) =$$

$$= \int_{Q\cap\Sigma(u_n)} \varphi([u_n](y), \nu_n(y))dH_{N-1}(y)$$

hence, by (5.11)

$$\mathcal{F}(u;Q) \le \limsup_{n\to+\infty} \left[f(0) + \int_Q f^\infty(\nabla u_n(y))dy + \int_{Q\cap\Sigma(u_n)} \varphi([u_n](y), \nu_n(y))dH_{N-1}(y) \right] =$$

$$= f(0) + h(\xi, e_N).$$

Now let $a_0 \in \mathbf{R}^N$ and $\eta > 0$ be arbitrary. Define

$$f_\eta(A) := f\left(\frac{A}{\eta}\right)$$

and let

$$\begin{aligned}\mathcal{F}_\eta(u;\Omega^*) \;=\; & \inf_{\{u_n\}} \left\{ \liminf_{n\to+\infty} \left[\int_{\Omega^*} f_\eta(\nabla u_n(x))dx + \frac{1}{\eta} \int_{\Omega^*\cap\Sigma(u_n)} \varphi([u_n](x), \nu_n(x))dH_{N-1}(x) \right] : \right. \\ & \left. u_n \in SBV(\Omega^*;\mathbf{R}^p),\ u_n \rightharpoonup u \text{ in } BV(\Omega^*;\mathbf{R}^p) \right\}.\end{aligned}$$

Setting

$$u_0(x) = \begin{cases} \xi & \text{if } x_N > 0 \\ 0 & \text{if } x_N < 0 \end{cases}$$

by the first case it follows that

$$\mathcal{F}_\eta(u_0; Q) \le f_\eta(0) + h_\eta(\xi, e_N).$$

Given any sequence $u_n \rightharpoonup u_0$ in $BV(Q;\mathbf{R}^p)$ we define for $x \in \Omega^*$

$$v_n(x) := u_n\left(\frac{x - a_0}{\eta}\right).$$

Then, since $u(a_0 + \eta y) = u_0(y)$, we have $v_n \rightharpoonup u$ in $BV(\Omega^*;\mathbf{R}^p)$ and thus

$$\mathcal{F}(u;\Omega^*) \le \liminf_{n\to+\infty}\left[\int_{\Omega^*} f(\nabla v_n(x))dx + \int_{\Omega^*\cap\Sigma(v_n)} \varphi([v_n](x), \mu_n(x))dH_{N-1}(x)\right] =$$

$$= \liminf_{n\to+\infty}\left[\int_{\Omega^*} f(\frac{1}{\eta}\nabla u_n(\frac{x-a_0}{\eta}))dx + \right.$$

$$\left. + \int_{\Omega^*\cap(a_0+\eta\Sigma(u_n))} \varphi([u_n](\frac{x-a_0}{\eta}), \nu_n(\frac{x-a_0}{\eta}))dH_{N-1}(x)\right] =$$

$$= \eta^N \liminf_{n\to+\infty}\left[\int_Q f_\eta(\nabla u_n(y))dy + \frac{1}{\eta}\int_{Q\cap\Sigma(u_n)} \varphi([u_n](y), \nu_n(y))dH_{N-1}(y)\right].$$

Given the arbitrariness of u_n we conclude that

$$\mathcal{F}(u;\Omega^*) \le \eta^N \mathcal{F}_\eta(u_0; Q) \le \eta^N \left[f_\eta(0) + h_\eta(\xi, e_N)\right]$$

where

$$f_\eta(0) = f(0) \text{ and } h_\eta(\xi, e_N) = \frac{1}{\eta}h(\xi, e_N)$$

since f^∞ is positively homogeneous of degree one. Hence

$$\mathcal{F}(u;\Omega^*) \le \eta^N f(0) + \eta^{N-1}h(\xi, e_N)$$

and (5.10) is proved.

We now turn to the general case where f and φ have an explicit dependence on x and we proceed with the proof of (5.9).

b) Assume first that u has planar interface i.e.

$$u(x) = \begin{cases} \xi & \text{if } (x - a_0)\cdot\nu > 0 \\ 0 & \text{if } (x - a_0)\cdot\nu < 0 \end{cases}$$

and let $\Omega^* = \alpha + \theta Q_\nu \subset\subset \Omega$ for some $\alpha \in \mathbf{R}^N$, $\theta > 0$. As in part a) without loss of generality we assume that $a_0 = 0$ and $\nu = e_N$, we denote Q_ν by Q and we let

$$\Omega' := \{x \in \Omega^* : x_N = 0\},\ Q' := \{x \in Q : x_N = 0\}.$$

Clearly $\Sigma(u) \cap \Omega^* = \Omega'$. Since Ω^* is a compact subset of Ω, fixing $\epsilon > 0$, it is possible to find a $\delta > 0$ such that properties (H2) and (H5) and Proposition 3.6 ii) are satisfied uniformly in Ω^* i.e.

$$x, y \in \Omega^*,\ |x - y| < \delta \Rightarrow |f(x, A) - f(y, A)| \leq \epsilon C(1 + ||A||),\ \forall A \in M^{p\times N}, \tag{5.12}$$

$$x, y \in \Omega^*,\ |x - y| < \delta \Rightarrow |\varphi(x, \xi, \nu) - \varphi(y, \xi, \nu)| \leq \epsilon C|\xi|,\ \forall(\xi, \nu) \in \mathbf{R}^p \times S^{N-1}, \tag{5.13}$$

$$x, y \in \Omega^*,\ |x - y| < \delta \Rightarrow |h(x, \xi, \nu) - h(y, \xi, \nu)| \leq \epsilon C(1 + |\xi|),\ \forall(\xi, \nu) \in \mathbf{R}^p \times S^{N-1}. \tag{5.14}$$

Let $m \in \mathbf{N}$ be such that

$$\eta := \frac{\theta}{m} < \delta \tag{5.15}$$

and partition Ω' into m^{N-1} (N-1)-dimensional cubes aligned according to the coordinate axes and with mutually disjoint interiors

$$\Omega' = \bigcup_{i=1}^{m^{N-1}} (a_i + \eta\overline{Q'}). \tag{5.16}$$

We write $Q_i' := a_i + \eta Q'$ and $Q_i := a_i + \eta Q$. For each $i = 1, \ldots, m^{N-1}$ let $u_n^{(i)} \to u$ in $L^1(Q_i; \mathbf{R}^p)$ be such that

$$\lim_{n\to+\infty} \left[\int_{Q_i} f(a_i, \nabla u_n^{(i)}(x))dx + \int_{Q_i\cap\Sigma(u_n^{(i)})} \varphi(a_i, [u_n^{(i)}](x), \nu_n^{(i)}(x))dH_{N-1}(x)\right] \leq$$
$$\leq \mathcal{F}(u; Q_i) + \frac{\epsilon}{m^{N-1}} \leq \eta^N f(a_i, 0) + \eta^{N-1} h(a_i, \xi, e_N) + \frac{\epsilon}{m^{N-1}}$$

by part a). Using the slicing method as in Lemma 3.1 and Remark 3.2 applied to

$$\mathcal{F}(u, Q_i) = \int_{Q_i} f(a_i, \nabla u(x))dx + \int_{Q_i\cap\Sigma(u)} \varphi(a_i, [u](x), \nu(x))dH_{N-1}(x)$$

we conclude that there exists a sequence $\{\xi_k^{(i)}\} \in \mathcal{A}(\xi, e_N)$ such that, setting $v_k = \rho_{n_k} * u$, $\xi_k^{(i)} = v_k$ on ∂Q_i, $\xi_k^{(i)} \to u$ in $L^1(Q_i; \mathbf{R}^p)$ and

$$\limsup_{k\to+\infty} \left[\int_{Q_i} f(a_i, \nabla\xi_k^{(i)}(x))dx + \int_{Q_i\cap\Sigma(\xi_k^{(i)})} \varphi(a_i, [\xi_k^{(i)}](x), \mu_k^{(i)}(x))dH_{N-1}(x)\right] \leq$$
$$\leq \liminf_{n\to+\infty} \left[\int_{Q_i} f(a_i, \nabla u_n^{(i)}(x))dx + \int_{Q_i\cap\Sigma(u_n^{(i)})} \varphi(a_i, [u_n^{(i)}](x), \nu_n^{(i)}(x))dH_{N-1}(x)\right]$$

so we conclude that

$$\limsup_{k\to+\infty} \left[\int_{Q_i} f(a_i, \nabla\xi_k^{(i)}(x))dx + \int_{Q_i\cap\Sigma(\xi_k^{(i)})} \varphi(a_i, [\xi_k^{(i)}](x), \mu_k^{(i)}(x))dH_{N-1}(x)\right] \leq$$
$$\leq \eta^N f(a_i, 0) + \eta^{N-1} h(a_i, \xi, e_N) + \frac{\epsilon}{m^{N-1}}. \tag{5.17}$$

Define the sequence $w_{k,\epsilon}$ as follows

$$w_{k,\epsilon}(x) := \begin{cases} \xi_k^{(i)}(x) & \text{if } x \in Q_i \\ \xi & \text{if } x_N > \frac{\eta}{2} \\ 0 & \text{if } x_N < -\frac{\eta}{2} \\ v_k(x) & \text{otherwise.} \end{cases}$$

Clearly $w_{k,\epsilon} \in SBV(\Omega^*;\mathbf{R}^p)$ and $\Sigma(w_{k,\epsilon}) = \cup_i \Sigma\left(\xi_k^{(i)}\right)$. As $\|v_k\|_\infty \leq const.$ and since

$$meas\left(\left(\Omega^* \cap \left\{|x_N| < \frac{\eta}{2}\right\}\right) \setminus \left(\bigcup_{i=1}^{m^{N-1}} Q_i\right)\right) = 0$$

we have

$$\|w_{k,\epsilon} - u\|_{L^1(\Omega^*;\mathbf{R}^p)} = \sum_{i=1}^{m^{N-1}} \|\xi_k^{(i)} - u\|_{L^1(Q_i;\mathbf{R}^p)}$$

and so

$$\lim_{k\to+\infty} \|w_{k,\epsilon} - u\|_{L^1(\Omega^*;\mathbf{R}^p)} = 0.$$

Hence,

$$\begin{aligned}
&\mathcal{F}(u;\Omega^*) \leq \\
&\leq \liminf_{k\to+\infty}\left[\int_{\Omega^*} f(x,\nabla w_{k,\epsilon}(x))dx + \int_{\Omega^*\cap\Sigma(w_{k,\epsilon})} \varphi(x,[w_{k,\epsilon}](x),\nu_{k,\epsilon}(x))dH_{N-1}(x)\right] \leq \\
&\leq \limsup_{k\to+\infty} \sum_{i=1}^{m^{N-1}} \left[\int_{Q_i} f(a_i,\nabla\xi_k^{(i)}(x))dx + \int_{Q_i\cap\Sigma(\xi_k^{(i)})} \varphi(a_i,[\xi_k^{(i)}](x),\mu_k^{(i)}(x))dH_{N-1}(x)\right] \\
&+ \limsup_{k\to+\infty} \sum_{i=1}^{m^{N-1}} \left[\int_{Q_i} f(x,\nabla\xi_k^{(i)}(x)) - f(a_i,\nabla\xi_k^{(i)}(x))dx\right] + \\
&+ \limsup_{k\to+\infty} \sum_{i=1}^{m^{N-1}} \left[\int_{Q_i\cap\Sigma(\xi_k^{(i)})} \varphi(x,[\xi_k^{(i)}](x),\mu_k^{(i)}(x)) - \varphi(a_i,[\xi_k^{(i)}](x),\mu_k^{(i)}(x))dH_{N-1}(x)\right] + \\
&+ \int_{\Omega^*\cap\{|x_N|>\frac{\eta}{2}\}} f(x,0)dx
\end{aligned}$$

where by (5.17)

$$\begin{aligned}
&\limsup_{k\to+\infty} \sum_{i=1}^{m^{N-1}} \left[\int_{Q_i} f(a_i,\nabla\xi_k^{(i)}(x))dx + \int_{Q_i\cap\Sigma(\xi_k^{(i)})} \varphi(a_i,[\xi_k^{(i)}](x),\mu_k^{(i)}(x))dH_{N-1}(x)\right] \leq \\
&\leq \eta^{N-1} \sum_{i=1}^{m^{N-1}} h(a_i,\xi,e_N) + \eta^N \sum_{i=1}^{m^{N-1}} f(a_i,0) + \epsilon.
\end{aligned}$$

Also, by (5.12), (5.15) and (H9)

$$\begin{aligned}
&\limsup_{k\to+\infty} \sum_{i=1}^{m^{N-1}} \left[\int_{Q_i} f(x,\nabla\xi_k^{(i)}(x)) - f(a_i,\nabla\xi_k^{(i)}(x))dx\right] \leq \\
&\leq \limsup_{k\to+\infty} \sum_{i=1}^{m^{N-1}} \int_{Q_i} \epsilon C(1 + \|\nabla\xi_k^{(i)}(x)\|)dx \leq \\
&\leq \limsup_{k\to+\infty} \sum_{i=1}^{m^{N-1}} \int_{Q_i} \epsilon C(1 + f(a_i,\nabla\xi_k^{(i)}(x)))dx = O(\epsilon)
\end{aligned}$$

since by (5.17) $\left\{\int_{Q_i} f(a_i,\nabla\xi_k^{(i)}(x))dx\right\}$ remains bounded and by (5.13), (5.15) and (H9)

$$\limsup_{k\to+\infty}\sum_{i=1}^{m^{N-1}}\left[\int_{Q_i\cap\Sigma(\xi_k^{(i)})}\varphi(x,[\xi_k^{(i)}](x),\mu_k^{(i)}(x))-\varphi(a_i,[\xi_k^{(i)}](x),\mu_k^{(i)}(x))dH_{N-1}(x)\right]\le$$

$$\le\limsup_{k\to+\infty}\sum_{i=1}^{m^{N-1}}\int_{Q_i\cap\Sigma(\xi_k^{(i)})}\epsilon C|[\xi_k^{(i)}](x)|dH_{N-1}(x)\le$$

$$\le\limsup_{k\to+\infty}\sum_{i=1}^{m^{N-1}}\int_{Q_i\cap\Sigma(\xi_k^{(i)})}\epsilon C\varphi(a_i,[\xi_k^{(i)}](x),\mu_k^{(i)}(x))dH_{N-1}(x)=O(\epsilon),$$

where we used the fact that, due to (5.17), $\left\{\int_{Q_i\cap\Sigma(\xi_k^{(i)})}\varphi(a_i,[\xi_k^{(i)}](x),\mu_k^{(i)}(x))dH_{N-1}(x)\right\}$ remains bounded. Finally we note that by (5.14)–(5.16)

$$\left|\int_{\Omega^*\cap\Sigma(u)}h(x,\xi,e_N)dH_{N-1}(x)-\eta^{N-1}\sum_{i=1}^{m^{N-1}}h(a_i,\xi,e_N)\right|\le$$

$$\le\sum_{i=1}^{m^{N-1}}\int_{Q_i'}|h(x,\xi,e_N)-h(a_i,\xi,e_N)|\,dH_{N-1}(x)\le$$

$$\le\sum_{i=1}^{m^{N-1}}\int_{Q_i'}\epsilon C(1+|\xi|)dH_{N-1}(x)=O(\epsilon)$$

and by (5.12)

$$\left|\int_{\Omega^*\cap\{|x_N|<\frac{\eta}{2}\}}f(x,0)dx-\eta^{N-1}\sum_{i=1}^{m^{N-1}}f(a_i,0)\right|\le$$

$$\le\sum_{i=1}^{m^{N-1}}\int_{Q_i}|f(x,0)-f(a_i,0)|\,dx\le\sum_{i=1}^{m^{N-1}}\int_{Q_i}\epsilon C dx=O(\epsilon).$$

Therefore we obtain

$$\mathcal{F}(u;\Omega^*)\le\int_{\Omega^*}f(x,0)dx+\int_{\Omega^*\cap\Sigma(u)}h(x,\xi,e_N)dH_{N-1}(x)+O(\epsilon)$$

so to conclude (5.9) it suffices to let $\epsilon\to0^+$.

c) Take u as in part b) but now let $\Omega^*\subset\Omega$ be an arbitrary open set. Let Π be the plane

$$\Pi:=\left\{x\in\mathbf{R}^N: x\cdot e_N=0\right\}=\Sigma(u).$$

It is clear that

$$\Omega^*=\bigcup_{+\infty}^{n=1}(\cup A_n)$$

where A_n is an increasing finite collection of non-overlapping (i.e. with disjoint interiors) cubes $\overline{Q}$ of the form $a_i+\epsilon\overline{Q}$ with edge length bigger than or equal to $\frac{1}{n}$ and such that

$$H_{N-1}(\partial Q\cap\Pi)=0. \tag{5.18}$$

Thus, applying part b) to a decreasing sequence of open cubes whose intersection is the closed cube $\overline{Q}$ one has

$$\mathcal{F}(u;\overline{Q}) \leq \int_{\overline{Q}} f(x,0)dx + \int_{\overline{Q}\cap\Sigma(u)} h(x,\xi,e_N)dH_{N-1}(x)$$

and so by Proposition 4.2 iii)

$$\mathcal{F}(u;\Omega^*) \leq \lim_{n\to+\infty} \mathcal{F}(u;\cup A_n) \leq \lim_{n\to+\infty} \sum_{\overline{Q}\in A_n} \mathcal{F}(u;\overline{Q}) \leq$$

$$\leq \liminf_{n\to+\infty} \sum_{\overline{Q}\in A_n} \left[\int_{\overline{Q}} f(x,0)dx + \int_{\overline{Q}\cap\Sigma(u)} h(x,\xi,e_N)dH_{N-1}(x)\right].$$

By (5.18) and Lebesgue's Monotone Convergence Theorem we conclude that

$$\mathcal{F}(u;\Omega^*) \leq \liminf_{n\to+\infty} \left[\int_{\cup A_n} f(x,0)dx + \int_{(\cup A_n)\cap\Sigma(u)} h(x,\xi,e_N)dH_{N-1}(x)\right] =$$

$$= \int_{\Omega^*} f(x,0)dx + \int_{\Omega^*\cap\Sigma(u)} h(x,\xi,e_N)dH_{N-1}(x).$$

d) Now suppose that u has polygonal interface i.e. $u = \xi\chi_E$ where E is a polyhedral set (i.e. E is a bounded, strongly Lipschitz domain and $\partial E = H_1 \cup \ldots \cup H_M$, H_i are closed subsets of hyperplanes of the type $\{x \in \mathbf{R}^N : x\cdot\nu_i = \alpha_i\}$). Let Ω^* be an open set contained in Ω and let

$$I := \{i \in \{1,\ldots,M\} : H_{N-1}(H_i \cap \Omega^*) > 0\}.$$

If $\Omega^* \cap \Sigma(u) = \emptyset$, i.e. if card$I = 0$, then $u \in W^{1,1}(\Omega^*;\mathbf{R}^p)$ and it suffices to consider $u_k = u$ to obtain

$$\mathcal{F}(u;\Omega^*) \leq \int_{\Omega^*} f(x,0)dx.$$

The case card$I = 1$ was studied in part c) where E is a large cube so that $\Omega \cap \Sigma(u)$ reduces to the flat interface $\{x \in \Omega : x\cdot\nu = 0\}$. Using an induction procedure, assume that (5.9) is true if card$I = k$ for $k \leq M-1$ and we prove it is still true if card$I = M$. Recall that

$$\partial E \cap \Omega^* = (H_1 \cap \Omega^*) \cup \ldots \cup (H_M \cap \Omega^*).$$

Consider the sets

$$S := \left\{x \in \mathbf{R}^N : dist(x,H_1) = dist(x, H_2 \cup \ldots \cup H_M)\right\}$$

and

$$\Omega_1 := \{x \in \Omega^* : dist(x,H_1) < dist(x,H_2 \cup \ldots \cup H_M)\}.$$

Notice that $H_{N-1}(S \cap \Sigma(u)) = 0$ because $H_{N-1}(H_i \cap H_j) = 0$ for $i \neq j$. Also Ω_1 is an open set and $\Omega_1 \cap (H_2 \cup \ldots \cup H_M) = \emptyset$. Fix $\delta > 0$ and let

$$U_\delta := \left\{x \in \mathbf{R}^N : dist(x,S) < \delta\right\},$$

$$U_\delta^- := \left\{x \in \mathbf{R}^N : dist(x,S) < \delta,\ dist(x,H_1) < dist(x,H_2 \cup \ldots \cup H_M)\right\},$$

$$U_\delta^+ := \left\{x \in \mathbf{R}^N : dist(x,S) < \delta,\ dist(x,H_1) > dist(x,H_2 \cup \ldots \cup H_M)\right\}.$$

Since Ω_1 contains only one interface and $\Omega_2 := \Omega^* \setminus \overline{\Omega_1}$ contains at most $M-1$ flat interfaces we can use the induction hypothesis to obtain sequences $u_n \in SBV(\Omega_1;\mathbf{R}^p)$, $v_n \in SBV(\Omega_2;\mathbf{R}^p)$ such that $u_n \rightharpoonup u$ in $BV(\Omega_1;\mathbf{R}^p)$, $v_n \rightharpoonup u$ in $BV(\Omega_2;\mathbf{R}^p)$ and

$$\lim_{n\to+\infty}\left[\int_{\Omega_1} f(x,\nabla u_n(x))dx + \int_{\Omega_1\cap\Sigma(u_n)} \varphi(x,[u_n](x),\nu_n(x))dH_{N-1}(x)\right] \le$$
$$\le \int_{\Omega_1} f(x,0)dx + \int_{\Omega_1\cap\Sigma(u)} h(x,\xi,\nu(x))dH_{N-1}(x) + \frac{\delta}{2};$$
$$\lim_{n\to+\infty}\left[\int_{\Omega_2} f(x,\nabla v_n(x))dx + \int_{\Omega_2\cap\Sigma(v_n)} \varphi(x,[v_n](x),\mu_n(x))dH_{N-1}(x)\right] \le$$
$$\le \int_{\Omega_2} f(x,0)dx + \int_{\Omega_2\cap\Sigma(u)} h(x,\xi,\nu(x))dH_{N-1}(x) + \frac{\delta}{2}.$$

As in Lemma 3.1 and Remark 3.2 we use the slicing method to connect u_n to w_n across $U_\delta^- \cap \Omega_1$ where $w_n(x) = (\rho_n * u)(x)$. We obtain a sequence $\bar{u}_n \to u$ in $L^1(\Omega_1;\mathbf{R}^p)$ such that $\bar{u}_n = w_n$ on $\partial\Omega_1 \cap S$ and

$$\limsup_{n\to+\infty}\left[\int_{\Omega_1} f(x,\nabla \bar{u}_n(x))dx + \int_{\Omega_1\cap\Sigma(\bar{u}_n)} \varphi(x,[\bar{u}_n](x),\bar{\nu}_n(x))dH_{N-1}(x)\right] \le$$
$$\le \int_{\Omega_1} f(x,0)dx + \int_{\Omega_1\cap\Sigma(u)} h(x,\xi,\nu(x))dH_{N-1}(x) + \frac{\delta}{2} +$$
$$+ C meas(\Omega_1 \cap U_\delta^-) + C H_{N-1}(U_\delta^- \cap \Omega_1 \cap \Sigma(u))$$

where we have also used Lemma 2.4 ii). Similarly, we may connect v_n to w_n across $U_\delta^+ \cap \Omega_2$ and we obtain a sequence $\bar{v}_n \to u$ in $L^1(\Omega_2;\mathbf{R}^p)$ such that $\bar{v}_n = w_n$ on $\partial\Omega_2 \cap S$ and

$$\limsup_{n\to+\infty}\left[\int_{\Omega_2} f(x,\nabla \bar{v}_n(x))dx + \int_{\Omega_2\cap\Sigma(\bar{v}_n)} \varphi(x,[\bar{v}_n](x),\bar{\mu}_n(x))dH_{N-1}(x)\right] \le$$
$$\le \int_{\Omega_2} f(x,0)dx + \int_{\Omega_2\cap\Sigma(u)} h(x,\xi,\nu(x))dH_{N-1}(x) + \frac{\delta}{2} +$$
$$+ C meas(\Omega_2 \cap U_\delta^+) + C H_{N-1}(U_\delta^+ \cap \Omega_2 \cap \Sigma(u)).$$

We set

$$\xi_n(x) := \begin{cases} \bar{u}_n(x) & \text{if } x \in \overline{\Omega_1} \cap \Omega^* \\ \bar{v}_n(x) & \text{if } x \in \overline{\Omega_2} \cap \Omega^*. \end{cases}$$

Clearly $\xi_n \in SBV(\Omega^*;\mathbf{R}^p)$ and

$$\lim_{n\to+\infty} \|\xi_n - u\|_{L^1(\Omega^*;\mathbf{R}^p)} = 0.$$

Hence, as $H_{N-1}(S \cap \Sigma(u)) = 0$ and $\Omega^* = \Omega_1 \cup \Omega_2 \cup (S \cap \Omega^*)$, it follows that

$$\mathcal{F}(u;\Omega^*) \le \liminf_{n\to+\infty}\left[\int_{\Omega^*} f(x,\nabla \xi_n(x))dx + \int_{\Omega^*\cap\Sigma(\xi_n)} \varphi(x,[\xi_n](x),\theta_n(x))dH_{N-1}(x)\right] \le$$

$$\leq \limsup_{n\to+\infty}\left[\int_{\Omega_1} f(x,\nabla\bar{u}_n(x))dx+\int_{\Omega_1\cap\Sigma(\bar{u}_n)}\varphi(x,[\bar{u}_n](x),\bar{\nu}_n(x))dH_{N-1}(x)\right]+$$
$$+\limsup_{n\to+\infty}\left[\int_{\Omega_2} f(x,\nabla\bar{v}_n(x))dx+\int_{\Omega_2\cap\Sigma(\bar{v}_n)}\varphi(x,[\bar{v}_n](x),\bar{\mu}_n(x))dH_{N-1}(x)\right]\leq$$
$$\leq \int_{\Omega^*} f(x,0)dx+\int_{\Omega^*\cap\Sigma(u)} h(x,\xi,\nu(x))dH_{N-1}(x)+\delta+$$
$$+C\,meas(\Omega^*\cap U_\delta)+CH_{N-1}(U_\delta\cap\Omega^*\cap\Sigma(u))$$

so letting $\delta\to 0^+$ we conclude the result.

e) Finally, if E is an arbitrary set of finite perimeter in Ω, by Theorem 2.11 there exists a sequence of polyhedral sets E_n such that $\chi_{E_n}\to\chi_E$ in $L^1(\Omega)$ and $Per_\Omega(E_n)\to Per_\Omega(E)$. By Proposition 3.6 iv) there exists a sequence of continuous functions $h_m:\Omega\times\mathbf{R}^N\to[0,+\infty)$ such that

$$h(x,\xi,y)\leq h_m(x,y)\leq C|y|,\ \forall(x,y)\in\Omega\times\mathbf{R}^N$$

and

$$h(x,\xi,y)=\inf_m h_m(x,y)$$

where we extended $h(x,\xi,\cdot)$ as a homogeneous function of degree one. Setting

$$u_n(x):=\xi\chi_{E_n}(x)$$

it turns out that $u_n\to u$ in $L^1(\Omega^*;\mathbf{R}^p)$ so by Proposition 4.2 ii) and iii) we have

$$\mathcal{F}(u;\Omega^*)\leq\liminf_{n\to+\infty}\mathcal{F}(u_n;\Omega^*)\leq$$
$$\leq\lim_{n\to+\infty}\left[\int_{\Omega^*} f(x,0)dx+\int_{\Omega^*\cap\Sigma(u_n)} h(x,\xi,\nu_n(x))dH_{N-1}(x)\right]\leq$$
$$\leq\int_{\Omega^*} f(x,0)dx+\lim_{n\to+\infty}\int_{\Omega^*\cap\Sigma(u_n)} h_m(x,\nu_n(x))dH_{N-1}(x)=$$
$$=\int_{\Omega^*} f(x,0)dx+\int_{\Omega^*\cap\Sigma(u)} h_m(x,\nu(x))dH_{N-1}(x),$$

where we have used the fact that $meas(E_n\Delta E)\to 0$, $Per(E_n)\to Per(E)$. Letting $m\to+\infty$ and using Lebesgue's Monotone Convergence Theorem we obtain

$$\mathcal{F}(u;\Omega^*)\leq\int_{\Omega^*} f(x,0)dx+\int_{\Omega^*\cap\Sigma(u)} h(x,\xi,\nu(x))dH_{N-1}(x)$$

and this concludes the proof of (5.9).

Inequality (5.9) together with Proposition 4.2 iii) yields

$$\mathcal{F}(u;\Sigma(u))\leq\inf\{\mathcal{F}(u;A):A\subset\Omega,\ A\text{ is open },\Sigma(u)\subset A\}\leq$$
$$\leq\inf\left\{\int_A f(x,0)dx+\int_{A\cap\Sigma(u)} h(x,\xi,\nu(x))dH_{N-1}(x):A\subset\Omega,\ A\text{ is open },\Sigma(u)\subset A\right\}=$$
$$=\int_{\Sigma(u)} h(x,\xi,\nu(x))dH_{N-1}(x)$$

and therefore we conclude (5.8) in case 1). The proof of cases 2) and 3) follow exactly as in [7] Proposition 4.8 Steps 1 and 2 respectively. ■

6 Characterization of the Density Functions and Relaxation for BV Functions in the Homogeneous Case

We now extend our relaxation result of Theorem 4.1 and Proposition 5.1 to arbitrary BV functions in the case where φ satisfies (H8) i.e. $\varphi(x,\cdot,\nu)$ is positively homogeneous of degree one. Recall that we consider an energy functional of the form

$$E(u) = \int_\Omega f(x, \nabla u(x))dx + \int_{\Sigma(u)} \varphi(x, [u](x), \nu(x))dH_{N-1}(x)$$

and we obtained the following integral representation for the relaxation

$$\mathcal{F}(u) = \inf_{\{u_n\}} \left\{ \liminf_{n\to+\infty} E(u_n) : u_n \in SBV(\Omega; \mathbf{R}^p),\ u_n \rightharpoonup u \text{ in } BV(\Omega; \mathbf{R}^p) \right\}$$

when $u \in SBV(\Omega; \mathbf{R}^p)$:

$$\mathcal{F}(u) = \int_\Omega g(x, \nabla u(x))dx + \int_{\Sigma(u)} h(x, [u](x), \nu(x))dH_{N-1}(x). \tag{6.1}$$

Now consider

$$\begin{aligned} \mathcal{F}^*(u) &= \int_\Omega g(x, \nabla u(x))dx + \int_{\Sigma(u)} h(x, [u](x), \nu(x))dH_{N-1}(x) + \\ &+ \int_\Omega g^\infty\left(x, \frac{dC(u)}{d|C(u)|}(x)\right) d|C(u)|(x). \end{aligned}$$

We will show that for $u \in BV(\Omega; \mathbf{R}^p)$

$$\mathcal{F}^*(u) = \mathcal{F}(u)$$

and we will characterize the densities g and h (see Propositions 6.3 and 6.4). This is done in a series of lemmas.

Lemma 6.1 *For every $(x_0, A) \in \Omega \times M^{p\times N}$*

$$g(x_0, A) \le f(x_0, A)$$

and for every $(x_0, \xi, \nu) \in \Omega \times \mathbf{R}^p \times S^{N-1}$

$$h(x_0, \xi, \nu) \le \min\{\varphi(x_0, \xi, \nu), f^\infty(x_0, \xi \otimes \nu)\}.$$

Proof. Define $\xi_0(x) := \xi(x\cdot\nu) + \frac{\xi}{2}$. Then $\xi_0 \in \mathcal{A}(\xi, \nu)$ so that

$$h(x_0, \xi, \nu) \le \int_{Q_\nu} f^\infty(x_0, \nabla\xi_0(x))dx = f^\infty(x_0, \xi \otimes \nu).$$

Now consider $\xi_1 \in \mathcal{A}(\xi, \nu)$ defined by

$$\xi_1(x) := \begin{cases} \xi & \text{if } x\cdot\nu > 0 \\ 0 & \text{if } x\cdot\nu < 0. \end{cases}$$

It follows that

$$h(x_0,\xi,\nu) \leq \int_{\Sigma(\xi_1)} \varphi(x_0,[\xi_1](x),\nu)dH_{N-1}(x) = \varphi(x_0,\xi,\nu).$$

On the other hand, letting $\xi(x) = Ax$ for every $x \in Q$, we have

$$g(x_0,A) \leq \int_Q f(x_0,\nabla\xi(x))dx = f(x_0,A).$$

∎

Lemma 6.2 *Let G be a nonnegative Borel measurable function $G : M^{p\times N} \to [0,+\infty)$ and let $H : \mathbf{R}^p \times S^{N-1} \to [0,+\infty)$ be a continuous function such that $H(\cdot,\nu)$ is positively homogeneous of degree one. Define*

$$\begin{aligned} G_\Omega(A) &:= \inf\left\{\frac{1}{meas(\Omega)}\left[\int_\Omega G(\nabla u(x))dx + \int_{\Omega\cap\Sigma(u)} H([u](x),\nu(x))dH_{N-1}(x)\right] :\right. \\ &\quad \left. u \in SBV(\Omega;\mathbf{R}^p),\ u|_{\partial\Omega}(x) = Ax\right\}. \end{aligned}$$

Then $G_\Omega(A) = G_Q(A)$ where $Q = \left(-\frac{1}{2},\frac{1}{2}\right)^N$.

Proof. By Vitali's Covering Theorem we may write

$$\Omega = \bigcup_{i=1}^{\infty}(a_i + \epsilon_i Q) \cup N$$

where $meas(N) = 0$ and $\sum_{i=1}^{\infty} \epsilon_i^N = meas(\Omega)$. Let $u \in SBV(Q;\mathbf{R}^p)$ be such that $u|_{\partial Q}(x) = Ax$ and define

$$u_\Omega(x) = \begin{cases} Ax + \epsilon_i(u - A)\left(\frac{x-a_i}{\epsilon_i}\right) & \text{if } x \in a_i + \epsilon_i Q \\ Ax & \text{otherwise.} \end{cases}$$

Since $H(\cdot,\nu)$ is positively homogeneous of degree one, $u_\Omega|_{\partial\Omega}(x) = Ax$ and $\Sigma(u_\Omega) = \cup_{i=1}^{\infty}(a_i + \epsilon_i Q) \cap (a_i + \epsilon_i\Sigma(u))$ it follows that

$$\begin{aligned} G_\Omega(A) &\leq \frac{1}{meas(\Omega)}\sum_{i=1}^{\infty}\left[\int_{a_i+\epsilon_i Q} G\left(\nabla u\left(\frac{x-a_i}{\epsilon_i}\right)\right)dx + \right. \\ &\quad \left. + \int_{(a_i+\epsilon_i Q)\cap(a_i+\epsilon_i\Sigma(u))} H\left(\epsilon_i[u]\left(\frac{x-a_i}{\epsilon_i}\right),\nu\left(\frac{x-a_i}{\epsilon_i}\right)\right)dH_{N-1}(x)\right] = \\ &= \frac{1}{meas(\Omega)}\sum_{i=1}^{\infty}\epsilon_i^N\left[\int_Q G(\nabla u(y))dy + \int_{Q\cap\Sigma(u)} H([u](y),\nu(y))dH_{N-1}(y)\right] = \\ &= \int_Q G(\nabla u(y))dy + \int_{Q\cap\Sigma(u)} H([u](y),\nu(y))dH_{N-1}(y). \end{aligned}$$

Taking the infimum over all such u we conclude that

$$G_\Omega(A) \leq G_Q(A).$$

A similar construction yields

$$G_Q(A) \le G_\Omega(A)$$

and so equality holds. ∎

In the next proposition we find an explicit formula for the function g. Precisely, we show that

$$g = Q(f\nabla\varphi_0)$$

i.e. g is the quasiconvexification of the inf-convolution of f and φ_0 given by

$$f\nabla\varphi_0(x,A) := \inf\left\{f(x, A - a\otimes b) + \varphi_0(x,a,b) : a \in \mathbf{R}^p,\ b \in \mathbf{R}^N\right\}.$$

Proposition 6.3 *Assume that hypotheses (H0)-(H7) and (H9) hold. Then*

$$g(x_0, A) = Q(f\nabla\varphi_0)(x_0, A)$$

for every $(x_0, A) \in \Omega \times M^{p\times N}$.

Proof. To show that $g(x_0, A) \le Q(f\nabla\varphi_0)(x_0, A)$ it suffices to prove that

$$g(x_0, A) \le f\nabla\varphi_0(x_0, A) \tag{6.2}$$

for any $(x_0, A) \in \Omega \times M^{p\times N}$ since $g(x_0, \cdot)$ is quasiconvex (cf. Proposition 3.3 i)). We must show that for any $\alpha \in \mathbf{R}^p$, $\beta \in \mathbf{R}^N$

$$g(x_0, A) \le f(x_0, A - \alpha\otimes\beta) + \varphi_0(x_0, \alpha, \beta).$$

Assume without loss of generality that $|\beta| = 1$, let $\{\beta_1, \dots, \beta_{N-1}, \beta\}$ be an orthonormal basis of $\mathbf{R}^N$ and consider

$$Q_n = \left\{x \in \mathbf{R}^N : |x\cdot\beta_i| < \frac{n}{2},\ |x\cdot\beta| < \frac{1}{2}\right\}$$

$$S_n = \left\{x \in \mathbf{R}^N : |x\cdot\beta_i| < \frac{n}{2} - \frac{1}{n^2},\ |x\cdot\beta| = 0\right\}.$$

We denote by Q_n^+ the trapezoid with bases S_n and $\{x \in Q_n : x\cdot\beta = \frac{1}{2}\}$ and Q_n^- the one with bases S_n and $\{x \in Q_n : x\cdot\beta = -\frac{1}{2}\}$. Let $R_n = Q_n \setminus (Q_n^- \cup Q_n^+)$ and let $\partial R_n^\pm$ denote the common boundary of R_n and $Q_n^\pm$, respectively. Then

$$meas(Q_n) = n^{N-1},\ H_{N-1}(S_n) = n^{N-1} + O(\frac{1}{n}) \tag{6.3}$$

$$meas(R_n) = O(n^{N-4}),\ H_{N-1}(\partial R_n^\pm) = O(n^{N-2}). \tag{6.4}$$

By Lemma 6.2 (with $G = f$ and $H = \varphi_0$) we have

$$g(x_0, A) \le \frac{1}{meas(Q_n)}\left[\int_{Q_n} f(x_0, \nabla v(x))dx + \int_{Q_n\cap\Sigma(v)} \varphi_0(x_0, [v](x), \nu(x))dH_{N-1}(x)\right]$$

for every $v \in SBV(Q_n;\mathbf{R}^p)$ such that $v|_{\partial Q_n}(x) = Ax - \frac{\alpha}{2}$. Define

$$v_n(x) = \begin{cases} (A - \alpha \otimes \beta)x & \text{if } x \in Q_n^+ \\ (A - \alpha \otimes \beta)x - \alpha & \text{if } x \in Q_n^- \\ Ax - \frac{\alpha}{2} & \text{if } x \in R_n. \end{cases}$$

Then $v_n|_{\partial Q_n}(x) = Ax - \frac{\alpha}{2}$ so

$$\begin{aligned} g(x_0, A) \leq \frac{1}{meas(Q_n)} \Big[\int_{Q_n^+} f(x_0, A - \alpha \otimes \beta) dx + \int_{Q_n^+} f(x_0, A - \alpha \otimes \beta) dx + \\ + \int_{R_n} f(x_0, A) dx + \int_{S_n} \varphi_0(x_0, \alpha, \beta) dH_{N-1}(x) + \\ + \int_{\partial R_n^+} \varphi_0(x_0, (\alpha \otimes \beta)x - \frac{\alpha}{2}, \nu) dH_{N-1}(x) + \\ + \int_{\partial R_n^-} \varphi_0(x_0, (\alpha \otimes \beta)x + \frac{\alpha}{2}, \nu) dH_{N-1}(x) \Big]. \end{aligned} \tag{6.5}$$

By (H4) and since $|x \cdot \beta| \leq \frac{1}{2}$ for every $x \in Q_n$, we have

$$\varphi_0 \left(x_0, (\alpha \otimes \beta)x \pm \frac{\alpha}{2}, \nu \right) \leq C$$

so from (6.5) using (6.3) and (6.4) we obtain

$$\begin{aligned} g(x_0, A) \leq \frac{1}{meas(Q_n)} \Big[\int_{Q_n} f(x_0, A - \alpha \otimes \beta) dx + f(x_0, A) O(n^{N-4}) + \\ + \left(n^{N-1} + O\left(\frac{1}{n}\right) \right) \varphi_0(x_0, \alpha, \beta) + C O(n^{N-2}) \Big] = \\ = f(x_0, A - \alpha \otimes \beta) + \frac{f(x_0, A)}{n^3} + \varphi_0(x_0, \alpha, \beta) + O(\frac{1}{n}). \end{aligned}$$

Letting $n \to +\infty$ we conclude that

$$g(x_0, A) \leq f(x_0, A - \alpha \otimes \beta) + \varphi_0(x_0, \alpha, \beta).$$

Conversely, for any $(x, A) \in \Omega \times M^{p \times N}$

$$f \nabla \varphi_0(x, A) \leq f(x, A) + \varphi_0(x, 0, 0) = f(x, A)$$

where we have used the fact that φ_0 is positively homogeneous of degree one and so $\varphi_0(x, a, b) = 0$ if $a = 0$. Thus

$$Q(f \nabla \varphi_0)(x, A) \leq f \nabla \varphi_0(x, A) \leq f(x, A). \tag{6.6}$$

On the other hand

$$f \nabla \varphi_0(x, \xi \otimes \nu) \leq f(x, 0) + \varphi_0(x, \xi, \nu)$$

and this implies that

$$Q(f \nabla \varphi_0)(x, \xi \otimes \nu) \leq f(x, 0) + \varphi_0(x, \xi, \nu)$$

and thus, since φ_0 is positively homogeneous of degree one

$$Q^\infty(f\nabla\varphi_0)(x,\xi\otimes\nu)\le\varphi_0(x,\xi,\nu). \tag{6.7}$$

Defining for $u\in SBV(\Omega;\mathbf{R}^p)$

$$F_0(u):=\int_\Omega Q(f\nabla\varphi_0)(x,\nabla u(x))dx+\int_{\Sigma(u)}Q^\infty(f\nabla\varphi_0)(x,[u](x)\otimes\nu(x))dH_{N-1}(x)$$

and

$$F_1(u):=\int_\Omega f(x,\nabla u(x))dx+\int_{\Sigma(u)}\varphi_0(x,[u](x)\otimes\nu(x))dH_{N-1}(x)$$

it follows from (6.6) and (6.7) that $F_0(u)\le F_1(u)$ and so

$$\mathcal{F}_0(u)\le\mathcal{F}_1(u) \tag{6.8}$$

for any $u\in SBV(\Omega;\mathbf{R}^p)$, where $\mathcal{F}_0$ (resp. $\mathcal{F}_1$) denotes the relaxation in SBV of F_0 (resp. F_1). However by Fonseca and Müller's result (see also [5]) F_0 is lower semi-continuous in SBV so it coincides with its relaxation i.e.

$$\mathcal{F}_0(u)=\int_\Omega Q(f\nabla\varphi_0)(x,\nabla u(x))dx+\int_{\Sigma(u)}Q^\infty(f\nabla\varphi_0)(x,[u](x)\otimes\nu(x))dH_{N-1}(x)$$

and, by Theorem 4.1 and Proposition 5.1

$$\mathcal{F}_1(u)=\int_\Omega\tilde{g}(x,\nabla u(x))dx+\int_{\Sigma(u)}\tilde{h}(x,[u](x)\otimes\nu(x))dH_{N-1}(x)$$

where

$$\begin{aligned}\tilde{g}(x_0,A) &= \inf\Big\{\int_Q f(x_0,\nabla v(x))dx+\int_{Q\cap\Sigma(v)}(\varphi_0)_0(x_0,[v](x),\nu(x))dH_{N-1}(x): \\ &\qquad v\in SBV(Q;\mathbf{R}^p),\ v|_{\partial Q}(x)=Ax\}\end{aligned}$$

and

$$\tilde{h}(x_0,\xi,\nu)=\inf\left\{\int_{Q_\nu}f^\infty(x_0,\nabla v(x))dx+\int_{Q_\nu\cap\Sigma(v)}\varphi_0(x_0,[v](x),\nu(x))dH_{N-1}(x):v\in\mathcal{A}(\xi,\nu)\right\}.$$

Since φ_0 is positively homogeneous of degree one, $(\varphi_0)_0=\varphi_0$ so it follows from (6.8) that

$$Q(f\nabla\varphi_0)(x_0,A)\le\tilde{g}(x_0,A)=g(x_0,A)$$

■

To prove the following proposition we assume that (H8) holds i.e. that $\varphi(x,\cdot,\nu)$ is positively homogeneous of degree one. Under this assumption $\varphi=\varphi_0$.

Proposition 6.4 *Under hypotheses (H0)-(H5) and (H8)-(H9)*

$$h(x,\xi,\nu)=g^\infty(x,\xi\otimes\nu)$$

for every $(x,\xi,\nu)\in\Omega\times\mathbf{R}^p\times S^{N-1}$.

Proof. Since $\mathcal{F}(\cdot)$ is lower semi-continuous in SBV it must coincide with its relaxation in SBV. Therefore since g satisfies (H0)–(H3), (H9) and h verifies (H4)–(H5) and (H8)–(H9), by Theorem 4.1 and Proposition 5.1 we have for $u \in SBV(\Omega; \mathbf{R}^p)$

$$\begin{aligned}\mathcal{F}(u) &= \int_\Omega g(x, \nabla u(x))dx + \int_{\Omega\cap\Sigma(u)} h(x, [u](x), \nu(x))dH_{N-1}(x) = \\ &= \int_\Omega \overline{g}(x, \nabla u(x))dx + \int_{\Omega\cap\Sigma(u)} \overline{h}(x, [u](x), \nu(x))dH_{N-1}(x)\end{aligned}$$

where

$$\begin{aligned}\overline{g}(x_0, A) &= \inf\left\{\int_Q g(x_0, \nabla v(x))dx + \int_{Q\cap\Sigma(v)} h(x_0, [v](x), \nu(x))dH_{N-1}(x)\right. \\ &\quad : v \in SBV(Q; \mathbf{R}^p),\ v|_{\partial Q}(x) = Ax\}\end{aligned}$$

and

$$\overline{h}(x_0, \xi, \nu) = \inf\left\{\int_{Q_\nu} g^\infty(x_0, \nabla v(x))dx + \int_{Q_\nu\cap\Sigma(v)} h(x_0, [v](x), \nu(x))dH_{N-1}(x) : v \in \mathcal{A}(\xi, \nu)\right\}.$$

This implies that $g = \overline{g}$ and $h = \overline{h}$. Let $\xi_0(x) = \xi(x \cdot \nu) + \frac{\xi}{2}$. Then $\xi_0 \in \mathcal{A}(\xi, \nu)$ so

$$h(x_0, \xi, \nu) = \overline{h}(x_0, \xi, \nu) \leq \int_{Q_\nu} g^\infty(x_0, \nabla\xi_0(x))dx = g^\infty(x_0, \xi \otimes \nu).$$

Using Lemma 6.2 we will now show that

$$g(x_0, \xi \otimes \nu) \leq h(x_0, \xi, \nu). \tag{6.9}$$

Since $h(x, \cdot, \nu)$ is positively homogeneous of degree one (6.9) will imply that

$$g^\infty(x_0, \xi \otimes \nu) \leq h(x_0, \xi, \nu)$$

and this will conclude the proof. We now prove (6.9). Fix $\xi \in \mathbf{R}^p$ and $\nu \in S^{N-1}$. Assume without loss of generality that $\nu = e_N$. Note that, by Lemma 6.2

$$\begin{aligned}&g(x_0, \xi \otimes e_N) = \overline{g}(x_0, \xi \otimes e_N) \leq \\ &\leq \frac{1}{meas(\Omega)}\left[\int_\Omega g(x_0, \nabla v(x))dx + \int_{\Omega\cap\Sigma(v)} h(x_0, [v](x), \nu(x))dH_{N-1}(x)\right]\end{aligned}$$

where $v \in SBV(\Omega; \mathbf{R}^p)$ is such that $v|_{\partial\Omega}(x) = (\xi \otimes e_N)x + \frac{\xi}{2}$. We use a construction similar to the one in the previous proof. Let

$$Q_n = \left(-\frac{n}{2}, \frac{n}{2}\right)^{N-1} \times \left(-\frac{1}{2}, \frac{1}{2}\right)$$

and

$$S_n = \left(-\frac{n}{2} + \frac{1}{n^2}, \frac{n}{2} - \frac{1}{n^2}\right)^{N-1} \times \{0\}.$$

We denote by Q_n^+ the trapezoid with bases S_n and $\{x \in Q_n : x_N = \frac{1}{2}\}$ and Q_n^- the one with bases S_n and $\{x \in Q_n : x_N = -\frac{1}{2}\}$. Let $R_n = Q_n \setminus (Q_n^- \cup Q_n^+)$ and let $\partial R_n^\pm$ denote the common boundary of R_n and $Q_n^\pm$, respectively. Then

$$meas(Q_n) = n^{N-1},\ H_{N-1}(S_n) = n^{N-1} + O(\frac{1}{n}) \tag{6.10}$$

$$meas(R_n) = O(n^{N-4}),\ H_{N-1}(\partial R_n^\pm) = O(n^{N-2}). \tag{6.11}$$

Define

$$v_n(x) = \begin{cases} (\xi \otimes e_N)x + \frac{\xi}{2} & \text{if } x \in R_n \\ \xi & \text{if } x \in Q_n^+ \\ 0 & \text{if } x \in Q_n^-. \end{cases}$$

Then $v_n|_{\partial Q_n}(x) = (\xi \otimes e_N)x + \frac{\xi}{2}$ and so

$$\begin{aligned} & g(x_0, \xi \otimes e_N) \leq \\ & \leq \frac{1}{meas(Q_n)} \left[\int_{Q_n} g(x_0, \nabla v_n(x))dx + \int_{Q_n \cap \Sigma(v_n)} h(x_0, [v_n](x), \nu_n(x)) dH_{N-1}(x) \right] = \\ & = \frac{1}{meas(Q_n)} \left[\int_{R_n} g(x_0, \xi \otimes e_N)dx + \int_{S_n} h(x_0, \xi, e_N) dH_{N-1}(x) + \right. \\ & + \int_{\partial R_n^+} h(x_0, (\xi \otimes e_N)x - \frac{\xi}{2}, \nu) dH_{N-1}(x) + \\ & \left. + \int_{\partial R_n^-} h(x_0, (\xi \otimes e_N)x + \frac{\xi}{2}, \nu) dH_{N-1}(x) \right]. \end{aligned} \tag{6.12}$$

By Proposition 3.6 i) and as for every $x \in Q_n$, $|x \cdot e_N| \leq \frac{1}{2}$, it follows that

$$h(x_0, (\xi \otimes e_N)x \pm \frac{\xi}{2}, \nu) \leq C$$

so from (6.12), using (6.10) and (6.11), we obtain

$$\begin{aligned} & g(x_0, \xi \otimes e_N) \leq \\ & \leq \frac{1}{n^{N-1}} \left[g(x_0, \xi \otimes e_N) O(n^{N-4}) + h(x_0, \xi, e_N) \left(n^{N-1} + O\left(\frac{1}{n}\right) \right) + CO\left(n^{N-2}\right) \right] = \\ & = g(x_0, \xi \otimes e_N) \frac{1}{n^3} + h(x_0, \xi, e_N) + O\left(\frac{1}{n}\right). \end{aligned}$$

Letting $n \to +\infty$ we conclude that

$$g(x_0, \xi \otimes e_N) \leq h(x_0, \xi, e_N).$$

■

Lemma 6.5 *If $h(x_0, \xi, \nu) = g^\infty(x_0, \xi \otimes \nu)$ for every $(x_0, \xi, \nu) \in \Omega \times \mathbf{R}^p \times S^{N-1}$ then $\mathcal{F}(u) = \mathcal{F}^*(u)$ for all $u \in BV(\Omega; \mathbf{R}^p)$.*

Proof. Under the above assumption, by Fonseca and Müller's result (see also [5]), $\mathcal{F}^*$ is lower semi-continuous in BV. Given any $u \in SBV(\Omega;\mathbf{R}^p)$, Lemma 6.1 yields

$$\begin{aligned}\mathcal{F}^*(u) &= \int_\Omega g(x,\nabla u(x))dx + \int_{\Sigma(u)} h(x,[u](x),\nu(x))dH_{N-1}(x) \le \\ &\le \int_\Omega f(x,\nabla u(x))dx + \int_{\Sigma(u_n)} \varphi(x,[u](x),\nu(x))dH_{N-1}(x) = E(u)\end{aligned}$$

so if u_n is any sequence in $SBV(\Omega;\mathbf{R}^p)$ converging to u in $L^1(\Omega;\mathbf{R}^p)$ with $\sup_n |Du_n|(\Omega) < +\infty$ by lower semi-continuity of $\mathcal{F}^*(\cdot)$ it follows that

$$\mathcal{F}^*(u) \le \liminf_{n\to+\infty} \mathcal{F}^*(u_n) \le \liminf_{n\to+\infty} E(u_n).$$

Taking the infimum over all such u_n we get

$$\mathcal{F}^*(u) \le \mathcal{F}(u).$$

Conversely for smooth u by (6.1)

$$\mathcal{F}(u) = \mathcal{F}^*(u) = \int_\Omega g(x,\nabla u(x))dx$$

so by lower semi-continuity of $\mathcal{F}(\cdot)$ in BV, $\mathcal{F}(u)$ must be less than or equal to the relaxation in BV of the functional

$$u \mapsto \int_\Omega g(x,\nabla u(x))dx.$$

Taking into account that g satisfies hypotheses (H0)–(H3) and (H9) (cf. Proposition 3.3 i)) and Fonseca and Müller's result (see also [5]) we obtain

$$\mathcal{F}(u) \le \mathcal{F}^*(u).$$

■

Clearly Lemma 6.5 and Proposition 6.4 yield

Theorem 6.6 *If f and φ satisfy (H0)-(H5) and (H8)-(H9) then*

$$\mathcal{F}(u) = \mathcal{F}^*(u)$$

for every $u \in BV(\Omega;\mathbf{R}^p)$.

Finally, by Propositions 6.3, 6.4 and the above theorem, we obtain

Theorem 6.7 *If f and φ satisfy (H0)-(H5) and (H8)-(H9) then*

$$\mathcal{F}(u) = \int_\Omega Q(f\nabla\varphi)(x, \nabla u(x))dx + \int_{\Sigma(u)} Q^\infty(f\nabla\varphi)(x, [u](x) \otimes \nu(x))dH_{N-1}(x) +$$
$$+ \int_\Omega Q^\infty(f\nabla\varphi)\left(x, \frac{dC(u)}{d|C(u)|}(x)\right) d|C(u)|(x) =$$
$$= \int_\Omega Q(f\nabla\varphi)(x, \nabla u(x))dx + \int_\Omega Q^\infty(f\nabla\varphi)(x, D_s u)$$

for every $u \in BV(\Omega; \mathbf{R}^p)$.

7 Relaxation for BV Functions : The General Case

Using the results obtained in the preceeding sections, we are now ready to prove Theorem 2.13, precisely we show that if (H0)–(H7) hold and if $u \in BV(\Omega; \mathbf{R}^p)$ then

$$\mathcal{F}(u) = \int_\Omega g(x, \nabla u(x))dx + \int_{\Sigma(u)} h(x, [u](x), \nu(x))dH_{N-1}(x) + \int_\Omega g^\infty(x, dC(u)) =: \mathcal{F}^*(u)$$

where g and h are as defined in Section 2. Under hypothesis (H9), we begin by extending the result of Theorem 6.6 to functions φ which are not necessarily positively homogeneous of degree one and in Proposition 7.2 we show that (H9) can be removed.

Proposition 7.1 *Assuming (H0)-(H7) and (H9) hold it follows that*

$$\mathcal{F}(u) = \mathcal{F}^*(u)$$

for any $u \in BV(\Omega; \mathbf{R}^p)$.

Proof. By (H6) $\varphi \leq \varphi_0$ so given any $u \in BV(\Omega; \mathbf{R}^p)$

$$E(u) \leq \int_\Omega f(x, \nabla u(x))dx + \int_{\Omega\cap\Sigma(u)} \varphi_0(x, [u](x), \nu(x))dH_{N-1}(x) =: E_1(u).$$

It follows that

$$\mathcal{F}(u; A) \leq \mathcal{F}_1(u; A) \text{ for every } A \in \mathcal{B}(\Omega) \tag{7.1}$$

where $\mathcal{F}_1(u)$ is the relaxation of $E_1(\cdot)$ from SBV to BV. On the other hand, as φ_0 is positively homogeneous of degree one, by Theorem 6.6

$$\mathcal{F}_1(u) = \int_\Omega g(x, \nabla u(x))dx + \int_\Omega g^\infty(x, D_s u) \tag{7.2}$$

for every $u \in BV(\Omega; \mathbf{R}^p)$. Hence, since $\mathcal{F}(u, \cdot)$ is a measure (cf. Proposition 4.2) and as by (5.8)

$$\mathcal{F}(u, \Omega \cap \Sigma(u)) \leq \int_{\Omega\cap\Sigma(u)} h(x, [u](x), \nu(x))dH_{N-1}(x)$$

we conclude that

$$\mathcal{F}(u) = \mathcal{F}(u, \Omega) \leq \mathcal{F}(u, \Omega \setminus \Sigma(u)) + \int_{\Omega\cap\Sigma(u)} h(x, [u](x), \nu(x))dH_{N-1}(x)$$

which, together with (7.1) and (7.2), yields

$$\mathcal{F}(u) \leq \int_\Omega g(x, \nabla u(x))dx + \int_{\Omega\setminus\Sigma(u)} g^\infty(x, D_s u) + \int_{\Omega\cap\Sigma(u)} h(x, [u](x), \nu(x))dH_{N-1}(x) = \mathcal{F}^*(u).$$

The converse inequality was obtained in Theorem 4.1. ∎

Proposition 7.2 *Under hypotheses (H0)–(H7) it follows that*

$$\mathcal{F}(u) = \mathcal{F}^*(u)$$

for any $u \in BV(\Omega; \mathbf{R}^p)$.

Proof. Consider an energy $E(\cdot)$ satisfying the initial hypotheses (H0)–(H7) and let $u \in BV(\Omega; \mathbf{R}^p)$, $u_n \in SBV(\Omega; \mathbf{R}^p)$ be such that $C = \sup_n |Du_n|(\Omega) < +\infty$ and $u_n \to u$ in $L^1(\Omega; \mathbf{R}^p)$. Fix $\epsilon > 0$ and let

$$E_\epsilon(u) := \int_\Omega [f(x, \nabla u(x)) + \epsilon||\nabla u(x)||]\, dx + \int_{\Sigma(u)} [\varphi(x, [u](x), \nu(x)) + \epsilon|[u](x)|)\, dH_{N-1}(x).$$

Clearly $E_\epsilon(\cdot)$ satisfies (H0)–(H7) and (H9) and so, by Proposition 7.1,

$$\begin{aligned} \mathcal{F}^*_\epsilon(u) &= \mathcal{F}_\epsilon(u) \leq \liminf_{n\to+\infty} E_\epsilon(u_n) = \liminf_{n\to+\infty} [E(u_n) + \epsilon|Du_n|(\Omega)] \leq \\ &\leq \liminf_{n\to+\infty} E(u_n) + \epsilon C. \end{aligned} \tag{7.3}$$

We claim that

$$\mathcal{F}^*_\epsilon(u) \geq \mathcal{F}^*(u) + \epsilon|Du|(\Omega). \tag{7.4}$$

To prove (7.4) it suffices to show that

$$g_\epsilon(x, A) \geq g(x, A) + \epsilon||A|| \tag{7.5}$$

$$h_\epsilon(x, \xi, \nu) \geq h(x, \xi, \nu) + \epsilon|\xi|. \tag{7.6}$$

Given $v \in SBV(\Omega; \mathbf{R}^p)$ consider the following functional, for which (H9) holds,

$$\int_\Omega ||\nabla v(x)||dx + \int_{\Sigma(v)} |[v](x)|dH_{N-1}(x).$$

By Example 2.16 i) its relaxation in BV is given by

$$\int_\Omega ||\nabla v(x)||dx + \int_{\Sigma(v)} |[v](x)|dH_{N-1}(x) + |C(v)|(\Omega)$$

and so, by Proposition 7.1, we conclude that

$$||A|| = \inf\left\{\int_Q ||\nabla u(x)||dx + \int_{\Sigma(u)\cap Q} |[u](x)|dH_{N-1}(x) : u \in SBV(Q; \mathbf{R}^p),\ u|_{\partial Q}(x) = Ax\right\} \tag{7.7}$$

and

$$|\xi| = \inf\left\{\int_{Q_\nu} ||\nabla u(x)||dx + \int_{\Sigma(u)\cap Q_\nu} |[u](x)|dH_{N-1}(x) : u \in \mathcal{A}(\xi, \nu)\right\}. \tag{7.8}$$

On the other hand,

$$g_\epsilon(x,A)=\inf\left\{\int_Q [f(x,\nabla u(y))+\epsilon\|\nabla u(y)\|]\,dy+\right.$$
$$\left.+\int_{\Sigma(u)\cap Q}[\varphi_0(x,[u](y),\nu(y))+\epsilon|[u](y)|]\,dH_{N-1}(y): u\in SBV(Q;\mathbf{R}^p),\ u|_{\partial Q}(x)=Ax\right\}$$

so, by (7.7) and if $u \in SBV(Q;\mathbf{R}^p)$ is such that $u|_{\partial Q}(x) = Ax$, then

$$\int_Q [f(x,\nabla u(y))+\epsilon\|\nabla u(y)\|]\,dy+\int_{\Sigma(u)\cap Q}[\varphi_0(x,[u](y),\nu(y))+\epsilon|[u](y)|]\,dH_{N-1}(y)\geq$$
$$\geq \int_Q f(x,\nabla u(y))dy+\int_{\Sigma(u)\cap Q}\varphi_0(x,[u](y),\nu(y))dH_{N-1}(y)+\epsilon\|A\|\geq$$
$$\geq g(x,A)+\epsilon\|A\|.$$

Taking the infimum over all such u we obtain (7.5). (7.6) is proved in a similar way using (7.8). Thus, by (7.3) and (7.4),

$$\begin{aligned}\mathcal{F}^*(u) &\leq \mathcal{F}^*(u)+\epsilon|Du|(\Omega)\leq \mathcal{F}^*_\epsilon(u)\leq\\ &\leq \liminf_{n\to+\infty} E(u_n)+\epsilon C.\end{aligned}$$

Let $\epsilon \to 0^+$ to get

$$\mathcal{F}^*(u)\leq \liminf_{n\to+\infty} E(u_n)$$

which, taking the infimum over u_n, yields

$$\mathcal{F}^*(u)\leq\mathcal{F}(u).$$

To obtain the reverse inequality we recall that $\mathcal{F}(u,\cdot)$ is a measure and

$$\mathcal{F}(u,A)\leq C\,(meas(A)+|Du|(A))$$

(cf. Proposition 4.2), hence $\mathcal{F}(u) \leq \mathcal{F}^*(u)$ if and only if

$$\frac{d\mathcal{F}(u)}{d\mathcal{L}_N}(x_0)\leq g(x_0,\nabla u(x_0)) \text{ for a.e. } x_0\in\Omega, \tag{7.9}$$

$$\frac{d\mathcal{F}(u)}{d|u^- - u^+|H_{N-1}\lfloor\Sigma(u)}(x_0)\leq\frac{h(x_0,[u](x_0),\nu(x_0))}{|u^-(x_0)-u^+(x_0)|} \text{ for } H_{N-1} \text{ a.e. } x_0\in\Omega\cap\Sigma(u), \tag{7.10}$$

$$\frac{d\mathcal{F}(u)}{d|C(u)|}(x_0)\leq g^\infty\left(x_0,\frac{dC(u)}{d|C(u)|}(x_0)\right) \text{ for } |C(u)| \text{ a.e. } x_0\in\Omega. \tag{7.11}$$

For every $\epsilon > 0$ $E(\cdot) \leq E_\epsilon(\cdot)$ so by Proposition 7.1

$$\mathcal{F}(u)\leq\mathcal{F}_\epsilon(u)=\mathcal{F}^*_\epsilon(u).$$

Therefore, for every $\epsilon > 0$

$$\frac{d\mathcal{F}(u)}{d\mathcal{L}_N}(x_0)\leq\frac{d\mathcal{F}^*_\epsilon(u)}{d\mathcal{L}_N}(x_0)=g_\epsilon(x_0,\nabla u(x_0)). \tag{7.12}$$

Given $\delta > 0$ let $\bar{u} \in SBV(Q;\mathbf{R}^p)$ be such that $\bar{u}(x) = \nabla u(x_0)x$ on ∂Q and

$$g(x_0, \nabla u(x_0)) + \delta \geq \int_Q f(x_0, \nabla \bar{u}(x))dx + \int_{Q\cap\Sigma(\bar{u})} \varphi_0(x_0, [\bar{u}](x), \nu(x))dH_{N-1}(x).$$

Then, for every $\epsilon > 0$,

$$\begin{aligned} g_\epsilon(x_0, \nabla u(x_0)) &= \inf\Big\{\int_Q f(x_0, \nabla v(x)) + \epsilon\|\nabla v(x)\|dx + \\ &\quad + \int_{Q\cap\Sigma(v)} \varphi_0(x_0, [v](x), \nu(x)) + \epsilon|[v](x)|dH_{N-1}(x) : \\ &\quad v \in SBV(Q;\mathbf{R}^p),\ v|_{\partial Q}(x) = \nabla u(x_0)x\Big\} \leq \\ &\quad \leq g(x_0, \nabla u(x_0)) + \delta + \epsilon|D\bar{u}|(Q) \end{aligned}$$

and so, from (7.12) one gets

$$\frac{d\mathcal{F}(u)}{d\mathcal{L}_N}(x_0) \leq \liminf_{\epsilon\to 0^+} g_\epsilon(x_0, \nabla u(x_0)) \leq g(x_0, \nabla u(x_0)) + \delta.$$

(7.9) now follows if we let $\delta \to 0^+$. (7.10) and (7.11) are proved in a similar way.

∎

8 Acknowledgements

The research of the first author was partially supported by the Science Program of the Junta Nacional de Investigação Científica e Tecnológica, Portugal, under scholarship BD/1612/91-RM.

The research of the third author is part of the project "EURHomogenization", contract SC1-CT91-0732 of the program SCIENCE of the Commission of the European Communities.

The research of the fourth author was partially supported by the National Science Foundation under Grant No. DMS-9201215.

This research was partially supported by the Army Research Office and the National Science Foundation, through the Center for Nonlinear Analysis (Carnegie Mellon University) and it was conducted in part during the visits of G. Bouchitté and G. Buttazzo to the Center for Nonlinear Analysis.

References

[1] Alberti, G. Rank-one property for derivatives of functions with bounded variation. *Proc. Royal Soc. Edin.* **A-123** (1993), 239-274.

[2] Ambrosio, L. On the lower semicontinuity of quasiconvex integrals in $SBV(\Omega;\mathbf{R}^k)$. *Nonlinear Anal.* (to appear).

[3] Ambrosio, L. and A. Braides. Functionals defined on partitions of sets of finite perimeter, I: Integral representation and Γ-convergence. *J. Math. Pures Appl.* **69** (1990), 285-305.

[4] Ambrosio, L. and A. Braides. Functionals defined on partitions of sets of finite perimeter, II: Semicontinuity, relaxation and homogenization. *J. Math. Pures Appl.* **69** (1990), 307-333.

[5] Ambrosio, L. and G. Dal Maso. On the relaxation in $BV(\Omega;\mathbf{R}^m)$ of quasiconvex integrals. *J. Funct. Anal.* **109** (1992), 76-97.

[6] Ambrosio, L. and E. De Giorgi. Un nuovo tipo di funzionale del calcolo delle variazioni. *Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **82** (1988), 199-210.

[7] Ambrosio, L., S. Mortola and V. M. Tortorelli. Functionals with linear growth defined on vector valued *BV* functions. *J. Math. Pures et Appl.* **70** (1991), 269-323.

[8] Baldo, S. Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids. *Ann. Inst. H. Poincaré - Analyse Non Linéaire* **7** (1990), 37-65.

[9] Bouchitté, G. and G. Buttazzo. New lower semicontinuity results for non convex functionals defined on measures. *Nonlinear Anal.* **15** (1990), 679-692.

[10] Bouchitté, G., A. Braides and G. Buttazzo. Relaxation results for some free discontinuity problems. Preprint SISSA, Trieste, 1992.

[11] Braides, A. and A. Coscia. The interaction between bulk energy and surface energy in multiple integrals. *Proc. Royal Soc. Edin.*, to appear.

[12] Buttazzo, G. *Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations.* Pitman Res. Notes Math. Ser. **207**, Longman, Harlow, 1989.

[13] Celada, P. and G. Dal Maso. In preparation.

[14] De Giorgi, E. and G. Letta. Une notion générale de convergence faible pour des fonctions croissantes d'ensemble. *Ann. Sc. Norm. Sup. Pisa Cl. Sci.* **4** (1977), 61-99.

[15] Evans, L. C. and R. F. Gariepy. *Measure Theory and Fine Properties of Functions.* CRC Press, 1992.

[16] Federer, H. *Geometric Measure Theory.* Springer, 1969.

[17] Fonseca, I. and D. Kinderlehrer. Lower semicontinuity and relaxation of surface energy functionals. (to appear).

[18] Fonseca, I. and S. Müller. Quasiconvex integrands and lower semicontinuity in L^1. *SIAM J. Math. Anal.* **23** (1992), 1081-1098.

[19] Fonseca, I. and S. Müller. Relaxation of quasiconvex functionals in $BV(\Omega;\mathbf{R}^p)$ for integrands $f(x,u,\nabla u)$. *Arch. Rat. Mech. Anal.* **123** (1993), 1-49.

[20] Fonseca, I. and P. Rybka. Relaxation of multiple integrals in the space $BV(\Omega;\mathbf{R}^p)$. *Proc. Royal Soc. Edin.* **121A** (1992), 321-348.

[21] Giusti, E. *Minimal Surfaces and Functions of Bounded Variation.* Birkhäuser, Boston, 1984.

[22] Morrey, C. B. Quasiconvexity and the semicontinuity of multiple integrals. *Pacific J. Math.* 2 (1952), 25-53.

[23] Morrey, C. B. *Multiple Integrals in the Calculus of Variations*, Springer-Verlag, Berlin, 1966.

[24] Vol'pert, A. I. Spaces BV and quasi-linear equations. *Math. USSR Sb.* **17** (1969), 225-267.

[25] Ziemer, W. P. *Weakly Differentiable Functions*. Springer-Verlag, Berlin, 1989.