MULTICOLORED TREES IN RANDOM GRAPHS

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1 INTRODUCTION

Let $G = (V, E)$ be a graph in which the edges are coloured. A set $S \subseteq E$ is said to be *multicoloured* if each edge of S is a different colour. A spanning tree of G is said to be multicoloured if its edge set is. In this paper we study

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the existence of a multicoloured spanning tree (MST) in a randomly coloured random graph.

In fact, our main result will concern a randomly coloured graph process. Here $e_1, e_2, \ldots, e_N$ is a random permutation of the edges of the complete graph K_n and so $N = \binom{n}{2}$. Each edge e independently chooses a random colour $c(e)$ from a given set of colours W, $|W| \geq n-1$.

The graph process consists of the sequence of random graphs $G_m, m = 1, 2, \ldots, N$, where $G_m = ([n], E_m)$ and $E_m = \{e_1, e_2, \ldots, e_m\}$. We identify the following events:

$\mathcal{C}_m = \{G_m \text{ is connected }\}$.

$\mathcal{N}_m = \{|c(E_m)| \geq n-1\}$, where $c(E_m)$ is the set of colours used by E_m.

$\mathcal{MT}_m = \{G_m \text{ has a multicoloured spanning tree }\}$.

Let $\mathcal{E}_m$ stand for one of the above three sequences of events and let

$$m_{\mathcal{E}} = \min\{m : \mathcal{E}_m \text{ occurs}\},$$

provided such an m exists. Clearly, if $m_{\mathcal{MT}}$ is defined,

$$m_{\mathcal{MT}} \geq \max\{m_{\mathcal{C}}, m_{\mathcal{N}}\},$$

and the main result of the paper is

Theorem 1 *In almost every (a.e.) randomly coloured graph process*

$$m_{\mathcal{MT}} = \max\{m_{\mathcal{C}}, m_{\mathcal{N}}\}.$$

□

To establish the existence of an MST we use a result of Edmonds [2] on the matroid intersection problem. In this scenario Mi; *M<i* are matroids over a common ground set *E* with rank functions ri,r2 respectively. Edmonds' general theorem on this problem is

$$\max(|/| : / \text{ is independent in both matroids}) = \min_{E_1 \cap E_2 = \emptyset} (ri(Ei) + r_2(£?2))- \tag{1}$$

For us *Mi* is the cycle matroid of a graph $G = G_m$ and M2 is the partition matroid associated with the colours. Thus for a set of edges £*, *r\(S)* = *n* — *K(S)* where K(5) is the number of components of the graph *Gs* = ([w], *S)* and *r2(S)* is the number of distinct colours occurring in 5. If *i* G *W* then C; denotes the set of edges of colour *i* and for / ⊆ *W*, *Cj* = $\backslash J_{ieI}$ *C{*. We will use Edmonds' theorem in the following form:

Theorem 2 *Suppose* *W*\ = n — 1. *Then a necessary and sufficient condition for the existence of an MST is that*

$$K(d) \le n - IJ| \qquad \textit{for all } I \subseteq W. \tag{2}$$

[To see this, w.l.o.g. restrict attention in (1) to *E2* of the form *Cj* and then take/ = *W*\ Jin (2).]

2 Proof of Theorem 1

Observe first that if *u* = *u(n)* —> oo slowly, then in a.e. randomly coloured graph process

$$m_C \ge m_0 = \lfloor \tfrac{1}{2} n(\ln n - \omega) \rfloor \text{ and } m_{\mathcal{N}} \le m_1 = \lceil n(\ln n + \omega) \rceil.$$

We will start by justifying a concentration on the case $|W| = n - 1$. We will describe a coupled process in which there are never more than $n-1$ colours used: from $m_{\mathcal{N}}$ onwards, the colours that have not yet been used are randomly changed to one of the $n-1$ colours that have appeared so far. The relevant properties of this coupled process are

1. For each $m \in [m_0, m_1]$ the edges of G_m are independently randomly coloured from a choice of $n-1$ colours.

2. If $m_{\mathcal{MT}} > \max\{m_{\mathcal{C}}, m_{\mathcal{N}}\}$ holds for the original process then it also holds for the coupled process.

Thus to prove our theorem we need only prove that

$$\Pr(m_{\mathcal{MT}} > \max\{m_{\mathcal{C}}, m_{\mathcal{N}}\}) = o(1).$$

where Pr refers to the coupled process.

Fix some m in the range $[m_0, m_1]$. We define the event

$$\mathcal{A}_k = \{\exists I \subseteq W, |I| = k : \kappa(C_I) \geq n - |I| + 1\}.$$

We know that if $|W| = n-1$, G_m is connected and each colour is used at least once and there is no MST then $\mathcal{A}_k$ occurs for some $k \in [3, n-2]$ ($\mathcal{A}_1 \cup \mathcal{A}_2$ cannot occur if all $n-1$ colours are used and $\mathcal{A}_{n-1}$ cannot occur if G_m is connected.) Take a minimal k, corresponding set I and let $S = C_I$.

Claim 1 *G_S has no bridges.*

Proof If there is a bridge, remove it and all edges of the same colour. Clearly $\mathcal{A}_{k-1}$ occurs, contradicting the minimality of k. □

With the notation of Claim 1 suppose then that G_S has i isolated vertices and $n-k+x-i$ non-trivial components, $x \geq 1$. Since non-trivial components without bridges have at least three vertices,

$$i + 3(n - k + x - i) \leq n \tag{3}$$

or

$$\begin{aligned} i &\geq n - \frac{3}{2}k + \frac{3}{2}x \\ &\geq n - \frac{3}{2}k + \frac{3}{2}. \end{aligned}$$

So now let B_k denote the event

$\{\exists I \subseteq W, |I| = k, T \subseteq [n] : \; t = |T| \leq 3(k-1)/2,$
all edges coloured with I are contained in T,
there are $u \geq \max\{k, t\}$ I-coloured edges$\}$.

Here T is the set of vertices in the non-trivial components of G_{C_I}. Thus if $|W| = n - 1$,

$$Mm \cap A_k \subseteq \bigcup_{i=3}^{k} B_i \qquad \text{for } k \geq 3. \tag{4}$$

For $k > 9n/10$ we consider a slightly different event.

We first rephrase (2) as

$$\kappa(C_W/J) \leq |J| + 1 \qquad \text{for all } J \subseteq W. \tag{5}$$

So if $|W| = n - 1$ and there is no MST then there exist $\ell \geq 1$ colours whose deletion produces $\lambda \geq \ell + 2$ components of sizes $n_1, \ldots, n_\lambda$.

Claim 2 *Some subsequence of the n_i's sums to between $\ell + 1$ and $n/2$.*

Proof Assume $n\backslash <\_ri2 <\_\bullet \bullet \bullet \leq$ ^A-

If $n\backslash \geq\_ £ + 1$, one of ni,..., n^-i and 71A suffices.

Suppose then that n,- $\leq$ £, 1 $\leq$ i $\leq$ A.

Choose r such that

$$\text{ni H——h } n_r \leq n/2, \quad \text{nH———h } n_{r+i} > n/2$$

and then

$$\begin{aligned} \text{ni H——h } n_r &> n/2 — n_r+i \\ &\geq n/2 - \ell \\ &\geq \ell. \end{aligned}$$

and we can take ni, ...,n_r. •

Note next that if J is minimal in (5) then each colour in J appears at least twice as an edge joining components of $G_{C_{W\backslash J}}$.

So if G_m is connected and there is no MST and Ak does not occur for $k \leq$ 9n/10 then there is a set L of 1 $\leq$ £ < n/10 colours and a set S of size s, £ + 1 $\leq$ s $\leq$ $n/2$ such that (i) all t = r/(5) = $\backslash(S : \bar{S})\backslash$ $\geq$ 1 edges are L-coloured, ((5* : $\bar{S}$) is the set of edges joining 5* and $\bar{S} = V \backslash S$), (ii) the lexicographically first max{2^—<,0} non-(5' : $\bar{S}$) edges joining up components (of the $W\backslash L$ coloured edges) are also L-coloured. Let T>£ denote this event. Then

$$\mathcal{C}_m \cap \left(\bigcup_{k=9n/10}^{n-2} \mathcal{A}_k \right) \subseteq \bigcup_{i=l}^{n/10} PT_m(V_£). \qquad (6)$$

It follows from (4) and (6) that

$\Pr(m_{\mathcal{MT}} > \max\{m_{\mathcal{N}}, m_{\mathcal{C}}\}) \leq$

$$o(1) + \sum_{m=m_0}^{m_1} \left[\sum_{k=3}^{9n/10} \Pr_m(\mathcal{B}_k) + \sum_{\ell=2}^{n/10} \Pr_m(\mathcal{D}_\ell) \right] + \Pr\left(\bigcup_{m=m_0}^{m_1} (\mathcal{C}_m \cap \mathcal{A}_{n-2}) \right). \quad (7)$$

Here $\Pr_m$ denotes probability w.r.t. G_m and the o(1) term is the probability that G_{m_0} is connected or that $m_{\mathcal{N}} > m_1$. (Our calculations force us to separate out $\mathcal{A}_{n-2}$.)

We must now estimate the individual probabilities in (7). It is easier to work with the independent model G_p, $p = m/N$, where each edge occurs independently with probability p and is then randomly coloured. For any event $\mathcal{E}$ we have (see Bollobás [1] Chapter II) the simple bound

$$\Pr_m(\mathcal{E}) \leq 3\sqrt{n \ln n} \Pr_p(\mathcal{E}). \quad (8)$$

where $\Pr_p$ denotes probability w.r.t. the model G_p.

Now, where $p = \alpha \ln n/n$, $1 - o(1) \leq \alpha \leq 2 + o(1)$,

$$\begin{aligned}
\Pr_p(\mathcal{B}_k) &\leq \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{t}{2}} \binom{n}{t}\binom{n-1}{k}\binom{\binom{t}{2}}{u}\left(1 - \frac{kp}{n-1}\right)^{\binom{n}{2}-u}\left(\frac{kp}{n-1}\right)^u \\
&\leq \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{t}{2}} \frac{n^t e^t}{t^t}\frac{n^k e^k}{k^k}\left(\frac{t^2 e}{2u}\right)^u n^{-k\alpha(\frac{1}{2}-o(1))}\left(\frac{\alpha k \ln n}{n^2}\right)^u. \quad (9)
\end{aligned}$$

Case 1: $3 \leq k \leq k_0 = n/(3 \ln n)$.

$$\begin{aligned}
\Pr_p(\mathcal{B}_k) &\leq \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{t}{2}} \left(\frac{e^3 n^{1-\alpha(\frac{1}{2}-o(1))}}{k}\right)^k \left(\frac{t}{n}\right)^{2u-t} \left(\frac{\alpha e k \ln n}{2u}\right)^u \\
&= \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{t}{2}} \left(\frac{e^3 n^{1-\alpha(\frac{1}{2}-o(1))}}{k}\right)^k \left(\frac{t}{n}\right)^{u-t} \left(\frac{\alpha e k t \ln n}{2un}\right)^u
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{t}{2}} \left(\frac{e^3 n^{1-\alpha(\frac{1}{2}-o(1))} \alpha e k \ln n}{2kn} \right)^k \left(\frac{t}{n} \right)^{u-t} \left(\frac{\alpha e k \ln n}{2n} \right)^{u-k} \\
&= O\left(\left(\frac{\ln n}{n^{\frac{1}{2}-o(1)}} \right)^k \right).
\end{aligned}
$$

It follows from this and (8) that

$$
\begin{aligned}
\sum_{m=m_0}^{m_1} \sum_{k=4}^{k_0} \Pr_m(\mathcal{B}_k) &= O((n \ln n)(\sqrt{n \ln n})((\ln n)^4 / n^{2-o(1)})) \\
&= o(1). \qquad (10)
\end{aligned}
$$

For $k = 3$ we compute $\Pr_m(\mathcal{B}_3)$ directly, but since now $u = t = k = 3$ is forced,

$$
\begin{aligned}
\Pr_m(\mathcal{B}_3) &\le \binom{n}{3}^2 \left(1 - \frac{3}{n-1}\right)^{m-3} \left(\frac{3}{n-1}\right)^3 \frac{\binom{N-3}{m-3}}{\binom{N}{m}} \\
&= O(e^{3\omega} (\ln n)^3 n^{-3/2})
\end{aligned}
$$

and so

$$
\sum_{m=m_0}^{m_1} \Pr_m(\mathcal{B}_3) = o(1). \qquad (11)
$$

Case 2: $k_0 < k \le n/2$.

We now write (9) as

$$
\begin{aligned}
\Pr_p(\mathcal{B}_k) &\le \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{t}{2}} \left(\frac{e^3 n^{1-\alpha(\frac{1}{2}-o(1))}}{k} \right)^k \left(\frac{t}{n} \right)^{u-t} \left(\frac{\alpha e k t \ln n}{2un} \right)^{u} \\
&\le \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{t}{2}} \left(\frac{e^3 n^{1-\alpha(\frac{1}{2}-o(1))}}{k} \right)^k \left(\frac{t}{n} \right)^{u-t} n^{\frac{\alpha t k}{2n}}
\end{aligned}
$$

(after maximising the last term over u)

$$= \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{t}{2}} \left(\frac{e^3 n^{1-\frac{\alpha}{2}(1-\frac{t}{n}-o(1))}}{k}\right)^k \left(\frac{t}{n}\right)^{u-t} \tag{12}$$

$$\le \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{t}{2}} \left(\frac{e^3 n^{1-\alpha(\frac{1}{8}-o(1))}}{k}\right)^k \tag{13}$$

since $t \le 3(k-1)/2 \le 3n/4$.

(13) and (8) clearly imply

$$\sum_{m=m_0}^{m_1} \sum_{k=k_0}^{n/2} \Pr_m(\mathcal{B}_k) = o(1). \tag{14}$$

Case 3: $n/2 < k \le 9n/10$

Claim 3 *Choose any constant $A > 0$. Then, in a.e. process, simultaneously for each $m \in [m_0, m_1]$, the sets of $s \le A$ vertices of G_m which span at least s edges together contain at most $(\ln n)^{A+1}$ vertices.*

Proof We need only prove this for G_{m_1} and since the property is monotone decreasing we need only prove it for G_{p_1}, $p_1 = m_1/N$ ([1], Chapter II.) But

$$E_{p_1}(\text{number of vertices}) \le \sum_{k=3}^{A} \binom{n}{k} \binom{\binom{k}{2}}{k} p_1^k$$
$$= O(e^{2A}(\ln n)^A).$$

Now use the Markov inequality. □

It follows that we may rewrite (3) as

$$i + 3(\ln n)^{A+1} + (A+1)(n-k+x-i) \le n$$

and so

$$
\begin{aligned}
i &\geq n - \frac{A+1}{A}k - O((\ln n)^{A+1}) \\
&\geq n - \frac{A}{A-1}k.
\end{aligned}
$$

By making A sufficiently large we see that if $k \leq 9n/10$ then $t \leq 19n/20$ in (12) and consequently

$$
\sum_{m=m_0}^{m_1} \sum_{k=n/2}^{9n/10} \Pr_m(\mathcal{B}_k) = o(1). \tag{15}
$$

Case 4: $k \geq 9n/10$

$\Pr_p(\mathcal{D}_\ell) \leq$

$$
\sum^{n/2} \binom{n}{\ } \binom{n-1}{\ }^{s(n-s)} \binom{s(n-s)}{\ } \ldots \left(\frac{\ell}{\ }\right)^{\max\{2\ell-t,0\}}.
$$

Let $u(s,\ell,t)$ denote the summand in the above and let $p = \alpha \ln n/n$ and note that $\alpha \in [1-\omega/\ln n, 2+\omega/\ln n]$.

Case 4.1: $t \leq 2\ell$

It will generally be convenient to split s into two ranges:

Case 4.1.1: $s \leq n^{1/10}$

$$
\begin{aligned}
u(s,\ell,t) &= \binom{n}{s}\binom{n-1}{\ell}\binom{s(n-s)}{t}p^t(1-p)^{s(n-s)-t}\left(\frac{\ell}{n-1}\right)^{2\ell} \\
&< \left(\frac{ne}{s}\right)^s \left(\frac{(n-1)e}{\ell}\right)^\ell \left(\frac{s(n-s)e^{1+p}\alpha \ln n}{tn}\right)^t n^{-\alpha s(n-s)/n}\left(\frac{\ell}{n-1}\right)^{2\ell} \\
&\leq \left(\frac{n^{1-\alpha+\alpha s/n}e}{s}\right)^s \left(\frac{\ell e}{n-1}\right)^\ell \left(\frac{e^2 s(n-s)\ln n}{tn}\right)^t \\
&\leq \left(\frac{n^{1-\alpha+\alpha s/n}e}{s}\right)^s \left(\frac{e^4 s^2(n-s)^2(\ln n)^2}{n^3\ell}\right)^\ell.
\end{aligned} \tag{16}
$$

Now

$$n^{1-\alpha+\alpha s/n} \leq (1+o(1))e^{\omega} \qquad (17)$$

where $\alpha \geq 1 - \omega/\ln n$ and $\omega \to \infty$ slowly.

So if $s \leq 3e^{\omega}$ then (16) implies that

$$u(s,\ell,t) \leq n^{-(1-o(1))\ell},$$

and if $s > 3e^{\omega}$

$$u(s,\ell,t) \leq \left(\frac{e^{1+5}(n-s)^2(\ln n)^2}{nH}\right)^{\ell}$$
$$= O\left(\left(\frac{s}{n^{1-o(1)}}\right)^{\ell}\right).$$

Case 4.1.2: $s > n^{1/10}$.

Claim 4 *In a.e. process, every* G_m, $m \in [m_0, m_1]$ *is such that* $\eta(S) \geq \gamma|S|\ln n$ *for all* $n^{1/10} \leq |S| \leq n/2$, *where* $\gamma > 0$ *is some absolute constant.*

Proof (outline) For $|S| \geq n^{2/3}$ one can use the Chernoff bounds on the tails of the binomial $\eta(S)$. If $|S| \leq n^{2/3}$ we use the fact that with high probability (i) G_{m_0} has $n^{\epsilon'}$ vertices of degree $\leq \delta\ln n$ where $\epsilon' = \epsilon'(\epsilon) \to 0$ with δ, and (ii) in G_{m_1} no set S of size $\leq n/(\ln n)^2$ contains $3|S|$ edges. •

So if $s \geq n^{1/10}$ then we can take $t \geq \gamma s\ln n > 2\ell$ for some constant $\gamma > 0$ and this case is vacuous.

Case 4.2 : $t > 2\ell$.

$$u(s,\ell,t) \leq \left(\frac{ne}{s}\right)^{s}\left(\frac{(n-1)e}{\ell}\right)^{\ell}\left(\frac{s(n-s)e^{1+p}\alpha\ell\ln n}{tn(n-1)}\right)^{t} n^{-\alpha s(n-s)/n}$$

$$= \left(\frac{n^{1-\alpha+\alpha s/n}e}{s}\right)^s \left(\frac{(n-l)e}{l}\right)^l \left(\frac{s(n-s)e^{\omega+1} a n n n}{tn(n-l)}\right)^t \tag{18}$$

Case 4.2.1: $t \le 2n$ and so $((n-l)e/l)^l \le (2ne/t)^{t/2}$.

$$u(s,\ell,t) \le \left(\frac{n^{1-\alpha+\alpha s/n}e}{s}\right)^s \left(\frac{20s\ell\ln n}{t^{3/2}n^{1/2}}\right)^t. \tag{19}$$

Case 4.2.1.1: $s < n^{1/10}$. Now (17) gives

$$\begin{aligned}\left(\frac{n^{1-\alpha+\alpha s/n}e}{s}\right)^s &\le \left(\frac{(1+o(1))e^{\omega+1}}{s}\right)^s \\ &\le e^{(1+o(1))e^{\omega}} \\ &= e^{\hat{\omega}}, \text{ say,}\end{aligned}$$

and so (19) implies

$$u(s,\ell,t) \le \left(\frac{s}{n^{\frac{1}{2}-o(1)}}\right)^t. \tag{20}$$

Case 4.2.1.2: $s \ge n^{1/10}$.

Using Claim 4 and (19),

$$u(s,\ell,t) \le n^{-s/11}\left(\frac{1}{n^{\frac{1}{2}-o(1)}\sqrt{s}}\right)^t.$$

Case 4.2.2: $t \ge 2n$ and so $(ne/\ell)^\ell \le e^n \le e^{t/2}$.

From (18),

$$u(s,\ell,t) \le \left(\frac{(1+o(1))e^{\omega+1}}{s}\right)^s \left(\frac{20s\ell\ln n}{tn}\right)^t.$$

Case 4.2.2.1: $s < n^{1/10}$.

Arguing as in (20),

$$u(s,\ell,t) \leq \left(\frac{s}{n^{1-o(1)}}\right)^t.$$

Case 4.2.2.2: $s \geq n^{1/10}$.

From Claim 4

$$u(s,\ell,t) \leq \left(\frac{(1+o(1))e^{\omega+1}}{s}\right)^s \left(\frac{A\ell}{n}\right)^t.$$

for some constant $A > 0$. Now this clearly implies

$$u(s,\ell,t) = O(2^{-n}) \tag{21}$$

for $\ell \leq n/(3A)$. For $\ell > n/(3A)$ we have $s \geq \ell$ and

$$u(s,\ell,t) \leq n^{-s/2}A^n$$

and so (21) holds here also.

Summarising,

$$\begin{aligned}
\Pr(\mathcal{D}_\ell) &= O\left(\sum_{t=1}^{2\ell}\sum_{s=\ell+1}^{n^{1/10}}\left(\frac{s}{n^{1-o(1)}}\right)^\ell + \sum_{t=2\ell+1}^{2n}\sum_{s=\ell+1}^{n^{1/10}}\left(\frac{s}{n^{\frac{1}{2}-o(1)}}\right)^t\right.\\
&\quad + \sum_{t=2\ell+1}^{2n}\sum_{s=n^{1/10}}^{n/2}\left(\frac{s}{n^{\frac{1}{2}-o(1)}\sqrt{s}}\right)^t + \sum_{s=1}^{n^{1/10}}\sum_{t=2n+1}^{s(n-s)}\left(\frac{s}{n^{\frac{1}{2}-o(1)}}\right)^t\\
&\quad \left.+ \sum_{s=n^{1/10}}^{n/2}\sum_{t=2n+1}^{s(n-s)} 2^{-n}\right)\\
&= O(\ell n^{-(.9-o(1))\ell}).
\end{aligned}$$

where the double summations correspond to the five cases enumerated above.

Thus, we see that

$$\begin{aligned}
\sum_{m=m_0}^{m_1}\sum_{\ell=2}^{n/10}\Pr_m(\mathcal{D}_\ell) &= O((n\ln n)(\sqrt{n\ln n})n^{-1.7})\\
&= o(1). \qquad (22)
\end{aligned}$$

We are thus left with $\Pr\left(\bigcup_{m=m_0}^{m_1}(\mathcal{C}_m \cap \mathcal{A}_{n-2})\right)$.

We consider G_{m_0}. We know that a.e. G_{m_0} consists of a giant connected component C plus $O(e^{\omega})$ isolated vertices T. If $\bigcup_{m=m_0}^{m_1}(\mathcal{C}_m \cap \mathcal{A}_{n-2})$ occurs at some time during the process then either

(i) there exist $u, v \in T$ such that the first edges of the process that are incident with each of u and v are the same colour,

OR

(ii) there exists a colour c and a set S, $2 \leq |S| \leq n/2$ such that in G_{m_0} the $t \geq 2$ $(S : \bar{S})$ edges are all of colour c.

(Suppose that deleting the edges of colour c from G_m produces at least three components. If colour k has not occurred by time m_0 then two of these components must be vertices from T, contradicting (i). If G_{m_0} has edges of colour c then deleting these edges must beak C into at least three pieces.)

Clearly

$$\Pr((i)) = o(1) + O(e^{2\omega}/n) = o(1).$$

Furthermore

$$\begin{aligned}
\Pr_p((ii)) &\leq \sum_{s=2}^{n/2} \binom{n}{s} n \sum_{t=2}^{s(n-s)} \binom{s(n-s)}{t} \left(\frac{p}{n}\right)^t (1-p)^{s(n-s)-t} \\
&\leq 2\sum_{s=2}^{n/2} \binom{n}{s} n \sum_{t=2}^{10\ln n} \frac{(s(n-s))^t}{t!} \left(\frac{\alpha \ln n}{n^2}\right)^t n^{-\alpha s} \\
&\leq n\sum_{s=2}^{n/2} \left(\frac{n^{1-\alpha}}{s}\right)^s \sum_{t=2}^{10\ln n} \left(\frac{s\alpha \ln n}{n}\right)^t \\
&= O(n^{-(1-o(1))}).
\end{aligned}$$

The upper bound is good enough to apply (8) and so $\Pr_{m_0}((ii)) = o(1)$. Thus

$$\Pr\left(\bigcup_{m=m_0}^{m_1} (\mathcal{C}_m \cap \mathcal{A}_{n-2})\right) = o(1). \tag{23}$$

Our theorem now follows from (7),(10),(11),(14),(15),(22) and (23).

References

[1] B.Bollobás, *Random graphs*, Academic press, 1985.

[2] J.Edmonds, *Submodular functions, matroids and certain polyhedra*, in Combinatorial Structures and their Applications, R.Guy et al, eds., Gordon and Breach, 1970, pp69-87.

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