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Irene Fonseca
David Kinderlehrer
Carnegie Mellon University

Pablo Pedregal
Universidad Complutense de Madrid

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ENERGY FUNCTIONALS DEPENDING ON ELASTIC STRAIN AND CHEMICAL COMPOSITION

Irene Fonseca†
David Kinderlehrer††
Pablo Pedregal†††

Abstract

In this paper we deal with energy functionals depending on elastic strain and chemical composition and we obtain lower semicontinuity results, existence theorems and relaxation in the spaces $H^{1,p}(\Omega;\mathbb{R}^n)\times L^q(\Omega;\mathbb{R}^d)$ with respect to weak convergence. Our proofs use parametrized measures associated with weakly converging sequences.

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†† Department of Mathematics, Carnegie Mellon University, Pittsburgh, PA 15213. The research was partially supported by the National Science Foundation uncer Grants No. DMs 911572, the AFOSR 91 0301, the ARO DAAL03 92 G 003 and also by the ARO and the NSF through the Center for Nonlinear Analysis.

††† Departamento de Matem tica Aplicada, Universidad Complutense de Madrid, 28040 Madrid, Spain. The research was supported by DGICYT (Spain) through "Programa de Perfeccionamiento y Movilidad del Personal Investigador" and through grant PB90-0245, by the Army Research Office and the National Science Foundation through the Center for Nonlinear Analysis and also by the project EurHomogenization SC1-CT91-0732 of the European Comunity.

1. Introduction

Systems depending on both deformation and auxiliary parameters such as concentration arise in several applications. These systems lead to the introduction of energy densities of the form

$$\psi(\nabla u, m)$$

with functionals

$$I(u,m) = \int_\Omega \psi(\nabla u, m)\,dx, \quad u \in H^{1,p}(\Omega;\mathbf{R}^n),\ m \in L^q(\Omega;\mathbf{R}^d),\ \Omega \subset \mathbf{R}^N.$$

We study the lower semicontinuity of such functionals in the case $1 \le p \le \infty$, $1 \le q \le \infty$. When lower semicontinuity fails, a sequence $\{u^k, m^k\}$ which drives I to a minimum develops oscillations, generally on successively finer scales. To study these oscillations, we shall examine the Young measures generated by sequences $\{\nabla u^k, m^k\}$. The novel feature of I is that u and m have no special relationship, like $u = m$, which ensures regularity or tameness in the second variable. Relaxation in the ∇u variable alone or separately in the ∇u and m variables does not produce the relaxation. Moreover, we may require the range of m to be constrained in some manner.

The relaxation question was motivated by the analysis of coherent thermochemical equilibria in a multiphase, multicomponent system [AJ], [JA], [LC1], [LC2]. Consider a binary alloy of components A and B which may exist in phase α (matrix) or in phase β (precipitate) and let ρ_A (respectively ρ_B) denote the molecular density per unit volume of the component A (respectively B). Then the network constraint is (see [LC1], [LC2], [JM])

$$\rho_A + \rho_B = \rho_0,$$

where ρ_0 is the density of lattice sites of the reference volume, and the caloric equations of state on phases α and β become

$$\begin{aligned} W_\alpha &= W_\alpha(F, \rho_A^\alpha, \rho_B^\alpha) \\ &= W_\alpha(F, \rho_A^\alpha, \rho_0 - \rho_A^\alpha) \\ &:= \psi_\alpha(F, c_\alpha) \end{aligned}$$

and, respectively,

$$W_\beta = \psi_\beta(F, c_\beta)$$

where all other thermodynamical quantities are held fixed, F denotes the deformation gradient,

$$\begin{aligned} \psi_\alpha(F,c) &:= W_\alpha(F, \rho_0 c_\alpha, \rho_0 - \rho_0 c_\alpha), \\ \psi_\beta(F,c) &:= W_\beta(F, \rho_0 c_\beta, \rho_0 - \rho_0 c_\beta), \end{aligned}$$

and the composition (concentration or mole fraction of component A) in phase α (respectively β) is defined by

$$c_\alpha := \frac{\rho_A^\alpha}{\rho_0} \quad (\text{respectively } c_\beta := \frac{\rho_A^\beta}{\rho_0}).$$

Assuming conservation of mass of each species gives rise to the condition

$$\int_{\Omega_\alpha} \rho_A^\alpha + \int_{\Omega_\beta} \rho_A^\beta = \Gamma$$

for fixed Γ, or, equivalently

$$\int_{\Omega_\alpha} c_\alpha + \int_{\Omega_\beta} c_\beta = \theta \,\text{meas}(\Omega)$$

for some fixed $\theta \in [0,1]$.

Equilibria are interpreted as minima of the energy functional

$$E(u,c,\varphi) = \int_\Omega [\varphi(x)\psi_\alpha(\nabla u(x), c_\alpha(x)) + (1-\varphi(x))\psi_\beta(\nabla u(x), c_\beta(x))]\, dx,$$

where $\varphi : \Omega \to \{0,1\}$ denotes the characteristic function of phase α, $c : \Omega \to [0,1]$ is the chemical composition of the system and

$$c_\alpha := \varphi c, \quad c_\beta := (1-\varphi)c.$$

Minimizing first in φ, we see easily that we are reduced to examining the functional

$$E^*(u,c) := \int_\Omega \psi(\nabla u(x), c(x))\, dx$$

where

$$\psi(F,c) := \min\{\psi_\alpha(F,c), \psi_\beta(F,c)\}$$

and

$$\int_\Omega c(x)\, dx = \theta \,\text{meas}(\Omega).$$

Kohn [K] obtained a formula for this relaxation in the case where composition is uniform, i. e. $\psi(F,c) := \psi^*(F)$, and for two linearly elastic phases with identical elastic moduli. We note, however, that in the systems discussed in [AJ],[JA],[JM],[LC1],[LC2] composition is not uniform (see [LC2]) and so we must address the problem of finding the effective energy in the case where it depends on the chemical composition c.

A second example is from magnetostriction. In the theory of linear magnetostriction [C], the stored energy density is assumed to depend on the linearized strain and direction of magnetization,

$$\epsilon = \frac{1}{2}\left(\nabla u + \nabla u^T\right) \text{ and } \alpha, \ |\alpha| = 1,$$

and has the form

$$\varphi(\epsilon,\alpha) = \varphi_{el}(\epsilon) + \varphi_{em}(\epsilon,\alpha) + \varphi_{an}(\alpha),$$

where $\varphi_{el}(\epsilon)$ is a linear elastic strain energy with cubic symmetry,

$$\varphi_{em}(\epsilon,\alpha) = b_0 \operatorname{tr}\epsilon + b_1 \sum_{1\le i\le 3} \epsilon_{ii}(\alpha_i)^2 + b_2 \sum_{1\le i,j\le 3} \epsilon_{ij}\alpha_i\alpha_j,$$

and $\varphi_{an}(\alpha)$ is the anisotropy energy, often a fourth order or higher degree polynomial in α. Owing to the constraint that $|\alpha| = 1$, φ is not its own relaxation even though it may be convex. Some special cases used in computation have been examined in [CKM]. See also [P2] for some questions related to relaxation in the context of micromagnetics.

Our principal results bind the lower semicontinuity of the functional to the joint quasiconvexity/convexity of the integrand. An integrand $\varphi(A,\lambda)$ gives rise to a weakly sequentially lower semicontinuous functional provided that

$$\varphi(A,\lambda) = \inf_{\mathcal{A}} \frac{1}{|\Omega|}\int_\Omega \varphi(A+\nabla u, \lambda + m)\, dx,$$

where

$$\mathcal{A} = \left\{(u,m) : u \in H^{1,p}(\Omega;\mathbf{R}^n), m \in L^q(\Omega;\mathbf{R}^d), \int_\Omega m\, dx = 0\right\},$$

subject to appropriate growth conditions, Theorem 4.4. In Section 6, these results are extended to the case of a Carathéodory energy density. Finally, in Section 5 we obtain the relaxation theorem (see Theorem 5.4) asserting that the two infima

$$\inf\left\{\int_\Omega \psi(\nabla u, m)\, dx : (u,m) \in H^{1,p}(\Omega;\mathbf{R}^n)\times L^q(\Omega;\mathbf{R}^d), u-u_0 \in H_0^{1,p}(\Omega;\mathbf{R}^n),\right.$$
$$\left.\int_\Omega m\, dx = |\Omega|\, m_0, m(x) \in K \text{ a.e. } x \in \Omega\right\},$$

and

$$\inf\left\{\int_\Omega \psi^{\sharp}(\nabla u, m)\, dx : (u,m) \in H^{1,p}(\Omega;\mathbf{R}^n)\times L^q(\Omega;\mathbf{R}^d), u-u_0 \in H_0^{1,p}(\Omega;\mathbf{R}^n),\right.$$
$$\left.\int_\Omega m\, dx = |\Omega|\, m_0, m(x) \in K \text{ a.e. } x \in \Omega\right\}$$

coincide, where the relaxed energy density is given by

$$\psi^{\sharp}(A,\lambda) = \inf\left\{\frac{1}{|\Omega|}\int_\Omega \psi(\nabla u, m)\, dx : u - Ax \in H_0^{1,p}(\Omega;\mathbf{R}^n), m \in L^q(\Omega;\mathbf{R}^d)\right.$$
$$\left.\int_\Omega m\, dx = |\Omega|\,\lambda, m(x) \in K \text{ a.e. } x \in \Omega\right\},$$

extending the well-known relaxation result of Dacorogna [D] to the case where ψ depends also on the chemical composition. In most of the paper K is not assumed to be a closed, convex (except on Section 5). Indeed, we are usually careful enough to specify $co(K)$, $\overline{co(K)}$, etc ...In the particular case discussed above $K = [0,1]$ since the function c should take values in $[0,1]$. In Theorem 5.1 we assert that under some suitable growth conditions, the existence of minimizers for the energy E^*.

In the scalar case $n = 1$, Ioffe [I] studied the lower semicontinuity of E^* in $H^{1,1}(\text{weak})\times L^1_{loc}$ (see also [Am] for a new proof of this result). Here, generalizing E^* to the case where c may take vector values and assuming that $N, n > 1$ we want to study the weak lower semicontinuity and relaxation of the energy E^*. An integral representation of the relaxed functional in $BV(\Omega,\mathbf{R}^n) \times L^\infty(\Omega,\mathbf{R}^d)$ with respect to the $L^1 \times L^\infty(\text{w-*})$ topology was obtained in [FKP].

In order to describe oscillatory behavior and macroscopic limits of sequences $\{\nabla u^k, m^k\}$ bounded in $L^p \times L^q$ we introduce appropriate Young measures or parametrized measures (see Theorems 3.3 and 3.5).

The context for our discussion is $H^{1,p}$-Young measures and L^q-Young measures as studied in [KP2]. Recall that a sequence $f^k \in L^p(\Omega; \mathbf{R}^s)$ with

$$\|f^k\|_{L^p(\Omega)} \leq M \tag{1.1}$$

generates a Young measure $\nu = \{\nu_x\}_{x\in\Omega}$, a family of probability measures on $\mathbf{R}^s$, if whenever

$$\psi(x, f^k) \rightharpoonup \overline{\psi} \text{ in } L^1(\Omega), \text{ for } \psi \in C(\Omega \times \mathbf{R}^s), \tag{1.2}$$

we have the representation

$$\overline{\psi}(x) = \int_{\mathbf{R}^s} \psi(\lambda, x)\, d\nu_x(\lambda) \text{ in } \Omega \text{ a.e.} \tag{1.3}$$

When $f^k = \nabla u^k$, we call ν a gradient Young measure.

Even though (1.1) is assumed, (1.2) need not hold for any subsequence of the $\{f^k\}$ when ψ has pth power growth, that is, when

$$|\psi(\lambda)| \leq C(1 + |\lambda|^p).$$

This would require either an additional condition on the sequence $\{f^k\}$, or some restriction of the class of functions ψ of the type of convergence utilized. In [KP1], [KP2] this situation is resolved by proving, among other things, that given a sequence $\{f^k\}$ satisfying (1.1) which generates a restricted Young measure in any of the senses mentioned above and if $p > 1$, there is another sequence $\left\{\tilde{f}^k\right\}$ satisfying (1.1) which generates the same Young measure and in addition provides the representation (1.2), (1.3) whenever ψ has pth power growth. In particular, there is a $g \in L^1(\Omega)$ such that

$$\left|\tilde{f}^k\right|^p \rightharpoonup g \text{ in } L^1(\Omega).$$

We refer to [KP1], [KP2] for details of this. If $f^k = \nabla u^k$, then the $\tilde{f}^k = \nabla \tilde{u}^k$ are also gradients.

A consequence of these properties of $H^{1,p}$-Young measures is, indeed, that a minimizing sequence for a functional may be chosen so that $\left\{\left|\nabla u^k\right|^p, \left|m^k\right|^q\right\}$ is weakly convergent in $L^1(\Omega)$. This permits us to exhibit quite simple and short proofs of our claims.

2. Preliminaries

We are going to study some questions concerning weak lower semicontinuity for this type of integrand using parametrized measures as in [P1]. We will use equally both terms: parametrized measures and Young measures. In this context and in order to describe nonlinear macroscopic limits, we want to associate parametrized measures to bounded sequences $\{\nabla u_k, m_k\}$ in $H^{1,p}(\Omega; \mathbf{R}^n) \times L^q(\Omega; \mathbf{R}^d)$ where $\Omega \subset \mathbf{R}^N$ is open and bounded. This is possible using the general framework for the study of oscillations described by Ball [B], Matos [M] and Tartar [T]. We recall

THEOREM 2.1([B]) *Let $\Omega \subset \mathbf{R}^N$ be open and bounded and let $z_j : \Omega \to \mathbf{R}^s$ be a sequence of measurable functions such that*

$$\sup_j \int_\Omega g(|z_j(x)|)\, dx < \infty,$$

for a non-decreasing function $g:[0,\infty)\to\mathbf{R}$ *with* $\lim_{t\to\infty} g(t)=\infty$. *Then there is a subsequence, again denoted by* $\{z_j\}$, *and a family of probability measures* $\{\nu_x\}_{x\in\Omega}$, *depending measurably on* x, *such that given any measurable* $E\subset\Omega$,

$$f(z_j)\rightharpoonup\langle\nu_x,f\rangle=\int_{\mathbf{R}^s} f(y)\,d\nu_x(y)\ \text{in}\ L^1(E) \tag{2.1}$$

for any continuous $f:\mathbf{R}^s\to\mathbf{R}$ *such that* $\{f(z_j)\}$ *is sequentially weakly relatively compact in* $L^1(E)$.

It is important for us to remember how the existence of $\{\nu_x\}_{x\in\Omega}$ is obtained in [B]. The space $L^1(\Omega;C_0(\mathbf{R}^s))$ of strongly measurable functions with finite norm

$$\int_\Omega \sup_{z\in\mathbf{R}^s}|\psi(x,z)|\,dx<\infty,$$

is a separable Banach space under this norm. The space $C_0(\mathbf{R}^s)$ is as usual

$$C_0(\mathbf{R}^s)=\left\{f\in C(\mathbf{R}^s):\lim_{|\lambda|\to\infty} f(\lambda)=0\right\}.$$

Let $\mathcal{L}$ denote a dense separable subset. Its dual space can be identified with the space $L^\infty_w(\Omega;\mathcal{M}(\mathbf{R}^s))$ of weakly measurable functions which are Radon measures at each point in Ω. The duality is given by

$$\langle\psi,\nu\rangle=\int_\Omega\int_{\mathbf{R}^s}\psi(x,z)\,d\nu_x(z)\,dx,$$

for $\psi\in L^1(\Omega;C_0(\mathbf{R}^s))$ and $\nu=\{\nu_x\}_{x\in\Omega}\in L^\infty_w(\Omega;\mathcal{M}(\mathbf{R}^s))$. Define $\nu^j\in L^\infty_w(\Omega;\mathcal{M}(\mathbf{R}^s))$ by $\nu^j_x=\delta_{z_j(x)}$ where δ is the usual Dirac mass. From this sequence we can extract a subsequence which converges weakly * to some $\nu=\{\nu_x\}_{x\in\Omega}\in L^\infty_w(\Omega;\mathcal{M}(\mathbf{R}^s))$, i.e.

$$\lim_{j\to\infty}\int_\Omega\psi(x,z_j(x))\,dx=\langle\psi,\nu\rangle=\int_\Omega\langle\nu_x,\psi(x,\cdot)\rangle\,dx. \tag{2.2}$$

We say that ν *is generated by* $\{z_j\}$ whenever (2.2) holds for every $\psi\in L^1(\Omega;C_0(\mathbf{R}^s))$. It can be shown (see [B]) that (2.2) is also true when ψ is a Carathéodory function such that $\{\psi(x,z_j)\}$ is weakly convergent in $L^1(\Omega)$ and this in turn implies (2.1). This fact has the important consequence that in order to determine a parametrized measure in the sense of Theorem 2.1 for a given sequence of functions, we only need to worry about the convergence (2.2) for $\psi\in L^1(\Omega;C_0(\mathbf{R}^s))$. Let us keep this fact in mind for future reference.

Consider next a sequence $\{\nabla u_k,m_k\}$ where $u_k:\Omega\subset\mathbf{R}^N\to\mathbf{R}^n$, $m_k:\Omega\to\mathbf{R}^d$ and $\{u_k,m_k\}$ is bounded in $H^{1,p}(\Omega;\mathbf{R}^n)\times L^q(\Omega;\mathbf{R}^d)$, $1\le p\le\infty$, $1\le q\le\infty$. Setting $g(t)=t^\alpha$ with $\alpha=\min\{p,q\}\ge 1$ if $\alpha<\infty$ and $g(t)=t$ if $\alpha=\infty$, we can conclude from Theorem 2.1 that, possibly for a subsequence, there is a family of probability measures $\{\nu_x\}_{x\in\Omega}$ such that whenever the compositions $\{\varphi(\nabla u_k,m_k)\}$, for $\varphi:\mathbf{M}\times\mathbf{R}^d\to\mathbf{R}$ a continuous function, converge weakly in $L^1(E)$ for some measurable $E\subset\Omega$, the limit is given by the integral representation

$$\overline{\varphi}(x)=\int_{\mathbf{M}\times\mathbf{R}^d}\varphi(A,\lambda)\,d\nu_x(A,\lambda),\quad \text{a.e. } x\in\Omega, \tag{2.3}$$

where $\mathbf{M}$ is the set of $N\times n$ matrices. For example, if φ is such that

$$|\varphi(A,\lambda)|\le C(1+|A|^s),$$

for some $C > 0$, and $1 \le s < \alpha$, then $\{\varphi(\nabla u_k, m_k)\}$ is weakly convergent in $L^1(\Omega)$ by Hölder's inequality and the Dunford-Pettis compactness criterion, and the integral representation (2.3) is valid for the weak limit. One particularly interesting situation that we should bear in mind throughout the paper, in which this integral representation is valid, holds when $\{|\nabla u_k|^p\}$, $\{|m_k|^q\}$ are weakly convergent sequences in $L^1(\Omega)$ and

$$|\psi(A,\lambda)| \le C(1+|A|^p+|\lambda|^q),$$

because by Dunford-Pettis, $\{\psi(\nabla u_k, m_k)\}$ is equiintegrable and thus weakly convergent in $L^1(\Omega)$.

For a bounded sequence in $L^1(\Omega)$ we may not have compactness in the weak topology. The best one can expect is biting convergence in the sense of Chacon's biting lemma ([Z]). We recall that the sequence $\{f^k\} \subset L^1(\Omega)$ *converges in the biting sense* to $f \in L^1(\Omega)$, and we write

$$f^k \xrightarrow{b} f \text{ in } L^1(\Omega),$$

if there is a non-increasing sequence of measurable sets $\{E^j\}$ such that $|E^j| \to 0$ and

$$f^k \rightharpoonup f \text{ in } L^1(\Omega \setminus E^j), \quad \forall j.$$

We may restate Chacon's biting lemma by saying that a uniformly bounded sequence in $L^1(\Omega)$ contains a subsequence converging in the biting sense to a function in $L^1(\Omega)$ ([BM], [BC]).

This lemma yields necessary and sufficient condition for biting convergence to become weak convergence. Its proof is elementary and can be found in [KP2].

LEMMA 2.2 *Let $f^k : \Omega \to \mathbf{R}^+$ ($f^k \ge 0$) be a sequence of measurable functions in $L^1(\Omega)$, converging in the biting sense to $f \in L^1(\Omega)$. A subsequence converges weakly in $L^1(\Omega)$ if and only if*

$$\liminf_{k\to\infty} \int_\Omega f^k(x)\,dx \le \int_\Omega f(x)\,dx.$$

Also, $\{f^k\}$ is weakly relatively compact in $L^1(\Omega)$ if and only if

$$\limsup_{k\to\infty} \int_\Omega f^k(x)\,dx \le \int_\Omega f(x)\,dx.$$

After Ball and Zhang [BZ], we identify biting limits with the help of Young measures.

LEMMA 2.3 *Let $w^k : \Omega \subset \mathbf{R}^N \to \mathbf{R}^n$ be a sequence of vector-valued functions for which we can apply Theorem 2.1 for some g. Let $\{\nu_x\}_{x\in\Omega}$ be its associated Young measure. If $\varphi : \mathbf{R}^n \to \mathbf{R}$ is continuous and if the sequence $\{\varphi(w^k)\}$ is uniformly bounded in $L^1(\Omega)$, then (possibly for a subsequence)*

$$\varphi(w^k) \xrightarrow{b} \overline{\varphi}(x) = \langle \nu_x, \varphi\rangle = \int_{\mathbf{R}^n} \varphi(\lambda)\,d\nu_x(\lambda)$$

The proof is nothing more than the fact that whenever $\varphi(w^k)$ converges weakly in $L^1(E)$, $E \subset \Omega$, the limit has to be $\overline{\varphi}(x)$ by Theorem 2.1. This weak convergence holds in $L^1(\Omega \setminus E_j)$ and $|E_j| \to 0$, so that the biting limit is equal to $\overline{\varphi}(x)$ a.e. $x \in \Omega$. Note that in particular $\overline{\varphi} \in L^1(\Omega)$ because

$$\begin{aligned}\int_{\Omega\setminus E_j} |\overline{\varphi}|\, dx &\leq \liminf_{k\to\infty} \int_{\Omega\setminus E_j} |\varphi(w^k)|\, dx \\ &\leq \liminf_{k\to\infty} \int_{\Omega} |\varphi(w^k)|\, dx \\ &\leq \text{Const.}\end{aligned}$$

Now let $j \to \infty$.

Another important fact that we will use in Sections 4 and 5 to guarantee the lower semicontinuity of the functional Ψ to be considered later is that for non-negative integrands, even though we may not have the parametrized measure representation (2.3), the "right" inequality still holds. This is a simple consequence of Lemmas 2.2 and 2.3 (see [P1]).

LEMMA 2.4 *Let* $g : \mathbf{R}^+ \to \mathbf{R}^+$ *be a continuous function with* $\lim_{t\to\infty} g(t) = \infty$, *and let* $z^j : \Omega \to \mathbf{R}^n$ *be a sequence of vector valued functions defined in an open bounded set* $\Omega \subset \mathbf{R}^N$, *such that*

$$\sup_j \int_\Omega g\left(|z^j|\right) dx < \infty.$$

If $\{\nu_x\}_{x\in\Omega}$ *is the parametrized measure associated to the* z^j*'s according to Theorem 2.1, then*

$$\liminf_{j\to\infty} \int_E \varphi\left(z^j\right) dx \geq \int_E \int_{\mathbf{R}^m} \varphi(\lambda)\, d\nu_x(\lambda)\, dx,$$

for any measurable $E \subset \Omega$ *and for every non-negative, continuous* φ.

Finally,

COROLLARY 2.5 *Under the same hypothesis of Lemma 2.4, assume that*

$$\lim_{j\to\infty} \int_\Omega \varphi(z^j)\, dx = \int_\Omega \int_{\mathbf{R}^m} \varphi(\lambda)\, d\nu_x(\lambda)\, dx.$$

Then the whole sequence $\{\varphi(z^j)\}$ *converges weakly in* $L^1(\Omega)$.

To prove this result it suffices to apply Lemma 2.4 to a given measurable $E \subset \Omega$ and its complement $\Omega \setminus E$, keeping in mind the hypothesis. This yields

$$\lim_{j\to\infty} \int_E \varphi(z^j)\, dx = \int_E \int_{\mathbf{R}^m} \varphi(\lambda)\, d\nu_x(\lambda)\, dx.$$

We also need to recall a few facts on $H^{1,p}$-Young measures, some of which have already been mentioned at the end of the introduction. For details, we again refer the reader to [KP2]. A $H^{1,p}$-Young measure is a parametrized measure in the sense of Theorem 2.1 associated to a sequence of gradients $\{\nabla u^j\}$, where $\{u^j\}$ is bounded in $H^{1,p}(\Omega;\mathbf{R}^n)$. If $p > 1$ we can always assume that $\{|\nabla u^j|^p\}$ is weakly convergent in $L^1(\Omega)$ and therefore it is an equiintegrable family of functions. Indeed, if $\{\nabla u^j\}$ generates $\{\nu_x\}_{x\in\Omega}$ then there is another sequence $\{\nabla v^j\}$, such that $\{|\nabla v^j|^p\}$ is weakly convergent in $L^1(\Omega)$ and whose parametrized measure is the same $\{\nu_x\}_{x\in\Omega}$. Another important fact is that in this situation each individual ν_x can be understood as a homogeneous (i.e. independent of the point in Ω) $H^{1,p}$-Young measure for a.e. $x \in \Omega$, so that there exists a sequence of gradients $\{\nabla u_x^j\}$, depending upon $x \in \Omega$, such that

$$\lim_{j\to\infty}\int_E \varphi(\nabla u_x^j(y))\,dy = |E|\int_{\mathbf{M}} \varphi(A)\,d\nu_x(A),$$

for any continuous $\varphi : \mathbf{M} \to \mathbf{R}$ with

$$|\varphi(A)| \leq (1 + |A|^p).$$

3. Characterization of parametrized measures

We want to understand the restrictions that govern parametrized measures arising from sequences $\{\nabla u_k, m_k\}$ where $u_k : \Omega \subset \mathbf{R}^N \to \mathbf{R}^n$, $m_k : \Omega \to K$ where K is some given subset of $\mathbf{R}^d$ and $\{u_k, m_k\}$ is a bounded sequence in $H^{1,p}(\Omega;\mathbf{R}^n) \times L^q(\Omega;\mathbf{R}^d)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$.

Here and throughout the paper, π_1 and π_2 will denote projections of $\mathbf{M} \times \mathbf{R}^d$ onto $\mathbf{M}$ and $\mathbf{R}^d$ respectively (or perhaps, projections of $\mathbf{R}^s \times \mathbf{R}^d$ onto $\mathbf{R}^s$ and $\mathbf{R}^d$ respectively). Our aim is to show that this family of Young measures is characterized by

i) $\{\pi_1\nu_x\}_{x\in\Omega}$ is an $H^{1,p}$-Young measure;

ii) $\{\pi_2\nu_x\}_{x\in\Omega}$ is a family of probability measures verifying

$$\int_\Omega\int_{\mathbf{R}^d} |\lambda|^q \, d\pi_2\nu_x(\lambda)\,dx < \infty, \quad \text{if } 1 \leq q < \infty,$$
$$\text{supp}\,(\pi_2\nu_x) \subset K, \text{ for a.e. } x \in \Omega,$$

and there exists a compact set $K' \subset K$ such that

$$\text{supp}\,(\pi_2\nu_x) \subset K', \text{ for a.e. } x \in \Omega, \quad \text{if } q = \infty.$$

Moreover any such parametrized measure may be generated by $\{\nabla\overline{u}_k, \overline{m}_k\}$ where $\{|\nabla\overline{u}_k|^p\}$, $\{|\overline{m}_k|^q\}$ are weakly compact sequences in $L^1(\Omega)$, and therefore equiintegrable.

The arguments throughout this section do not depend upon the fact that we are dealing with gradients ∇u_k. All we need to know to apply the results below to this particular situation is contained in [KP1], and we will state precisely the conclusions in Theorems 3.3 and 3.5 below. Meanwhile we consider sequences $\{v_k, m_k\}$ for $v_k : \Omega \to \mathbf{R}^s$ and $m_k : \Omega \to K$, as before. We will deal first with the case $q = \infty$ and deduce from this and a standard truncation argument, the case corresponding to finite q. Let us assume that K is bounded.

THEOREM 3.1 *Let $\nu = \{\nu_x\}_{x\in\Omega}$ be a family of probability measures such that $\{\pi_1\nu_x\}_{x\in\Omega}$ is generated by $\{\overline{v}_k\}$ with $\{|\overline{v}_k|^p\}$ weakly compact in $L^1(\Omega)$ and $\text{supp}\,(\pi_2\nu_x) \subset K$ for a.e. $x \in \Omega$. Then there is a sequence*

$\{v_j, m_j\} \in L^p(\Omega;\mathbf{R}^s) \times L^\infty(\Omega;\mathbf{R}^d)$ *whose parametrized measure is* $\{\nu_x\}_{x\in\Omega}$, $\{v_j\}$ *is a subsequence of* $\{\overline{v}_k\}$, *and* $m_j(x) \in K$ *for a.e.* $x \in \Omega$ *and all* j.

Proof. Introduce the set

$$\begin{gathered}\mathcal{A} = \{\mu = \{\mu_x\}_{x\in\Omega} \in L^\infty_w(\Omega;\mathcal{M}(\mathbf{R}^s \times \mathbf{R}^d)) : \text{for every}\\ \text{subsequence of } \{\overline{v}_k\} \text{ there is a further subsequence } \{\overline{v}_j\}\\ \text{and } \{w_j\} \subset L^\infty(\Omega;K) \text{ such that}\\ \mu \text{ is generated by } \{\overline{v}_j, w_j\}\}.\end{gathered}$$

Step 1. $\mathcal{A}$ is convex.

Take μ^1 and μ^2 in $\mathcal{A}$ and for $\rho \in (0,1)$, let $\mu = \rho\mu^1 + (1-\rho)\mu^2$. Let $\{\overline{v}_k\}$ be any subsequence, not relabelled, of the original $\{\overline{v}_k\}$. Then μ^i can be generated by $\left\{\overline{v}_k^{(i)}, w_k^{(i)}\right\}$ for $i = 1,2$, where

$$\left\{\overline{v}_k^{(2)}\right\} \subset \left\{\overline{v}_k^{(1)}\right\} \subset \{\overline{v}_k\},\qquad \overline{v}_k^{(2)} = \overline{v}_{j(k)}^{(1)},$$

and by inclusion we mean as subsequences.

It is well-known (see [D]) that we can always find a characteristic function χ of some subset of Ω such that

$$\chi(nx) \overset{*}{\rightharpoonup} |\Omega|\,\rho, \quad \text{in } L^\infty(\Omega).$$

Let

$$w^{k,n}(x) = \chi(nx) w_{j(k)}^{(1)}(x) + (1-\chi(nx))\, w_k^{(2)}(x), \quad w^{k,n}(x) \in K \text{ a.e.} x \in \Omega.$$

For $\psi \in L^1(\Omega;\mathcal{C}_0(\mathbf{R}^s \times \mathbf{R}^d))$,

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{|\Omega|}\int_\Omega \psi(x,\overline{v}_k^{(2)}, w^{k,n})\,dx\\
&= \lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{|\Omega|}\int_\Omega \left(\chi(nx)\psi(x,\overline{v}_k^{(2)}, w_{j(k)}^{(1)}(x)) + (1-\chi(nx))\,\psi(x,\overline{v}_k^{(2)}, w_k^{(2)}(x))\right)\,dx\\
&= \lim_{k\to\infty}\int_\Omega \left(\rho\psi(x,\overline{v}_k^{(2)}, w_{j(k)}^{(1)}(x)) + (1-\rho)\psi(x,\overline{v}_k^{(2)}, w_k^{(2)}(x))\right)\,dx\\
&= \int_\Omega\int_{\mathbf{R}^s\times K} \psi(x,a,\lambda)\,d\mu_x(a,\lambda)\,dx.
\end{aligned}$$

Taking suitable diagonal subsequences of the pairs (k,n) for all $\psi \in \mathcal{L}$ we obtain that $\mu \in \mathcal{A}$.

Step 2. $\nu \in \overline{\mathcal{A}}$.

We use the Hahn-Banach Theorem. Assume that for some $\psi \in L^1(\Omega;\mathcal{C}_0(\mathbf{R}^s \times \mathbf{R}^d))$ and for all $\mu \in \mathcal{A}$,

$$\int_\Omega\int_{\mathbf{R}^s\times\mathbf{R}^d} \psi(x,v,w)\,d\mu_x(v,w)\,dx \geq 0.$$

It is easy to see that then

$$\liminf_{k\to\infty}\int_\Omega \psi(x,\overline{v}_k,w_k)\,dx \geq 0,$$

whenever $w_k(x)\in K$ for a.e. $x\in\Omega$ and all k. Define

$$\overline{\psi}(x,a)=\min_{\lambda\in K}\psi(x,a,\lambda)\leq\psi(x,a,\lambda),$$

and $w_k\in L^\infty(\Omega;\mathbf{R}^d)$ by

$$w_k(x)=\lambda \quad \text{if} \quad \psi(x,\overline{u}_k,\lambda)=\overline{\psi}(x,\overline{u}_k),$$

so that

$$\psi(x,\overline{u}_k,w_k)=\overline{\psi}(x,\overline{u}_k), \qquad \text{a.e. } x\in\Omega.$$

Then

$$\begin{aligned}\int_\Omega\int_{\mathbf{R}^s\times K}\overline{\psi}(x,a)\,d\nu_x(a,\lambda)\,dx &= \lim_{k\to\infty}\int_\Omega\overline{\psi}(x,\overline{u}_k)\,dx\\ &= \lim_{k\to\infty}\int_\Omega\psi(x,\overline{u}_k,w_k)\,dx\\ &\geq 0,\end{aligned}$$

by hypothesis. Finally,

$$\int_\Omega\int_{\mathbf{R}^s\times K}\psi(x,a,\lambda)\,d\nu_x(a,\lambda)\,dx=\int_\Omega\int_{\mathbf{R}^s\times K}(\psi-\overline{\psi})\,d\nu_x\,dx+\int_\Omega\int_{\mathbf{R}^s\times K}\overline{\psi}\,d\nu_x\,dx\geq 0.$$

Step 3. $\mathcal{A}$ is weak * closed in $L^\infty_w(\Omega;\mathcal{M}(\mathbf{R}^s\times\mathbf{R}^d))$.

Let $\mu^{(k)}\in\mathcal{A}$ converge to μ weak * in $L^\infty_w(\Omega;\mathcal{M}(\mathbf{R}^s\times\mathbf{R}^d))$ and let $\{v_n\}$ denote a subsequence of $\{v_k\}$. Then each $\mu^{(k)}$ is generated by $\left\{v_n^{(k)},w_n^{(k)}\right\}$ where

$$\ldots\subset\left\{v_n^{(k)}\right\}\subset\left\{v_n^{(k-1)}\right\}\subset\ldots\subset\left\{v_n^{(1)}\right\}\subset\{v_n\}.$$

Let $\mathcal{L}=\{\psi_j\}$ be a countable dense set in $L^1(\Omega;C_0(\mathbf{R}^s\times\mathbf{R}^d))$. For j fixed, choose $n(k,j)>n(k-1,j)$ such that

$$\left|\lim_{n\to\infty}\int_\Omega\psi_j(x,v_n^{(k)},w_n^{(k)})\,dx-\int_\Omega\psi_j(x,v_{n(k,j)}^{(k)},w_{n(k,j)}^{(k)})\,dx\right|\leq\frac{1}{k}.$$

In this way, and keeping in mind the weak * convergence of the μ^k's to μ, we have

$$\lim_{k\to\infty}\int_\Omega\psi_j(x,v_{n(k,j)}^{(k)},w_{n(k,j)}^{(k)})\,dx=\int_\Omega\int_{\mathbf{R}^s\times\mathbf{R}^d}\psi_j(x,v,w)\,d\mu_x(v,w)\,dx.$$

We conclude by a routine density argument that for every $\psi\in L^1(\Omega;C_0(\mathbf{R}^s\times\mathbf{R}^d))$,

$$\lim_{k\to\infty}\int_\Omega\psi(x,v_{n(k,k)}^{(k)},w_{n(k,k)}^{(k)})\,dx=\int_\Omega\int_{\mathbf{R}^s\times\mathbf{R}^d}\psi(x,v,w)\,d\mu_x(v,w)\,dx.$$

Step 4. Conclusion.

The fact that $\nu \in \mathcal{A}$ implies that there is a sequence $\{v_j, m_j\}$ where $\{v_j\}$ is a subsequence of $\{\overline{v}_k\}$ such that

$$\lim_{j\to\infty} \int_\Omega \psi(x, v_j(x), m_j(x))\, dx = \int_\Omega \int_{\mathbf{R}^s\times K} \psi(x, a, \lambda)\, d\nu_x(a, \lambda)\, dx,$$

for all $\psi \in L^1(\Omega; C_0(\mathbf{R}^s \times \mathbf{R}^d))$. By the comments made after Theorem 2.1 (in particular after (2.2)) we can conclude that ν is the parametrized measure associated to $\{v_j, m_j\}$. ∎

We now deal with the case $q < \infty$ using a truncation argument.

THEOREM 3.2 *Let $\nu = \{\nu_x\}_{x\in\Omega}$ be a family of probability measures such that*

i) $\{\pi_1\nu_x\}_{x\in\Omega}$ can be generated by $\{\overline{v}_k\}$ with $\{|\overline{v}_k|^p\}$ weakly compact in $L^1(\Omega)$;

ii) $\{\pi_2\nu_x\}_{x\in\Omega}$ is a family of probability measures with supp$(\pi_2\nu_x) \subset K$ for a.e. $x \in \Omega$ and

$$\int_\Omega \int_K |\lambda|^q\, d\pi_2\nu_x(\lambda)\, dx < \infty.$$

Then there is a sequence $\{v_j, m_j\} \in L^p(\Omega; \mathbf{R}^s) \times L^q(\Omega; \mathbf{R}^d)$ whose parametrized measure is $\{\nu_x\}_{x\in\Omega}$, $\{v_j\}$ is a subsequence of $\{\overline{v}_k\}$, $\{|m_j|^q\}$ are weakly compact in $L^1(\Omega)$ and $m_j(x) \in K$ for a.e. $x \in \Omega$.

Proof. Let χ_n be the characteristic functions of balls B_n centered at the origin with increasing radii tending to ∞, choose $\lambda_n \in \partial B_n \cap K$ if $\partial B_n \cap K \neq \emptyset$ or $\lambda_n = k_0$ if $\partial B_n \cap K = \emptyset$ where $k_0 \in K$ is fixed, and define $\nu^n = \{\nu^n_x\}_{x\in\Omega}$ through the formula

$$\langle \nu^n, \psi\rangle = \int_\Omega \int_{\mathbf{R}^s\times K} (\chi_n(\lambda)\psi(x, a, \lambda) + (1 - \chi_n(\lambda))\psi(x, a, \lambda_n))\, d\nu_x(a, \lambda)\, dx,$$

for $\psi \in L^1(\Omega; C_0(\mathbf{R}^s \times \mathbf{R}^d))$. In other words, ν^n concentrates the mass outside B_n on $\lambda_n \in \partial B_n \cap K$ or on k_0. We need three basic properties of these truncations:

i) $\pi_1\nu^n_x = \nu_x$ for all n and a.e. $x \in \Omega$;

ii) $\nu^n \overset{*}{\rightharpoonup} \nu$ in $L^\infty_w(\Omega; \mathcal{M}(\mathbf{R}^s \times \mathbf{R}^d))$;

iii) $\langle \nu^n, \psi_0\rangle \to \langle \nu, \psi_0\rangle$ for $\psi_0(x, a, \lambda) = |\lambda|^q$.

The first one is an immediate consequence of the definition of ν^n. For the second and third, notice that if either $\psi \in L^1(\Omega; C_0(\mathbf{R}^s \times \mathbf{R}^d))$ or $\psi = \psi_0$ then

$$\begin{aligned}\langle \nu^n - \nu, \psi\rangle &= \int_\Omega \int_{\mathbf{R}^s\times K} (\psi(x, a, \lambda_n) - \psi(x, a, \lambda))(1 - \chi_n(\lambda))\, d\nu_x(A, \lambda)\, dx \\ &=: \int_\Omega \int_{\mathbf{R}^s\times K} \psi^n(x, a, \lambda)\, d\nu_x(a, \lambda)\, dx.\end{aligned}$$

Since $\psi^n \to 0$ pointwise $\nu \otimes dx$ a.e., as

$$|\psi^n_0| \leq |\lambda|^q, \quad \forall n,$$

with
$$\int_\Omega \int_{\mathbf{R}^s \times K} |\lambda|^q \, d\nu_x(a,\lambda)\, dx < \infty,$$
by hypothesis, and since if $\psi \in L^1(\Omega; C_0(\mathbf{R}^s \times \mathbf{R}^d))$ then
$$|\psi^n| \leq 2 \sup_{a,\lambda} |\psi(x,a,\lambda)| := \overline{\psi}(x) \in L^1(\Omega),$$
we conclude ii) and iii) using Lebesgue's Dominated Convergence Theorem.

Since $\operatorname{supp} \pi_2 \nu^n \subset (B_n \cap K) \cup \{k_0\}$ we can now apply Theorem 3.1 successively to each ν^n and find sequences $\{v_k^n, m_k^n\}$ such that $m_k^n(x) \in K$ for a.e. $x \in \Omega$ (recall $\lambda_n \in K$) and each $\{v_k^n\}$ is a subsequence of the previous $\{v_k^{n-1}\}$ and therefore all of them are subsequences of the initial $\{\overline{v}_k\}$. This can be done because of property i) above. Finally we can find a subsequence $k(n)$ such that if $\{v_j, m_j\} = \left\{v_{k(n)}^n, m_{k(n)}^n\right\}$ then
$$\lim_{j\to\infty} \int_\Omega \psi(x, v_j(x), m_j(x))\, dx = \int_\Omega \int_{\mathbf{R}^s \times K} \psi(x,a,\lambda)\, d\nu_x(a,\lambda)\, dx,$$
for all $\psi \in L^1(\Omega; C_0(\mathbf{R}^s \times \mathbf{R}^d)) \cup \{\psi_0\}$ just as we did at the end of the proof of Theorem 3.1. By applying Corollary 2.5 to ψ_0 we obtain that $\{|m_j|^q\}$ is weakly compact in $L^1(\Omega)$. ∎

We now can replace v_k by gradients ∇u_k and obtain the following

THEOREM 3.3 *Let $\nu = \{\nu_x\}_{x\in\Omega}$ be a family of probability measures such that*

i) $\{\pi_1 \nu_x\}_{x\in\Omega}$ is a $H^{1,p}$-Young measure, $p > 1$;

ii) $\{\pi_2 \nu_x\}_{x\in\Omega}$ is a family of probability measures with supp$(\pi_2\nu_x) \subset K$ for a.e. $x \in \Omega$ and
$$\int_\Omega \int_K |\lambda|^q \, d\pi_2\nu_x(\lambda)\, dx < \infty.$$

Then there is a sequence $\{\nabla u_j, m_j\} \in H^{1,p}(\Omega; \mathbf{R}^n) \times L^q(\Omega; \mathbf{R}^d)$ whose parametrized measure is ν, $\{|\nabla u_j|^p\}$, $\{|m_j|^q\}$ are weakly compact in $L^1(\Omega)$ and $m_j(x) \in K$ for a.e. $x \in \Omega$.

Proof. We know ([KP1], [KP2]) that $\{\pi_1\nu_x\}_{x\in\Omega}$, being an $H^{1,p}$-Young measurewith $p > 1$, is generated by a sequence of gradients $\{\nabla\overline{u}_k\}$ whose pth power is weakly compact in $L^1(\Omega)$. Hence applying Theorem 3.2 we have the result. ∎

Due to the equiintegrability of $\{|\nabla u_j|^p + |m_j|^q\}$, by Theorems 5.3 and 2.1 the integral representation in terms of ν is valid for any φ in the space
$$X^{p,q} = \left\{\varphi \in C(\mathbf{M} \times \mathbf{R}^d) : |\varphi| \leq C(1 + |A|^p + |\lambda|^q)\right\}.$$

We would like to complete Theorem 3.3 by adding the integral constraint
$$\frac{1}{|\Omega|}\int_\Omega m_j(x)\, dx = \frac{1}{|\Omega|}\int_\Omega\int_K \lambda\, d\pi_2\nu_x(\lambda)\, dx, \qquad \text{for all } j.$$

For this we establish first a basic fact from elementary convexity.

In what follows, if K is a subset of $\mathbf{R}^d$ we fix $a \in K$ and we set $L(K) = a + < K - a >$, where $< V >$ denotes the linear manifold spanned by V. Also, $co(K) = \{\theta x + (1-\theta)y | \theta \in [0,1], x, y \in K\}$.

LEMMA 3.4 *Let K be any subset of $\mathbf{R}^d$ and $L(K)$ the linear manifold spanned by K. The set*

$$\left\{ \int_K \lambda \, d\nu(\lambda) : \operatorname{supp} \nu = K, \nu \text{ is a probability measure} \right\}$$

is convex and is contained in the interior of $co(K)$ relative to $L(K)$, i.e. if ν is a probability measure with support K,

$$\alpha = \int_K \lambda \, d\nu(\lambda),$$

then there exists $\epsilon > 0$ such that

$$B(\alpha, \epsilon) \cap L(K) \subset co(K),$$

where $B(\alpha, \epsilon)$ is the ball centered at α and radius ϵ.

Proof. We may assume that $L(K)$ is all of $\mathbf{R}^d$. Otherwise we restrict attention to that linear manifold.

The convexity is clear. Suppose that

$$\alpha = \int_K \lambda \, d\nu(\lambda) \in \partial(\text{co}(K)),$$

and $\operatorname{supp} \nu = K$. Then there is a vector $a \in \mathbf{R}^d$ such that

$$\alpha \cdot a = 0,$$
$$\lambda \cdot a \geq 0, \quad \forall \lambda \in K.$$

Therefore

$$0 = \alpha \cdot a = \int_K \lambda \cdot a \, d\nu(\lambda),$$

which implies $\lambda \cdot a = 0$, $\forall \lambda \in \operatorname{supp} \nu$ and $K = \operatorname{supp} \nu$ is contained in the hyperplane determined by a. This is a contradiction. ■

THEOREM 3.5 *Let $\nu = \{\nu_x\}_{x \in \Omega}$ be a family of probability measures such that*

i) $\{\pi_1 \nu_x\}_{x \in \Omega}$ is a $H^{1,p}$-Young measure, $p > 1$;

ii) $\{\pi_2 \nu_x\}_{x \in \Omega}$ is a family of probability measures with $\operatorname{supp}(\pi_2 \nu_x) \subset K$ for a.e. $x \in \Omega$ and

$$\int_\Omega \int_K |\lambda|^q \, d\pi_2 \nu_x(\lambda) \, dx < \infty.$$

Then there is a sequence $\{\nabla u_j, m_j\} \in H^{1,p}(\Omega; \mathbf{R}^n) \times L^q(\Omega; \mathbf{R}^d)$ whose parametrized measure is ν, $\{|\nabla u_j|^p\}$, $\{|m_j|^q\}$ are weakly compact in $L^1(\Omega)$, $m_j(x) \in K$ for a.e. $x \in \Omega$ and

$$\int_\Omega m_j(x) \, dx = \int_\Omega \int_K \lambda \, d\pi_2 \nu_x(\lambda) \, dx \tag{3.1}$$

for all j.

Proof. First, we would like to keep K as small as possible but having the property $\operatorname{supp}(\pi_2\nu_x) \subset K$ for a.e. $x \in \Omega$. This is easily done by introducing the probability measure $\overline{\nu}_2$ defined by

$$\langle \overline{\nu}_2, \varphi \rangle = \frac{1}{|\Omega|} \int_\Omega \int_K \varphi(\lambda)\, d\pi_2\nu_x(\lambda)\, dx,$$

for $\varphi \in \mathcal{C}_0(\mathbf{R}^d)$, and replacing K by $\operatorname{supp} \overline{\nu}_2$. Let us assume then that $K = \operatorname{supp} \overline{\nu}_2$.

By Theorem 3.3, there exists a sequence $\{u_k, m_k\} \in H^{1,p}(\Omega; \mathbf{R}^n) \times L^q(\Omega; \mathbf{R}^d)$ whose parametrized measure is ν, $\{|\nabla u_k|^p + |m_k|^q\}$ is equiintegrable and $m_k(x) \in K$ a.e. $x \in \Omega$. Our purpose is to redefine m_k so as to have the average constraint (3.1).

Let

$$\alpha := \frac{1}{|\Omega|} \int_\Omega \int_K \lambda\, d\pi_2\nu_x(\lambda)\, dx.$$

Then

$$\alpha = \int_K \lambda\, d\overline{\nu}_2(\lambda),$$

and by Lemma 3.4,

$$\alpha \in \operatorname{int} \operatorname{co}(K) \cap L(K).$$

Without loss of generality we may assume that $L(K) = \mathbf{R}^d$ or else we restrict our attention to that linear manifold.

Let $m_k \rightharpoonup \overline{m}$ in $L^q(\Omega; \mathbf{R}^d)$, $\overline{m}(x) \in \overline{\operatorname{co}(K)}$ a.e. $x \in \Omega$. In fact

$$\overline{m}(x) = \int_K \lambda\, d\pi_2\nu_x(\lambda), \quad \text{a.e. } x \in \Omega.$$

Then

$$\frac{1}{|\Omega|} \int_\Omega \overline{m}(x)\, dx = \alpha \in \operatorname{int} \operatorname{co}(K) \tag{3.2}$$

and so we may find $x_1, x_2, \ldots, x_s$ Lebesgue points for $\overline{m}$ (changing $\overline{m}$ on a set of measure zero, we may suppose without loss of generality that all its points are Lebesgue points) and $\theta_1, \theta_2, \ldots, \theta_s \in (0,1)$, $\sum_{i=1}^s \theta_i = 1$ such that

$$\alpha_0 = \sum_{i=1}^s \theta_i \overline{m}(x_i) \in \operatorname{int} \operatorname{co}(K).$$

Otherwise, since α can be approximated by sums of the type

$$\sum_{i=1}^l \theta_i \overline{m}(x_i)$$

for $x_i \in \Omega$, $l \in \mathbf{N}$, $\theta_i \in (0,1)$, $\sum_{i=1}^l \theta_i = 1$, we would have $\alpha \in \partial \operatorname{co}(K)$ contrary to (3.2).

Let

$$\Omega_j = \bigcup_{i=1}^s B(x_i, \theta_i^{1/N}/j) \subset \Omega$$

which is a disjoint union for j large, $|\Omega_j| \to 0$ and set

$$\alpha_{j,k} = \frac{|\Omega|}{|\Omega_j|}\alpha - \frac{1}{|\Omega_j|}\int_{\Omega\setminus\Omega_j} m_k(x)\,dx.$$

Since

$$\begin{aligned}
\lim_{j\to\infty}\lim_{k\to\infty}|\alpha_0 - \alpha_{j,k}| &= \lim_{j\to\infty}\left|\frac{|\Omega|}{|\Omega_j|}\alpha - \frac{1}{|\Omega_j|}\int_{\Omega\setminus\Omega_j}\overline{m}\,dx - \alpha_0\right| \\
&= \lim_{j\to\infty}\left|\frac{1}{|\Omega_j|}\int_{\Omega_j}\overline{m}\,dx - \alpha_0\right| \\
&= \lim_{j\to\infty}\left|\sum_{i=1}^{s}\theta_i\frac{1}{|B(x_i,\theta_i^{1/N}/j)|}\int_{B(x_i,\theta_i^{1/N}/j)}\overline{m}\,dx - \alpha_0\right| \\
&= \left|\sum_{i=1}^{s}\theta_i\overline{m}(x_i) - \alpha_0\right| \\
&= 0,
\end{aligned}$$

we may extract a diagonal subsequence $k(j)$ such that

$$|\alpha_0 - \alpha_{j,k(j)}| < \frac{1}{j}.$$

Since $\alpha_0 \in$int co(K), there is a $\delta > 0$ such that

$$\alpha_0 + [-\delta,\delta]^d \subset \mathrm{co}(K),$$

and so, by Carathéodory's theorem, each of the $2d$ vertices of the type $\alpha_0 \pm \delta e_i$ (where e_i are the vectors for the standard basis in $\mathbf{R}^d$) can be written as a convex combination of at most $d+1$ elements of K. On the other hand, each point in the cube $\alpha_0 + [-\delta,\delta]^d$ can be written as a convex combination of the vertices and so, we conclude that there exist $k_1, k_2, \ldots, k_{2d(d+1)} \in K$ such that

$$\alpha_0 + [-\delta,\delta]^d \subset \mathrm{co}(\{k_1, k_2, \ldots, k_{2d(d+1)}\}).$$

Hence, since $\alpha_{j,k(j)} \to \alpha_0$, for j large enough and, $m = 2d(d+1)$,

$$\alpha_{j,k(j)} = \sum_{i=1}^{m}\lambda_i^{(j)}k_i, \quad \lambda_i^{(j)} \in [0,1], \sum_{i=1}^{m}\lambda_i^{(j)} = 1.$$

Now we set

$$\overline{u}_j = u_{k(j)},$$
$$\overline{m}_j(x) = \begin{cases} m_{k(j)}(x), & \text{if } x \notin \Omega_j, \\ k_i, & \text{if } x \in \Omega_{j,i}, \end{cases}$$

where $\Omega_j = \bigcup_i \Omega_{j,i}$ and $|\Omega_{j,i}| = \lambda_i^{(j)}|\Omega_j|$. Clearly

$$\overline{m}_j \in K \text{ a.e. } x \in \Omega,$$
$$\int_\Omega \overline{m}_j(x)\,dx = \int_{\Omega\setminus\Omega_j} m_{k(j)}(x)\,dx + |\Omega_j|\,\alpha_{j,k(j)} = |\Omega|\,\alpha.$$

In addition, for any measurable subset E of Ω,

$$\int_E |\overline{m}_j|^p \, dx \le \int_E |m_{k(j)}|^p \, dx + \sum_{i=1}^{m} |E \cap \Omega_{j,i}| \, |k_i|^p$$
$$\le \int_E |m_{k(j)}|^p \, dx + C\,|E|,$$

and so $\{|\overline{m}_j|^p\}$ is equiintegrable. It remains to verify that $\{\nabla\overline{u}_j, \overline{m}_j\}$ generates ν. Consider $\varphi \in X^{p,q}$ and $E \subset \Omega$ measurable. We want to show that

$$\lim_{j\to\infty} \int_E \varphi(\nabla\overline{u}_j, \overline{m}_j)\, dx = \int_E \int_{\mathbf{M}\times\mathbf{R}^d} \varphi(A,\lambda)\, d\nu_x(A,\lambda)\, dx.$$

Indeed, due to the equiintegrability,

$$\left|\int_E \varphi(\nabla\overline{u}_j, \overline{m}_j)\, dx - \int_E \varphi(\nabla u_{k(j)}, m_{k(j)})\, dx\right| \le C \int_{\Omega_j} (1 + |\nabla u_{k(j)}|^p + |m_{k(j)}|^q)\, dx \to 0.$$

Hence

$$\lim_{j\to\infty} \int_E \varphi(\nabla\overline{u}_j, \overline{m}_j)\, dx = \lim_{j\to\infty} \int_E \varphi(\nabla u_{k(j)}, m_{k(j)})\, dx$$
$$= \int_E \int_{\mathbf{M}\times\mathbf{R}^d} \varphi(A,\lambda)\, d\nu_x(A,\lambda)\, dx$$

■

4. Weak lower semicontinuity

With the parametrized measure device we are going to reduce weak lower semicontinuity to a local level. Using the language in [FM2], [FM1], this is done by "blowing-up" the initial sequence near the point $y \in \Omega$ (see also [KP1], [KP2]).

LEMMA 4.1 *Let $\{\nu_x\}_{x\in\Omega}$ be the parametrized measure generated by a sequence $\{\nabla u_k, m_k\}$, bounded in $H^{1,p}(\Omega;\mathbf{R}^n)\times L^q(\Omega;\mathbf{R}^d)$. Then for a.e. $y \in \Omega$, ν_y is a homogeneous (i.e. spatially independent) parametrized measure associated with $\{\nabla u_k^y, m_k^y\}$, bounded in $H^{1,p}(\Omega;\mathbf{R}^n)\times L^q(\Omega;\mathbf{R}^d)$.*

Proof. We know that $\{\pi_1\nu_x\}$ is a $H^{1,p}$-Young measure. Therefore by the comments made at the end of Section 2, $\pi_1\nu_y$ is an homogeneous $H^{1,p}$-Young measure for a.e. $y \in \Omega$ (see [KP2]). On the other hand, by Lemma 2.4 applied to $\varphi(A,\lambda) = |\lambda|^q$ we have

$$\int_\Omega \int_{\mathbf{R}^d} |\lambda|^q \, d\pi_2\nu_x(\lambda)\, dx < \infty$$

whence

$$\int_{\mathbf{R}^d} |\lambda|^q \, d\pi_2\nu_y(\lambda) < \infty$$

for a.e. $y \in \Omega$. Now we can conclude from Theorem 3.3 that for a.e. $y \in \Omega$, ν_y is a homogeneous parametrized measure associated with one of the appropriate sequences. ■

In the following lemma we give the condition which enables us to show weak lower semicontinuity at this local level.

LEMMA 4.2 *Assume that $G : \mathbf{M} \times \mathbf{R}^d \to \mathbf{R}$ is a continuous function such that*

$$|G(A,\lambda)| \le C(1+|A|^p+|\lambda|^q), \quad C>0, 1\le p, 1\le q<\infty,$$
$$|G(A,\lambda)| \le g(\lambda)(1+|A|^p), \quad g \in L^\infty_{loc}(\mathbf{R}^d), 1<p, q=\infty.$$

If

$$G(A,\lambda) \le \frac{1}{|\Omega|}\int_\Omega G(A+\nabla u, \lambda+m)\,dx,$$

for every $(A,\lambda) \in \mathbf{M} \times \overline{co(K)}$ and $(u,m) \in H_0^{1,p}(\Omega;\mathbf{R}^n) \times L^q(\Omega;\mathbf{R}^d)$, $\int_\Omega m\,dx = 0$, $\lambda + m(x) \in K$ a.e. $x \in \Omega$, then

$$G(A,\lambda) \le \liminf_{k\to\infty} \frac{1}{|\Omega|}\int_\Omega G(\nabla u_k, m_k)\,dx,$$

whenever $u_k \rightharpoonup Ax$ in $H^{1,p}(\Omega;\mathbf{R}^n)$, $m_k \rightharpoonup \lambda$ in $L^q(\Omega;\mathbf{R}^d)$ (or $m_k \overset{*}{\rightharpoonup} \lambda$ in $L^\infty(\Omega;\mathbf{R}^d)$), $m_k(x) \in K$ a.e. $x \in \Omega$ and $\{|\nabla u_k|^p\}$, $\{|m_k|^q\}$ are weakly convergent in $L^1(\Omega)$.

Proof. Let η_j be a sequence of cut-off functions such that

$$\eta_j = 1 \text{ when } \operatorname{dist}(x,\partial\Omega) \ge \frac{1}{j},$$
$$\eta_j = 0 \text{ on } \partial\Omega,$$
$$|\nabla\eta_j| \le Cj.$$

Let

$$\Omega_j = \left\{x \in \Omega : \operatorname{dist}(x,\partial\Omega) \le \frac{1}{j}\right\}.$$

Set

$$u_{k,j}(x) = \eta_j u_k(x) + (1-\eta_j)Ax,$$
$$\lambda_k = \frac{1}{|\Omega|}\int_\Omega m_k(x)\,dx.$$

By the hypothesis on G,

$$\begin{aligned}|\Omega|G(A,\lambda_k) &\le \int_\Omega G(\nabla u_{k,j}, m_k)\,dx \\ &= \int_\Omega G(\nabla u_k, m_k)\,dx - \int_{\Omega_j} G(\nabla u_k, m_k)\,dx + \int_{\Omega_j} G(\nabla u_{k,j}, m_k)\,dx \\ &= \int_\Omega G(\nabla u_k, m_k)\,dx + I_{k,j} + II_{k,j}.\end{aligned}$$

We next show that $|I_{k,j}|$ and $|II_{k,j}|$ can be made small.

1)

$$|I_{k,j}| \le \int_{\Omega_j} C(1+|\nabla u_k|^p + |m_k|^q)\,dx.$$

By hypothesis, the integrands are equiintegrable in all of Ω, so that for j sufficiently large we can make $|I_{k,j}|$ small uniformly in k. Same argument for $q = \infty$.

2) For fixed j, take k sufficiently large so that

$$\int_{\Omega_j} |\nabla \eta_j \otimes (u_k - Ax)|^p \, dx \leq \frac{1}{j}.$$

This is possible because

$$\int_{\Omega} |u_k - Ax|^p \, dx \to 0, \; k \to \infty.$$

For such a subsequence, since

$$\nabla u_{k,j} = \eta_j \nabla u_k + (1 - \eta_j) A + \nabla \eta_j \otimes (u_k - Ax),$$

we have

$$|II_{k,j}| \leq C \int_{\Omega_j} (1 + |\nabla u_k|^p + |A|^p + |\nabla \eta_j \otimes (u_k - Ax)|^p + |m_k|^q) \, dx,$$

and therefore, due to the equiintegrability of $\{|\nabla u_k|^p, |m_k|^q\}$,

$$|II_{k,j}| \to 0, \quad j \to \infty,$$

for both cases corresponding to $q < \infty$ and $q = \infty$. Therefore

$$\liminf_{k \to \infty} \int_{\Omega} G(\nabla u_k, m_k) \, dx \geq |\Omega| \liminf_{k \to \infty} G(A, \lambda_k) = |\Omega| \, G(A, \lambda),$$

because $\lambda_k \to \lambda$ by hypothesis and G is continuous. ∎

In the proof of Theorem 3.5 we used the average of a Young measure. Let $\{\nu_x\}_{x \in \Omega}$ be a Young measure associated to $\{\nabla u_k, m_k\}$, a sequence bounded in $H^{1,p}(\Omega; \mathbf{R}^n) \times L^q(\Omega; \mathbf{R}^d)$ with $m_k(x) \in K$ for a.e. $x \in \Omega$. Assume that the associated underlying deformation u with deformation gradient

$$\nabla u(x) = \int_{\mathbf{M}} A \, d\pi_1 \nu_x(A),$$

is affine on $\partial\Omega$, i.e., $u(x) = Fx$, $x \in \partial\Omega$. The average $\overline{\nu}$ is defined by

$$\langle \overline{\nu}, \psi \rangle = \frac{1}{|\Omega|} \int_{\Omega} \int_{\mathbf{M} \times K} \psi(A, \lambda) \, d\nu_x(A, \lambda) \, dx,$$

for $\psi \in C(\mathbf{M} \times \mathbf{R}^d)$.

Lemma 4.3 *$\overline{\nu}$ is a parametrized measure generated by some sequence $\{\nabla \overline{u}_k, \overline{m}_k\}$ with $\{|\nabla \overline{u}_k|^p\}$, $\{|\overline{m}_k|^q\}$ weakly compact in $L^1(\Omega)$, $\overline{u}_k(x) = Fx$, $x \in \partial\Omega$, for all k and $\overline{m}_k(x) \in K$ for a.e. $x \in \Omega$.*

Notice that we are asserting here that under the additional condition of affine boundary values for the deformations, the average defined above is itself a parametrized measure generated by a sequence with the same properties i) and ii) in Theorem 3.3. Indeed, for the proof we rely again on Theorem 3.3.

It is easy to check that

$$\pi_1\bar{\nu} = \overline{\pi_1\nu}$$

and we know that this is a homogeneous $H^{1,p}$-Young measure precisely because we have affine boundary conditions (see [KP2]). On the other hand,

$$\int_K |\lambda|^q \, d\pi_2\bar{\nu}(\lambda) = \frac{1}{|\Omega|}\int_\Omega \int_K |\lambda|^q \, d\pi_2\nu_x(\lambda)\, dx < \infty.$$

We conclude by Theorem 3.3. ∎

We are ready for weak lower semicontinuity.

THEOREM 4.4 *Let $G : \mathbf{M} \times \mathbf{R}^d \to \mathbf{R}$ be a continuous function such that*

$$0 \le G(A,\lambda) \le C(1 + |A|^p + |\lambda|^q), \quad C > 0, 1 \le p < \infty, 1 \le q < \infty,$$
$$0 \le G(A,\lambda) \le g(\lambda)(1 + |A|^p), \quad g \in L^\infty_{loc}(\mathbf{R}^d), 1 \le p, q = \infty,$$
$$0 \le G(A,\lambda) \le g(A)(1 + |\lambda|^q), \quad g \in L^\infty_{loc}(\mathbf{M}), p = \infty, 1 \le q < \infty.$$

Then the weak lower semicontinuity property

$$\int_E G(\nabla u, m)\, dx \le \liminf_{k\to\infty} \int_E G(\nabla u_k, m_k)\, dx$$

holds for any measurable $E \subset \Omega$ whenever $u_k \rightharpoonup u$ in $H^{1,p}(\Omega;\mathbf{R}^n)$ and $m_k \rightharpoonup m$ in $L^q(\Omega;\mathbf{R}^d)$, $m_k(x) \in K$ a.e. $x \in \Omega$ (or $m_k \overset{*}{\rightharpoonup} m$ in $L^\infty(\Omega;\mathbf{R}^d)$) if and only if

$$G(A,\lambda) = \inf_{\mathcal{A}} \left\{ \frac{1}{|\Omega|}\int_\Omega G(A + \nabla u, \lambda + m)\, dx \right\}, \tag{4.1}$$

for all $(A,\lambda) \in \mathbf{M} \times \overline{co(K)}$ where

$$\mathcal{A} = \left\{ (u,m) \in H^{1,p}_0(\Omega;\mathbf{R}^n) \times L^q(\Omega;\mathbf{R}^d) : \int_\Omega m\, dx = 0, \lambda + m(x) \in K \text{ a.e. } x \in \Omega \right\}.$$

Proof. We will not make the distinction between the cases $q < \infty$ and $q = \infty$ or $p < \infty$ and $p = \infty$. We start by showing that (4.1) is a necessary condition for lower semicontinuity.

As usual (see [D]), we may assume without loss of generality that Ω is the unit cube $(0,1)^N$. Given $(u,m) \in \mathcal{A}$, we extend ∇u and m to $\mathbf{R}^N$ as periodic functions of period one. Let $(A,\lambda) \in \mathbf{M} \times \overline{\text{co}(K)}$ and consider

$$u_k(x) = Ax + \frac{1}{k}u(kx),$$
$$m_k(x) = \lambda + m(kx).$$

Then

$$u_k \rightharpoonup Ax \quad \text{in } H^{1,p}(\Omega;\mathbf{R}^n)$$

and

$$m_k \rightharpoonup \lambda + \int_\Omega m(y)\,dy = \lambda \quad \text{in } L^q(\Omega;\mathbf{R}^d).$$

Hence, as the functional is lower semicontinuous we conclude that

$$\begin{aligned} G(A,\lambda) &\le \liminf_{k\to\infty} \int_\Omega G(\nabla u_k, m_k)\,dx \\ &= \liminf_{k\to\infty} \int_\Omega G(A+\nabla u(kx), \lambda + m(kx))\,dx \\ &= \int_\Omega G(A+\nabla u(y), \lambda + m(y))\,dy. \end{aligned}$$

Conversely, assume we have $u_k \rightharpoonup u$ in $H^{1,p}(\Omega;\mathbf{R}^n)$ and $m_k \rightharpoonup m$ in $L^q(\Omega;\mathbf{R}^d)$, $m_k(x) \in K$ a.e. $x \in \Omega$, and let $\{\nu_x\}_{x\in\Omega}$ be the associated parametrized measure. Observe that $\operatorname{supp} \pi_2\nu_x \subset \overline{K}$ a.e. $x \in \Omega$. By Lemma 4.1, for almost every $y \in \Omega$, ν_y may be regarded as a homogeneous parametrized measure which by Theorem 3.3 is generated by some sequence $\{\nabla u_k^y, m_k^y\}$ with the properties that $\{|\nabla u_k^y|^p\}$, $\{|m_k^y|^q\}$ are equiintegrable families in Ω and $m_k^y(x) \in \overline{K}$ a.e. $x \in \Omega$. In this case the Young measure representation is valid since $\{G(\nabla \overline{u}_k^y, \overline{m}_k^y)\}$ (or a subsequence of it) is weakly convergent in $L^1(\Omega)$ (by Dunford-Pettis) and by Lemma 4.2 (take $K = \overline{K}$),

$$\begin{aligned} \int_{\mathbf{M}\times\overline{K}} G(A,\lambda)\,d\nu_y(A,\lambda) &= \lim_{k\to\infty} \frac{1}{|\Omega|}\int_\Omega G(\nabla \overline{u}_k^y, \overline{m}_k^y)\,dx \\ &\ge G(\nabla u(y), m(y)). \end{aligned}$$

Since this is true for a.e. $y \in \Omega$,

$$\int_E \int_{\mathbf{M}\times\overline{K}} G(A,\lambda)\,d\nu_x(A,\lambda)\,dx \ge \int_E G(\nabla u, m)\,dx.$$

Use Lemma 2.4 to conclude the proof. ∎

5. Existence theorems and relaxation

We present several results and applications of the preceding facts related to existence theorems, regularity properties of minimizing sequences and relaxation, as well as some examples of functions verifying the "convexity" condition (4.1).

Let $\psi : \mathbf{M}\times\mathbf{R}^d \to \mathbf{R}$ be continuous and

$$c\max(|A|^p + |\lambda|^q - 1, 0) \le \psi(A,\lambda) \le C(1+|A|^p+|\lambda|^q), \quad A\in\mathbf{M}, \lambda\in\mathbf{R}^d, \tag{5.1}$$

where $1 < p \le \infty$ and $1 < q \le \infty$. For very well-known reasons, we have to restrict attention here to $p, q > 1$. When $p = 1$ one needs to work on spaces of functions of bounded variation ([FKP]). We will be dealing in this section with the functional

$$\Psi(u,m) = \int_\Omega \psi(\nabla u, m)\,dx,$$

defined on an admissible class $\mathcal{A}$, where

$$\mathcal{A} = \Big\{(u,m) \in H^{1,p}(\Omega;\mathbf{R}^n) \times L^q(\Omega;\mathbf{R}^d) : u - u_0 \in H_0^{1,p}(\Omega;\mathbf{R}^n), \\ \int_\Omega m\,dx = |\Omega|\,m_0, m(x) \in K \text{ a.e. } x \in \Omega\Big\},$$

for $u_0 \in H^{1,p}(\Omega;\mathbf{R}^n)$ and $m_0 \in \mathbf{R}^d$. We also assume K to be closed and convex (although not necessarily bounded) so that $\mathcal{A}$ is closed under weak convergence.

THEOREM 5.1 *If ψ satisfies (5.1) and (4.1), the problem*

$$\min_{\mathcal{A}} \Psi(u,m),$$

admits minimizers.

Proof. The proof is standard once we have lower semicontinuity. First of all, notice that the functional Ψ is well-defined and finite on $\mathcal{A}$. If we choose a minimizing sequence (u_k, m_k), by the lower bound in (5.1) we can extract a weakly convergent subsequence in $H^{1,p}(\Omega;\mathbf{R}^n) \times L^q(\Omega;\mathbf{R}^d)$ to some $(u,m) \in \mathcal{A}$. By the weak lower semicontinuity obtained in Theorem 4.4 this provides a minimizer of our problem. ■

THEOREM 5.2 *Assume that ψ satisfies (4.1) and*

$$0 \le \psi(A,\lambda) \le C(1 + |A|^p + |\lambda|^q), \quad A \in \mathbf{M}, \lambda \in \mathbf{R}^d,$$

where $1 \le p,q \le \infty$. Suppose that

$$(u_k, m_k) \rightharpoonup (u,m) \quad \text{in } H^{1,p}(\Omega;\mathbf{R}^n) \times L^q(\Omega;\mathbf{R}^d), \\ \int_\Omega \psi(\nabla u, m)\,dx = \lim_{k\to\infty} \int_\Omega \psi(\nabla u_k, m_k)\,dx. \tag{5.2}$$

Then

$$\psi(\nabla u_k, m_k) \rightharpoonup \psi(\nabla u, m) \quad \text{in } L^1(\Omega).$$

Proof. Given any measurable set $E \subset \Omega$ we apply the weak lower semicontinuity property to both E and $\Omega \setminus E$. Keeping in mind (5.2) we conclude that

$$\int_E \psi(\nabla u, m)\,dx = \lim_{k\to\infty} \int_E \psi(\nabla u_k, m_k)\,dx.$$

■

As a consequence of both theorems, we have that for any minimizing sequence (u_k, m_k) converging to a minimizer (u,m), $\psi(\nabla u_k, m_k)$ converges weakly in $L^1(\Omega)$ to $\psi(\nabla u, m)$, and in particular, at least for a subsequence, $\{|\nabla u_k|^p\}$, $\{|m_k|^q\}$ are weakly convergent in $L^1(\Omega)$ (we are assuming (5.1)). This property is still true for some minimizing sequences even if we do not assume condition (4.1).

Theorem 5.3 *Let ψ satisfy the growth assumptions in (5.1). Then the problem*

$$\inf_{\mathcal{A}} \Psi(u,m),$$

admits a minimizing sequence $(u_k, m_k) \in \mathcal{A}$ such that $\{|\nabla u_k|^p\}$, $\{|m_k|^q\}$ are weakly convergent in $L^1(\Omega)$. Moreover, if $(\overline{u}_k, \overline{m}_k)$ is a minimizing sequence with Young measure $\{\nu_x\}_{x\in\Omega}$ then the parametrized measure representation for ψ holds, i.e.

$$\inf_{\mathcal{A}} \Psi(u,m) = \int_\Omega \int_{\mathbf{M}\times\mathbf{R}^d} \psi(A,\lambda)\, d\nu_x(A,\lambda)\, dx$$

and $\operatorname{supp} \pi_2 \nu_x \subset K$ a.e. $x \in \Omega$.

Proof. Let $(\overline{u}_k, \overline{m}_k)$ be a minimizing sequence in $\mathcal{A}$. By Theorem 3.5, we may assume that $(\overline{u}_k, \overline{m}_k)$ is such that $\{|\nabla \overline{u}_k|^p\}$, $\{|\overline{m}_k|^q\}$ are weakly convergent in $L^1(\Omega)$, $\overline{m}_k(x) \in K$ for a.e. $x \in \Omega$ and the average value of $\overline{m}_k$ is m_0 for every k. However, in making this change we might no longer have that the new sequence is in $\mathcal{A}$ since the trace of $\overline{u}_k$ might not coincide with u_0. Since both sequences share the same Young measure, the weak limit for both coincide and so let $(u,m) \in \mathcal{A}$ denote this common weak limit. We can arrange the boundary values of $\overline{u}_k$ in the usual way using cut-off functions as in the proof of Lemma 4.2 (notice that u should replace Ax) and we find a sequence $\{\nabla u_j, m_j\}$ for $(u_j, m_j) \in \mathcal{A}$ such that $\{|\nabla u_j|^p\}$, $\{|m_j|^q\}$ are weakly convergent in $L^1(\Omega)$ and the parametrized measures for $\{\nabla u_j, m_j\}$ and $\{\nabla \overline{u}_k, \overline{m}_k\}$ are the same (this is also an easy exercise left to the reader). Then, because $\{\nabla \overline{u}_k, \overline{m}_k\}$ is a minimizing sequence and for $\{\nabla u_j, m_j\}$ the Young measure representation is valid,

$$\lim_{k\to\infty} \int_\Omega \psi(\nabla \overline{u}_k, \overline{m}_k)\, dx \le \lim_{j\to\infty} \int_\Omega \psi(\nabla u_j, m_j)\, dx = \int_\Omega \int_{\mathbf{M}\times\mathbf{R}^d} \psi(A,\lambda)\, d\nu_x(A,\lambda)\, dx.$$

By Lemma 2.4 we conclude that (u_j, m_j) is a minimizing sequence. ∎

For a continuous function ψ, we define the relaxed energy density by

$$\psi^\sharp(A,\lambda) = \inf \left\{ \frac{1}{|\Omega|}\Psi(u,m) : u - Ax \in H_0^{1,p}(\Omega;\mathbf{R}^n), m \in L^q(\Omega;\mathbf{R}^d) \right.$$
$$\left. \int_\Omega m\, dx = |\Omega|\, \lambda, m(x) \in K \text{ a.e. } x \in \Omega \right\},$$

where, as before,

$$\Psi(u,m) = \int_\Omega \psi(\nabla u, m)\, dx,$$

and set

$$\Psi^\sharp(u,m) = \int_\Omega \psi^\sharp(\nabla u, m)\, dx.$$

Relaxation Theorem 5.4 *Suppose that ψ satisfies the growth assumption (5.1), and for $(u_0, m_0) \in H^{1,p}(\Omega;\mathbf{R}^n) \times \mathbf{R}^d$ let*

$$\mathcal{A} = \left\{ (u,m) \in H^{1,p}(\Omega;\mathbf{R}^n) \times L^q(\Omega;\mathbf{R}^d) : u - u_0 \in H_0^{1,p}(\Omega;\mathbf{R}^n), \right.$$
$$\left. \int_\Omega m\, dx = |\Omega|\, m_0, m(x) \in K \text{ a.e. } x \in \Omega \right\}.$$

Then

$$\inf_{\mathcal{A}} \{\Psi(u,m)\} = \inf_{\mathcal{A}} \{\Psi^{\mathbf{l}}(u,m)\}.$$

Proof. Clearly $\Psi^{\mathbf{l}} \leq \Psi$ and so

$$\inf_{\mathcal{A}} \{\Psi(u,m)\} \leq \inf_{\mathcal{A}} \{\Psi^{\mathbf{l}}(u,m)\}.$$

It remains to show that for any $(\overline{u},\overline{m}) \in \mathcal{A}$ there is a sequence $(u_k,m_k) \in \mathcal{A}$ such that

$$\lim_{k\to\infty} \Psi(u_k,m_k) = \Psi^{\mathbf{l}}(\overline{u},\overline{m}).$$

Let $y \in \Omega$, and consider the minimization principle

$$\psi^{\mathbf{l}}(\nabla\overline{u}(y),\overline{m}(y)) = \inf\left\{\frac{1}{|\Omega|}\Psi(u,m) : u - \nabla\overline{u}(y) \in H_0^{1,p}(\Omega;\mathbf{R}^n)\right.$$
$$\left. m \in L^q(\Omega;\mathbf{R}^d), \frac{1}{|\Omega|}\int_\Omega m\,dx = \overline{m}(y), m(x) \in K \text{ a.e. } x \in \Omega\right\}.$$

By Theorem 5.3,

$$\psi^{\mathbf{l}}(\nabla\overline{u}(y),\overline{m}(y)) = \frac{1}{|\Omega|}\int_\Omega\int_{\mathbf{M}\times\mathbf{R}^d} \psi(A,\lambda)\,d\nu_x^y(A,\lambda)\,dx,$$

where $\{\nu_x^y\}_{x\in\Omega}$ is the parametrized measure generated by a minimizing sequence. Furthermore by Lemma 4.3 we may assume that this Young measure is homogeneous, i.e. $\nu_x^y = \nu^y$ for a.e. $x \in \Omega$. This is so because we have affine boundary conditions. Then

$$\psi^{\mathbf{l}}(\nabla\overline{u}(y),\overline{m}(y)) = \frac{1}{|\Omega|}\int_\Omega\int_{\mathbf{M}\times\mathbf{R}^d} \psi(A,\lambda)\,d\nu^y(A,\lambda)\,dx = \int_{\mathbf{M}\times\mathbf{R}^d} \psi(A,\lambda)\,d\nu^y(A,\lambda). \tag{5.3}$$

Since this holds for a.e. $y \in \Omega$, we can consider the family of probability measures $\{\nu^y\}_{y\in\Omega}$. By construction, the projection $\{\pi_1\nu^y\}_{y\in\Omega}$ is a $H^{1,p}$-Young measure. We refer the reader to [KP2] for details. Regarding the second projection, if $\{u_k^y, m_k^y\}$ stands for a generating sequence for ν^y, then

$$\begin{aligned}\int_K |\lambda|^q\,d\pi_2\nu^y(\lambda) &= \lim_{k\to\infty}\int_\Omega |m_k^y|^q\,dx \\ &\leq \lim_{k\to\infty} c\Psi(u_k^y,m_k^y) + c\,|\Omega| \\ &= c\psi^{\mathbf{l}}(\nabla\overline{u}(y),\overline{m}(y)) + c\,|\Omega|,\end{aligned}$$

whence

$$\int_\Omega\int_{\mathbf{R}^d} |\lambda|^q\,d\pi_2\nu^y(\lambda)\,dy \leq c\int_\Omega \psi^{\mathbf{l}}(\nabla\overline{u}(y),\overline{m}(y))\,dy + M < \infty.$$

By Theorem 3.5, there is a sequence $(u_k,m_k) \in \mathcal{A}$ whose parametrized measure is $\{\nu^y\}_{y\in\Omega}$ and for ψ the integral representation is valid. Finally by (5.3)

$$\begin{aligned}\lim_{k\to\infty} \Psi(u_k,m_k) &= \int_\Omega\int_{\mathbf{M}\times\mathbf{R}^d} \psi(A,\lambda)\,d\nu^y(A,\lambda)\,dy \\ &= \int_\Omega \psi^{\mathbf{l}}(\nabla\overline{u}(y),\overline{m}(y))\,dy \\ &= \Psi^{\mathbf{l}}(\overline{u},\overline{m}).\end{aligned}$$

∎

It is not easy to find explicit examples of functions verifying the convexity condition (4.1) other than the usual convex functions. There is however a source of such functions in taking

$$\psi(A,\lambda) = g(M(A),\lambda),$$

where $M(A)$ is the vector of all minors of the matrix A and g is a convex function of all its arguments. This is the analogue of polyconvexity. In particular, it is interesting to ask whether there might be more "null-lagrangians" than affine functions of $M(A)$ and λ, that is to say, to determine all functions G such that

$$G(A,\lambda) = \int_\Omega G(A+\nabla u, \lambda + m)\, dx, \tag{5.4}$$

for all $u \in H_0^{1,p}(\Omega;\mathbf{R}^n)$ and $m \in L^q(\Omega;\mathbf{R}^d)$, $\int_\Omega m\, dx = 0$ and all $A \in \mathbf{M}$ and $\lambda \in \mathbf{R}^d$ where $|\Omega| = 1$. We claim that the only null-lagrangians in this context are truly affine functions of $M(A)$ and λ.

If we take $u = 0$ in (5.4), then we come to the conclusion that G should be affine in λ, i.e.

$$G(A,\lambda) = g(A)\cdot\lambda + f(A).$$

Take this G back to (5.4), and let $\lambda = m = 0$ to find that $f(A)$ is some affine function of $M(A)$. Now (5.4) reduces to

$$\begin{aligned} g(A)\cdot\lambda &= \int_\Omega g(A+\nabla u)\cdot(\lambda+m)\, dx \\ &= \int_\Omega g(A+\nabla u)\, dx \cdot \lambda + \int_\Omega g(A+\nabla u)\cdot m\, dx \end{aligned} \tag{5.5}$$

for all $u \in H_0^{1,p}(\Omega;\mathbf{R}^n)$ and $m \in L^q(\Omega;\mathbf{R}^d)$, $\int_\Omega m\, dx = 0$ and all $A \in \mathbf{M}$ and $\lambda \in \mathbf{R}^d$. If we take $m = 0$ we conclude that $g(A)$ should be an affine function of $M(A)$. Let us write $g(A) = \overline{g}(A) + C$ where $\overline{g}(A)$ is a linear function of $M(A)$ and C is some constant. For $A = 0$ (5.5) becomes

$$\int_\Omega g(\nabla u)\cdot m\, dx = 0$$

for all $u \in H_0^{1,p}(\Omega;\mathbf{R}^n)$ and $m \in L^q(\Omega;\mathbf{R}^d)$, $\int_\Omega m\, dx = 0$. In particular, for any such u we can take $m(x) = \overline{g}(\nabla u)$ because by the well-known properties of null-lagrangians $\int_\Omega m\, dx = 0$. Hence

$$\int_\Omega (m(x) + C)\cdot m(x)\, dx = 0,$$

and therefore $\overline{g}(\nabla u) = 0$ for all $u \in H_0^{1,p}(\Omega;\mathbf{R}^n)$. Thus $\overline{g} = 0$ and g is constant, as desired.

Each one of the following is a sufficient condition for G to satisfy (4.1), but we do not know what type of restrictions these place on G. Those conditions are

i) G quasiconvex in A and

$$\int_\Omega G(A+\nabla u, \lambda + m)\, dx \geq \int_\Omega G(A+\nabla u, \lambda)\, dx,$$

for all $u \in H_0^{1,p}(\Omega;\mathbf{R}^n)$ and $\int_\Omega m\,dx = 0$.

ii) G quasiconvex in A, convex in λ and

$$\int_\Omega\int_\Omega G(A+\nabla u(x),\lambda+m(y))\,dx\,dy \le \int_\Omega G(A+\nabla u(z),\lambda+m(z))\,dz,$$

for all $u \in H_0^{1,p}(\Omega;\mathbf{R}^n)$ and $\int_\Omega m\,dx = 0$.

6. Weak lower semicontinuity for Carathéodory functions

Once we have the weak lower semicontinuity property on any measurable $E \subset \Omega$ we can prove weak lower semicontinuity for any Carathéodory function. The proof of this case is reduced to the homogeneous case (no dependence on x) via a standard localization procedure.

Let just deal with the case $1 < p, q < \infty$ and leave the obvious adaptations for $p = \infty$ or $q = \infty$ to the reader.

THEOREM 6.1 *Let $G : \Omega \times \mathbf{M} \times \mathbf{R}^d \to \mathbf{R}$ be a Carathéodory function, i.e. continuous on (A,λ) for a.e. $x \in \Omega$ and measurable on x for all $(A,\lambda) \in \mathbf{M}\times\mathbf{R}^d$. Assume that*

$$0 \le G(x,A,\lambda) \le c(1+|A|^p+|\lambda|^q),$$

and that $G(x,\cdot,\cdot)$ verifies the "convexity" condition (4.1) for a.e. $x \in \Omega$. Then

$$\liminf_{k\to\infty}\int_\Omega G(x,\nabla u_n,m_n)\,dx \ge \int_\Omega G(x,\nabla u,m)\,dx, \tag{6.1}$$

whenever $(u_n,m_n) \rightharpoonup (u,m)$ in $H^{1,p}(\Omega;\mathbf{R}^n)\times L^q(\Omega;\mathbf{R}^d)$.

Proof. We divide the proof in several steps.

1. Since $\{|\nabla u_n|^p + |m_n|^q\}$ is bounded in $L^1(\Omega)$, by the biting lemma there exists a decreasing sequence of sets $\{E_k\}$, $|E_k| \to 0$ such that $\{|\nabla u_n|^p + |m_n|^q\}$ is equiintegrable in $\Omega\setminus E_k$. Observe that we can consider an appropriate subsequence yielding the liminf in (6.1).

Fix k and $\epsilon < 0$. Let

$$X_{n,t} = \{x \in \Omega : |\nabla u_n(x)| > t \text{ or } |m_n(x)| > t\}.$$

Then

$$\begin{aligned}|X_{n,t}| &\le |\{x\in\Omega : |\nabla u_n(x)|^p > t^p\}| + |\{x \in \Omega : |m_n(x)|^q > t^q\}|\\ &\le \frac{1}{t^p}\|\nabla u_n\|_p^p + \frac{1}{t^q}\|m_n\|_q^q \to 0 \text{ as } t\to\infty.\end{aligned}$$

Choose t large enough so that due to the equiintegrability

$$\int_{E_k\cap X_{n,t}} (1+|\nabla u_n|^p+|m_n|^q)\,dx \le \epsilon, \tag{6.2}$$

and write $X_n \equiv X_{n,t}$.

2. Choose $\delta > 0$ so that $\delta(1 + t^p + t^q) \leq \epsilon$ and by the Scorza-Dragoni Theorem for Carathéodory functions (see [D]), let $K \subset \Omega$ be a compact set so that $|\Omega \setminus K| \leq \delta$ and G restricted to $K \times \overline{B}(0, t+1) \times \overline{B}(0, t+1)$ is uniformly continuous.

Let $\eta > 0$ be such that whenever

$$(x, A, \lambda), (x', A', \lambda') \in K \times \overline{B}(0, t+1) \times \overline{B}(0, t+1)$$
$$|x - x'| + |A - A'| + |\lambda - \lambda'| \leq \eta$$

then

$$|G(x, A, \lambda) - G(x', A', \lambda')| \leq \epsilon. \tag{6.3}$$

By Vitali's covering lemma, write

$$\Omega = \cup_i (a_i + \epsilon_i B(0,1)) \cup N,$$

where $|N| = 0$ and $\epsilon_i \leq \eta$. Choose $x_i \in (a_i + \epsilon_i B(0,1)) \cap K$ if this set has positive measure. Set $B \equiv B(0,1)$.

3. Fix $j \in \mathbf{N}$ and let $[g]_t$ stand for the truncation $g\chi_{|g| \leq t}$ for any function g. We decompose the left hand side of (6.1) in several terms

$$\begin{aligned}
\int_\Omega G(x, \nabla u_n, m_n)\, dx \geq & \sum_{i=1}^{j} \int_{E_k \cap (a_i + \epsilon_i B)} (G(x, \nabla u_n, m_n) - G(x, [\nabla u_n]_t, [m_n]_t))\, dx \\
& + \sum_{i=1}^{j} \int_{E_k \cap (a_i + \epsilon_i B) \cap K} (G(x, [\nabla u_n]_t, [m_n]_t) - G(x_i, [\nabla u_n]_t, [m_n]_t))\, dx \\
& + \sum_{i=1}^{j} \int_{E_k \cap (a_i + \epsilon_i B) \cap K} (G(x_i, [\nabla u_n]_t, [m_n]_t) - G(x_i, \nabla u_n, m_n))\, dx \\
& + \sum_{i=1}^{j} \int_{E_k \cap (a_i + \epsilon_i B) - K} G(x, [\nabla u_n]_t, [m_n]_t)\, dx \\
& + \sum_{i=1}^{j} \int_{E_k \cap (a_i + \epsilon_i B) \cap K} (G(x_i, \nabla u_n, m_n) - G(x_i, \nabla u, m))\, dx \\
& + \sum_{i=1}^{j} \int_{E_k \cap (a_i + \epsilon_i B) \cap K} (G(x_i, \nabla u, m) - G(x_i, [\nabla u]_t, [m]_t))\, dx \\
& + \sum_{i=1}^{j} \int_{E_k \cap (a_i + \epsilon_i B) \cap K} (G(x_i, [\nabla u]_t, [m]_t) - G(x, [\nabla u]_t, [m]_t))\, dx \\
& + \sum_{i=1}^{j} \int_{E_k \cap (a_i + \epsilon_i B) \cap K} (G(x, [\nabla u]_t, [m]_t) - G(x, \nabla u, m))\, dx \\
& + \sum_{i=1}^{j} \int_{E_k \cap (a_i + \epsilon_i B) \cap K} G(x, \nabla u, m)\, dx \\
& = A + B + C + D + E + F + G + H + I
\end{aligned}$$

4. Estimates.

$$
\begin{aligned}
A &\le c\int_{E_k\cap X_{n,t}} (1+|\nabla u_n|^p+|m_n|^q)\,dx\\
&\le c\epsilon \quad \text{by (6.2)},\\
B &\le |\Omega|\epsilon \quad \text{by (6.3)},\\
C &\le c\int_{E_k\cap X_{n,t}} (1+|\nabla u_n|^p+|m_n|^q)\,dx\\
&\le c\epsilon \quad \text{by (6.2)},\\
D &\le c|\Omega\setminus K|(1+t^p+t^q)\\
&\le c\delta(1+t^p+t^q)\\
&< c\epsilon \quad \text{by choice of } \delta \text{ in Step 2},
\end{aligned}
$$

F is like C,

G is like B,

H is like A.

5. Conclusion. When $n \to \infty$, because of all the previous estimates, all those terms go to 0. Regarding E we have

$$
\begin{aligned}
&\liminf_{n\to\infty}\sum_{i=1}^{j}\int_{E_k\cap(a_i+\epsilon_i B)\cap K} (G(x_i,\nabla u_n,m_n)-G(x_i,\nabla u,m))\,dx\\
&\ge \sum_{i=1}^{j}\liminf_{n\to\infty}\int_{E_k\cap(a_i+\epsilon_i B)\cap K} (G(x_i,\nabla u_n,m_n)-G(x_i,\nabla u,m))\,dx\\
&\ge 0
\end{aligned}
$$

by Theorem 5.3. We conclude that

$$\liminf_{n\to\infty}\int_\Omega G(x,\nabla u_n,m_n)\,dx \ge \liminf_{n\to\infty} I \ge \sum_{i=1}^{j}\int_{E_k\cap(a_i+\epsilon_i B)\cap K} G(x,\nabla u,m)\,dx.$$

Finally, if we now let $\eta \to 0$, $k\to\infty$ and $j\to\infty$ in this order, using the monotone convergence theorem we conclude that

$$\liminf_{n\to\infty}\int_\Omega G(x,\nabla u_n,m_n)\,dx \ge \int_\Omega G(x,\nabla u,m)\,dx.$$

■

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