Lower Semicontinuity Problems in Sobolev Spaces with Respect to a Measure

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Abstract. For every finite nonnegative measure μ we introduce the Sobolev spaces $W^{1,p}_\mu(\Omega,\mathbf{R}^k)$ and we study the lower semicontinuity of functionals of the form

$$F(u)=\int_\Omega f\left(\frac{dDu}{d\mu}\right)d\mu$$

where the integrand f is quasiconvex.

1. Introduction

Recently much attention has been devoted to the study of nonconvex functionals and relaxation techniques, motivated in part by the fact that these play an important role in the understanding of phase transition phenomena (see for instance Barroso & Fonseca [8], Modica [15] and the references therein). As it is well known, the space $BV(\Omega,\mathbf{R}^k)$ seems to be a suitable function space for describing incoherent phase transitions, and the study of lower semicontinuity and relaxation properties for functionals with linear growth defined on $BV(\Omega,\mathbf{R}^k)$ was initiated in a systematic way by the works of Ambrosio & Dal Maso [4] and Fonseca & Müller [13].

We recall that if $u\in BV(\Omega,\mathbf{R}^k)$ then its distributional derivative Du can be decomposed as

$$Du=\nabla u\cdot\mathcal{L}^n+(u^+-u^-)\otimes\nu\cdot\mathcal{H}^{n-1}\llcorner S_u+Cu, \tag{1.1}$$

where $\nabla u(x)$ is the density of the absolutely continuous part of Du with respect to the n-dimensional Lebesgue measure $\mathcal{L}^n$, S_u is the jump set of u, $\nu(x)\in\mathbf{R}^n$ is a unit vector normal to S_u, $u^+(x)$ and $u^-(x)$ are the traces of u near the jump point $x\in S_u$, and Cu is the so-called Cantor part of the measure Du. According to De Giorgi & Ambrosio [11] a function u is said to belong to the space $SBV(\Omega,\mathbf{R}^k)$ if the Cantor part Cu in (1.1) is zero.

Recently Ambrosio established in [3] the L^1-lower semicontinuity for the functional

$$\int_\Omega f(x,u,\nabla u)\,dx+\int_{S_u}\varphi(u^+,u^-,\nu)\,d\mathcal{H}^{n-1}, \tag{1.2}$$

where $u \in SBV(\Omega,\mathbf{R}^k)$ and $f(x,u,\cdot)$ is a quasiconvex function with growth $p > 1$, while the case $p = 1$ is treated by Barroso, Bouchitté, Buttazzo & Fonseca in [6] (see also Braides & Coscia [9]). In these papers, although the formation of jumps is penalized via the surface energy contribution, we do not know a priori where the jumps, if any, are going to occur.

Here we study problems where the crack site is imposed a priori. For instance, if $\mu = \mathcal{L}^n + \mathcal{H}^{n-1} \llcorner \Gamma$ for some piecewise C^1 surface $\Gamma \subset \Omega$, we may minimize functionals of the type (1.2) on all functions $u \in BV(\Omega,\mathbf{R}^k)$ such that $|Du| << \mu$. It turns out that any function u with this property belongs to $SBV(\Omega,\mathbf{R}^k)$ and $S_u \subset \Gamma$ up to a $\mathcal{H}^{n-1}$-negligible set (compare with Vol'pert & Hudjaev [18]). More generally, for any finite nonnegative measure μ in Ω and any $p > 1$ we define

$$W^{1,p}_\mu(\Omega,\mathbf{R}^k) = \left\{ u \in BV(\Omega,\mathbf{R}^k) \ : \ |Du| << \mu, \ \int_\Omega \left| \frac{Du}{\mu} \right|^p d\mu < +\infty \right\}$$

and we consider the energy functional on $W^{1,p}_\mu(\Omega,\mathbf{R}^k)$

$$F(u) = \int_\Omega f\left(\frac{Du}{\mu}\right) d\mu,$$

where for simplicity we denoted by Du/μ the Radon-Nikodym derivative $\frac{dDu}{d\mu}$ of Du with respect to μ.

It is well known (see for instance Dacorogna [10] and Acerbi & Fusco [1]) that, under appropriate growth conditions, F is weakly lower semicontinuous in $W^{1,p}_\mu(\Omega,\mathbf{R}^k)$ for $\mu = \mathcal{L}^n$ provided f is a quasiconvex function, i.e.

$$f(z) \leq \fint_\Omega f(z + D\varphi(y))\, dy \qquad \forall z \in \mathbf{M}^{n\times k},\ \forall \varphi \in W^{1,\infty}_0(\Omega;\mathbf{R}^k). \tag{1.3}$$

We prove that this result is still true for a large class of measures μ. Precisely, let $a \cdot \mathcal{L}^n + \mu^s$ be the Radon-Nikokym decomposition of μ into absolutely continuous and singular parts with respect to $\mathcal{L}^n$. Then we can write $F = F^a + F^s$, where

$$F^a(u) = \int_\Omega f\left(\frac{\nabla u}{a}\right) a\, dx, \qquad F^s(u) = \int_\Omega f\left(\frac{Du}{\mu^s}\right) d\mu^s.$$

Since $p > 1$, we are able to show that the lower semicontinuity properties of F^a and F^s can be studied separately. The lower semicontinuity properties of F^a follows by adapting the Lipschitz approximation and blow up arguments of [3] to the present situation and we need only to assume that $a \in L^\infty(\Omega)$. The lower semicontinuity of F^s relies on the rank-one theorem of Alberti [2], which enables us to show that

$$\frac{Du}{\mu^s} = \phi_u \otimes \eta \qquad \forall u \in W^{1,p}_\mu(\Omega,\mathbf{R}^k),$$

for a suitable $\phi_u \in L^1_{\mu^s}(\Omega;\mathbf{R}^k)$, where $\eta(x) \in \mathbf{R}^n$ is a unit vector depending only on μ. Since f is rank-one convex, it follows that F^s is a convex functional of ϕ_u, and this easily leads to the lower semicontinuity result.

2. Sobolev Spaces with respect to a Measure

In this section we introduce the Sobolev spaces $W^{1,p}_\mu(\Omega, \mathbf{R}^k)$ and study their main properties. In the following Ω will denote a bounded, open subset of $\mathbf{R}^n$ with a Lipschitz boundary, μ a nonnegative finite measure on Ω, and $p \geq 1$ a real number. The Sobolev space $W^{1,p}_\mu(\Omega, \mathbf{R}^k)$ is defined as the class of all functions $u \in BV(\Omega, \mathbf{R}^k)$ such that the first order distributional gradient Du is a $k \times n$ matrix of measures absolutely continuous with respect to μ and

$$\int_\Omega \left|\frac{Du}{\mu}\right|^p d\mu < +\infty.$$

In order to study lower semicontinuity properties of the integral functionals on $W^{1,p}_\mu(\Omega, \mathbf{R}^k)$ we need to know the structure of the measures Du for functions u belonging to $W^{1,p}_\mu(\Omega, \mathbf{R}^k)$. To this aim, according to Alberti [2] we introduce for every nonnegative finite measure λ on Ω and every $x \in \operatorname{supp}\lambda$ the set

$$E(\lambda, x) = \left\{ v \in \mathbf{R}^n \ : \ \lim_{\rho \to 0} \frac{|Du - v\lambda|(B_\rho(x))}{\lambda(B_\rho(x))} = 0 \text{ for some } u \in BV(\Omega) \right\}$$

and we define $E(\lambda, x) = \{0\}$ if $x \notin \operatorname{supp}\lambda$. The main properties of $E(\lambda, x)$ are the following (see Alberti [2]):

(i) $E(\lambda, x)$ is a linear space;

(ii) $\dim E(\lambda, x) \leq 1$ for λ-a.e. $x \in \Omega$ whenever λ is singular with respect to $\mathcal{L}^n$;

(iii) for every $u \in BV(\Omega)$ we have $Du = f \cdot \lambda + \theta$ with $\theta \perp \lambda$ and $f \in L^1(\Omega; \mathbf{R}^n)$ with $f(x) \in E(\lambda, x)$ for λ-a.e. $x \in \Omega$.

From the properties above we can deduce easily a structural property for functions in $W^{1,p}_\mu(\Omega)$. As usual, given $u \in BV(\Omega)$ we denote by ∇u the absolutely continuous part of Du with respect to the Lebesgue measure $\mathcal{L}^n$, and for every measure λ we use the notation $\lambda = \lambda^a + \lambda^s$ for the Lebesgue-Nikodym decomposition of λ into absolutely continuous and singular parts with respect to $\mathcal{L}^n$.

Proposition 2.1. *Let $\eta \in L^\infty_{\mu^s}(\Omega; \mathbf{R}^n)$ be such that*

$$|\eta(x)| = 1 \quad \text{and} \quad E(\mu^s, x) \subset \operatorname{span}\{\eta(x)\} \quad \text{for } \mu^s\text{-a.e. } x \in \Omega. \tag{2.1}$$

Then, for every $u \in W^{1,p}_\mu(\Omega)$ there exists a unique $\phi_u \in L^p_{\mu^s}(\Omega)$ such that

$$Du = \nabla u \cdot \mathcal{L}^n + \phi_u \eta \cdot \mu^s. \tag{2.2}$$

Proof. By property (ii) above for μ^s-a.e. $x \in \Omega$ we can find a unit vector $\eta(x)$ such that (2.1) holds. Moreover, (see [2]) by using the Aumann's measurable selection theorem we can assume that $x \mapsto \eta(x)$ is a Borel map. Consider now the Lebesgue-Nikodym decomposition of Du with respect to $\mathcal{L}^n$

$$Du = \nabla u \cdot \mathcal{L}^n + D^s u.$$

By property (iii)

$$Du = \phi_u \eta \cdot \mu^s + \theta \tag{2.3}$$

for suitable $\phi_u \in L^1_{\mu^s}(\Omega)$ and $\theta \perp \mu^s$. Taking the singular parts in (2.3) gives

$$D^s u = \phi_u \eta \cdot \mu^s + \theta^s$$

which implies $\theta^s = 0$ because, being $u \in W^{1,p}_\mu(\Omega)$, we have $|Du| << \mu$ and so $|D^s u| << \mu^s$. ■

Remark 2.2. In the vector valued case $u \in W^{1,p}_\mu(\Omega; \mathbf{R}^k)$ equality (2.2) reads

$$Du = \nabla u \cdot \mathcal{L}^n + \phi_u \otimes \eta \cdot \mu^s$$

with $\phi_u \in L^p_{\mu^s}(\Omega; \mathbf{R}^k)$ and η satisfying (2.1).

Remark 2.3. It is not hard to see that the usual Sobolev space $W^{1,p}(\Omega, \mathbf{R}^k)$ coincides with $W^{1,p}_{\mathcal{L}^n}(\Omega, \mathbf{R}^k)$. Moreover, since BV functions do not charge H^{n-1}-negligible sets, we have $W^{1,p}_{\mu_1}(\Omega) = W^{1,p}_{\mu_2}(\Omega)$ whenever $|\mu_1 - \mu_2|(B) = 0$ and $H^{n-1}(\Omega \setminus B) = 0$. In particular, if

$$E = \left\{ x \in \Omega \ : \ \limsup_{\rho \to 0} \frac{\mu(B_\rho(x))}{\rho^{n-1}} < +\infty \right\}$$

and $\nu = \mu \llcorner E$, as $H^{n-1}(\Omega \setminus E) = 0$ (see for instance Ziemer [19]), we have $W^{1,p}_\mu(\Omega) = W^{1,p}_\nu(\Omega)$.

We consider now some compactness properties of the spaces $W^{1,p}_\mu(\Omega)$.

Proposition 2.4. *Let $p > 1$ and let (u_h) be a sequence in $W^{1,p}_\mu(\Omega)$ such that*

$$\left| \int_\Omega u_h \, dx \right| + \int_\Omega \left| \frac{Du_h}{\mu} \right|^p d\mu \leq c. \tag{2.4}$$

Then there exists a subsequence (u_{h_k}) and a function $u \in W^{1,p}_\mu(\Omega)$ such that

$$\begin{cases} u_{h_k} \to u & \text{strongly in } L^1(\Omega) \\ Du_{h_k}/\mu \to Du/\mu & \text{weakly in } L^p_\mu(\Omega; \mathbf{R}^n). \end{cases} \tag{2.5}$$

Proof. By (2.4) we get that (u_h) is bounded in $BV(\Omega)$, hence by the standard compact embedding theorem, a subsequence (which we still denote by (u_h)) converges in $L^1(\Omega)$. Moreover, Du_h/μ are bounded in $L^p_\mu(\Omega; \mathbf{R}^n)$ so that we may assume (a subsequence of) Du_h/μ converges weakly in $L^p_\mu(\Omega)$ to some $w \in L^p_\mu(\Omega)$. The equality $w = Du/\mu$ follows from the fact that, being (u_h) bounded in $BV(\Omega)$, we have $Du_h \to Du$ in the weak* convergence of measures, so that for every $\phi \in C^1_0(\Omega; \mathbf{R}^n)$

$$\begin{aligned} \int_\Omega w\phi \, d\mu &= \lim_{h \to +\infty} \int_\Omega \frac{Du_h}{\mu} \phi \, d\mu = \lim_{h \to +\infty} \langle Du_h, \phi \rangle \\ &= \langle Du, \phi \rangle = \int_\Omega \frac{Du}{\mu} \phi \, d\mu. \quad ■ \end{aligned}$$

Remark 2.5. From (2.5) we get, using the notation of Proposition 2.1

$$\begin{cases} \phi_{u_{h_k}} \to \phi_u & \text{weakly in } L^p_{\mu^s}(\Omega) \\ \nabla u_{h_k}/a \to \nabla u/a & \text{weakly in } L^p_{\mu^a}(\Omega; \mathbf{R}^n). \end{cases}$$

3. Lower Semicontinuity

In this section we prove our main lower semicontinuity result. Many of the technical tools needed for the proof will be proved in the next section.

Theorem 3.1. *Let $p > 1$ and $f : \mathbf{M}^{n\times k} \to \mathbf{R}$ be a quasiconvex function such that*

$$0 \le f(z) \le C(1 + |z|^p) \qquad \forall z \in \mathbf{M}^{n\times k}$$

for a suitable constant $C > 0$. Assume that the Lebesgue-Nikodym decomposition of μ with respect to $\mathcal{L}^n$ is $\mu = a \cdot \mathcal{L}^n + \mu^s$ with $a \in L^\infty(\Omega)$. Then the functional

$$F(u) = \int_\Omega f\left(\frac{Du}{\mu}\right) d\mu$$

is sequentially lower semicontinuous on $W^{1,p}_\mu(\Omega; \mathbf{R}^k)$ with respect to the weak convergence defined by

$$u_h \to u \quad \text{weakly in } W^{1,p}_\mu(\Omega;\mathbf{R}^k) \quad \Longleftrightarrow \quad \begin{cases} u_h \to u \quad \text{strongly in } L^1(\Omega;\mathbf{R}^k) \\ Du_h/\mu \to Du/\mu \quad \text{weakly in } L^p_\mu(\Omega;\mathbf{M}^{n\times k}). \end{cases}$$

Remark 3.2. The assumption $a \in L^\infty(\Omega)$ in Theorem 3.1 is technical; we do not know if it is necessary.

The proof of Theorem 3.1 is based on the splitting of the functional F into two parts, namely $F = F^a + F^s$ where

$$F^a(u) = \int_\Omega f\left(\frac{Du}{\mu^a}\right) d\mu^a = \int_\Omega f\left(\frac{Du}{a}\right) a(x)\, dx$$
$$F^s(u) = \int_\Omega f\left(\frac{Du}{\mu^s}\right) d\mu^s.$$

The lower semicontinuity of F^s follows easily from Remarks 2.2 and 2.5 . Indeed, we have $F^s(u) = \Phi(\phi_u)$ with

$$\Phi(\phi) = \int_\Omega f(\phi \otimes \eta)\, d\mu^s,$$

and since f is rank-one convex, Φ is convex and lower semicontinuous in $L^p_{\mu^s}(\Omega, \mathbf{R}^k)$. The proof of the lower semicontinuity of F^a will be obtained using the blow-up technique. The main tool will be the following inequality.

Theorem 3.3. [Approximate quasiconvexity inequality] *Let $M > 0$, $p > 1$, and let f be a quasiconvex function satisfying*

$$\gamma |z|^p \le f(z) \le C(1 + |z|^p) \qquad \forall z \in \mathbf{M}^{n\times k} \tag{3.1}$$

with $\gamma > 0$. For any $\epsilon > 0$ there exists a constant $\delta > 0$ depending only on M, p, f with the following property: for any choice of $\rho > 0$, $c_0 > 0$, $z_0 \in \mathbf{M}^{n\times k}$, $u \in BV(B_\rho)$, and $a \in L^1(B_\rho, \mathbf{R}^+)$, the approximate quasiconvexity inequality

$$\epsilon\, \mathrm{meas}(B_\rho) + \int_{B_\rho} f\left(\frac{\nabla u}{a}\right) a\, dx \ge \int_{B_\rho} f\left(\frac{z_0}{c_0}\right) c_0\, dx$$

holds provided $|z_0| \le M$, $c_0 \ge M^{-1}$, $\|a\|_\infty \le M$, *and*

$$\rho^{-n}|D^s u|(B_\rho) + \fint_{B_\rho} |a - c_0|\, dx + \rho^{-1} \fint_{B_\rho} |u(x) - z_0(x)|\, dx < \delta.$$

Heuristically, if a is sufficiently close to a constant, u is sufficiently close to a linear function and the singular part of u is small, then an inequality similar to (1.3) holds. The proof of Theorem 3.3 will be given in Section 4 using the method of Lipschitz approximations of [1] and [3]. Next we show how Theorem 3.3 and a covering type argument yield the lower semicontinuity of F^a.

Proof of lower semicontinuity of F^a. Fix a sequence (u_h) bounded in $W^{1,p}_\mu(\Omega, \mathbf{R}^k)$ and converging in L^1 to some $u \in W^{1,p}_\mu(\Omega, \mathbf{R}^k)$. We have to prove that

$$\int_\Omega f\left(\frac{\nabla u}{a}\right) a\, dx \le \liminf_{h \to +\infty} \int_\Omega f\left(\frac{\nabla u_h}{a}\right) a\, dx. \tag{3.2}$$

We may assume that the liminf in (3.2) is a finite limit, say L. Since $|\nabla u_h|^p/a^{p-1}$ is bounded in $L^1(\Omega)$, by adding $\gamma|z|^p$ to $f(z)$ we can assume that the stronger condition (3.1) is satisfied for some $\gamma > 0$. Let ψ_h be the densities of $|D^s u_h|$ with respect to μ^s. Since (ψ_h) is bounded in $L^p(\mu^s)$ we can assume that ψ_h weakly converges in $L^p(\mu^s)$ to some function ψ. By our growth assumption on f, the function $\phi = f(\nabla u/a)a$ belongs to $L^1(\Omega)$. We recall (see for instance Federer [12], 4.5.9) that any BV function is approximately differentiable, i.e.,

$$\lim_{\rho \to 0+} \fint_{B_\rho(x)} \frac{|u(y) - u(x) - \langle \nabla u(x), y - x\rangle|}{|y - x|}\, dy = 0 \tag{3.3}$$

for $\mathcal{L}^n$-a.e. $x \in \Omega$. We denote by E the set of Lebesgue points x of a and ϕ such that

$$\lim_{\rho \to 0+} \rho^{-1} \fint_{B_\rho(z)} |u(x) - u(z) - \nabla u(z)(x - z)|\, dx = 0.$$

We define also

$$F = \left\{ z \in \Omega \ :\ \limsup_{\rho \to 0+} \rho^{-n} \int_{B_\rho(z)} \psi\, d\mu^s > 0 \right\}$$
$$\Omega_M = \left\{ z \in E \setminus F \ :\ a(z) \ge M^{-1},\ |\nabla u(z)| \le M \right\} \qquad \text{for } M \ge \|a\|_\infty.$$

By (3.3), $\Omega \setminus E$ is negligible. Moreover, since F is negligible, we have

$$\mathrm{meas}(\{x \in \Omega : a(x) > 0\}) = \lim_{M \to +\infty} \mathrm{meas}(\Omega_M).$$

In particular, we need only to show the inequality

$$\liminf_{h \to +\infty} \int_\Omega f\left(\frac{\nabla u_h}{a}\right) a\, dx \ge \int_{\Omega_M} f\left(\frac{\nabla u}{a}\right) a\, dx \tag{3.4}$$

for any $M \geq \|a\|_\infty$. Let $\epsilon > 0$, and let $\delta > 0$ be given by Theorem 3.3. We denote by $\mathcal{F}_\epsilon$ the collection of all closed balls $\overline{B}_\rho(z) \subset \Omega$ centered at points $z \in \Omega_M$, such that

$$\rho^{-n} \int_{\overline{B}_\rho(z)} \psi \, d\mu^s + \fint_{B_\rho(z)} |a - a(z)| \, dx + \rho^{-1} \fint_{B_\rho(z)} |u(x) - u(z) - \nabla u(z)(x-z)| \, dx < \delta$$

and

$$\int_{B_\rho(z)} \phi(x) \, dx \leq \int_{B_\rho(z)} (\phi(z) + \epsilon) \, dx.$$

By our definition of Ω_M, we have that for any $z \in \Omega_M$ the ball $\overline{B}_\rho(z)$ belongs to $\mathcal{F}_\epsilon$ for $\rho > 0$ small enough. By Besicovitch's theorem there is a family of pairwise disjoint balls $B_i = \overline{B}_{\rho_i}(z_i) \in \mathcal{F}_\epsilon$, indexed by $I \subset \mathbf{N}$, which covers almost all of Ω_M. Since ψ_h converges weakly to ψ in $L^p(\mu^s)$ and u_h converges in $L^1_{\text{loc}}(\Omega, \mathbf{R}^k)$ to u, for any finite set $J \subset I$ we can find $h_0 \in \mathbf{N}$ such that

$$\rho_i^{-n} \int_{B_i} \psi_h \, d\mu^s + \fint_{B_i} |a - a(z_i)| \, dx + \rho_i^{-1} \fint_{B_i} |u_h(x) - u(z_i) - \nabla u(z_i)(x - z_i)| \, dx < \delta$$

for any $h \geq h_0$ and any $i \in J$. By Theorem 3.3 we get

$$\begin{aligned}
\int_\Omega f\left(\frac{\nabla u_h}{a}\right) a \, dx &\geq \sum_{i \in J} \int_{B_i} f\left(\frac{\nabla u_h}{a}\right) a \, dx \\
&\geq -\epsilon \sum_{i \in J} \text{meas}(B_i) + \sum_{i \in J} \int_{B_i} f\left(\frac{\nabla u(z_i)}{a(z_i)}\right) a(z_i) \, dx \\
&\geq -2\epsilon \sum_{i \in J} \text{meas}(B_i) + \sum_{i \in J} \int_{B_i} f\left(\frac{\nabla u}{a}\right) a \, dx.
\end{aligned}$$

Hence, letting $h \to +\infty$ we get

$$2\epsilon \, \text{meas}(\Omega) + L \geq \sum_{i \in J} \int_{B_i} f\left(\frac{\nabla u}{a}\right) a \, dx.$$

By letting first $J \uparrow I$ then $\epsilon \to 0$, (3.4) follows. ■

Remark 3.4. More generally, the methods of this paper and of [3] apply to the lower semicontinuity of functionals of the form

$$\int_\Omega f(x, u, \nabla u/a) d\mu^a + \int_\Omega g(x, \varphi_u) \, d\mu^s \qquad u \in W^{1,p}_\mu(\Omega, \mathbf{R}^k)$$

assuming that $a \in L^\infty(\Omega)$, $f(x, s, z)$ and $g(x, y)$ are Carathéodory functions, f is quasiconvex with respect to z, $g \geq 0$ is convex with respect to y and

$$0 \leq f(x, s, z) \leq C\big(b(x) + |s|^p + |z|^p\big) \qquad \forall (x, s, z) \in \Omega \times \mathbf{R}^k \times \mathbf{M}^{n \times k}$$

for some $b \in L^1(\Omega)$, $C > 0$ and $p > 1$.

4. Proof of the Approximate Quasiconvexity Inequality

In this section we prove the approximate quasiconvexity inequality of Theorem 3.3. We start by introducing some preliminary notions.

Let θ be a nonnegative, finite measure in the unit ball B of $\mathbf{R}^n$. The (local) maximal function $M(\theta)$ of θ is defined by

$$M(\theta)(x) = \sup\left\{\frac{\theta(B_\rho(x))}{\operatorname{meas}(B_\rho(x))}\ :\ 0<\rho<1-|x|\right\}. \tag{4.1}$$

If θ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^n$ and ϕ is its density, we set $M(\phi) = M(\theta)$. In the following proposition we recall well known properties of the maximal function (see for instance [3], [17]).

Proposition 4.1. *Let θ be as above. Then,*

$$\operatorname{meas}(\{x \in B : M(\theta)(x) > \lambda\}) \le \frac{\xi(n)\theta(B)}{\lambda} \qquad \forall\lambda>0, \tag{4.2}$$

with $\xi(n)$ depending only on n. Moreover, if θ is absolutely continuous with respect to $\mathcal{L}^n$ and its density ϕ belongs to $L^p(B)$ for some $p > 1$, then

$$\int_B M^p(\theta)\,dx \le \xi(n,p)\int_B \phi^p\,dx$$

with $\xi(n,p) = p2^p/(p-1)$.

The following theorem shows that BV functions can be approximated (in the sense of Lusin) by Lipschitz functions in the regions where the maximal function of the total variation is controlled. The proof is based on the inequality

$$\fint_{B_\rho(x)} \frac{|u(z)-u(x)|}{|z-x|}\,dz \le \int_0^1 \frac{|Du|(B_{t\rho}(x))}{\operatorname{meas}(B_{t\rho}(x))}\,dt \le M(|Du|)(x),$$

which holds for any Lebesgue point x of u (see [3]).

Theorem 4.2. *Let $\lambda > 0$, $u \in BV(B,\mathbf{R}^k)\cap L^\infty(B,\mathbf{R}^k)$, and let*

$$E = \{x \in B\ :\ M(|Du|)(x) \le \lambda\}.$$

Then, for any $\rho \in (0,1)$ there exists a Lipschitz function $v : B_\rho \mapsto \mathbf{R}^k$ such that $u(x) = v(x)$ for $\mathcal{L}^n$-almost every $x \in E \cap B_\rho$ and

$$\operatorname{Lip}(v, B_\rho) \le c(n)k\lambda + \frac{2k\|u\|_\infty}{1-\rho}.$$

Finally, we will need a weak equi-integrability property of sequences bounded in L^1 (the so-called biting lemma, see [1]) and a truncation lemma (see [3]).

Lemma 4.3. *Let (ϕ_h) be a bounded sequence in $L^1(B)$. Then, for every $\epsilon > 0$ there exist a Borel subset C_ϵ of B, $\delta \in (0,\epsilon)$, and an infinite set $S \subset \mathbf{N}$ such that* $\text{meas}(C_\epsilon) < \epsilon$ *and*

$$\text{meas}(C) < \delta, \quad C \cap C_\epsilon = \emptyset \qquad \Rightarrow \qquad \int_C |\phi_h|\,dx < \epsilon$$

for any $h \in S$.

Proposition 4.4. *Let (u_h) be a sequence in $BV(B,\mathbf{R}^k)$ converging in $L^1(B,\mathbf{R}^k)$ to $u \in L^\infty(B,\mathbf{R}^k)$. Let us assume that $|\nabla u_h|$ are equi-integrable in B and*

$$\lim_{h\to+\infty} |D^s u_h|(B) = 0. \tag{4.3}$$

Then, there exists a sequence of sets of finite perimeter $E_h \subset B$ such that

$$\lim_{h\to+\infty} \left[\text{meas}(E_h) + |D\chi_{E_h}|(B)\right] = 0$$

and $|u_h|(x) \le (1 + \|u\|_\infty)$ for every $x \in B \setminus E_h$.

Remark 4.5. Let u_h, E_h be as in Proposition 4.4 and let $\phi : \mathbf{R}^k \mapsto \mathbf{R}^k$ be a bounded Lipschitz function such that $\phi(s) = 1$ for any s with $|s| \le (M+1)$, $M \ge \|u\|_\infty$. Since $\phi(u_h)\chi_{B\setminus E_h} = u_h\chi_{B\setminus E_h}$, we infer (by [18]) that the functions $v_h = u_h\chi_{B\setminus E_h}$ belong to $BV(B,\mathbf{R}^k)$, are bounded in $L^\infty(B,\mathbf{R}^k)$, converge to u in $L^1(B,\mathbf{R}^k)$. Moreover (3.3) implies that if $w \in BV(\Omega)$ then $\nabla w(x)$ is zero for $\mathcal{L}^n$-a.e. in a level set of w and so

$$\nabla v_h(x) = \begin{cases} \nabla u_h(x) & \text{if } x \in B \setminus E_h \\ 0 & \text{if } x \in E_h \end{cases} \tag{4.4}$$

for almost every $x \in B$. Finally, by the inequality (see [18])

$$|D^s v_h|(B) \le |D^s u_h|(B) + \|\phi\|_\infty |D\chi_{E_h}|(B)$$

we obtain that $|D^s v_h|(B)$ tends to 0 as $h \to +\infty$.

The proof of Theorem 3.3 will be achieved by a contradiction argument. The contradiction will be a consequence of the following lower semicontinuity theorem.

Theorem 4.6. *Let $f : \mathbf{M}^{n\times k} \to \mathbf{R}$ be a quasiconvex function and let $(u_h) \subset BV(B,\mathbf{R}^k)$ and $(a_h) \subset L^1(B,\mathbf{R}^+)$ be sequences converging in $L^1(B)$ respectively to a linear function $u(x) = z_0(x)$ and to a constant $c_0 > 0$. Assume that*

$$M = \sup_{h\in\mathbf{N}} \|u_h\|_\infty + \|a_h\|_\infty < +\infty \qquad \text{and} \qquad \lim_{h\to+\infty} |D^s u_h|(B) = 0$$

and (3.1) holds. Then we have

$$\liminf_{h\to+\infty} \int_B f\left(\frac{\nabla u_h}{a_h}\right) a_h\,dx \ge \int_B f\left(\frac{\nabla u}{c_0}\right) c_0\,dx. \tag{4.5}$$

Proof. It is not restrictive to assume that the liminf in (4.5) is a finite limit, say L. We denote by Γ a constant such that $|Du_h|(B) \le \Gamma$ for any $h \in \mathbf{N}$. Since

$$\gamma \int_B M^p(|\nabla u_h|)\,dx \le \gamma\xi(n,p)\int_B |\nabla u_h|^p\,dx \le \xi(n,p)M^{p-1}\int_B f\left(\frac{\nabla u_h}{a_h}\right)a_h\,dx,$$

the sequence $\phi_h = M^p(|\nabla u_h|)$ is bounded in $L^1(B)$. Let $\epsilon > 0$ be fixed, and let $C_\epsilon \subset B$, $\delta \in]0,\epsilon[$, $S \subset \mathbf{N}$ be given by Lemma 4.3 so that for any Borel set $C \subset B \setminus C_\epsilon$ such that $\mathrm{meas}(C) < \delta$ we have

$$\int_C M^p(|\nabla u_h|)\,dx < \epsilon \qquad \forall h \in S. \tag{4.6}$$

Since $|\nabla u_h|$ is bounded in $L^1(B)$, by (4.2) we can choose $\lambda_\epsilon \ge 1$ such that for every $\lambda \ge \lambda_\epsilon$

$$\mathrm{meas}(\{x \in B \;:\; M(|\nabla u_h|)(x) > \lambda\}) < \delta \qquad \forall h \in \mathbf{N}, \tag{4.7}$$

$$\frac{2kM}{\epsilon} < 2\lambda. \tag{4.8}$$

For any $\lambda \ge \lambda_\epsilon$, we apply Theorem 4.2 with $\rho = 1 - \epsilon$ to obtain Lipschitz functions $u_{h,\lambda} : B_\rho \longmapsto \mathbf{R}^k$ whose Lipschitz constant does not exceed $\lambda' = (1 + c(n)k)2\lambda$, such that $u_{h,\lambda} = u_h$ $\mathcal{L}^n$-a.e. in $B_\rho \setminus E_{h,\lambda}$, where

$$E_{h,\lambda} = \{x \in B : M(|Du_h|) > 2\lambda\}.$$

Let

$$C_h = C_\epsilon \cup \{x \in B : a_h(x) \le c_0/2\}$$
$$E'_{h,\lambda} = \{x \in B \setminus C_h : M(|\nabla u_h|) > \lambda\}$$
$$E''_{h,\lambda} = \{x \in B \setminus C_h : M(|D^s u_h|) > \lambda\}.$$

Since $\nabla u_h = \nabla u_{h,\lambda}$ a.e. in $B_\rho \setminus E_{h,\lambda}$ we get

$$\int_{B_\rho \setminus C_h} f\left(\frac{\nabla u_h}{a_h}\right)a_h\,dx \le \int_B f\left(\frac{\nabla u_h}{a_h}\right)a_h\,dx + C\int_{E_{h,\lambda}\setminus C_h}\left(a_h + \frac{(\lambda')^p}{a_h^{p-1}}\right)dx. \tag{4.9}$$

By the inclusion $E_{h,\lambda} \setminus C_h \subset E'_{h,\lambda} \cup E''_{h,\lambda}$ we get

$$\int_{E_{h,\lambda}\setminus C_h}\left(a_h + \frac{(\lambda')^p}{a_h^{p-1}}\right)dx \le \int_{E'_{h,\lambda}}\left(c_0/2 + \frac{(\lambda')^p}{(c_0/2)^{p-1}}\right)dx + \int_{E''_{h,\lambda}}\left(c_0/2 + \frac{(\lambda')^p}{(c_0/2)^{p-1}}\right)dx.$$

The second term in the right hand side vanishes as $h \to +\infty$ because

$$\lambda\,\mathrm{meas}(E''_{h,\lambda}) \le \xi(n)|D^s u_h|(B).$$

To estimate the first term we replace λ^p by $M^p(|\nabla u_h|)$:

$$\int_{E'_{h,\lambda}}\left(c_0/2 + \frac{(\lambda')^p}{(c_0/2)^{p-1}}\right)dx \le \int_{E'_{h,\lambda}}\left(c_0/2 + 2^p(1+c(n)k)^p\frac{M^p(|\nabla u_h|)}{(c_0/2)^{p-1}}\right)dx.$$

If $h \in S$, by (4.6) we obtain

$$\int_{E'_{h,\lambda}}\left(c_0/2 + \frac{(\lambda')^p}{(c_0/2)^{p-1}}\right)dx \le c_0\delta + 2^{2p}(1+c(n)k)^p c_0^{1-p}\epsilon.$$

By letting $h \in S$ go to $+\infty$, (4.9) yields

$$\liminf_{h\to+\infty} \int_{B_\rho\setminus C_h} f\left(\frac{\nabla u_{h,\lambda}}{a_h}\right) a_h\, dx \le L + C\left(c_0\delta + 2^{2p}(1+c(n)k)^p c_0^{1-p}\epsilon\right) \tag{4.10}$$

for any $\lambda \ge \lambda_\epsilon$. We now claim that there are Lipschitz function $v_\lambda : B_\rho \mapsto \mathbf{R}^k$ such that

$$\operatorname{meas}(\{x \in B_\rho : u(x) \ne v_\lambda(x)\}) \le \frac{\Gamma\xi(n)}{2\lambda} \tag{4.11}$$

$$\int_{B_\rho\setminus C_\epsilon} f\left(\frac{\nabla v_\lambda}{c_0}\right) c_0\, dx \le \liminf_{h\to+\infty} \int_{B_\rho\setminus C_h} f\left(\frac{\nabla u_{h,\lambda}}{a_h}\right) a_h\, dx. \tag{4.12}$$

Indeed, let $\lambda \ge \lambda_\epsilon$ be fixed and let $v_k = u_{h_k,\lambda}$ and $a_k = a_{h_k}$ such that

$$\lim_{k\to+\infty} \int_{B_\rho\setminus C_h} f\left(\frac{\nabla v_k}{a_k}\right) a_k\, dx = \liminf_{h\to+\infty} \int_{B_\rho\setminus C_h} f\left(\frac{\nabla u_{h,\lambda}}{a_h}\right) a_h\, dx.$$

Possibly extracting a subsequence, we can assume with no loss of generality that v_k converges in the weak* topology of $W^{1,\infty}(B_\rho)$ to a Lipschitz function $v_\lambda : B_\rho \mapsto \mathbf{R}^k$. By Lemma 4.7 below we get (4.12). On the other hand, Proposition 4.1 yields

$$\operatorname{meas}(\{x \in B_\rho : u_{h_k}(x) \ne v_k(x)\}) \le \operatorname{meas}(E_{h_k,\lambda}) \le \frac{\Gamma\xi(n)}{2\lambda}.$$

By the lower semicontinuity of the functional $w \mapsto \operatorname{meas}(\{x \in B_\rho : w(x) \ne 0\})$ we obtain (4.11) and this shows the claim. By (4.10) and (4.12) we get

$$\int_{B_\rho\setminus C_{\epsilon,\lambda}} f\left(\frac{z_0}{c_0}\right) c_0\, dx \le L + C\left(c_0\delta + 2^{2p}(1+c(n)k)^p c_0^{1-p}\epsilon\right)$$

with $C_{\epsilon,\lambda} = C_\epsilon \cup \{x \in B_\rho : u(x) \ne v_\lambda(x)\}$ and $\lambda \ge \lambda_\epsilon$. By letting $\lambda \to +\infty$, (4.11) yields

$$\int_{B_\rho\setminus C_\epsilon} f\left(\frac{z_0}{c_0}\right) c_0\, dx \le L + C\left(c_0\delta + 2^{2p}(1+c(n)k)^p c_0^{1-p}\epsilon\right).$$

The conclusion follows now by letting $\epsilon \to 0$. ■

Lemma 4.7. *Let $f : \mathbf{M}^{n\times k} \to \mathbf{R}$ be a quasiconvex function, let $(v_h) \subset W^{1,\infty}(B,\mathbf{R}^k)$ be a sequence converging in the weak* topology of $W^{1,\infty}(B,\mathbf{R}^k)$ to v, and let $(a_h) \subset L^\infty(B,\mathbf{R}^+)$ be a bounded sequence converging in $L^1(B)$ to some constant $c_0 > 0$. Then, for every Borel subset D of B, setting*

$$D_h = \{x \in D \;:\; a_h \le c_0/2\}$$

we have

$$\liminf_{h\to+\infty} \int_{D\setminus D_h} f\left(\frac{\nabla v_h}{a_h}\right) a_h\, dx \ge \int_D f\left(\frac{\nabla v}{c_0}\right) c_0\, dx.$$

Proof. Let $\tilde a_h = \max\{a_h, c_0/2\}$. Since $\operatorname{meas}(D_h) \to 0$, $\tilde a_h$ still converges in $L^1(B)$ to c_0. It is well known (see for instance [10]) that

$$\liminf_{h\to+\infty} \int_D f\left(\frac{\nabla v_h}{c_0}\right) c_0\, dx \ge \int_D f\left(\frac{\nabla v}{c_0}\right) c_0\, dx.$$

On the other hand, since $\tilde{a}_h$ is bounded away from 0 and $+\infty$, and f is locally Lipschitz continuous in $\mathbf{M}^{n\times k}$

$$\lim_{h\to+\infty}\int_D f\left(\frac{\nabla v_h}{c_0}\right)c_0\,dx-\int_D f\left(\frac{\nabla v_h}{\tilde{a}_h}\right)\tilde{a}_h\,dx=0,$$

so that

$$\liminf_{h\to+\infty}\int_D f\left(\frac{\nabla v_h}{\tilde{a}_h}\right)\tilde{a}_h\,dx\geq\int_D f\left(\frac{\nabla v}{c_0}\right)c_0\,dx.$$

Then, the statement follows by the inequality

$$\int_D f\left(\frac{\nabla v_h}{\tilde{a}_h}\right)\tilde{a}_h\,dx\leq\int_{D\setminus D_h} f\left(\frac{\nabla v_h}{a_h}\right)a_h\,dx+C\int_{D_h}\left(a_h+\frac{|\nabla v_h|^p}{(c_0/2)^{p-1}}\right)dx. \quad \blacksquare$$

Proof of Theorem 3.3. We argue by contradiction. If the claim was false, it would be possible to find $\epsilon>0$, a sequence δ_h converging to 0, and sequences ρ_h, c_h, z_h, u_h satisfying all the conditions in the statement of the theorem, such that

$$\epsilon\,\mathrm{meas}(B_{\rho_h})+\int_{B_{\rho_h}} f\left(\frac{\nabla u_h}{a_h}\right)a_h\,dx<\int_{B_{\rho_h}} f\left(\frac{z_h}{c_h}\right)c_h\,dx. \tag{4.13}$$

Defining, by a standard scaling argument ,

$$\hat{u}_h(y)=\frac{u(\rho_h y)}{\rho_h}$$

and setting $\hat{\rho}_h=1$, $\hat{c}_h=c_h$, $\hat{z}_h=z_h$, we may assume that $\rho_h=1$ for any $h\in\mathbf{N}$. Possibly extracting a subsequence, we may assume that z_h converges to $z_0\in\mathbf{M}^{n\times k}$ (with $|z_0|\leq M$) and c_h converges to c_0 (with $M\geq c_0\geq M^{-1}$). Now we claim that u_h fulfil the assumptions of Proposition 4.4. Indeed

$$|D^s u_h|(B)+\int_B|u_h(x)-z_h(x)|\,dx<\delta_h$$

implies that

$$\lim_{h\to+\infty}|D^s u_h|(B)+\int_B|u_h(x)-z_0(x)|\,dx=0.$$

Moreover, by (3.1) and (4.13) we get

$$\gamma\int_B|\nabla u_h|^p\,dx\leq M^{p-1}\int_B f\left(\frac{\nabla u_h}{a_h}\right)a_h\,dx\leq M^{p-1}\int_B f\left(\frac{z_h}{c_h}\right)c_h\,dx,$$

hence, as $p>1$, $|\nabla u_h|$ are equi-integrable in Ω. Let E_h be given by Proposition 4.4, and $v_h=u_h\chi_{B\setminus E_h}$. By Remark 4.5 we get

$$\int_B f\left(\frac{\nabla v_h}{a_h}\right)a_h\,dx\leq\int_B f\left(\frac{\nabla u_h}{a_h}\right)a_h\,dx+C\int_{E_h}a_h\,dx.$$

Since $\mathrm{meas}(E_h)\to 0$, by (4.13) we get

$$\epsilon\,\mathrm{meas}(B)+\limsup_{h\to+\infty}\int_B f\left(\frac{\nabla v_h}{a_h}\right)a_h\,dx\leq\int_B f\left(\frac{z_0}{c_0}\right)c_0\,dx. \tag{4.14}$$

On the other hand, since $\|v_h\|_\infty\leq(M+1)$ and

$$\lim_{h\to+\infty}|D^s v_h|(B)+\int_B|v_h(x)-z_0(x)|\,dx=0,$$

we can apply the lower semicontinuity Theorem 4.6 to obtain

$$\liminf_{h\to+\infty} \int_B f\left(\frac{\nabla v_h}{a_h}\right) a_h \, dx \geq \int_B f\left(\frac{z_0}{c_0}\right) c_0 \, dx.$$

This gives a contradiction with (4.14). ■

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