Ginzburg-Landau Equation and Motion by Mean Curvature, II: Development of the Initial Interface

Halil Mete Soner
Carnegie Mellon University

Research Report No. 93-NA-027

August 1993

Sponsors

U.S. Army Research Office
Research Triangle Park
NC 27709

National Science Foundation
1800 G Street, N.W.
Washington, DC 20550

Ginzburg-Landau equation and motion by mean curvature, II : development of the initial interface

H. Mete Soner[1]
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213-3890

April 27, 1993

[1]Partially supported by the NSF grant DMS-9200801 and by the Army Research Office through the Center for Nonlinear Analysis

1 Introduction

In an earlier paper [12], I have studied the asymptotic behavior of Ginzburg-Landau equation,

$$(1.1) \qquad u^\epsilon_t - \Delta u^\epsilon + \frac{1}{\epsilon^2} f(u^\epsilon) = 0, \qquad (0,\infty) \times \mathcal{R}^d,$$

$$(1.2) \qquad u^\epsilon(0,x) = u^\epsilon_0(x), \qquad x \in \mathcal{R}^d.$$

The nonlinearity f is the derivative of a bi-stable potential W :

$$(1.3) \qquad W(u) = \frac{1}{2}(u^2-1)^2, \quad f(u) = W'(u) = 2u(u^2-1).$$

In [12], I proved that there are two open, disjoint subsets $\mathcal{P}, \mathcal{N}$ of $(0,\infty) \times \mathcal{R}^d$ and a subsequence ϵ_n satisfying,

a. $u^{\epsilon_n} \rightarrow 1$,uniformly on bounded subsets of $\mathcal{P}$,

b. $u^{\epsilon_n} \rightarrow -1$, uniformly on bounded subsets of $\mathcal{N}$

c. Γ=complement of $(P \cup \mathcal{N})$ has Hausdorff dimension d and it moves by mean curvature in the sense defined in [12], [1].

This convergence result generalizes the previous results of Rubinstein, Steinberg and Keller [10], DeMottoni and Schatzman [8], Chen [2], Evans, Soner and Souganidis [4], Barles, Soner and Souganidis [1] and Ilmanen [7]. For more information on the Ginzburg-Landau equation, the weak theories for the mean curvature flow and other related topics we refer the reader to the introduction of the companion paper [12] and the references therein.

The above result was proved under the assumption (c.f. (2.6) in [12]) that for every $\delta > 0$ there are positive constants K_δ and η such that for every continuous function φ,

$$\text{(A)} \qquad \sup\{\int |\varphi(x)|\mu^\epsilon(dx;t) \quad : \quad \epsilon \in (0,1), t \in [\delta, \frac{1}{\delta}]\},$$
$$\leq \quad K_\delta \sup\{|\varphi(x)|e^{\eta|x|} \ : x \in \mathcal{R}^d\}$$

where

$$\mu^\epsilon(dx;t) = \left[\frac{\epsilon}{2}|Du^\epsilon(t,x)|^2 + \frac{1}{\epsilon}W(u^\epsilon(t,x))\right] dx. \tag{1.4}$$

The main purpose of this paper is to verify (A) under some reasonable conditions on the initial data u_0^ϵ. This analysis requires a detailed description of $u^\epsilon(t,x)$ near the initial interface. Such an analysis have already been carried out by DeMottoni and Schatzman [9] and by Chen [2]. However, the condition (A) can not be directly obtained from the results of [2, 9].

There are two key estimates in the proof of (A). The first one is a detailed description of $u^\epsilon(t,x)$ near the initial interface; Theorem 4.1, below. This result is a sharper version of a result of DeMottoni and Schatzman [9] and its proof is similar to Lemma 4.1 in [5]. The description obtained in Theorem 4.1 is of independent interest. The second key step in the proof of (A) is a gradient estimate; Theorem 5.1, below.

The paper is organized as follows. In the next section the main result of this paper is described. In section 3, a result of DeMottoni and Schatzman is recalled and an easy gradient bound is proved. The behavior of $u^\epsilon(t,x)$ near the initial interface is analyzed in Section 4 and a second gradient estimate is obtained in Section 5. A proof of the main theorem is given in the last section.

2 Main Result

Multiply (1.1) by ϵu_t^ϵ, integrate and use integration by parts to obtain,

$$E^\epsilon(t_1) - E^\epsilon(t_2) = -\epsilon \int_{t_1}^{t_2} \int_{\mathcal{R}^d} (u_t^\epsilon)^2 dx dt, \qquad t_1 > t_2, \tag{2.1}$$

where

$$E^\epsilon(t) = \mu^\epsilon(\mathcal{R}^d; t) = \int_{\mathcal{R}^d} \left[\frac{\epsilon}{2} |Du^\epsilon(t,x)|^2 + \frac{1}{\epsilon} W(u^\epsilon(t,x)) \right] dx.$$

Hence (A) holds with $\eta = 0$ provided that $E^\epsilon(0)$ is bounded in ϵ. In particular, an elementary computation shows that $E^\epsilon(0)$ is bounded in ϵ, if there are a function z_0^ϵ, a constant $\lambda \geq 1$ and a bounded open set Ω of finite perimeter (c.f. [3, 6]) satisfying,

$$u_0^\epsilon(x) = q(\frac{z_0^\epsilon(x)}{\epsilon}), \quad q(r) = \tanh(r),$$

$$|Dz_0^\epsilon| \leq \lambda, \qquad \frac{1}{\lambda} d(x) \leq z_0^\epsilon(x) \leq \lambda d(x),$$

where $d(x)$ is the signed distance between x and the boundary of Ω.

When u_0^ϵ is independent of ϵ, we generally do not expect $E^\epsilon(0)$ to be bounded in ϵ. Indeed let $u_0^\epsilon \equiv \beta$ for some constant $\beta \neq \pm 1$. Then $u^\epsilon(t,x) = w^\epsilon(t)$ and $E^\epsilon(t) = +\infty$ for every $t \geq 0$ and $\epsilon > 0$. However, the condition (A) holds with any $\eta > 0$.

In the remainder of this paper, we assume that:

(2.2a) u_0^ϵ is independent of ϵ, i.e. $u_0^\epsilon = u_0$,

(2.2b) $u_0 \in C_b^3(\mathcal{R}^d)$, $|u_0(x)| < 1$,

(2.2c) $\Gamma_0 = \{x \in \mathcal{R}^d \ : \ u_0(x) = 0\}$ is bounded,

(2.2d) $\inf_{\Gamma_0} |Du_0| > 0$,

(2.2e) $\limsup_{R \to 0} \ \inf_{|x| \geq R} |u_0(x)| > 0$,

where $C_b^3(\mathcal{R}^d)$ is the set of all bounded functions that are thrice continuously differentiable with bounded derivatives. Observe that (2.2b,c,d) imply that Γ_0 is a C^2 manifold. The main goal of this paper is to prove (A) under the above hypotheses, see Theorem 6.1 below.

3 Preliminaries

Let $d_0(x)$ be the signed distance between x and Γ_0. Choose $\lambda > 0$ such that

(3.1) $$d_0 \in C^2(\Omega_\lambda), \quad \Omega_\lambda = \{x \in \mathcal{R}^d \ : \ |d_0(x)| < 2\lambda\}.$$

We now recall a result of DeMottoni and Schatzman [9, Theorem 5].

Theorem 3.1 *For every $\delta, m > 0$ there are $C_1, C_2 > 0$ such that for every*

(3.2) $$t \in I_\epsilon := [C_1 \epsilon^2 \ln(\frac{1}{\epsilon}) \, , \, C_2 \epsilon^{\frac{2}{3}}],$$

we have,

(3.3) $$|u^\epsilon(t,x) - q(\frac{d_0(x)}{\epsilon})| \leq \delta, \qquad \text{if} \quad |d_0(x)| \leq \lambda,$$

(3.4) $$|u^\epsilon(t,x) - sign[u_0(x)]| \leq \epsilon^m, \qquad \text{if} \quad |d_0(x)| \geq \lambda.$$

Recall that $q(r) = \tanh(r)$. In the remainder of this paper C_1, C_2 denote the constants constructed in Theorem 3.1 with $m = 2$ and $\delta = 1/8$. Also set

(3.5) $$C_3 = q^{-1}(7/8).$$

Fix $t \in I_\epsilon$. Then whenever $d(x) \in [\epsilon C_3, \lambda]$, (3.3) yields

$$u^\epsilon(t,x) \geq q^{-1}(\frac{d(x)}{\epsilon}) - \delta \geq \frac{3}{4}$$

Also if $d(x) \geq \lambda$, (3.4) implies the above inequality, provided that $\epsilon^2 < 1/4$. Hence

(3.6) $$u^\epsilon(t,x) \geq 3/4, \quad \forall \epsilon \leq 1/2, \ \ t \in I_\epsilon, \ \ d(x) \geq \epsilon C_3.$$

Similarly,

(3.7) $$u^\epsilon(t,x) \leq -3/4, \quad \forall \epsilon \leq 1/2, \ \ t \in I_\epsilon, \ \ d(x) \leq -\epsilon C_3.$$

We close this section with a simple gradient estimate.

Lemma 3.1 *There is a constant K, independent of ϵ, satisfying,*

(3.8) $$|Du^\epsilon(t,x)| \leq \frac{K}{\epsilon}.$$

Proof: Since $|u_0| \leq 1$, $|u^\epsilon(t,x)| \leq 1$ for all (t,x). Set

$$g(t,x) = \frac{1}{\epsilon^2} f(u^\epsilon(t,x)).$$

Then for all $0 \leq \tau \leq t$,

(3.9) $$\begin{aligned} u^\epsilon(t,x) &= [G(t-\tau,\cdot) * u^\epsilon(\tau,\cdot)](x) \\ &+ \int_\tau^t [G(t-s-\tau,\cdot) * g(s,\cdot)](x)ds, \end{aligned}$$

where * denotes the convolution and G is the heat kernel, i.e.,

$$G(\tau,y) = (4\pi\tau)^{-\frac{n}{2}} \exp(-\frac{|y|^2}{4\tau}).$$

Now, differentiate (3.9) with respect x_j and use the properties of the convolution and the heat kernel to obtain,

$$\begin{aligned} |u^\epsilon_{x_j}(t,x)| &\leq \|D_jG(t-\tau,\cdot)\|_{L^1} \ \|u^\epsilon(\tau,\cdot)\|_{L^\infty} + \int_\tau^t \|D_j(t-s-\tau,\cdot)\|_{L^1} \|g\|_{L^\infty} dx, \\ &\leq \frac{C}{\sqrt{t-\tau}} + \frac{C}{\epsilon^2}\sqrt{t-\tau}\ , \end{aligned}$$

where C is an appropriate constant. Choose $\tau = t - \epsilon^2$ to obtain (3.8).

■

4 Behaviour near the interface

In this section we prove a sharper version of (3.3), (3.4). Our approach is very similar to [5, Lemma 4.1]. Let λ be as in (3.1) and set

$$t_1 = C_1\epsilon^2 \ln(\frac{1}{\epsilon}). \tag{4.1}$$

Theorem 4.1 *There are $\mu, K > 0$ such that for sufficiently small $\epsilon > 0$,*

$$u^\epsilon(t,x) \geq W(t-t_1, d_0(x)), \quad \forall t \in I_\epsilon, d_0(x) \in [\epsilon C_3, \lambda], \tag{4.2}$$

$$u^\epsilon(t,x) \leq -W(t-t_1, |d_0(x)|), \quad \forall t \in I_\epsilon, d_0(x) \in [-\lambda, -\epsilon C_3], \tag{4.3}$$

where

$$W(t,d) = \max\{q(\frac{d-Kt}{\epsilon} - K) - K\epsilon - \frac{1}{4}\exp(-\frac{\mu t}{\epsilon}), \frac{3}{4}\}.$$

Proof: We will prove only (4.2). The proof of (4.3) is similar.

1. In view of (3.1) there is $d \in C_b^2(\mathcal{R}^d)$ satisfying

$$d(x) = d_0(x), \text{ if } |d_0(x)| \leq \lambda, \tag{4.4}$$

$$|d(x)| \geq \lambda, \text{ if } |d_0(x)| \geq \lambda, \tag{4.5}$$

$$|Dd(x)| \leq 1, \ \forall x. \tag{4.6}$$

For $\xi(t), p(t) \geq 0$ (to be determined later) define

$$v(t,x) = q(\frac{d(x) - \epsilon C_3 - \xi(\frac{t}{\epsilon})}{\epsilon}) - p(\frac{t}{\epsilon}),$$

where C_3 is as in (3.5).

We will show that for appropriately chosen $\xi(\cdot), p(\cdot)$ and a sufficiently small $\epsilon > 0$, v is a subsolution of (1.1) on $\{v \geq 0\}$. Indeed a direct computation shows that,

$$\begin{aligned} I &:= v_t - \Delta v + \frac{1}{\epsilon^2} f(v), \\ &= \frac{1}{\epsilon} q'(\cdots)[-\frac{1}{\epsilon}\xi'(\frac{t}{\epsilon}) - \Delta d(x)] - \frac{1}{\epsilon} p'(\frac{t}{\epsilon}), \\ &+ \frac{1}{\epsilon^2}[f(v) - q''(\cdots)|Dd|^2], \end{aligned}$$

where $(\cdots)$ denotes $[d(x) - \epsilon C_3 - \xi(\frac{t}{\epsilon})]/\epsilon$.

2. Since $q(\cdots) = v + p$ and $p \geq 0$, $q(\cdots) \geq 0$ whenever $v(t,x) \geq 0$. Therefore on $\{v \geq 0\}$, $q''(\cdots) \leq 0$ and (4.6) yields

$$q''(\cdots)|Dd|^2 \geq q''(\cdots) = f(q(\cdots)).$$

So on $\{v \geq 0\}$ we have,

$$I \leq -\frac{1}{\epsilon^2} q'(\cdots)\xi'(\frac{t}{\epsilon}) - \frac{1}{\epsilon} p'(\frac{t}{\epsilon}) + \frac{1}{\epsilon^2}[f(v) - f(q(\cdots))] + \frac{\beta}{\epsilon}, \tag{4.7}$$

where $\beta := \|q'\|_\infty \| \Delta d\|_\infty$.

3. Set

$$\mu = f'(\frac{5}{8}) = \min\{f'(u) \ : \ u \geq \frac{5}{8}\} > 0, \tag{4.8}$$

and

$$p(\tau) = \frac{\epsilon\beta}{\mu} + (\frac{1}{4} - \frac{\epsilon\beta}{\mu}) \exp(-\frac{\mu\tau}{\epsilon}), \ \tau \geq 0. \tag{4.9}$$

We will choose $\xi \geq 0$ in step 5 satisfying,

$$\xi' \geq 0. \tag{4.10}$$

4. Suppose that

$$q(\cdots) \in [\frac{7}{8}, 1] \tag{4.11}$$

The case $q(\cdots) \leq \frac{7}{8}$ will be analyzed in the next step. Since $|p(\tau)| \leq \frac{1}{4}$,(4.11) implies that

$$v(t,x) = q(\cdots) - p(\frac{t}{\epsilon}) \geq \frac{5}{8}.$$

Since $v = q(\cdots) - p \leq q(\cdots)$,(4.8) yields

$$f(v(t,x)) - f(q(\cdots)) \leq -\mu p(\frac{t}{\epsilon}).$$

Use (4.9), (4.10) and the above inequality in (4.7) to obtain,

$$I \leq \frac{\beta}{\epsilon} - \frac{1}{\epsilon} p'(\frac{t}{\epsilon}) - \frac{\mu}{\epsilon^2} p(\frac{t}{\epsilon}) = 0\,,$$

on $\{v \geq 0\}$.

5. Suppose that (4.11) does not hold, i.e.,

$$q(\cdots) \leq \frac{7}{8}.$$

Then on $\{v \geq 0\}$, $q(\cdots) \in [0, \frac{7}{8}]$ and

$$q'(\cdots) = (1 - q(\cdots)^2) \geq (1 - (\frac{7}{8})^2) := \gamma. \tag{4.12}$$

Set

$$\alpha := \max\{|f'(u)| \ : \ u \in [0,1]\}.$$

Since $v \leq 1$, on $\{v \geq 0\}$ we have,

$$f(v) - f(q(\cdots)) \ \leq \ \alpha |v(t,x) - q(\cdots)| = \alpha p(\frac{t}{\epsilon}).$$

Use the above inequality and (4.12) in (4.7) to obtain,

$$I \leq -\frac{\gamma}{\epsilon^2}\xi'(\frac{t}{\epsilon}) - \frac{1}{\epsilon}p'(\frac{t}{\epsilon}) + \frac{\alpha}{\epsilon^2}p(\frac{t}{\epsilon}) + \frac{\beta}{\epsilon}.$$

We now choose $\xi(\cdot)$ satisfying $\xi(0) = 0$ and

$$\xi'(\tau) = \frac{1}{\gamma}\{\beta\epsilon + \alpha p(\tau) - \epsilon p'(\tau)\} = \frac{\alpha + \mu}{\gamma} p(\tau), \ \ \tau \geq 0.$$

Using (4.9) we integrate the above equation,

$$\xi(\tau) = \frac{\epsilon}{\gamma}(1 + \frac{\alpha}{\mu})[\beta\tau + (\frac{1}{4} - \frac{\epsilon\beta}{\mu})(1 - \exp(-\frac{\mu\tau}{\epsilon}))].$$

Observe that this choice of ξ satisfies (4.10).

6. By the previous two steps,

$$I \leq 0 \quad \text{on} \quad \{v \geq 0\}.$$

Also by (3.6)

$$u^\epsilon(t,x) \geq \frac{3}{4}, \qquad \forall t \in I_\epsilon, \quad d_0(x) \geq \epsilon C_3.$$

In particular,

$$v(0,x) = q(\cdots) - \frac{1}{4} \leq \frac{3}{4} \leq u^\epsilon(t_1,x), \qquad \forall d_0(x) \geq \epsilon C_3,$$

and since $p, \xi > 0$,

$$v(t - t_1, x) \leq q(0) = 0 \leq u^\epsilon(t,x), \quad \forall t \in I_\epsilon, \quad \forall d_0(x) = \epsilon C_3.$$

Since $u^\epsilon(t,x) \geq 0$ for all $t \in I_\epsilon$ and $d_0(x) \geq \epsilon C_3$, the maximum principle yields,

$$u^\epsilon(t,x) \geq v(t - t_1, x), \qquad \forall t \in I_\epsilon, \; d_0(x) \geq \epsilon C_3. \tag{4.13}$$

Now (4.2) follows from (4.13), (4.4), (3.6) and the definitions of p and ξ.

■

6 Conclusion

Theorem 6.1 *Assume* (2.2). *Then* (A) *holds.*

Proof: Let A be a Borel subset of $\mathcal{R}^d$ with a finite Lebesgue measure. Set

$$\begin{aligned}
\Omega_1 &= \{x \in \mathcal{R}^d : \ |d_0(x)| \le \epsilon C_3\}, \\
\Omega_2 &= \{x \in \mathcal{R}^d : \ |d_0(x)| \in [\epsilon C_3, \lambda]\}, \\
\Omega_3 &= \{x \in \mathcal{R}^d : \ |d_0(x)| \ge \lambda\}, \\
A_i &= A \cap \Omega_i, i = 1,2,3, \\
I_i(t) &= \int_{A_i} \frac{\epsilon}{2} |Du^\epsilon(t,x)|^2 dx, \ \ i = 1,2,3, \quad t \ge 0, \\
J_i(t) &= \int_{A_i} \frac{1}{\epsilon} W(u^\epsilon(t,x)) dx, \ \ i = 1,2,3, \quad t \ge 0,
\end{aligned}$$

where λ, C_3 are as in (3.1) and (3.5), respectively. In the following steps we will estimate I_i and J_i's separately.

1. By Lemma 3.1,

$$I_1(t) + J_1(t) \le \int_{A_1} \frac{\epsilon}{2}\frac{K^2}{\epsilon^2} + \frac{1}{\epsilon} = (\frac{K^2}{2} + 1)\frac{|\Omega_1|}{\epsilon}.$$

Since Γ_0 is smooth and bounded, for sufficiently small $\epsilon > 0$, $|\Omega_1| \le \epsilon \hat{C}$ for an appropriate constant $\hat{C}$. Hence

$$I_1(t) + J_1(t) \le \hat{C}(1 + \frac{K^2}{2}), \quad \forall t \ge 0.$$

2. Set

$$C_4 = C_1 + \frac{1}{\delta}, \quad t_4 = C_4 \epsilon^2 \ln(\frac{1}{\epsilon}),$$

where $\delta > 0$ is the constant appearing in (5.1) and C_1 is as in Theorem 3.1. Then for all $t \in I_\epsilon \cap [t_4, \infty)$, by (5.1) we have,

$$|Du^\epsilon(t,x)|^2 \le \frac{K^2}{\epsilon^2}\left[\epsilon + \exp(-\frac{\alpha}{\epsilon}(|d(x)| - \epsilon C_3))\right].$$

Therefore,

$$I_2(t) \le \frac{K^2}{2}|A_2| + \frac{K^2}{2\epsilon}\int_{\Omega_2} \exp(-\frac{\alpha}{\epsilon}(|d(x)| - \epsilon C_3))dx.$$

By (4.4), $d_0 = d$ on A_2. In the above integral we use local orthogonal coordinates w, with $w_1 = d_0(x)$. Since d_0 is smooth in Ω_2, there is a constant C, depending on the $(d-1)$ dimensional measure of Γ_0, such that

$$\begin{aligned} I_2(t) &\le \frac{K^2}{2}|A_2| + \frac{K^2}{2\epsilon}C\int_{\epsilon C_3}^{\lambda} e^{-\frac{\alpha}{\epsilon}(w_1 - \epsilon C_3)}dw_1 \\ &\le \frac{K^2}{2}(|A_2| + \hat{C}), \quad \forall t \in I_\epsilon \cap [t_4, \infty), \end{aligned}$$

where $\hat{C}$ is an appropriate constant, possibly different than the constant appearing in the first step.

3. For $t \in I_\epsilon \cap [t_4,\infty)$ and $|d_0(x) \ge \lambda$,(5.1) and (4.5) yield,

$$|Du^\epsilon(t,x)|^2 \le \frac{K^2}{\epsilon^2}[\epsilon + e^{-\frac{\alpha}{\epsilon}(\lambda - \epsilon C_3)}].$$

Therefore for sufficiently small $\epsilon \ge 0$,

$$I_3(t) \le \frac{K^2}{2}(|A_3| + \hat{C}), \qquad \forall t \in I_\epsilon \cap [t_4, \infty),$$

for an appropriate constant $\hat{C}$, again possibly different than the constant appearing in the previous steps.

4. Recall that we have chosen C_1, C_2 satisfying (3.4) with $m = 2$. Hence for all $|d_0(x)| \ge \lambda$, and $t \in I_3$,

$$\begin{aligned} W(u^\epsilon) &= \frac{1}{2}(1-u^\epsilon)^2(1+u^\epsilon)^2 \\ &\le 2(u^\epsilon - sign(u_0))^2 \le 2\epsilon^4. \end{aligned}$$

Therefore,

$$J_3(t) \le 2\epsilon^3|A_3|, \qquad \forall t \in I_\epsilon.$$

5. Set

$$C_5 = C_1 + \frac{1}{\mu}, \qquad t_5 = C_5\epsilon^2 \ln(\frac{1}{\epsilon}),$$

where μ is the constant appearing in Theorem 4.1 and C_1 is as in Theorem 3.1. Then (4.2) and (4.3) imply that for all $t \in I_\epsilon \cap [t_5, \infty)$, $|d_0(x)| \in [\epsilon C_3, \lambda]$,

$$|u^\epsilon(t,x)| \ge \left[q(\frac{|d_0(x)| - Kt}{\epsilon} - K) - K\epsilon - \frac{1}{4}\epsilon\right]^+,$$

where $(a)^+ = \max\{a, 0\}$. Since $|W'(u)| \le 1$ for $|u| \le 1$, for sufficiently small $\epsilon > 0$ we have,

$$\begin{aligned} J_2(t) &\le \int_{A_2} \frac{1}{\epsilon} W([q(\frac{|d_0(x)| - Kt}{\epsilon} - K) - \epsilon(K + \frac{1}{4})]^+)dx \\ &\le \int_{A_2} \frac{1}{\epsilon} W([q(\frac{|d_0(x)| - Kt}{\epsilon} - K)]^+)dx + (K + \frac{1}{4})|A_2| \\ &\le \int_{\Omega_2} \frac{1}{\epsilon} W([q(\frac{|d_0(x)| - 2K\epsilon}{\epsilon})]^+)dx + (K + \frac{1}{4})|A_2|, \end{aligned}$$

for all $t \in I_\epsilon \cap [t_5, \epsilon]$. Now using the same change of variables used in step 2 we obtain,

$$\begin{aligned} J_2(t) &\le \frac{C}{\epsilon}\int_{\epsilon C_3}^{\lambda} W([q(\frac{w_1 - 2K}{\epsilon})]^+)dw_1 + (K + \frac{1}{4})|A_2| \\ &\le \frac{C}{\epsilon}\int_{\epsilon C_3}^{2\epsilon K} W(0)dw_1 + (K + \frac{1}{4})|A_2| \\ &+ C\int_0^{\lambda/\epsilon - 2K} W(q(r))dr. \end{aligned}$$

Since

$$W(q(r)) = \frac{(q'(r))^2}{2} = \frac{8e^{4r}}{(e^{2r}+1)^4},$$

$$J_2(t) \le \hat{C}(|A_2| + 1).$$

6. Combining the previous steps we conclude that

$$\mu^\epsilon(A;t) = \sum_{i=1}^{3}(I_i(t)+J_i(t)) \tag{6.1}$$
$$\leq \hat{C}(|A|+1),$$

for all $t \geq 0$ satisfying,

$$t \in I_\epsilon, \quad t \geq t_4, \quad t \geq t_5, \quad t \leq \epsilon, \tag{6.2}$$

and sufficiently small $\epsilon \geq 0$.

7. Let Ψ be a smooth positive function decaying exponentially as $|x| \to \infty$. Then using 1.1 we obtain,

$$\begin{aligned}
\frac{d}{dt}\int \Psi(x)\mu^\epsilon(dx;t) &= -\epsilon\int \Psi(-\Delta u^\epsilon + \frac{1}{\epsilon^2}f(u^\epsilon))^2 dx \\
&+ \epsilon\int D\Psi\cdot Du^\epsilon(-\Delta u^\epsilon + \frac{1}{\epsilon^2}f(u^\epsilon))dx \\
&\leq -\epsilon\int \Psi(-\Delta u^\epsilon + \frac{1}{\epsilon^2}f(u^\epsilon) - \frac{D\Psi\cdot Du^\epsilon}{2\Psi})^2 \\
&+ \epsilon\int |Du^\epsilon|^2\,\frac{|D\Psi|^2}{4\Psi}dx \\
&\leq \epsilon\int |Du^\epsilon|^2\frac{|D\Psi|^2}{4\Psi}\,dx.
\end{aligned}$$

Let

$$\hat{\Psi}(x) = \exp\left(-\sqrt{1+|x|^2}\right).$$

Then $|D\hat{\Psi}| \leq \hat{\Psi}$ and

$$\begin{aligned}
\frac{d}{dt}\int \hat{\Psi}(x)\mu^\epsilon(dx;t) &\leq \frac{1}{2}\int \frac{\epsilon}{2}|Du^\epsilon|^2\hat{\Psi}dx \\
&\leq \frac{1}{2}\int \hat{\Psi}(x)\mu^\epsilon(dx;t)
\end{aligned}$$

Therefore for any $t \geq t_0 \geq 0$,

$$\int \hat{\Psi}(x)\mu^\epsilon(dx;t) \leq \int \hat{\Psi}(x)\mu^\epsilon(dx;t_0)\ e^{\frac{t-t_0}{2}} \tag{6.3}$$

8. Let t_0 be a point satisfying (6.2). Then (6.1) yields,

$$\begin{aligned}
\int \hat{\Psi}(x)\mu^\epsilon(dx;t_0) &\leq \sum_{i=1}^{\infty} e^{-i}\mu^\epsilon(\{|x| \in [i-1,i)\};t_0) \\
&\leq \hat{C}w_d \sum_{i=1}^{\infty} e^{-i}(1+(i)^d-(i-1)^d) \\
&\leq \tilde{C},
\end{aligned}$$

where w_d is the volume of the unit space in $\mathcal{R}^d$ and $\tilde{C}$ is an appropriate constant. Then by (6.3)

$$\int \hat{\Psi}(x)\mu^\epsilon(dx;t) \leq \tilde{C}e^{\frac{t}{2}},$$

for every sufficiently small ϵ and

$$t \geq \max\{C_1, C_4, C_5\}\epsilon^2 \ln(\frac{1}{\epsilon}). \tag{6.4}$$

9. Now let ϕ be any continuous function satisfying,

$$\Lambda := \sup\{|\phi(x)|e^{\sqrt{2}(1+|x|)}\ :\ x \in \mathcal{R}^d\} < \infty.$$

Then $|\phi(x)| \leq \Lambda\hat{\Psi}(x)$, and

$$\int |\phi(x)|\mu^\epsilon(dx;t) \leq \tilde{C}\ \Lambda\ e^{\frac{t}{2}}, \tag{6.5}$$

for all t satisfying (6.4), and sufficiently small $\epsilon > 0$. Since for every $\epsilon > 0$, by (3.9)

$$\mu^\epsilon(dx;t) \leq \frac{1}{\epsilon}(\frac{K^2}{2}+1)dx.$$

Hence for every $t \geq 0$,

$$\int |\phi(x)|\mu^\epsilon(dx;t) \leq \frac{\Lambda}{\epsilon}(\frac{K^2}{2}+1)\int \hat{\Psi}(x)dx. \tag{6.6}$$

Now (A) follows from (6.5) and (6.6) with $\eta = \sqrt{2}$.

■

References

[1] G. Barles, H.M. Soner, and P.E. Souganidis, Front propagation and phase field theory, SIAM. J. Cont. Opt., issue dedicated to W.H. Fleming, 31 (1993), 439-469.

[2] X.-Y. Chen, Generation and propagation of the interface for reaction-diffusion equations, Jour. Diff. Equations, 96 (1992), 116-141.

[3] L.C. Evans and R.F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press (1992), Boca Raton, Ann Arbor, London.

[4] L.C. Evans, H.M. Soner and P.E. Souganidis, Phase transitions and generalized motion by mean curvature, Comm. Pure Appl. Math., 45 (1992), 1097-1123.

[5] P.C. Fife and B. McLeod, The approach of solutions of nonlinear diffusion equation to travelling front solutions, Arc. Rat. Mech. An., 65 (1977), 335-361.

[6] E. Guisti, Minimal Surfaces and Functions of Bounded Variation, Birkhäser (1984), Boston.

[7] T. Ilmanen, Convergence of the Allen-Cahn equation to the Brakke's motion by mean curvature, (1991), preprint.

[8] P. deMottoni and M. Schatzman, Geometrical evolution of developed interfaces, Trans. AMS, to appear, (announcement: Evolution géométric d'interfaces, C.R. Acad. Sci. Sér. I. Math., 309 (1989), 453-458.)

[9] P. deMottoni and M. Schatzman, Development of surfaces in $\mathcal{R}^d$, Proc. Royal Edinburgh Sect. A, 116 (1990), 207-220.

[10] J. Rubinstein, P. Sternberg and J.B. Keller, Fast reaction, slow diffusion and curve shortening, SIAM J. Appl. Math., 49 (1989), 116-133.

[11] H.M. Soner, Motion of a set by the curvature of its boundary, Jour. Diff. Equations, 101/2 (1993), 313-372.

[12] H.M. Soner, Ginzburg-Landau equation and motion by mean curvature, I:convergence, prepint.