ON THE SIZE OF A RANDOM MAXIMAMAL GRAPH

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Abstract

Let P be a graph property which is preserved by removal of edges. A *random maximal P-graph* is obtained from n independent vertices by randomly adding edges, at each stage choosing uniformly among edges whose inclusion would not destroy property P, until no further edges can be added.

We address the question of the number of edges of a random maximal P-graph for several properties P, in particular the cases of "bipartite" and "triangle-free". A variety of techniques are used to show that the size of the random maximal bipartite graph is quadratic in n but of order only $n^{3/2}$ in the triangle-free case. Along the way we obtain a slight improvement in the lower bound of the Ramsey number $r(3,t)$.

1 Introduction

The problem of describing random graphs with a particular property (e.g. regular, triangle-free) has been much studied since the seminal work of Erdős and Renyi [2] on evolution

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of random graphs. Here we are interested in graphs which are *maximal* with respect to a property P, that is, graphs $G = \langle V, E\rangle$ which satisfy P but such that $\langle V, E'\rangle$ fails to satisfy P for all E' properly containing E. By closing downward, we may assume P itself is preserved by removal of edges; typical such properties include "k-colorable," "planar," "K_m-free," "disconnected," "girth $> k$."

As is often the case with random structures, there is a probability measure which is both more tractable and more natural (we think) than the uniform measure on random maximal P-graphs, namely that given by building the graph via uniform edge-choice subject to preserving property P. In the next section we describe three equivalent ways to obtain our random maximal P-graphs, which we will denote by $\mathbf{M}_n(P)$.

Even with all these constructions for $\mathbf{M}_n(P)$, however, it is far from clear how to divine its properties, even for rather simple P. In this work we confine ourselves to the very basic question "how many edges does $\mathbf{M}_n(P)$ have?". Even there we shall not pursue sharp concentration results nor address any non-trivial cases other than for the two properties "bipartite" and "triangle-free."

Let us note that the size (that is, number of edges) of $\mathbf{M}_n(P)$ is not necessarily an interesting parameter. If P is "planar," for example, then $\mathbf{M}_n(P)$ is a triangulated planar graph with exactly $3n - 6$ edges. If P is "disconnected" then $\mathbf{M}_n(P)$ will nearly always consist of a clique on $n - 1$ points and a single isolated vertex, thus $\binom{n-1}{2}$ edges, since with probability $1 - o(1)$ a standard graph process will reach a point where the "giant component" is just one vertex short of swallowing the whole graph (see e.g. Bollobás [1]).

The cases of "bipartite" and "triangle-free" present, on the other hand, an intriguing challenge and an instructive contrast. In both cases, the size of a maximal graph may vary from $n - 1$ for the star $K_{1,n-1}$ to $\lfloor n^2/4 \rfloor$ for the balanced complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$. Thus the size of $\mathbf{M}_n(P)$ is somewhere between linear and quadratic in n, but where?

One might expect that "bipartite," being the more restrictive of the two conditions, would produce fewer edges (indeed it does, in the early stages of the construction). But readers of this work will not be misled, and in fact we shall see that the random maximal bipartite graphs are vastly larger than their triangle-free cousins.

2 Constructions

The following equivalent constructions of $\mathbf{M}_n(P)$ are offered both because they are useful in the sequel and because readers may find one construction more intuitive than another. The first construction is merely a precise restatement of our definition.

Construction A

1. Fix $V = \{1, 2, \ldots, n\}$, K_n = complete graph on vertex set V, E^0 = initial edge set = $\emptyset$;

2. For each $i \geq 0$, if $F^i = \{e \in K_n : \langle V, E^i \cup \{e\}\rangle \text{ satisfies } P\}$, then $E^{i+1} = E^i \cup \{e\}$ where e is chosen randomly and uniformly from F^i;

3. If $F^i = \emptyset$ then $\mathbf{M}_n(P) = \langle V, E^i\rangle$ and the construction is complete.

Construction A suffers from two apparent shortcomings: it entails an unknown number of steps (which happens to be the parameter we seek) and the number $|F^i|$ of choices at each step is also variable. We therefore generally prefer the following process instead.

Construction B

1. Choose a random permutation $\pi = (e_1, e_2, \ldots, e_m)$ from the uniform distribution on the $m!$ permutations of the edges of K_n, where $m = \binom{n}{2}$;

2. let $E_0 = \emptyset$;

3. for $0 \leq i < m$, set $E_{i+1} = E_i \cup \{e_{i+1}\}$ if $\langle V, E_i \cup \{e_{i+1}\}\rangle$ satisfies P, and $E_{i+1} = E_i$ otherwise;

4. $\mathbf{M}_n(P) = \langle V, E_m\rangle$.

Construction B is thus a "graph process" in the sense of Bollobás [1] but with some edges discarded along the way. To see that it is equivalent to Construction A, observe that an edge which is rejected from F_i at some point in A need never be considered again.

The last construction is just a "real-time" version of B.

Construction C

1. Set $E(0) = \emptyset$.
2. For each edge e of K_n, let e appear in $E(t)$ independently at exponential rate $1/(1-t)$, for $t \in [0,1)$, provided property P is not destroyed. Multiple occurrences of an edge are ignored.
3. $\mathbf{M}_n(P) = \langle V, E(1) \rangle$.

Since the probability that a particular edge e has *not* appeared by time t is $1-t$, $\langle V, E(t) \rangle$ is the classic random graph $G_{n,t}$ of Erdős and Renyi [2] when P is the "improper" property possessed by all graphs.

3 The Bipartite Case

When P is "bipartite" $\mathbf{M}_n(P)$ turns out to be nearly balanced.

Theorem 3.1 *The expected size of a random maximal bipartite graph is greater than* $(n^2 - n)/4$.

Proof. Let us imagine that after construction of $\mathbf{M}_n(P)$, we color vertex 1 randomly red or blue (each with probability $\frac{1}{2}$ and then extend uniquely to a proper red-blue coloring of the whole graph.

For each vertex u, let the random variable $\mathbf{X}_u$ be given by $\mathbf{X}_u = 1$ if u is colored red, -1 if blue. Then trivially $\mathbf{Pr}(\mathbf{X}_u = 1) = \mathbf{Pr}(\mathbf{X}_u = -1) = \frac{1}{2}$ for all u.

Were the $\mathbf{X}_u$'s independent, we would have a binomial distribution for the number of red vertices, hence expected number of edges of $\mathbf{M}_n(P)$ is

$$\begin{aligned}
& \mathbf{E}(|\{u : \mathbf{X}_u = 1\}| \cdot |\{u : \mathbf{X}_u = -1\}|) \\
= \; & \mathbf{E}\left(\left(\frac{n}{2} + \frac{\sum \mathbf{X}_u}{2}\right) \cdot \left(\frac{n}{2} - \frac{\sum \mathbf{X}_u}{2}\right)\right) \\
= \; & \frac{n^2}{4} - \frac{1}{4}\mathbf{E}((\sum \mathbf{X}_u)^2)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{n^2}{4} - \frac{n}{4}\sigma^2(\mathbf{X}_u) \\
&= \frac{n^2 - n}{4} \,.
\end{aligned}
$$

In fact, the $\mathbf{X}_u$'s are negatively correlated. To see this, fix vertices u and u' and let i be the last stage of Construction B at which u and u' are still disconnected; that is, u and u' lie in distinct connected components C and C' of $\langle V, E_i \rangle$ but are connected by a path in $\langle V, E_{i+1} \rangle$.

Let A and B be the proper color classes of C, of cardinalities a and b respectively, and similarly for A', B', a' and b'. Let $\{v, v'\} = e_{i+1}$ be the edge which connected C with C', where $v \in C$, $v' \in C'$. Set

$$
p = \frac{a}{a+b} \cdot \frac{a'}{a'+b'} + \frac{b}{a+b} \cdot \frac{b'}{a'+b'}
$$

and

$$
q = 1 - p = \frac{a}{a+b} \cdot \frac{b'}{a'+b'} + \frac{b}{a+b} \cdot \frac{a'}{a'+b'}.
$$

We say that two vertices are "of the same type" if they are both in $A \cup A'$, or both in $B \cup B'$. Then we have by symmetry that

$$
\begin{aligned}
& \mathbf{Pr}(u \text{ and } u' \text{ are of the same type}) \\
= \; & \mathbf{Pr}(v \text{ and } v' \text{ are of the same type}) \\
= \; & p \,.
\end{aligned}
$$

When the components C and C' join, the classes A and A' will unite to become a single color class just when v and v' are of *different* types; hence u and u' will end up with the same color in $\mathbf{M}_n(P)$ iff either they are of the same type and v and v' are of different types, or vice-versa. Thus we have

$$
\mathbf{Pr}(\mathbf{X}_u = \mathbf{X}_{u'}) = 2pq \le \frac{1}{2}
$$

regardless of the values of a, b, a' and b'. Since p may not be equal to $\frac{1}{2}$ we have, overall, the strict inequality

$$
\mathbf{E}(\mathbf{X}_u \mathbf{X}_{u'}) < 0
$$

for all u and u'.

Finally, we have that the expected number of edges of $\mathbf{M}_n(P)$ is

$$\begin{aligned}
& \frac{n^2}{4} - \frac{1}{4}\mathbf{E}\left((\sum \mathbf{X}_u)^2\right) \\
= & \frac{n^2}{4} - \frac{1}{4}\mathbf{E}\left(\sum(\mathbf{X}_u^2)\right) - \frac{1}{4}\mathbf{E}\left(\sum_{u \neq v}(\mathbf{X}_u\mathbf{X}_v)\right) \\
> & \frac{n^2}{4} - \frac{n}{4}
\end{aligned}$$

completing the proof of the theorem. □

The construction of $\mathbf{M}_n(P)$ is not essentially different if P is the property "contains *no* cycles" instead of "contains no odd cycles"; the result is then a tree, whose color balance is the same as for our random complete bipartite graph. This random tree is *not* uniform, but is instead equivalent in distribution to the minimum spanning tree for a copy of K_n with edge-weights which have been independently selected from some fixed continuous distribution.

4 The Triangle-Free Case

In this section our effort will be devoted to obtaining upper and lower bounds for the size of the random maximal triangle-free graph $\mathbf{M}_n(P)$, or $\mathbf{M}_n$ for short. It turns out that a lower bound of the right order can be obtained fairly easily by considering a slightly different random graph $\mathbf{M}'_n$, obtained as follows.

Construction B above is followed as before, but now an edge is rejected at stage i not merely if it makes a triangle with two edges from $\langle V, E_i \rangle$ but with any two previously considered edges. Thus, rule 3. is replaced by:

3′. For $0 \leq i < m$, set $E_{i+1} = E_i \cup \{e_{i+1}\}$ if $\langle V, \{e_1, e_2, \ldots, e_i, e_{i+1}\}\rangle$ satisfies P, and $E_{i+1} = E_i$ otherwise.

The edge-set of $\mathbf{M}'_n$ will thus be a subset of the edge-set of $\mathbf{M}_n$, for any permutation π. Incidentally, if $r(\pi)$ denotes the reverse of π, then we may also take $\mathbf{M}'_n$ as the graph obtained by an edge deletion process in which edges in the complete graph K_n on $[n]$ are examined sequentially as in $r(\pi)$ and an edge e is deleted if and only if it is in a triangle contained in the current graph.

Theorem 4.1 *Let e be an edge of K_n. Then*

$$\mathbf{Pr}(\textit{edge } e \textit{ is in } \mathbf{M}'_n) \sim \frac{\sqrt{\pi n}}{2}.$$

Proof. We may as well take $e = \{1,2\}$. Let A_i be the event that edges $\{1,i\}$ and $\{2,i\}$ both arrive after edge $\{1,2\}$, and B_i be the event that exactly one of the edges $\{1,i\}$ and $\{2,i\}$ arrive after edge $\{1,2\}$. Now the event that edge e is in $\mathbf{M}'_n$ is equivalent to

$$\bigcup_{i=0}^{n-2} (A_i \cup B_i).$$

Since for each i, A_i and B_i are disjoint events, we have

$$\begin{aligned}
&\mathbf{Pr}(\text{edge } e \text{ is in } \mathbf{M}'_n)\\
= &\sum_{m=0}^{n-2} \binom{n-2}{m} \mathbf{Pr}(B_1 \cap \cdots \cap B_m \cap A_{m+1} \cap \cdots \cap A_{n-2})\\
= &\sum_{m=0}^{n-2} \binom{n-2}{m} 2^m\, \mathbf{Pr}(\text{edges } \{1,3\}, \{1,4\}, \ldots, \{1,m+2\} \text{ arrive before edge } \{1,2\} \text{ and}\\
&\qquad\qquad \text{all other edges with end points 1 or 2 arrive after } \{1,2\})\\
= &\sum_{m=0}^{n-2} \binom{n-2}{m} 2^m \frac{m!(2n-4-m)!}{(2n-3)!}\\
= &\frac{1}{2n-3} \sum_{m=0}^{n-2} \prod_{i=1}^{m-1} \left(1 - \frac{i}{2n-4-i}\right)\\
\sim &\frac{\sqrt{\pi n}}{2}.
\end{aligned}$$

□

Thus theorem 4.1 implies that if $\mathbf{Z}'_n$ (resp. $\mathbf{Z}_n$) is the number of edges in $\mathbf{M}'_n$ (resp. $\mathbf{M}_n$), then $\mathbf{E}[\mathbf{Z}_n] \geq \mathbf{E}[\mathbf{Z}'_n] \sim \frac{\sqrt{\pi}}{4} n^{3/2}$. In fact it is possible to show a slightly better result for $\mathbf{Z}_n$ by considering vertex degrees of $\mathbf{M}_n$; this is what we are prepared to do next. However, in order to avoid some dependence considerations, we employ Construction C above. It will be convenient to denote by $M(t)$ the graph $\langle V, E(t) \rangle$ present at time t, and by $G(t)$ the result of allowing edges to appear regardless of their effect on property P; thus $G(t)$ is an ordinary graph process coupled to $M(t)$. In particular, $M(t) \subset G(t)$, $M(1) = \mathbf{M}_n$ and $G(1) = K_n$. We shall obtain the following theorem by considering $M(t/\sqrt{n})$ as $t \to \infty$.

Theorem 4.2 *Let $d_n(v)$ denote the degree of vertex v in $\mathbf{M}_n$. Then for any $\epsilon_1 > 0$,*

$$\mathbf{Pr}(d_n(v) \leq f^*\sqrt{n} - \epsilon_1\sqrt{n}) = o(1/n) \text{ as } n \to \infty,$$

where

$$f^* = \lim_{x\to\infty} f(x),$$

and $f(x)$ satisfies the differential equation

$$\frac{df}{dx} = \exp(-xf), \qquad f(0) = 0.$$

Numerical results show that f^* is about 1.13. Theorem 4.2 implies that for any $\epsilon_1 > 0$, every vertex in almost all $\mathbf{M}_n$ has degree at least $(1-\epsilon_1)f^*\sqrt{n}$. An immediate consequence of this is that for any $\epsilon_1 > 0$,

$$\mathbf{Pr}(\mathbf{Z}_n \leq \frac{(1-\epsilon_1)}{2} f^* n^{3/2}) \to 0 \text{ as } n \to \infty.$$

Proof. We first note that for any $\epsilon_2 \in (0, \epsilon_1/2)$, f^* can be approximated by $| f^* - \nu_j | \leq \epsilon_2$ by choosing sufficiently small $\delta > 0$ and large integer j, where ν_k approximates $f(k\delta)$ and satisfies the recurrence relation:

$$\nu_k = \nu_{k-1} + \delta e^{-k\delta\nu_{k-1} - k\delta^2}, \quad k \geq 1, \quad \nu_0 = 0.$$

Although it is more natural to leave out the term $-k\delta^2$ in the exponent in the above equation, its inclusion makes our approximation a little simpler as we shall see.

Let $\tau = \delta n^{-1/2}$. Since $M(t)$ is a subgraph of $\mathbf{M}_n$ for all t, our theorem follows from

$$\mathbf{Pr}(\text{degree of vertex } v \text{ in } M(j\tau) \leq \nu_j\sqrt{n} - \epsilon_2\sqrt{n}) = o(1/n). \tag{1}$$

To show (1), we consider (for each k) the growth of the degree of vertex v in $M(t)$ where $t \in (k\tau - \tau, k\tau]$. Let $D(t)$ be the degree of vertex v in $G(t)$ and use $\triangle_k D$ to denote $D(k\tau) - D(k\tau - \tau)$. Note that $\triangle_k D$ is a binomial variable with parameters $n-1$ and $\tau = \delta n^{-1/2}$. Hence for any $\epsilon \in (0,1)$, there is $\rho \in (0,1)$ such that for large n,

$$\mathbf{Pr}(| \triangle_k D - \delta\sqrt{n} | \leq \epsilon\delta\sqrt{n}) \leq \rho^{\sqrt{n}}. \tag{2}$$

Corresponding to $D(t)$, we next define $Y(t)$ which is a lower bound of the degree of vertex v in $M(t)$. Suppose that for some ν, $\eta > 0$, $Y(k\tau - \tau) = \nu\sqrt{n}$ and $\triangle_k D = \eta\sqrt{n}$. (We may mentally regard ν as ν_{k-1} and η as δ.) Consider $t \in (k\tau - \tau, k\tau]$ at which $D(t)$ is increased by 1, and assume inductively that $Y(t-)$ is a lower bound of the degree of v in $M(t-)$. Then an edge $e = \{v, w\}$ is added to $G(t-)$. Use d to denote the degree of vertex v in $M(t-)$ and $\{v, v_1\}, \{v, v_2\}, \ldots, \{v, v_d\}$ to denote the edges incident to v in $M(t-)$. Note that edge $\{v, w\}$ is added to $M(t-)$ if and only if $M(t-)$ does not contain any of the edges $\{w, v_1\}, \{w, v_2\}, \ldots, \{w, v_d\}$. There are two cases:

(1) If $(\nu + \eta)\sqrt{n} \leq d$, then we arbitrarily pick $(\nu + \eta)\sqrt{n}$ vertices that are adjacent (in $M(t-)$) to vertex v. $Y(t)$ is then increased by 1 if and only if $G(k\tau)$ does not contain any of the edges joining w to the $(\nu + \eta)\sqrt{n}$ chosen vertices. It is clear that $Y(t)$ remains a lower bound.

(2) If $(\nu + \eta)\sqrt{n} > d$, then we assume that there are additional $(\nu + \eta)\sqrt{n} - d$ independent coin tosses, each with probability of success equal to $1 - k\tau$. $Y(t)$ is increased by 1 if and only if $G(k\tau)$ does not contain any of the edges $\{w, v_1\}, \{w, v_2\}, \ldots, \{w, v_d\}$ and all the coin tosses are successful. It is also clear that Y_t remains a lower bound in this case.

Note that the edges in cases (1) and (2) above are inspected once in the entire construction of $\{Y(t)\}$. Let $\triangle_k Y = Y(k\tau) - Y(k\tau - \tau)$. Then given that $Y(k\tau - \tau) = \nu\sqrt{n}$ and $\triangle_k D = \eta\sqrt{n}$, $\triangle_k Y$ has a binomial distribution with parameters $\eta\sqrt{n}$ and $(1 - k\tau)^{(\nu+\eta)\sqrt{n}} \sim e^{-k\delta\nu - k\delta\eta}$. Hence, for any $\epsilon \in (0, 1)$, there is $\rho \in (0, 1)$ such that for large n,

$$\mathbf{Pr}(|\triangle_k Y - \mu\sqrt{n}| \leq \epsilon\mu\sqrt{n} \mid Y(k\tau - \tau) = \nu\sqrt{n},\ \triangle_k D = \eta\sqrt{n}) \leq \rho^{\sqrt{n}}, \tag{3}$$

where we used μ to denote $\eta e^{-k\delta\nu - k\delta\eta}$. Now (2) and (3) suggest that we can approximate $Y(j\tau)$ in j successive stages. To be specific, let $\epsilon \in (0, 1)$. Define ϵ'_k and ϵ''_k recursively by

$$1 + \epsilon'_k = (1 + \epsilon'_{k-1})^2 \exp(k\epsilon''_{k-1}\delta(\nu_{k-1} + \delta)),\ k \geq 1,\quad \epsilon'_0 = \epsilon,$$

$$1 - \epsilon''_k = (1 - \epsilon''_{k-1})^2 \exp(-k\epsilon'_{k-1}\delta(\nu_{k-1} + \delta)),\ k \geq 1,\quad \epsilon''_0 = \epsilon.$$

Then it is not difficult to deduce from (2) and (3) that there is $\rho_k \in (0, 1)$ such that for large n,

$$\mathbf{Pr}((1 - \epsilon''_k)\nu_k\sqrt{n} \leq Y(k\tau) \leq (1 + \epsilon'_k)\nu_k\sqrt{n}) \geq 1 - \rho_k^{\sqrt{n}}. \tag{4}$$

Indeed, we have from (2) that there is $\rho \in (0, 1)$ such that for large n,

$$\mathbf{Pr}((1 - \epsilon''_{k-1})\delta\sqrt{n} \leq \triangle_k D \leq (1 + \epsilon'_{k-1})\delta\sqrt{n}) \geq 1 - \rho^{\sqrt{n}}.$$

Assume inductively that for large n,

$$\mathbf{Pr}((1 - \epsilon''_{k-1})\nu_{k-1}\sqrt{n} \leq Y(k\tau - \tau) \leq (1 + \epsilon'_{k-1})\nu_{k-1}\sqrt{n}) \geq 1 - \rho_{k-1}^{\sqrt{n}}.$$

By taking $\epsilon = \min\{\epsilon'_{k-1}, \epsilon''_{k-1}\}$ in (3), we have, as $n \to \infty$ and with probability at least $1-\rho_k^{\sqrt{n}}$ for some $\rho_k \in (0,1)$,

$$\begin{aligned}
n^{-1/2}\triangle_k Y &\leq (1+\epsilon'_{k-1})\mu \\
&\leq (1+\epsilon'_{k-1})^2 \delta e^{-k\delta[(1-\epsilon''_{k-1})\nu_{k-1}+(1-\epsilon''_{k-1})\delta]} \\
&= (1+\epsilon'_{k-1})^2 \delta e^{k\delta\epsilon''_{k-1}(\nu_{k-1}+\delta)} e^{-k\delta\nu_{k-1}-k\delta^2} \\
&= (1+\epsilon'_k) e^{-k\delta\nu_{k-1}-k\delta^2},
\end{aligned}$$

and similarly,

$$\triangle_k Y \geq (1-\epsilon''_k) e^{-k\delta\nu_{k-1}-k\delta^2}\sqrt{n}.$$

Inequality (4) now follows easily from the induction hypothesis. Since j is finite, by putting $k = j$ in (4) and by choosing sufficiently small ϵ in the definitions of ϵ'_k and ϵ''_k so that $\epsilon''_j \nu_j \leq \epsilon_2$, we have, as $n \to \infty$,

$$\mathbf{Pr}(Y(j\tau) \leq \nu_j\sqrt{n} - \epsilon_2\sqrt{n}) \leq \rho_j^{\sqrt{n}},$$

which implies (1). $\square$

Using similar arguments, it is possible to show that for any $\epsilon \in (0,1)$, the degree of any vertex in $\mathbf{M}'_n$ is bounded between $(1-\epsilon)\sqrt{\pi n}/2$ and $(1+\epsilon)\sqrt{\pi n}/2$ with probability at least $1-\rho^{\sqrt{n}}$ for some $\rho \in (0,1)$.

We next turn our attention to the size of the largest independent set of $\mathbf{M}_n$. Let β_n be the independence number of $\mathbf{M}_n$. Note that since the neighbours of a vertex in a triangle-free graph form an independent set, theorem 4.2 shows that for almost all $\mathbf{M}_n$, $\beta_n = \Omega(n^{1/2})$. We next would like to find an upper bound for β_n.

Theorem 4.3 *For any $A > 3/2$, let $i = \lfloor An^{1/2}\log n\rfloor$. Then as $n \to \infty$,*

$$\mathbf{Pr}(\mathbf{M}_n \textit{ contains an independent set of size } i) = \mathrm{o}(1).$$

It turns out that this theorem is related to a result attributed to Paul Erdős (see for example Bollobás [1]) concerning the Ramsey numbers $r(3,t)$. In finding a lower bound for $r(3,t)$, it is sufficient to establish that there are triangle-free graphs with $r(3,t)$ vertices containing no independent set of a certain size. The technique is to delete edges from the standard random graph $G_{n,p}$ to obtain a maximal triangle-free graph, and then prove that the resulting triangle-free graph does not contain an independent set of a given size. Using our construction of $\mathbf{M}_n$, coupled with $G(t) = G_{n,t}$, we are able to do slightly better. Note that our proof of theorem 4.3 shows that $r(3,t) > \frac{1}{9}(1-\epsilon)(t/\log t)^2$, for any $\epsilon > 0$. (The lower bound given in Bollobás [1] is $\frac{1}{27}(t/\log t)^2$.)

Proof. Since $M(t)$ is a subgraph of $\mathbf{M}_n$, we have for $I = \{1, 2, \ldots, i\}$,

$$\begin{aligned} &\mathbf{Pr}(\mathbf{M}_n \text{ contains an independent set of size } i) \\ \leq\ & \binom{n}{i} \mathbf{Pr}(I \text{ is an independent set of } \mathbf{M}_n) \\ \leq\ & \binom{n}{i} \mathbf{Pr}(\forall t \in (0,1), I \text{ is an independent set of } M(t)). \end{aligned}$$

We shall show the theorem by considering the graph process $\{M(t)\}$ for $t \leq j\tau < n^{-1/2}$ for some suitably chosen j and δ (note that $\tau = \delta n^{-1/2}$ as before).

Let K_i be the set of all unordered pairs $\{x, y\}$ of vertices $x, y \in I$. For $k = 1, 2, \ldots$ let A_k be the event that I is an independent set of $M(k\tau)$. Also, let H_k denote the set

$$H_k = \{e = \{x, y\} \in K_i : \exists w \notin I \text{ s.t. } G(k\tau) \text{ contains edges } \{w, x\} \text{ and } \{w, y\}\}.$$

Use B_k to denote the event that for all $e \in K_i - H_k$, edge e does not appear in the process $\{G(t)\}$ for $t \in (k\tau - \tau, k\tau)$. Observe that the event A_j implies A_k ($k \leq j$) and that A_k implies B_k. Hence, event A_j implies $B_1 \cap \cdots \cap B_{j-1} \cap B_j$, and so

$$\mathbf{Pr}(A_j) \leq \mathbf{Pr}(B_1 \cap \cdots \cap B_{j-1} \cap B_j). \tag{5}$$

Next, since the probability that an edge does not appear in (t_1, t_2) is $(1-t_2)/(1-t_1)$ and the events that an edge appears in disjoint intervals are independent, we have

$$\begin{aligned} &\mathbf{Pr}(B_1 \cap \cdots \cap B_{j-1} \cap B_j) \\ =\ & \sum_{S_1,\ldots,S_j} \prod_{k=1}^{j} \left(\frac{1-k\tau}{1-k\tau-\tau}\right)^{\binom{i}{2}-|S_k|} \mathbf{Pr}(H_1 = S_1, \ldots, H_j = S_j) \\ \leq\ & \mathbf{E}\left[\exp\left(-j\tau\binom{i}{2} + \tau\sum_{k=1}^{j} \mid H_k \mid\right)\right]. \end{aligned} \tag{6}$$

We next would like to estimate $|H_k|$. For $w \notin I$, note that if $D_k(w)$ is the number of neighbours of w in I in $G(k\tau)$, then

$$|H_k| \le \frac{1}{2}\sum_{w\notin I} D_k(w)^2 = \frac{1}{2}F_k, \quad \text{say.}$$

Since each $D_k(w)$ is a sum of i independent Bernoulli variables with parameter $k\tau$, we see that

$$\mathbf{E}[D_k(w)] = ik\tau = \mathrm{O}(\log n),$$

and that

$$\frac{D_k(w)^2}{i^2k^2\tau^2} \text{ converges to 1 in probability, as } n \to \infty.$$

Note also that $\{D_k(w)^2 : w \notin I\}$ is a set of independent random variables. Hence it is not difficult to show that for any $\epsilon > 0$, $\exists \rho \in (0,1)$ such that for large n,

$$\mathbf{Pr}\left(\left|\frac{F_k}{i^2k^2\tau^2} - (n-i)\right| \ge \epsilon(n-i)\right) \le \rho^{n-i}.$$

Since $i = \mathrm{o}(n)$, the above implies that for any $\epsilon > 0$, $\exists \rho \in (0,1)$ such that for large n,

$$\mathbf{Pr}(F_k \ge (1+\epsilon)ni^2k^2\tau^2) \le \rho^n.$$

Hence for any $\epsilon > 0$, $\exists \rho \in (0,1)$ such that for large n,

$$\begin{aligned}
&\mathbf{E}\left[\exp\left(\tau\sum_{k=1}^{j}|H_k|\right)\right] \\
\le\ &\mathbf{E}\left[\exp\left(\tau\sum_{k=1}^{j}F_k/2\right)\right] \\
\le\ &\mathbf{E}\left[\exp\left(\frac{1}{2}(1+\epsilon)i^2n\tau^3\sum_{k=1}^{j}k^2\right)\right] + j\rho^n.
\end{aligned}$$

It therefore follows from (5) and (6) that for $\epsilon \in (0,1)$, there is $\rho \in (0,1)$ such that for large n,

$$\mathbf{Pr}(A_j) \le \exp\left(-\frac{1}{2}A^2(j\delta - (1+\epsilon)\sum_{k=1}^{j}k^2\delta^3)\sqrt{n}\log^2 n\right) + j\rho^n.$$

Since $A > 3/2$, it is possible to choose sufficiently small $\delta > 0$, large $j < 1/\delta$ and small $\epsilon > 0$ such that

$$A(j\delta - (1+\epsilon)\sum_{k=1}^{j}k^2\delta^3) = \eta > 1.$$

Hence, for $A > 3/2$, there is $\eta > 1$ such that for large n,

$$\mathbf{Pr}(A_j) \leq \exp\left(-\frac{A}{2}\eta\sqrt{n}\log^2 n\right) + j\rho^n.$$

Lastly, we note that for large n,

$$\binom{n}{i} \leq \frac{n^i}{i!} \leq \exp\left(\frac{A}{2}(1+o(1))\sqrt{n}\log^2 n\right).$$

and since $\eta > 1$, we have

$$\begin{aligned} & \mathbf{Pr}(\mathbf{M}_n \text{ contains an independent set of size } i) \\ \leq\ & \binom{n}{i}\mathbf{Pr}(A_j) \\ \leq\ & \rho_1^{\sqrt{n}\log^2 n}, \qquad \text{for some } \rho_1 \in (0,1). \end{aligned}$$

□

Since the neighbours of a vertex in a triangle-free graph form an independent set, the previous result shows that for any $\epsilon > 0$, every vertex in almost all $\mathbf{M}_n$ has a degree at most $(1+\epsilon)\frac{3}{2}\sqrt{n}\log n$. This gives a corresponding upper bound on the number of edges in $\mathbf{M}_n$.

References

[1] B. Bollobás, *Random Graphs*, Academic Press, 1985.

[2] P. Erdős & A. Renyi, On the evolution of random graphs, *Publ. Math. Inst. Hungar. Acad. Sci.* **5** (1960), 17-61.