# QUOTIENT–TOPOLOGICAL COMPLETIONS AND HULLS OF CONCRETE CATEGORIES

by

Oswald Wyler

Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

Research Report No. 91–136 2

September, 1991

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OSWALD WYLER

## Introduction

Categories in this paper will always be concrete over a fixed base category S, with faithful and amnestic forgetful functors to S, and functors will be concrete functors. A completion of a concrete category A is a full and dense concrete embedding $G : A \longrightarrow B$ of A into a concrete category B which is complete in an appropriate sense. Completions of A with specified properties usually define a (quasi-)category, with full concrete embeddings as morphisms. An initial object of such a category is called a hull of A.

Topological completions and hulls with various properties have been studied intensively; we refer to the papers in the Bibliography for important contributions to this theory, and for further references. As H. HERRLICH [17] has pointed out, "smaller is better" for completions: a small completion of a category A is likely to retain more of the desirable properties of A than a large one. This makes hulls particularly desirable, and it suggests that the word "dense" should be used in a reasonably restricted sense. Beginning with P. ANTOINE [12], "dense" has often been used in the least restrictive meaning of finally dense. J. PENON in [22] introduced epidense extensions, and the author in [29] discussed colimit-dense extensions over a monosieve-complete quasitopos S as base category. Colimit-dense extensions are too restrictive for a more general base category; we replace them by quotient-dense extensions which have almost all basic properties established in the literature for other types of dense extensions.

The existence of at least three different meanings for "dense extension" calls for a general theory of dense extensions, completions and hulls. Sections 1 through 3 of this paper develop such a theory for an arbitrary concrete category A, based on factorizations of sinks in the base category S. We specialize in Sections 4 through 6 to quotient-dense extensions and quotient-topological completions. This material could also be generalized, but the finally dense case is well-known, and other cases (if any) seem to be less interesting. Section 4 deals with cartesian closed completions and hulls, and Section 5 with completions and hulls with (strong) partial morphisms represented. Section 6 deals with quasitopos completions and hulls, and with initial lifts for finite sources. We show that almost all results obtained in the literature for dense completions of some kind remain valid in our theory.

For every reasonable meaning of the word "dense", there is a largest dense completion of a concrete category A, a terminal object in the category of dense extensions of A, and completions of A can only be as complete as this largest completion. Thus dense completions are in general not topological; we can only expect them to be quotient-topological, i.e. to admit final lifts for quotient sinks. This raises the problem of existence of initial lifts for sources in dense completions. We discuss this for finite sources in Section 6; very little is known about the general situation.

Our terminology follows mostly [6] and [29]; these books will be cited as [ACC] and [TQT].

## 1. Factorizations of Sinks

**1.1. Sink factorization structures.** Throughout this paper, categories will be concrete categories (A,P) over a fixed base category S, and functors will be concrete functors over S. As in [TQT] 11.2, we define a *sink* for *P* or *P-sink,* with *codomain E* in S, as a class 4> of pairs *(A,u)* with *A* an object of A and *u : PA —> E* in S. A sink for the functor Id S will be called a sink in S.

Sinks thus defined are structured sinks as defined dually to [ACC] 17.1. For sinks in S, classes of pairs *(A,u)* with *u : A —> E* in S may be replaced by classes of morphisms of S with codomain *E.*

We denote by /$ the composition of a sink $ at an object *E* of S and a morphism / : *E* —> $E^1$ of S; this is the sink at $E^f$ consisting of all pairs *(A, fu)* with *(A,u)* in $.

For a collection E of sinks in S and a class M of morphisms in the category S, both closed under composition with isomorphisms in S, we say that S *has* (E,M) *-factorizations,* or that S is an (E, M)- *category,* if the following conditions are satisfied.

(1) Every sink $ in E factors $ = m* with * in E and *m* G M.

(2) If *g$* = m\P for sinks $ and $ in S and morphisms *m* and *g* of S, with $ in E and *rn* £ M, then *g* = *mt* and * = *t$* for a unique morphism *t* of S.

1.2. With our definition of sinks as classes, instead of discrete diagrams as in [ACC], the proof of the dual of [ACC] 15.4 is not valid. We do not know whether *M* must consist of monomorphisms of S if S is an (E,M)- category, but we note the following result.

**Proposition.** *If* **S** *is an* (E,M)- *category, then the following are equivalent.*

(i) *M consists of monomorphisms of* S.

(ii) *Every morphism f of* S *factors f — me in S, with {e} in* E *and m* £ *M.*

(iii) {id#} *is in* E *for every object EofS.*

PROOF. If {/} = *m§* with *m* monomorphic, then $ is a singleton; thus (i)=>(ii). If we factor id# = *me* by (ii), then it follows easily from 1.1.(2) that *e* and *m* are inverse isomorphisms; thus (ii)=>(iii). If *ma* = *mb* in S with m in M, then we can factor *m{a,b}* = ma{idj5;} for the common domain *E* of *a* and 6. If {id^} is in E, then *{a,b}* = *{t}* for a morphism *t* with *mt* = *ma.* Thus *a—b,* and (iii) ⟹ (i).

**1.3. Example and** remarks. Every category S is an (E,M)- category for E consisting of all sinks in S and *M* the class of all isomorphisms of S.

For an (E, M)- category structure of S, the collection E is determined by M; it consists of all sinks $ which satisfy (2) for all factorizations *g$* — m\£ with *m* 6 *M.*

a largest finally dense full extension of $\mathbf{A}$. We may find it convenient to denote a $P$-sieve by a single letter, in which case $|X|$ denotes the underlying $\mathbf{S}$-object of a $P$-sieve $X$.

If $(\mathbf{B}, Q)$ is a concrete category, then a sink $\Phi$ at an object $B$ of $\mathbf{B}$ induces a $Q$-sink at $QB$, consisting of all pairs $(B', u)$ with $u : B' \to B$ in $\Phi$, and a $Q$-sink has an underlying sink in $\mathbf{S}$. We shall say that a sink in $\mathbf{B}$ or a $Q$-sink is $\mathbf{E}$-*dense* if the underlying sink in $\mathbf{S}$ is in $\mathbf{E}$. This definition applies in particular to $P$-sieves, and we denote by $\mathbf{A}^{\mathrm{cd}}$ the full subcategory of $\mathbf{A}^{\mathrm{cr}}$ with $\mathbf{E}$-dense $P$-sieves as its objects.

**2.2. Factorization of $P$-sieves.** If we factor the underlying sink of a $P$-sieve $(E, \Phi)$ as $m\Psi$, with $m : E' \to E$ in $\mathcal{M}$ and $\Psi$ in $\mathbf{E}$, then $\Psi$ is the underlying sink of an $\mathbf{E}$-dense $P$-sieve $\Phi'$ at $E'$, consisting of all pairs $(A, u')$ with $(A, mu')$ in $\Phi$, and $m$ becomes a coarse ([TQT] 11.8) monomorphism $m : (E', \Phi') \to (E, \Phi)$ of $\mathbf{A}^{\mathrm{cr}}$.

We denote by $\mathcal{M}^{\mathrm{gr}}$ the class of all coarse monomorphisms $m : X' \to X$ in $\mathbf{A}^{\mathrm{cr}}$ with $m$ in $\mathcal{M}$. With this notation, $\mathbf{A}^{\mathrm{cr}}$ becomes an ($\mathbf{E}$-dense $P$-sieve,$\mathcal{M}^{\mathrm{gr}}$) category, and we have the following result.

**Proposition.** *$\mathbf{E}$-dense $P$-sieves define an $\mathcal{M}^{\mathrm{gr}}$-coreflective full subcategory $\mathbf{A}^{\mathrm{cd}}$ of $\mathbf{A}^{\mathrm{cr}}$.*

PROOF. For the morphism $m : (E', \Phi') \to (E, \Phi)$ in $\mathcal{M}^{\mathrm{gr}}$ constructed above, and a morphism $g : (F, \Psi) \to (E, \Phi)$ in $\mathbf{A}^{\mathrm{cr}}$, we have a factorization $gv = mv'$ in $\mathbf{E}$ for every pair $(A, v)$ in $\Psi$. Then $g = mt$ for a unique $t : F \to E'$ if $\Psi$ is $\mathbf{E}$-dense, and we have $t : (F, \Psi) \to (E', \Phi')$ in $\mathbf{A}^{\mathrm{cd}}$ since $m$ is coarse. Thus $m$ is the desired coreflection.

**2.3. E-dense extensions and E-cotopological completions.** We recall that a full concrete embedding $G : (\mathbf{A}, P) \to (\mathbf{B}, Q)$ is *finally dense* ([ACC] 10.72, [TQT] 68.1) if every object $B$ of $\mathbf{B}$ has the final structure for the sink of morphisms $u : GA \to B$ in $\mathbf{B}$. We say that $G$, or by *abus de langage* $(\mathbf{B}, Q)$, is an $\mathbf{E}$-*dense extension* of $(\mathbf{A}, P)$ if $G$ is finally dense, and for every object $B$ of $\mathbf{B}$, the $P$-sieve of pairs (A,u) with $u : GA \to B$ in $\mathbf{B}$ is $\mathbf{E}$-dense. In this situation, we say that $G$ or $(\mathbf{B}, Q)$ is an $\mathbf{E}$-*cotopological completion* of $(\mathbf{A}, P)$ if $(\mathbf{B}, Q)$ is $\mathbf{E}$-cotopological, *i.e.* if every $\mathbf{E}$-sink of objects $B_i$ of $\mathbf{B}$ and morphisms $f_i : QB_i \to E$ in $\mathbf{S}$ has a final lift $B$ in $\mathbf{B}$, with $QB = E$. It follows that $\mathbf{E}$-cotopological completions are transportable.

$\mathbf{E}$-dense extensions $G : (\mathbf{A}, P) \to (\mathbf{B}, Q)$ are the objects of a category, with full concrete embeddings $I$ satisfying $IG = H$ as morphisms $I : G \to H$.

For $\mathbf{E}$ the collection of all sinks, we get finally dense extensions, and $\mathbf{E}$-cotopological completions are topological categories. For $\mathbf{E}$ the collection of all episinks or all quotient sinks, we get epidense and quotient-dense extensions, and $\mathbf{E}$-cotopological categories are dual to $\mathbf{M}$-topological categories for $\mathbf{M}$ the collections of all monosinks and all strong monosinks respectively.

**2.4. Proposition** ([ACC] 10.71). *For $\mathbf{E}$-dense extensions $G : (\mathbf{A}, P) \to (\mathbf{B}, Q)$ and $H : (\mathbf{A}, P) \to (\mathbf{C}, R)$, every morphism $I : G \to H$ preserves initial lifts of sources.*

Proof. If objects $B_i$ of $\mathbf{B}$ and morphisms $f_i : E \to QB_i$ define a source for $Q$ with initial lift $B$ in $\mathbf{B}$, then $g : C \to IB$ in $\mathbf{C}$, for $g : RC \to E$ in $\mathbf{S}$, iff $gu : HA \to IB$ for every $u : HA \to C$ in $\mathbf{C}$. This is the case iff $gu : GA \to B$ in $\mathbf{B}$ for every such $u$, hence iff $f_i gu : GA \to B_i$ in $\mathbf{B}$ for every $u$ and every $f_i$, hence iff $f_i gu : HA \to IB_i$ in $\mathbf{C}$ for every $u$ and every $f_i$, hence iff $f_i g : C \to IB_i$ in $\mathbf{C}$ for every $f_i$. Thus $IB$ is an initial lift in $\mathbf{C}$ for the objects $IB_i$ and morphisms $f_i$ .

**2.5. Theorem.** *For a morphism $I : G \to H$ of $\mathbf{E}$-dense extensions of $(\mathbf{A}, P)$, with $H : (\mathbf{A}, P) \to (\mathbf{C}, R)$ an $\mathbf{E}$-cotopological completion and $G : (\mathbf{A}, P) \to (\mathbf{B}, Q)$, the following are equivalent.*

(i) *$G$ is an $\mathbf{E}$-cotopological completion of $(\mathbf{A}, P)$.*

(ii) *The functor $I$ has a left-inverse concrete left adjoint $J$.*

(iii) *The functor $I$ creates initial lifts,* i.e. *a source of objects $B_i$ of $\mathbf{B}$ and morphisms $f_i : E \to QB_i$ of $\mathbf{S}$ has an initial lift in $\mathbf{B}$, preserved by $I$, if the source of objects $IB_i$ of $\mathbf{C}$ and morphisms $f_i$ has an initial lift in $\mathbf{C}$.*

Proof. For an object $C$ of $\mathbf{C}$, the $Q$-sink of pairs $(GA, u)$ with $u : HA \to C$ in $\mathbf{C}$ is an $\mathbf{E}$-sink. If (i) holds, let $JC$ be the final lift of this sink in $\mathbf{B}$. Then it is easily seen that $f : JC \to B$ in $\mathbf{B}$, for $f : RC \to QB$ in $\mathbf{S}$, iff $f : C \to IB$ in $\mathbf{C}$. Thus objects $JC$ define a concrete left adjoint $J$ of $I$, with $J$ left inverse to $I$ because $I$ is a full embedding and $G$ amnestic.

If morphisms $f_i : E \to RIB_i$ have an initial lift $C$ in $\mathbf{C}$, and if (ii) is valid, then $f_i : JC \to B_i$; we claim that $JC$ is an initial lift for this source. If $g : QX \to E$ in $\mathbf{S}$ with $f_i g : X \to B_i$ in $\mathbf{B}$ for every $f_i$, then also $f_i g : IX \to IB_i$ in $\mathbf{C}$ for every $f_i$, and thus $g : IX \to C$ in $\mathbf{C}$. But then $g : X \to JC$ in $\mathbf{B}$ since $JIX = X$.

For an $\mathbf{E}$-sink $\Sigma$ of objects $B_i$ of $\mathbf{B}$ and morphisms $f_i : QB_i \to E$ in $\mathbf{S}$, consider the source $\Sigma'$ of all pairs $(u, B)$ with $u : E \to QB$ in $\mathbf{S}$ and $uf_i : B_i \to B$ in $\mathbf{B}$ for each $(B_i, f_i)$ in $\Sigma$, and the sink $\Sigma''$ of pairs $(B', v)$ with $v : QB' \to E$ and $uv : B' \to B$ for every pair $(u, B)$ in $\Sigma'$. Then $\Sigma''$ is an $\mathbf{E}$-sink containing $\Sigma$; let $C$ be its final lift in $\mathbf{C}$. If $g : RX \to E$ in $\mathbf{S}$ with $ug : X \to IB$ in $\mathbf{C}$ for every pair $(u, B)$ in $\Sigma'$, then $(GA, gx)$ is in $\Sigma''$, and $gx : HA \to C$ in $\mathbf{C}$, for every $x : HA \to X$ in $\mathbf{C}$, with $A$ an object of $\mathbf{A}$. But then $g : X \to C$ in $\mathbf{C}$; thus $C$ is an initial lift for the source of morphism $u : E \to RIB$ with $(u, B)$ in $\Sigma'$. An initial lift of $\Sigma'$ in $\mathbf{B}$ is clearly a final lift for $\Sigma$; thus (iii)$\Longrightarrow$(i).

**2.6. The Antoine functor.** For an object $A$ of $\mathbf{A}$, we denote by $\mathsf{Y}A = (PA, \Upsilon A)$ the *Antoine sieve* with $(X, u) \in \Upsilon A$ iff $u : X \to A$ in $\mathbf{A}$. The sieve $\Upsilon A$ is $\mathbf{E}$-dense since $(A, \mathrm{id}_{PA}) \in \Upsilon A$. Antoine sieves clearly define a concrete functor $\mathsf{Y} : \mathbf{A} \to \mathbf{A}^{\mathrm{cd}}$, an $\mathbf{E}$-dense embedding since $(A, u) \in \Phi$ for a $P$-sieve $(E, \Phi)$ iff $u : \mathsf{Y}A \to (E, \Phi)$.

**Theorem.** *The Antoine functor $\mathsf{Y} : \mathbf{A} \to \mathbf{A}^{\mathrm{cd}}$ is an $\mathbf{E}$-cotopological completion, and a terminal object in the category of $\mathbf{E}$-dense extensions of $(\mathbf{A}, P)$.*

Proof. For an $\mathbf{E}$-sink of morphisms $f_i : E_i \to E$ of $\mathbf{S}$ and $\mathbf{E}$-dense $P$-sieves $(E_i, \Phi_i)$, it is easily seen that the final lift $(E, \Phi)$ in $\mathbf{A}^{\mathrm{cr}}$, consisting of all pairs of the form $(A, f_i u)$

for some /;, with (A, *u)* G 3>i, is E-dense. Thus Y is an E-cotopological completion. For an E-dense extension *G* : (A,P) —> (B,<2) and a full concrete embedding *K :* B —» $A^{cd}$ with iiTG = Y, and for *u : PA* -> QB in S, we have

(A,u)<E ifJ9 <=> u:YA->iTB <^> u: *GA* -> £.

This determines if uniquely, with each *KB* E-dense since *G* is E-dense.

2.7. If C? : (A,P) —» (B,<5) is an E-dense extension, with *IG* = Y for a concrete full embedding *I* : B -* $A^{cd}$, then every source of objects *B{* of B and morphisms *fi'.E—* QBi* of S induces a source of objects *IB{* of $A^{cd}$ and morphisms *f{:E—> \IBi* of S. We note the following result.

**Proposition.** *If G* : **(A,P)** —> (B,<2) is an **E-** *cotopological completion, then the following are equivalent for* a *source of objects B{ of* B *and morphisms fc : E —> QB{ in* S.

(i) *The source has a lift in* B .

(ii) The *source has an initial lift in* B.

(iii) *The induced source of morphisms fi'.E-* \IB{\ has a lift in* $A^{cd}$.

(iv) The *induced source has an initial lift in* $A^{cd}$.

PROOF. If morphisms *fa : B —> B{* form a lift of the given source, then consider the *P-* sieve of all morphisms *u* : *PA —> E oi S* with *fau : GA —> B{* in B for every /;. This sieve includes all morphisms *u* : *GA* —> *B* of B; thus it is E-dense. If *Bo* is a final lift in B for this sieve and *g* : *QB' —> E* satisfies *fig* : $B^f$ —» *B{* for every /{, then *fogy,* : *GA* —» i?j for every /^ and every u : *GA* —> J9'. But then *gu : GA* —> JBO for every u : *GA* —> J5', and *g : B'* -* $B_o$ follows. Thus $B_o$ is the desired initial lift, and (i)=> (ii).

(ii)=4> (iv) because the functor / preserves initial lifts, and (iv)=>(iii) trivially.

If *X* is a lift of the induced source, with *fa : X —> IBi* in $A^{cd}$ for each /{, then *fi* : *JX —* B{* for each /{, for the concrete left adjoint *J* of /; thus (iii)=>(i).

**2.8. Concrete** monomorphisms, **epimorphisms and colimits.** All limits in $A^{cr}$ are concrete, i.e. preserved by the forgetful functor to S. Limits in $A^{cd}$ are corefiections of limits in $A^{cr}$, hence not necessarily concrete. Concrete limits in an E-cotopological completion B of A are initial lifts of sources, and thus preserved and created by the full concrete embedding B —> $A^{cd}$.

Monomorphisms in an E-cotopological completion B of A are reflected by the faithful forgetful functor of B, but not necessarily preserved. The dual of the category of HausdorfF spaces is an example for this; see 5.2. We say that a monomorphism preserved by the forgetful functor is *concrete;* we shall discuss this more fully in Section 6. The concrete full embedding B —» $A^{cd}$ preserves and reflects monomoprhisms and concrete monomorphisms.

For epimorphisms and colimits, we have the following result.

**Proposition.** *If* (B, *Q) is an E-cotopological completion of* (A, *P), then the forgetful functor QofB preserves and reflects epimorphisms, and Q preserves and lifts colimits.*

PROOF. The faithful functor $Q$ reflects monomorphisms and epimorphisms. If $e : B \to C$ is an epimorphism of $\mathbf{B}$ and $ae = be$ in $\mathbf{S}$, then we can factor $a = m\alpha$, $b = m\beta$, with $\alpha$ and $\beta$ forming a sink in $\mathbf{E}$. This sink has a final lift in $\mathbf{B}$, and $\alpha e = \beta e$ in $\mathbf{B}$ for the lifted morphisms. But then $\alpha = \beta$, and $e$ is epimorphic in $\mathbf{S}$.

If $D$ is a diagram in $\mathbf{B}$ with a colimit cone $\tau : D \to B$, factor $\tau = m\sigma$ in $\mathbf{S}$, with $m$ in $\mathcal{M}$ and $\sigma$ a sink in $\mathbf{E}$. Then $\sigma$ has a final lift $X$ to a cone $\sigma : D \to X$ in $\mathbf{B}$, with $m : X \to B$ in $\mathbf{B}$, and with $\sigma = s\tau$ for a morphism $s : B \to X$. But then $ms = \mathrm{id}_B$ in $\mathbf{B}$, and $m$ is an isomorphism. Thus the underlying sink of $\tau$ is in $\mathbf{E}$. If $\lambda : QD \to E$ is a cone in $\mathbf{S}$, factor $\lambda = m\rho$ with $m$ monomorphic and $\rho$ a quotient cone. Then $\rho$ can be lifted to $\mathbf{B}$. Thus $\rho = r\tau$ for a morphism $r$, and $\lambda = mr \cdot Q\tau$. If also $\lambda = g \cdot Q\tau$, then $g = ms$ for a unique morphism $s$ of $\mathbf{S}$, with $s \cdot Q\tau = \rho$ since $m$ is monomorphic. But then $s = r$, and $g = mr$. Thus $Q\tau$ is a colimit cone in $\mathbf{S}$.

Conversely, if the diagram $QD$ in $\mathbf{S}$ has a colimit cone $\sigma : QD \to E$ in $\mathbf{S}$ and we factor the underlying sink of $\sigma$ as $m\sigma'$ with $\sigma'$ in $\mathbf{E}$, then $\sigma'$ is a cone with domain $QD$. Thus $\sigma' = t\sigma$ for a morphism $t$ of $\mathbf{S}$, with $mt = \mathrm{id}_E$. Now $m$ is an isomorphism, and the underlying sink of $\sigma$ is in $\mathbf{E}$. But then $\sigma$ has a final lift to a cone $\sigma : D \to B$ in $\mathbf{B}$, and this final lift is clearly a colimit cone of $D$ in $\mathbf{B}$.

**2.9. E-dense full sieves.** We denote by $\Omega_E$ the full $P$-sieve at an object $E$ of $\mathbf{S}$, consisting of all pairs $(A, u)$ with $A$ an object of $\mathbf{A}$ and $u : PA \to E$ in $\mathbf{S}$. Full $P$-sieves need not be $\mathbf{E}$-dense; we note however that if there is an $\mathbf{E}$-dense $P$-sieve $\Phi$ at an object $E$ of $\mathbf{E}$, then every $P$-sieve at $E$ coarser than $\Phi$, and in particular $\Omega_E$, is $\mathbf{E}$-dense. This is the case for every object $PA$ of $\mathbf{S}$.

The following result shows that there is no essential loss of generality if we assume that all $P$-sieves $\Omega_E$ are $\mathbf{E}$-dense.

**Proposition.** *Objects $E$ of $\mathbf{S}$ with an $\mathbf{E}$-dense $P$-sieve at $E$ define an $\mathcal{M}$-coreflective full subcategory of $\mathbf{S}$, with objects including all underlying objects of objects of $\mathbf{A}$.*

We denote this full subcategory of $\mathbf{S}$ by $\mathbf{S}^{\mathrm{cd}}$.

PROOF. If $m : (E', \Phi') \to (E, \Omega_E)$ is a coreflection for $\mathbf{A}^{\mathrm{cd}}$, then clearly $\Phi' = \Omega_{E'}$, and we claim that $m : E' \to E$ is a coreflection for $\mathbf{S}^{\mathrm{cd}}$. If $g : F \to E$ in $\mathbf{S}$, then we have a factorization $gv = mv'$ for every pair $(A, v)$ in $\Omega_F$. If $\Omega_F$ is $\mathbf{E}$-dense, it follows that $g = mg'$ for a (unique) morphism $g' : F \to E'$.

## 3. E-cotopological hulls

**3.1. Definition.** By 2.6, every $\mathbf{E}$-dense extension of $(\mathbf{A}, P)$ over $\mathbf{S}$ is concretely isomorphic to a full subcategory $\mathbf{B}$ of $\mathbf{A}^{\mathrm{cd}}$, containing all objects $\mathsf{Y}A$ for objects $A$ of $\mathbf{A}$. If we restrict ourselves to full subcategories of $\mathbf{A}^{\mathrm{cd}}$, then morphisms $I : G \to H$ of quotient extensions become full subcategory embeddings. If $\mathcal{C}$ is a class of $\mathbf{E}$-dense $P$-sieves, then a smallest $\mathbf{E}$-cotopological completion of $(\mathbf{A}, P)$ with $\mathcal{C}$ contained in its class of objects is called an $\mathbf{E}$-*cotopological hull* of $\mathcal{C}$ in $\mathbf{A}^{\mathrm{cd}}$.

**3.2. Theorem.** *For a concrete category A over an* (E, *M) category* S, *every collection C of E-dense P-sieves has an* E- *cotopological hull If every object YA* of $A^{cd}$ *is an initial lift of a source of morphisms fi : YA —> X{ in* $A^{cd}$ *with every* $X_t$- *in Q, then the objects of this hull are all initial lifts of sources />·: E —>* |Xi| in $A^{cd}$, with *each* X; *in* C.

PROOF. By 2.5 and 2.7, E-cotopological completions of A are E-dense extensions which, regarded as full subcategories of $A^{cd}$, are closed under initial lifts of sources. Intersections preserve this property; thus every class C of E-dense P-sieves has an E-cotopological hull. The collection of all objects X of $A^{cd}$ with the initial structure for a source of morphisms /; : *X* -> X; of $A^{cd}$, with each X; in C, consists of objects of the E-cotopological hull and is closed under initial lifts of sources. Thus it is the class of all objects of the E- cotopological hull of C if it contains all objects *YA.*

3.3. Fibres. We recall that the *fibre* of an object *E* of S, in a concrete category B over S, is the class of all objects *X* of B with underlying object |X| — *E.* It is considered desirable that concrete categories should have small fibres. We investigate this property for E-cotopological hulls.

We assume that every object YA of $A^{cd}$ admits an intial source of morphisms *f{* : *YA* —• X{, with each *X{* in C. This is obviously no essential loss of generality.

For pairs (A, *u*) and (J?,v), with *A* and *B* objects of A and with *u : PA* —> *E* and *v : PB* —> *E* in S for an object *E* of S, we put *(B,v) -<e (A,u)* if for every pair (/,X) with X in C and / : *E* -> |X| in S, and with *fu* : YA -> X in $A^{cd}$, we also have *fv:YB ^X* in $\mathbf{A}^{cd}$.

For a fixed pair *(A,u)* with *u* : *PA* —• *E* in S, the pairs *(B,v)* with *(B,v) ~<e {A,u)* clearly form a *P*- sieve (22, $). This sieve has the initial structure in $A^{cr}$ for the morphisms / : *E* -> |X| in S with X in *Q* and *fu* : YA -> X in $A^{cr}$, but it need not be an E-sieve.

We put (A,u) -j^e *(A\u')* if *(A,u)* -$<_e$ *(A',u^f)* and (A',u') -<e *{A,u),* with *u : PA* —• *E* and *v! : PA!* —> *E.* This clearly defines an equivalence relation, with (A,u) *~E,e {A',v!)* iff the *P*- sieve *(E,* $) constructed in the preceding paragraph is the same for *(A,u)* and for *(A\u^f).*

**3.4. Proposition.** *If the collection of equivalence classes for the relation ^E,e* is *small for every object EofS, then the* E- *cotopological hull of Q in* $A^{cd}$ has *small fibres. Conversely, if* S *is well-powered for monomorphisms in M and the Ft-cotopological hull of* C *in* $A^{cd}$ *has small fibres, then the collection of equivalence classes for ^E,e is small for every object E of* S .

PROOF. For every object (2?, $) of the E-cotopological hull, the P-sieve § is a union of equivalence classes for ~£7,e- If the collection of equivalence classes is small, it follows that the collection of their unions, and hence the fibre of *E* in the E-cotopological hull, is small.

Conversely, the equivalence classes for ~£,e correspond bijectively to the *P*- sieves (J5, $) constructed in 3.3. If S is well-powered for monomorphism in M, then there is a set-indexed family of morphism m; : *Ei* —> *E* in *M* such that for every P-sieve (J5, $)

of this kind there is a coarse morphism $m_i : (E_i, \Phi_i) \to (E, \Phi)$, with $(E_i, \Phi_i)$ $\mathbf{E}$-dense and hence in the $\mathbf{E}$-cotopological hull of $\mathcal{C}$. If this $\mathbf{E}$-cotopological hull has small fibres, it follows that the collection of sieves $(E, \Phi)$ is small.

**3.5. Remarks.** The results of Sections 2 and 3 are valid in particular for finally dense extensions, with $\mathbf{E}$ the collection of all sinks in $\mathbf{S}$. For this case, Theorem 2.5 shows that $\mathbf{E}$-cotopological completions are topological categories, with initial lifts for all sources, and with forgetful functors preserving and lifting monomorphisms and strong monomorphisms, as well as all limits and colimits. Thus the forgetful functor of a completion $\mathbf{C}$ preserves all limits and colimits. If $D$ is a diagram in $\mathbf{C}$ for which the underlying diagram has a limit in $\mathbf{S}$, then the initial lift of the source induced by the limit cone is a limit of $D$ in $\mathbf{C}$, and colimits in $\mathbf{C}$ are obtained dually.

These properties remain valid for colimits in an $\mathbf{E}$-cotopological completion $\mathbf{B}$ of $\mathbf{A}$ if coarse $P$-sieves are $\mathbf{E}$-dense, but limits of a diagram $D$ in $\mathbf{B}$ are preserved and lifted by the forgetful functor of $\mathbf{B}$ only if the source induced by a limit cone of $D$ in $\mathbf{S}$ has a lift in $\mathbf{A}^{\mathrm{cd}}$. By 2.7, $\mathbf{E}$-cotopological completions admit initial lifts only for those sources which admit a lift in $\mathbf{A}^{\mathrm{cd}}$.

We observe that collections $\mathcal{C}$ of $\mathbf{E}$-dense $P$-sieves can be replaced by collections of $P$-sieves, since every coreflection for $\mathbf{A}^{\mathrm{cd}}$ in $\mathbf{A}^{\mathrm{cr}}$ is a coarse monomorphism in $\mathbf{A}^{\mathrm{cr}}$. Theorem 3.2 then characterizes the $\mathbf{E}$-cotopological hull of the $\mathbf{A}^{\mathrm{cd}}$-coreflections of the $P$-sieves in $\mathcal{C}$. If $\mathbf{E}$ is the collection of all sinks in $\mathbf{S}$, then the hull of $\mathcal{C}$ in $\mathbf{A}^{\mathrm{cr}}$, for a class $\mathcal{C}$ of $P$-sieves, has small fibres iff the collection of equivalence classes is small for every equivalence relation $\sim_{E,\mathcal{C}}$, by 3.4 with $\mathcal{M}$ the class of all isomorphisms of $\mathbf{S}$. Every known criterion for fibre smallness of topological or $\mathbf{E}$-cotopological hulls follows directly from 3.4 and these observations.

## 4. Cartesian closed quotient-topological hulls

**4.1.** We assume from now on that $\mathbf{S}$ is a (quotient sink,mono) category. Then $\mathbf{S}$ has finite limits, by 1.7, if $\mathbf{S}$ has finite products. For $\mathbf{E}$ the collection of all quotient sinks, $\mathbf{E}$-cotopological categories will be called *quotient-topological*, and the category quotient-dense $P$-sieves wii be denoted by $\mathbf{A}^{\mathrm{cq}}$. We recall from [TQT] 60.2 that $\mathbf{A}^{\mathrm{cr}}$ is cartesian closed if (and only if) $\mathbf{S}$ is cartesian closed. If $\mathbf{S}$ has function space objects $F^E$, then $\mathbf{A}^{\mathrm{cr}}$ has function space objects $(F^E, [\Phi, \Psi])$ for objects $(E, \Phi)$ and $(F, \Psi)$, with $(A, \hat{\varphi})$ in $[\Phi, \Psi]$, for an object $A$ of $\mathbf{A}$ and $\hat{\varphi} : PA \to F^E$ exponentially adjoint to $\varphi : PA \times E \to F$ in $\mathbf{S}$, iff $\varphi : \mathrm{Y}A \times (E, \Phi) \to (F, \Psi)$ in $\mathbf{A}^{\mathrm{cr}}$.

**4.2.** We note that the second hypothesis of the following result is always satisfied if $\mathbf{A}$ has finite products and the forgetful functor $P$ preserves them, because then the functor $\mathrm{Y}$ also preserves finite products.

**Theorem.** *If $\mathbf{S}$ is cartesian closed, and finite products of objects $\mathrm{Y}A$ in $\mathbf{A}^{\mathrm{cr}}$ are quotient-dense, then $\mathbf{A}^{\mathrm{cq}}$ is closed under finite products in $\mathbf{A}^{\mathrm{cr}}$, and cartesian closed.*

*In this situation, function space objects [Y,Z] in* $A^{cq}$ are *coreSections of function space objects* $Z^Y$ *in* $A^{cr}$, and we have *concrete* (2.8) *natural monomorphisms fiy,z • ·[Y, Z] —•* $Z^Y$, *with evy,z(n>y,z* X idy) an *evaluation in* $A^{cq}$ for an *evaluation evy,z in* $A^{cr}$, *for objects Y and Z of* $A^{cq}$. *Conversely, if all full P-sieves are quotient-dense, and if* $A^{cq}$ *is cartesian closed and closed under finite products in* $A^{cr}$, *then* S *is cartesian closed.*

PROOF. For objects *(E,* $) and (F, *) of $A^{cq}$, we have pullback squares

$$\begin{array}{ccccc}
\mathrm{Y}A \times \mathrm{Y}'_B & \xrightarrow{\mathrm{UXId}} & \mathbf{(JB,\$)\ x\ } YB & \mathbf{\text{-}\wedge U} & \mathrm{Y}B \\
\downarrow \mathrm{I\,id\,X\,t?} & & \downarrow \mathrm{id\ x\ u} & & \downarrow v \\
\mathrm{Y}A \times (F,\Psi) & \xrightarrow{u \times \mathrm{id}} & (E,\Phi) \times (F,\Psi) & \xrightarrow{q} & (F,\Psi) \\
\downarrow p & & \downarrow p & & \\
\mathrm{Y\,A} & \xrightarrow{u} & \mathbf{(JE,\$)} & &
\end{array} ,$$

with projections p and *q* of products, and with (A, it) in $ and *(B,v)* in \f. If S is cartesian closed, then the morphisms *u* x id form a quotient sink by 1.7, and thus P-sieves *(E,* $) x Y2? are quotient-dense if the P-sieves YA x Yi? are. In the same way, (22, $) x (JF, *) is quotient-dense if the sieves (E, $) X YJB are.

Now $A^{cq}$ is a full coreflective subcategory of a cartesian closed category $A^{cr}$, closed under products 1 x 7 in $A^{cr}$. It is well known that $A^{cq}$ is cartesian closed in this situation, with coreflections of function space objects in $A^{cr}$ as function space objects. These coreflections are concrete natural monomorphisms *fiy,z* by 2.2, with *fiY^^* exponentially adjoint to *<p : X* x *Y* —• *Z* in $A^{cr}$ if *(p* is exponentially adjoint to *(p* in $A^{cq}$.

The converse follows immediately from the main result of [13], since the forgetful functor of $A^{cq}$ preserves finite products, and objects (JB, *QE)* provide a full embedding S —> $A^{cq}$, right adjoint to the forgetful functor.

**4.3. Corollary.** *If* **S** *is cartesian closed and products YA* X *YB in* $A^{cr}$ are *quotient-dense, then every quotient-topological completion* (B,<2) of (A,P) has *concrete products BxC.*

PROOF. A concrete product *B* X *C* in B is an initial lift for the source of projections *p* : *QB* X *QC* -> *QB* and *q* : *QB* X *QC* -» *QC.* With the given assumptions, $A^{cq}$ has concrete products *X* X *Y,* and it follows immediately from 2.7 that B has concrete products *B xC.*

4.4. **Theorem.** *If* S *is cartesian closed and finite products of P-sieves YA are quotient-dense, then the following conditions are equivalent for a quotient-topological completion* (B,Q) *of* (A,P).

(i) B *is cartesian closed, with function space objects preserved up to isomorphism by the full concrete embedding I :* B —> $A^{cq}$.

(ii) *Every product functor* — X *C in* B *preserves final lifts of quotient sinks.*

(iii) *The concrete left adjoint* $J$ *:* $A^{cq}$ —• B *of the full concrete embedding* $I$ *:* B —> $A^{cq}$ *preserves products* $X$ X $Y$.

*If the forgetful functor of* $A^{cq}$ *preserves monomorphisms, then these conditions are also equivalent to:*

(iv) B *is cartesian closed.*

PROOF. If $B$ has the final structure for a quotient sink of morphisms *U{ : Bi* —> *B* in B and (i) holds, then $Q$ preserves products, and the morphisms *U{* x idc form a quotient sink. Now consider *<p : QB* x *QC* —> *QX* with *(p-(uiX* idc) : *Bi* X *C* —> *X* in B for every *ui.* Let *rfri : B{* -> [C, X] be exponentially adjoint to *tp-{ui* X idc) in B, and let <p$^{\#}$ : *QB* -> $QX^{QC}$ be exponentially adjoint to *(p* in S. We have commutative diagrams

$$\begin{array}{ccc} QB_i & \xrightarrow{-\wedge-} & QB \\ \downarrow{\psi_i} & & \downarrow{\varphi^{\#}} \\ Q[C,X] & \xrightarrow{\mu_{C,X}} & QX^{QC} \end{array}$$

in S, one for each i*;. It follows that *(p#* factors /xc,X^$_9$ with ^u$_z$- = ^$_t$* : *Bi* —• [C, X] in B. But then <$\dot{p}$ *: B* —> [C, X] in B, exponentially adjoint to ⋦ : *B* x (7 —* X, and (ii) holds.

Every object X of $A^{cq}$ has the final structure for morphisms *Ui : IBi* —> X which form a quotient sink, and then JX has the final structure for the morphisms *U{* : *Bi* —* JX. By (ii) for $A^{cq}$, X X /C has the final structure for the morphisms *Ui* X idc which form a quotient sink, and so J(X X *IC)* has the final structure for the morphisms u; x idc with domains *Bi* x *C.* If (ii) is valid for B, then *JX* X *C* has the final structure for the same quotient sink; thus *J(X* x *IC)* = JX X C. Now if an object *Y* of $A^{cq}$ has the final structure for a quotient sink of morphisms *Vj* : *ICj* —> *Y,* then *J(X* X F) has the final structure for morphisms idx X *Vj : J(X* x *ICj)* -^ J(X X F), and *JX* X /Y" for morphisms idx x t?j : *JX* X Cy —> JX X *JY.* But these morphisms have the same domains, hence the same final structure, and (iii) follows.

The step (iii)=>(i) follows immediately from the main result of [13].

(i)=>(iv) trivially. For the converse, let *H(Y,Z)* denote a function space object in B. Commutative diagrams

$$\begin{array}{ccccc} B(X \times F, Z) & \text{—U} & A^{cq}(JX, I(Y \times Z) & \xrightarrow{\text{id}} & A^{cq}(/X,/F \times IZ) \\ \downarrow\cong & & & & \downarrow\cong \\ \mathbf{B}(X,H(Y,Z)) & \xrightarrow{I} & \mathbf{A}^{cq}(IX,IH(Y,Z)) & \xrightarrow{\rho_{Y,Z}} & \mathbf{A}^{cq}(IX,[IY,IZ]) \end{array},$$

with exponential adjunctions as vertical arrows, define natural maps *py,z* •·*IH{Y, Z)* —> *[IY,IZ]* in $A^{cq}$, by the general construction of adjoint natural transformations, with *PY.zôc* : JX -» *[IYJZ]* exponentially adjoint in $A^{cq}$ to a : JX -> *IY* x *IZ* if â : X -> *H(Y,Z)* is exponentially adjoint to a : X x F — > Z in B. Now if *py,z&* = *PY,z*$\hat{P}$ with d,/$\hat{3}$ : X —> *IH(Y, Z),* then the equation remains valid for the morphisms d and $\hat{0}$ from JJA^ -> *IH{Y,Z).* But then d and /$\hat{3}$ from JX to *H(Y,Z)* are adjoints of the same morphism $JX \times Y \longrightarrow Z$, and $\hat{\alpha} = \hat{\beta}$ follows. Thus py^ is monomorphic in $A^{cq}$. The quotient sieve of morphisms $YA \longrightarrow [IY,IZ]$ clearly factors through /?y,#; thus $py_t$^ is an isomorphism if *py,z* is monomorphic in S.

For our next theorem, we need the following two lemmas.

4.5. Lemma. *If a quotient-topological completion* (B,Q) of (A,P) *satisfies the conditions of Theorem* 4.4, *and an object X of* $A^{cq}$ has *the final structure for a quotient sink of morphisms* $v_+ : IB_i \longrightarrow X$, *for objects* $B_i$ *of* B, then *[X,IC] has the initial structure for the morphisms* $[u_i,id_c] : [X,IC] \to I[B_i,C]$ of $A^{cq}$, for every object C of B.

PROOF. Consider $\varphi : |Y| \to |[XJC]|$ in S, with $[u_i,id_c]\xi : F \to I[B_i,C]$ in $A^{cq}$ for each $U_i$. If fixjc<P is exponentially adjoint to $(p : |Y| \times |X| \longrightarrow QC$ in S, then W£;,ic[$^u$i?idc]£ = (idQc)$^{U_i}$MX,/c^ is exponentially adjoint to v?(idy X *ui).* Since F x *X* has the final structure for the morphisms idy X u^, it follows that $\varphi : Y \times X \longrightarrow C$. The exponential adjoint of this morphism in $A^{cq}$ is $\xi : Y \longrightarrow [X,IC]$. Thus *[X,IC]* has the claimed initial structure.

**4.6. Lemma.** *For every object Y of* $A^{cq}$, the *endofunctor* [F,—] of $A^{cq}$ preserves *initial structures for sources.*

PROOF. Assume that *B* has the initial structure for morphisms $U_i : B \longrightarrow B_i$ of $A^{cq}$, and consider $\varphi : |X| \to |[Y,B]|$ in S with $[id_Y,U_i]\varphi : X \to [Y,B_i]$ in $A^{cq}$ for every $u_i$. If My,B^ is the exponential adjoint of <p : \X\ X|F| —> |JB|, then //y^Jidy, $U_i]\hat{\varphi}$ = ^i'$^y$'/^y,B^ is the exponential adjoint of $u_i\varphi$ in S, and it follows that [idy,Uj]y is the exponential adjoint of $u_i\varphi : X \times Y \to B_i$ in $A^{cq}$. But then $\varphi : X \times Y \to B$ *in* $A^{cq}$, with exponential adjoint $\hat{\varphi} : X \to [Y,B]$.

**4.7. Theorem.** *If* **S** is *cartesian closed, and finite products of P-sieves YA in* $A^{cr}$ are *quotient-dense, then every P-sieve YA has the initial structure for morphisms* $f : YA \to [YB,YC]$ in $A^{cq}$, and *the quotient-topological hull of the class of objects [YB,YC] of* $A^{cq}$, *for objects B and C of* A, *is cartesian closed. If the forgetful functor from* $A^{cq}$ *to* S *preserves monomorphisms, then this quotient-topological hull is the cartesian closed quotient-topological hull of* A *over* S .

PROOF. Let C be the quotient-topological hull of the Theorem. By assumption, the terminal object (l,fii) of $A^{cr}$ is quotient-dense, and thus a terminal object of $A^{cq}$, with the final structure for a quotient sink of morphisms $U_i : YA_i \longrightarrow$ (l,fii). By 4.5, every object [(l,ft!),Y;4] is in C, with the initial structure for the morphisms [u{,id^]. This proves the first part of the Theorem since *YA* is isomorphic to [(l,!^), *YA].*

Again by 4.5, every function space object [X, YC] is in C. Since a product $X \times Y$ in $A^{cq}$ has the initial structure for its projections, C is closed under finite products in $A^{cq}$. Now if *Y* has the initial structure for morphisms $U_i : Y \longrightarrow [YB_i, YC_i]$, then *[X,Y]* has the initial structure for the morphisms [idjr,Ui] with codomains [X, [Y2?{, YCY]], by 4.6. These codomains are isomorphic to the objects $[X \times \mathrm{Y}B_i, \mathrm{Y}C_i]$ of $\mathbf{C}$, and thus objects of $\mathbf{C}$. But then $[X, Y]$ is an object of $\mathbf{C}$ whenever $Y$ is an object of $\mathbf{C}$, and $\mathbf{C}$ is cartesian closed.

The category $\mathbf{C}$ is clearly the smallest cartesian closed quotient-topological completion $\mathbf{C}$ of $\mathbf{A}$ for which the full embedding $I : \mathbf{C} \to \mathbf{A}^{\mathrm{cq}}$ preserves function space objects. By 4.4.(iv), this is also the cartesian closed quotient-topological hull of $\mathbf{A}$ if the forgetful functor of $\mathbf{A}^{\mathrm{cq}}$ preserves monomorphisms.

**4.8. Remarks.** It is easily seen that the objects of the quotient-topological hull of the class of function spaces $[\mathrm{Y}B, \mathrm{Y}C]$ are the quotient-dense $P$-sieves which are power-closed in the sense of [11]; thus 2.4 becomes Theorem 1.13 of [11] for this case.

If $\mathbf{S}$ is the category of sets and $\mathbf{A}$ has constant maps, then $\mathbf{A}^{\mathrm{cr}}$ and $\mathbf{A}^{\mathrm{cq}}$ have constant maps, and $\mathbf{A}^{\mathrm{cq}}$ is topological over $\mathbf{S}$. In this situation, the underlying set of a function space object $[X, Y]$ in $\mathbf{A}^{\mathrm{cq}}$ is the set $\mathbf{A}^{\mathrm{cq}}(X, Y)$.

Similar remarks apply to Sections 5 and 6.

## 5. Representing partial morphisms in completions

**5.1.** We recall that a (strong) *partial morphism* in a category $\mathbf{C}$, with domain $A$ and codomain $B$, is a span $A \xleftarrow{m} \cdot \xrightarrow{f} B$ in $\mathbf{C}$ with $m$ a strong monomorphism of $\mathbf{C}$. A strong monomorphism $\vartheta_B : B \to \tilde{B}$ in $\mathbf{C}$ *represents partial morphisms* with codomain $B$ if for every partial morphism $A \xleftarrow{m} \cdot \xrightarrow{f} B$ in $\mathbf{C}$, there is a unique morphism $\bar{f} : A \to \tilde{B}$ in $\mathbf{C}$ such that

$$\begin{array}{ccc} \cdot & \xrightarrow{f} & B \\ \big\downarrow m & & \big\downarrow \vartheta_B \\ A & \xrightarrow{\bar{f}} & \tilde{B} \end{array}$$

is a pullback square in $\mathbf{C}$. Categories with partial morphisms represented for every codomain have been called *hereditary* [17] or *extensionable* [25]; we note that "hereditary" often has another meaning.

In the context of the present paper, we have the following basic result.

**Proposition** (J. PENON). *Partial morphisms in* $\mathbf{A}^{\mathrm{cr}}$ *are represented if and only if partial morphisms in* $\mathbf{S}$ *are represented, and then the forgetful functor* $\mathbf{A}^{\mathrm{cr}} \to \mathbf{S}$ *preserves representiations of partial morphisms.*

PROOF. If $\tau_E : E \to E^{\#}$ represents partial morphisms in $\mathbf{S}$, then $\tau_E : (E, \Phi) \to (E^{\#}, \Phi^{\#})$ represents partial morphisms in $\mathbf{A}^{\mathrm{cr}}$ if $\Phi^{\#}$ consists of all pairs $(A, \bar{u})$ with $u : m^*\mathrm{Y}A \to (E, \Phi)$ in $\mathbf{A}^{\mathrm{cr}}$, *i.e.* if $(X, uv) \in \Phi$ for every $v : PX \to F$ in $\mathbf{S}$ with

$mv : X \to A$ in $\mathbf{A}$, for a pullback

$$\begin{array}{ccc} F & \xrightarrow{u} & E \\ \Big\downarrow m & & \Big\downarrow \tau_E \\ PA & \xrightarrow{\bar{u}} & E^{\#} \end{array}$$

in $\mathbf{S}$; see [TQT] 60.3. Conversely, if $\tau_E : (E, \Omega_E) \to (E^{\#}, \bar{\Phi})$ represents partial morphisms in $\mathbf{A}^{\mathrm{cr}}$, then it is easily seen that $\bar{\Phi} = \Omega_{E^{\#}}$, and that $\tau_E : E \to E^{\#}$ represents partial morphisms in $\mathbf{S}$.

**5.2. Embeddings.** We recall that an *embedding* in a concrete category $\mathbf{C}$ over $\mathbf{S}$ is an initial lift $m : X \to Y$ of a strong monomorphism $m : |X| \to |Y|$ of $\mathbf{S}$, and we say that $\mathbf{C}$ *has embeddings* if every strong monomorphism $m : E \to |Y|$ of $\mathbf{S}$, with $Y$ an object of $\mathbf{C}$, has an initial lift $m : X \to Y$ in $\mathbf{C}$.

If the forgetful functor $\mathbf{C} \to \mathbf{S}$ preserves epimorphisms, then embeddings in $\mathbf{C}$ are strong monomorphisms. Conversely, if $\mathbf{S}$ has (epi, strong mono) factorizations and $\mathbf{C}$ has embeddings, then all strong monomorphisms in $\mathbf{C}$ are embeddings. Hausdorff spaces are an example of a concrete category over sets with embeddings, but with epimorphisms not preserved by the forgetful functor. In this example, epimorphisms are maps $f : X \to Y$ with $f^{\to}(X)$ dense in $Y$, and only closed embeddings are strong monomorphisms.

If $\mathbf{C}$ has embeddings, then it is easily seen that pullbacks

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & Y \\ \Big\downarrow m' & & \Big\downarrow m \\ X' & \xrightarrow{f} & X \end{array}$$

in $\mathbf{C}$ with $m : Y \to X$ an embedding are concrete, *i.e.* lifted from pullbacks in $\mathbf{S}$, with $m' : Y' \to X'$ an embedding.

We note that $\mathbf{A}^{\mathrm{cr}}$ always has embeddings, and that $\mathbf{A}^{\mathrm{cq}}$ has embeddings iff $X$ is quotient-dense for every embedding $m : X \to Y$ in $\mathbf{A}^{\mathrm{cr}}$ with $Y$ quotient-dense. We say that $\mathbf{A}^{\mathrm{cq}}$ is *closed under embeddings* in $\mathbf{A}^{\mathrm{cr}}$ if this is the case.

**5.3.** The second hypothesis of the following result is satisfied in particular if $\mathbf{A}$ has embeddings.

**Theorem.** *If partial morphisms in* $\mathbf{S}$ *are represented, and the domain* $X$ *of every embedding* $m : X \to \mathrm{Y}A$ *in* $\mathbf{A}^{\mathrm{cr}}$ *is quotient-dense, then partial morphisms in* $\mathbf{A}^{\mathrm{cq}}$ *are represented, and* $\mathbf{A}^{\mathrm{cq}}$ *is closed under embeddings in* $\mathbf{A}^{\mathrm{cr}}$*. If these conditions are satisfied, and if* $\tau_X : X \to X^{\#}$ *and* $\vartheta_X : X \to \tilde{X}$ *represent partial morphisms in* $\mathbf{A}^{\mathrm{cr}}$ *and in* $\mathbf{A}^{\mathrm{cq}}$*, with* $X$ *quotient-dense, then* $\tau_X = \nu_X \vartheta_X$ *with* $\nu_X$ *a coreflection for* $X^{\#}$ *and* $\mathbf{A}^{\mathrm{cq}}$*. Conversely, if partial morphisms in* $\mathbf{A}^{\mathrm{cq}}$ *are represented, with* $\mathbf{A}^{\mathrm{cq}}$ *closed under embeddings in* $\mathbf{A}^{\mathrm{cr}}$*, and every coarse* $P$*-sieve* $\Omega_E$ *is quotient-dense, then partial morphisms in* $\mathbf{S}$ *are represented.*

PROOF. If $m : (E', m^*\Phi) \to (E, \Phi)$ is an embedding, then consider pullbacks

$$\begin{array}{ccc} X & \xrightarrow{u'} & (E', m^*\Phi) \\ \downarrow{\scriptstyle m'} & & \downarrow{\scriptstyle m} \\ \mathsf{Y}A & \xrightarrow{u} & (E, \Phi) \end{array}$$

in $\mathbf{A}^{\mathrm{cr}}$ with $(A, u) \in \Phi$. The morphisms $u$ in these pullback squares form a quotient sink if $(E, \Phi)$ is quotient-dense, and so do their pullbacks $u'$ by the strong monomorphism $m$ if partial morphisms in $\mathbf{S}$ are represented. The morphisms $m' : X \to \mathsf{Y}A$ in the pullback squares are embeddings, and $m^*\Phi$ is quotient-dense if each $X$ is quotient-dense and the $u'$ form a quotient sink.

Now let $\tau_Y : Y \to Y^\#$ represent partial morphisms in $\mathbf{A}^{\mathrm{cr}}$, with coreflection $\nu_Y : \tilde{Y} \to Y^\#$ for $\mathbf{A}^{\mathrm{cq}}$. Every morphism $X \to Y^\#$ with $X$ an object of $\mathbf{A}^{\mathrm{cq}}$ factors uniquely through $\nu_Y$. In particular, $\tau_Y$ factors $\tau_Y = \nu_Y \vartheta_Y$, with $\vartheta_Y$ an embedding. If $\mathbf{A}^{\mathrm{cq}}$ is closed under embeddings in $\mathbf{A}^{\mathrm{cr}}$, then partial morphisms $X \xleftarrow{m} \cdot \xrightarrow{f} Y$ in $\mathbf{A}^{\mathrm{cq}}$ are partial morphisms in $\mathbf{A}^{\mathrm{cr}}$ with domain and codomain in $\mathbf{A}^{\mathrm{cq}}$. Now in a diagram

$$\begin{array}{ccccc} \cdot & \xrightarrow{f} & Y & \xrightarrow{\mathrm{id}_Y} & Y \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \vartheta_Y} & & \downarrow{\scriptstyle \tau_Y} \\ X & \xrightarrow{\bar{f}} & \tilde{Y} & \xrightarrow{\nu_Y} & Y^\# \end{array},$$

the righthand square is a pullback; thus the lefthand square is a pullback iff the outer rectangle is one. The morphism $X \to Y^\#$ in the diagram determines $\bar{f}$ uniquely; it follows that $\vartheta_Y$ represents partial morphisms in $\mathbf{A}^{\mathrm{cq}}$ if $\tau_Y$ represents partial morphisms in $\mathbf{A}^{\mathrm{cr}}$.

We observe for the converse that partial morphisms with codomain $(E, \Omega_E)$, in $\mathbf{A}^{\mathrm{cr}}$ or in $\mathbf{A}^{\mathrm{cq}}$, are represented by embeddings $\tau_E : (E, \Omega_E) \to (E^\#, \Omega_{E^\#})$, and then $\tau_E : E \to E^\#$ clearly represents partial morphisms in $\mathbf{S}$.

**5.4. Corollary.** *If partial morphisms in* $\mathbf{S}$ *are represented, and the domain* $X$ *of every embedding* $m : X \to \mathsf{Y}A$ *in* $\mathbf{A}^{\mathrm{cr}}$ *is quotient-dense, then every quotient-topological completion* $(\mathbf{B}, Q)$ *of* $(\mathbf{A}, P)$ *has embeddings.*

PROOF. Since $\mathbf{A}^{\mathrm{cq}}$ has embeddings under the assumptions of 5.3, this follows immediately from the fact, proved in 2.4 and 2.5, that the full concrete embedding $I : \mathbf{B} \to \mathbf{A}^{\mathrm{cq}}$ preserves and creates initial lifts for sources.

**5.5. Theorem.** *If partial morphisms in* $\mathbf{S}$ *are represented, and* $X$ *is quotient-dense for every embedding* $m : X \to \mathsf{Y}A$ *in* $\mathbf{A}^{\mathrm{cr}}$, *then the following conditions are equivalent for a quotient-topological completion* $G : (\mathbf{A}, P) \to (\mathbf{B}, Q)$ *of* $(\mathbf{A}, P)$.

(i) *Partial morphisms in* $\mathbf{B}$ *are represented, with representations preserved up to isomorphism by the full concrete embedding* $I : \mathbf{B} \to \mathbf{A}^{\mathrm{cq}}$ *with* $IG = \mathsf{Y}$.

(ii) *Every pullback functor* $m^*$ *by an embedding* $m$ *in* $\mathbf{B}$ *preserves final lifts of quotient sinks.*

(iii) The *concrete left adjoint* $J$ : $A^{cq}$ -+ B *of the full concrete embedding* $I$: B -> $A^{cq}$ *preserves embeddings.*

(iv) *For the concrete left adjoint* $J$ : $A^{cq}$ —» B *of the full concrete embedding* $I$: B —> $A^{cq}$ and ever/ *object BofH, the morphism tijB '.B -* JIB of 3 is an embedding.*

*If the forgetful functor from* $A^{cq}$ *to* S *preserves monomorphisms, then these conditions are equivalent to:*

(v) *Partial morphisms in 3 are represented.*

PROOF. For (i)=>(ii), consider pullback squares

$$\begin{array}{ccc} d & \text{-^-+} & C \\ \downarrow m_i & & \downarrow m \\ B_{\{} & \text{-*-•} & B \end{array},$$

for a quotient sink of morphisms *U{* with final lift *B* and an embedding *m : C* —• *B* in $A^{cq}$. These pullbacks lift pullbacks in S, and the morphisms *V{* form a quotient sink since partial morphisms in S are represented. For / : *\C* —> *\D* in S with /v; : *C{* —> 1? in B for every ?;;, pullback squares

$$\begin{array}{ccc} C< & \xrightarrow{v_i f} & I? \\ \downarrow m_i & & \downarrow \vartheta D \\ ft\ i & & X) \end{array} \quad \text{and} \quad \begin{array}{ccc} |C| & \text{^}U & \backslash D\backslash \\ \downarrow \backslash m & & \downarrow \tau_D \\ |B| & \xrightarrow{g} & |D^{\#}| \end{array}$$

determine morphisms /i- : ft -» J9 in B and ^ : |.B| —• l^^l i$^n$ S uniquely, with $v_D h_i$ = *gu{* for each *U{*. It follows that *g* factors *g* = *yryt* in S, with *hi* = *tu{* for each u^. But then *t : B* -> $\tilde{D}$ in $A^{cq}$. As *m = i?£>/ and i?/) is an embedding, we have / : *C* —> JD in $A^{cq}$. Thus C has the final structure for the quotient sink of morphisms *V{*.

Now let *m* : F —• *X* be an embedding in $A^{cq}$, with X the final lift in $A^{cq}$ for a quotient sink of morphisms *U{ : IGA{* —» *X* in $A^{cq}$. Then *JX* is the final lift in B for the sink of morphisms *U{ : GA{* —> JX in B. Pulling back the *U{* by m, we get pullback squares

$$\begin{array}{ccc} /a^* & \text{^^} & r \\ \downarrow m_i & & \downarrow m \\ YAi & \xrightarrow{u_i} & X \end{array}$$

in $A^{cq}$, with a quotient sink of morphisms *Vi* : /ft —> F in $A^{cq}$, by (ii) for $A^{cq}$ and 5.4. Now *J* preserves final lifts of quotient sinks; thus *JX* and *JY* are the final lifts in B for quotient sinks *Ui : GAi* —> *JX* and *Vi* : ft -+ *JY*. If Z is the initial lift in B for *m* and X, then *Z* is the final lift for the quotient sink of morphisms *V{ : Bi* —> *Z* in B if (ii) is valid for B. But then *Z* -- *JY,* and B satisfies (iii).

(iii)$\Longrightarrow$(iv) trivially, since $JIB = B$. If (iv) is valid, then we have pullback squares

$$\begin{array}{ccccccc} IB & \xrightarrow{\mathrm{id}_{IB}} & IB & & QB & \xrightarrow{\mathrm{id}_{QB}} & QB \\ \downarrow{\vartheta_{IB}} & & \downarrow{\vartheta_{IB}} & \text{and} & \downarrow{\vartheta_{IB}} & & \downarrow{\tau_{QB}} \\ IJ\widetilde{IB} & \xrightarrow{h} & \widetilde{IB} & & |\widetilde{IB}| & \xrightarrow{\nu_{IB}h} & (QB)^{\#} \end{array},$$

in $\mathbf{A}^{\mathrm{cq}}$ and in $\mathbf{S}$. But then $\nu_{IB}h = \nu_{IB}$, and $h = \mathrm{id}_{|X|}$ follows for $X = \widetilde{IB}$. Thus $\mathrm{id}_{|X|} : IJX \to X$ for this $X$. As $\mathrm{id}_{|X|} : X \to IJX$ in any case, we get $IJ\widetilde{IB} = \widetilde{IB}$. Since $I$ preserves embeddings and pullbacks by embeddings, it follows that (i) is valid, with partial morphisms in $\mathbf{B}$ represented by $\vartheta_{IB} : B \to J\widetilde{IB}$.

Finally, (i)$\Longrightarrow$(v) trivially. Conversely, if partial morphisms in $\mathbf{B}$ and in $\mathbf{A}^{\mathrm{cq}}$ are represented by $\tau_Y : Y \to Y^*$ and by $\vartheta_Y : Y \to \tilde{Y}$, then there are commutative squares

$$\begin{array}{ccc} IY & \xrightarrow{\mathrm{id}_Y} & IY \\ \downarrow{\tau_Y} & & \downarrow{\vartheta_{IY}} \\ IY^* & \xrightarrow{\sigma_Y} & \widetilde{IY} \end{array}$$

in $\mathbf{A}^{\mathrm{cq}}$. These squares are pullback squares; thus if a partial morphism $(m, f) : X \to Y$ is represented by $\bar{f} : X \to Y^*$ in $\mathbf{B}$, then $(m, f) : IX \to IY$ is represented by $\sigma_Y \bar{f}$ in $\mathbf{A}^{\mathrm{cq}}$. Now if $\sigma_Y \alpha = \sigma_Y \beta$ in $\mathbf{A}^{\mathrm{cq}}$ for morphisms $X \to IY$, then this remains true for $\alpha, \beta : IJX \to IY$. It follows that $\alpha$ and $\beta$ represent the same partial morphism $(m, f) : JX \to Y$; thus $\sigma_Y$ is monomorphic in $\mathbf{A}^{\mathrm{cq}}$. If $\sigma_Y$ is monomorphic in $\mathbf{S}$, then it follows as in the proof of 4.4 that $\sigma_Y$ is an isomorphism.

**5.6.** For a source of morphisms $u_i : X \to X_i$ in $\mathbf{A}^{\mathrm{cq}}$, we have pullback squares

(1)
$$\begin{array}{ccccc} X & \xrightarrow{\vartheta_X} & \tilde{X} & \xrightarrow{\nu_X} & X^{\#} \\ \downarrow{u_i} & & \downarrow{\bar{u}_i} & & \downarrow{u_i^{\#}} \\ X_i & \xrightarrow{\vartheta_{X_i}} & \widetilde{X_i} & \xrightarrow{\nu_{X_i}} & (X_i)^{\#} \end{array}$$

in $\mathbf{A}^{\mathrm{cr}}$ if partial morphisms in $\mathbf{A}^{\mathrm{cq}}$ are represented.

**Lemma.** *If $X$ has the initial structure for the morphisms $u_i$ in pullback squares* (1), *then $\tilde{X}$ has the initial structure for the morphisms $\bar{u}_i$.*

PROOF. Consider $f : |Y| \to |\tilde{X}|$, with $\bar{u}_i f : Y \to \widetilde{X_i}$ for every $u_i$. We have pullback squares

$$\begin{array}{ccccc} \cdot & \xrightarrow{g} & |X| & \xrightarrow{\mathrm{id}_{|X|}} & |X| \\ \downarrow{m} & & \downarrow{\vartheta_X} & & \downarrow{\tau_X} \\ |Y| & \xrightarrow{f} & |\tilde{X}| & \xrightarrow{\nu_X} & |X|^{\#} \end{array}$$

in S, and it follows with (1) that we have pullbacks

$$\begin{array}{ccc} Z & \xrightarrow{\ g u_i\ } & X_i \\ \big\downarrow m & & \big\downarrow d_{X_i} \\ Y & \xrightarrow{\ f \tilde u_i\ } & \tilde X_i \end{array}$$

in $A^{cq}$ for every $u_i$, with $m : Z \to Y$ an embedding. But then $g : Z \to X$ in $A^{cq}$, and it follows that $\tilde f : Y \to \tilde X$ in $A^{cq}$, representing the partial morphism $Y \leftarrow Z \to X$.

5.7. Lemma ([1]). *If* $A^{cq}$ *is closed under embeddings in* $A^{cr}$, and *partial morphisms in* S *and in* $A^{cq}$ are *represented, then for every object* $Y$ *of* $A^{cq}$*, the object* $\tilde Y$ *of* $A^{cq}$ has *the initial structure for a morphism* $z_Y : \tilde{\tilde Y} \to \tilde Y$ *of* $A^{cq}$*, with* $z_Y d_{\tilde Y} = \mathrm{id}_{\tilde Y}$.

PROOF. The morphism $z_Y$ is constructed by the pullback square at left in (1).

(1)
$$\begin{array}{ccc} Y & \xrightarrow{\ \mathrm{id}_Y\ } & Y \\ \big\downarrow d_{\tilde Y} d_Y & & \big\downarrow d_Y \\ \tilde{\tilde Y} & \xrightarrow{\ z_Y\ } & \tilde Y \end{array} \qquad \begin{array}{ccccc} Y & \xrightarrow{\ \mathrm{id}_Y\ } & Y & \xrightarrow{\ \mathrm{id}_Y\ } & Y \\ \big\downarrow d_Y & & \big\downarrow & & \big\downarrow d_{\tilde Y} d_Y \quad \big\downarrow d_Y \\ \tilde Y & \xrightarrow{\ d_{\tilde Y}\ } & \tilde{\tilde Y} & \xrightarrow{\ z_Y\ } & \tilde Y \end{array}$$

The pullbacks at right show that then $z_Y d_{\tilde Y} = \mathrm{id}_{\tilde Y}$.

Now let $Z$ be the initial lift of $|\tilde{\tilde Y}|$ for $z_Y$ with codomain $\tilde Y$. Then $d_{\tilde Y} : \tilde Y \to Z$ is a strong monomorphism in $A^{cq}$ since $z_Y d_{\tilde Y} = \mathrm{id}_{\tilde Y}$. The partial morphism $(d_{\tilde Y}, \mathrm{id}_{\tilde Y})$ is represented in S by $z_Y : |Z| \to |\tilde Y|^\sim$, and thus $(d_{\tilde Y}, \mathrm{id}_{\tilde Y}) : Z \to \tilde Y$ is represented in $A^{cq}$ by $\mathrm{id}_{\tilde{\tilde Y}} : Z \to \tilde{\tilde Y}$. It follows that $Z = \tilde{\tilde Y}$, as claimed.

5.8. Theorem. *If partial morphisms in* S are *represented, and* $X$ *is quotient-dense for every embedding* $m : X \to YA$ in $A^{cr}$, then the *quotient-topological hull of all objects* $\widetilde{YA}$ *in* $A^{cq}$ — *which includes all objects* $YA$ — has *partial morphisms represented. If the forgetful functor from* $A^{cq}$ *to* S preserves *monomorphisms, then this quotient-topological hull is the quotient-topological hull of* A *for completions with partial morphisms represented.*

PROOF. Let C be this quotient-topological hull. If X has the initial structure in $A^{cq}$ for morphisms $u_i : X \to \widetilde{YA_i}$, then by 5.6 and 5.7, the object $\tilde X$ has the initial structure for the morphisms $\tilde u_i$, with $Y_i = \widetilde{YA_i}$ and with $\tilde u_i$ constructed in 5.6. Thus C has partial morphisms represented, with representations inherited from $A^{cq}$. If the forgetful functor from $A^{cq}$ to S preserves monomorphisms, then it follows from 5.5.(v) that C is the quotient-topological hull of A for completions with partial morphisms represented.

## 6. Completions and hulls over a quasitopos

6.1. Results for quasitopos completions and quasitopos hulls are obtained by combining results of Sections 4 and 5. We begin with the following result.

**Theorem.** *If* S *is a quasitopos, finite products of P-sieves* $YA$ *are quotient dense, and the domain of every embedding* $m : X \to YA$ *in* $A^{cr}$ *is quotient dense, then* $A^{cq}$ *is a quasitopos, and the embedding* $A^{cq} \to A^{cr}$ *is the inverse image part of a geometric morphism. In this situation, the forgetful functor* $P^{cq} : A^{cq} \to S$ *preserves monomorphisms, strong monomorphisms, and finite limits.*

PROOF. Under the assumptions of the Theorem, $A^{cq}$ is cartesian closed, and closed under finite products in $A^{cr}$, by 4.2, and $A^{cq}$ has partial morphisms represented and is closed under embeddings in $A^{cr}$ by 5.3. Since finite limits in $A^{cr}$ are strong subobjects of finite products, it follows that $A^{cq}$ is closed under finite limits in $A^{cr}$, and the embedding $A^{cq} \to A^{cr}$ preserves finite limits. It follows that the forgetful functor $A^{cq} \to S$ also preserves finite limits, and hence monomorphisms. Since $A^{cq}$ has embeddings, this forgetful functor also preserves strong monomorphisms.

**6.2. Theorem.** *Under the assumptions of Theorem* 6.1, the *following conditions are equivalent for a quotient-topological completion* (B,Q) of (A,P).

(i) B *is a quasitopos.*

(ii) *Every pullback functor* $f^*$ *in* B *preserves final lifts of quotient sinks.*

(iii) *The concrete left adjoint* $J : A^{cq} \to B$ *of the full embedding* $I : B \to A^{cq}$ *preserves embeddings and finite products.*

*If these conditions are satisfied, then the adjunction* $J \dashv I$ *is a geometric morphism, and the functor I preserves function space objects and representations of partial morphisms, up to isomorphisms.*

PROOF. With the observation that the forgetful functor of $A^{cq}$ preserves monomorphisms, this follows immediately from 4.4 and 5.5, using for (ii) the fact that every morphism of $A^{cq}$ is the composition of an embedding and a projection of a product.

**6.3. Theorem.** *Under the assumptions of Theorem* 6.1, the *quotient-topological hull of the objects* $[YA, \widetilde{YB}]$ *in* $A^{cq}$, *for objects* $A$ *and* $B$ *of A, is the quotient-topological quasitopos hull of* ***A***.

PROOF. Let C be the quotient-topological hull described. By 6.2, a quotient-topological quasitopos completion of A has the same function space objects and the same representations of partial morphisms as $A^{cq}$; thus C is contained in the quotient-topological quasitopos hull of A.

Since t9j3 : $B \to \tilde{B}$ is always an embedding, C contains all function space objects $[YA, YB]$, and hence all objects $YA$ of $A^{cq}$. As in the proof of 4.7, it follows from 4.5 and 4.6 that C contains all objects $[X,YB]$ and $[X,\widetilde{YB}]$, and is cartesian closed.

Now put $T = [X, \tilde{Y}]$ for objects $X, Y$ of $\mathbf{A}^{\mathrm{cq}}$. By [9], we have a commutative diagram

$$(1)\qquad \begin{array}{ccccc} T \times X & \xrightarrow{\vartheta_T \times \mathrm{id}_X} & \tilde{T} \times X & \xrightarrow{k \times \mathrm{id}_X} & T \times X \\ \downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \mathrm{ev}} \\ \tilde{Y} & \xrightarrow{\vartheta_{\tilde{Y}}} & \tilde{\tilde{Y}} & \xrightarrow{z_Y} & \tilde{Y} \end{array} ,$$

with $h$ representing the partial morphism $(\vartheta_T \times \mathrm{id}_Y, \mathrm{ev})$, with $z_Y$ given by 5.7, and with $k$ exponentially adjoint to $z_Y h$. It follows from this and $z_Y \vartheta_{\tilde{Y}} = \mathrm{id}_{\tilde{Y}}$ that $k\vartheta_T = \mathrm{id}_T$. Now $\tilde{T}$ has the initial structure for $k$, as in the proof of 5.7, and it follows as in the proof of 5.8 that partial morphisms in $\mathbf{C}$ are represented. Thus $\mathbf{C}$ is the quotient-topological quasitopos hull of $\mathbf{A}$, as claimed.

**6.4. Remark.** It may be noted that diagram 6.3.(1) can be constructed for objects $X$ and $Y$ of an arbitrary quasitopos, with the additional property that if $h$ factors $h = h'e$, with $e$ monomorphic and epimorphic, then $e$ is an isomorphism. This can be used for proving that $\tilde{T}$ has the initial structure in $\mathbf{A}^{\mathrm{cq}}$ for $h$, and similarly that $\tilde{\tilde{Y}}$ in 5.7 has the initial structure for $z_Y$.

**6.5. Lifting finite sources.** It is well known, and easily seen, that the forgetful functor $\mathbf{A}^{\mathrm{cr}} \to \mathbf{S}$ preserves and creates limits. Limits in $\mathbf{A}^{\mathrm{cq}}$ are coreflections of limits in $\mathbf{A}^{\mathrm{cr}}$; it follows that a limit in $\mathbf{A}^{\mathrm{cq}}$ is preserved by the forgetful functor to $\mathbf{S}$ iff it is preserved by the embedding $\mathbf{A}^{\mathrm{cq}} \to \mathbf{A}^{\mathrm{cr}}$, and hence iff it is concrete.

**Theorem.** *If* $\mathbf{S}$ *has finite limits, then the following statements are logically equivalent for the forgetful functor* $P^{\mathrm{cq}} : \mathbf{A}^{\mathrm{cq}} \to \mathbf{S}$.

(i) *Every finite* $P^{\mathrm{cq}}$*-source admits a lift in* $\mathbf{A}^{\mathrm{cq}}$.

(ii) *Every finite* $P^{\mathrm{cq}}$*-source admits an initial lift in* $\mathbf{A}^{\mathrm{cq}}$.

(iii) $\mathbf{A}^{\mathrm{cq}}$ *has concrete finite products and embeddings, and all coarse* $P$*-sieves are quotient-dense.*

(iv) $\mathbf{A}^{\mathrm{cq}}$ *has concrete finite limits, and all coarse* $P$*-sieves are quotient-dense.*

(v) $\mathbf{A}^{\mathrm{cq}}$ *is closed under finite limits in* $\mathbf{A}^{\mathrm{cr}}$, *and all coarse* $P$*-sieves are quotient-dense.*

Proof. (ii)$\Longrightarrow$(i) trivially, and if (i) is valid, then for finite $P^{\mathrm{cq}}$-sources, initial lifts in $\mathbf{A}^{\mathrm{cr}}$ are quotient dense, and (i) holds. (ii)$\Longrightarrow$(iii) since concrete products, embeddings and coarse sieves in $\mathbf{A}^{\mathrm{cq}}$ are initial lifts of finite sources. Clearly (iv) $\Longleftrightarrow$ (v), and (iii)$\Longrightarrow$(iv) and (v) since concrete finite limits in $\mathbf{A}^{\mathrm{cr}}$ are objects embedded into finite products.

For a $P^{\mathrm{cq}}$-source of morphisms $f_i : S \to |X_i|$, an initial lift in $\mathbf{A}^{\mathrm{cr}}$ is a limit of a diagram of morphism $f_i : \Omega_S \to \Omega_{|X_i|}$ and of morphisms $\mathrm{id}_{|X_i|} : X_i \to \Omega_{|X_i|}$. With this observation, (ii) follows immediately from (iv) and (v).

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