MORSE INDEX CONCENTRATION FOR ELLIPTIC PROBLEMS WITH SOBOLEV EXPONENT

by

Gabriella Tarantello

Carnegie Mellon University
Department of Mathematics
Pittsburgh, PA 15213 U.S.A.

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2N .
where p = TTZV *S the ^)est exponent in the Sobolev embedding.
Our approximating sequence will satisfy,

$$-\Delta u_n = |u_n|^{p-2}u_n + \lambda u_n + f_n \text{ in } H^{-1} \qquad (1)_{\lambda,n}$$

with f_n —• 0 strongly in H"1, (H"1 denotes the dual of HJ (fi)).

By the blow up techniques of Sacks—Uhlenbeck (see [S—U]), one can describe precisely the concentration behavior of the given sequence via the solutions of the following problem:

(*) - A U = |u|P~2u inR^N
u e D 1 ' 2 (R^N)

with D1j2(R^N) = {u € L^p (R^N) : |Vu| € L^2 (R^N)|. See [St.].
A first difficulty is that nothing is known about the changing sign solutions of (*) except their existence (see [D]) (even their decay properties at infinity is not understood). The only known facts concern positive solutions of (*) which are completely described as the extremal function of the Sobolev embedding. To state our result we need to introduce some notations.

For u € L^p(R^n) u # 0, define:
v_∞(u) = number of eigenvalues p € (0, 1], counted with their multiplicity, of the eigenvalue problem:

- A V = (i M*~2v inR^N
vED 1 ' 2 (R^N)

Since M1*""2 € L^N/2(R^N) it is well known that *vjn)* is finite (see [Si]). In fact the beautiful semiclassical inequality of Cwickel [C], Lieb [L] and Rosenbljum [Ro] gives an universal constant a_N (depending only on N) such that,

$$\nu_\infty(u) \le a_N \|u\|_p^p.$$

Notice that if $u \neq 0$ satisfies $(*)$ then $\nu_\infty(u) \ge 1$. Given $m \ge 1$, define the values,

$$b_{N,m} = \inf \left\{ \frac{\| u \|_p^p}{\min \{\nu_\infty(u),\ m\}},\ u \neq 0 \ \text{solution of } (*) \right\}.$$

Clearly $b_{N,1} \ge b_{N,2} \ge b_{N,3} \ge \dots\dots \ge \frac{1}{a_N}$.

Furthermore, it is not difficult to see that $b_{1,N} = b_{2,N} = S^{N/2}$ where S is the best constant in the Sobolev embedding (see [A]). It is an interesting open problem to determine whether or not $b_{N,m} = S^{N/2}$ for $m \ge 3$.

Notice that, this would be the case if,

$$a_N^{-1} = S^{N/2} \tag{0.1}$$

On the other hand, we are certain that (0.1) is false for $N \ge 7$ where we have:

$$a_N^{-1} \le c_N^{-1} < S^{N/2}$$

with $c_N = (2\pi)^{-N} w_N$ the "classical" value (w_N = volume of the unit ball in $\mathbb{R}^N$). See [Si] and [G–G–M] where the stronger inequality $a_N^{-1} < c_N^{-1}$ is established. However a long standing conjecture of Lieb–Thirring [L–T] asserts that (0.1) should hold for $N = 3,\dots,6$. (This conjecture has been proved in [G–G–M] for radial potentials and $N = 4$). The knowledge of the precise value of a_N is important since it would led to an improved constant in the proof of "stability" of matter by Lieb–Thirring. Estimates for the a_N's have been obtained by several authors. See for instance [L], [L,1], [A–L], [G–G–M], [L–Y] and [M]; they also include estimates for the higher order moments of eigenvalues of the Schrodinger operator.

As we have seen our interest on the Lieb–Thirring conjecture has different origin. In fact, a consequence of our result will show how to use the L.–T. conjecture to obtain compactness (see Corollary 3). It would be interesting to see if this could contribute in any way to a better understanding of the conjecture itself. For instance, we point out, that

(although disappointing) our result could be used to disprove it.

Let $A_x < A_2 \leq A_3 \leq ... \leq A_n < ...$ be the sequence of eigenvalues of -A in $H^1_Q(n)$. Given u € L^p(fi), u * 0 and s > 0 define:

i/(u, s) = number of eigenvalues *fi* € (0, s], counted with multiplicity, for the (weighted) eigenvalue problem:

$$-(\Delta v + \lambda v) = \mu |u|^{p-2} v \quad \text{in } H^1_0(\Omega).$$

Set *{u, s) = 0 if no such eigenvalue exist.

To clarify the role of i/(u, s) noticed that if $0 \leq A < Aj$ and u is a nontrivial solution of (1)^, then *v(u,* p - 1): = ni*(u) gives exactly the augmented Morse index of u.

We have:

Theorem 1: Let $\{u_n\}$ c $H^1_Q(Q)$ be a sequence satisfying $(1)^\wedge{}_{,n}$ with A # A^ ; k = 1, 2, ... If,

(i) there exist m 6 IN and $\{s_n\}$ C (0, + OD):

$$\wedge^{u}{}_{n,}{}^{S} J \geq m \text{ and } s_n \sim *1;$$

(ii) u_n - ^ u_0 weakly in $H^1_0(\Omega)$;

then

$$\lim_{n \to +\infty} \| \nabla (u_n - u_0) \|_2^2 \geq (m - {}^f i/J\, b_N \ldots \qquad (0.2)$$

where u_Q = j<o0f 1).

D

Notice that in (0.2) we have posed $b_{N,-k} = 0$ for $k \geq 0$.

From Theorem 1 it is easy to derive a compactness result. To this purpose let *T* ($\{u_n\}$) the set of weak limit point of $\{u_n\}$. That is, u € *T* ($\{u_n\}$) if and only if there exists a

subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \rightharpoonup u$ weakly in $H_0^1(\Omega)$.

We have:

Corollary 1: Let $\{u_n\} \subset H_0^1(\Omega)$ satisfy $(1)_{\lambda,n}$ $\lambda \neq \lambda_k$, (i) and

$$\text{(ii)}^* \quad \lim_{n \to +\infty} \| \nabla u_n \|_2^2 = c.$$

If, for $u \in \Gamma(\{u_n\}) \cap \left\{\| \nabla u \|_2^2 < c\right\}$ we have:

$$c < \| \nabla u \|_2^2 + (m - \nu(u))\, b_{N,m-\nu(u)} \tag{0.3}$$

$(\nu(u) = \nu(u, 1))$

then $\{u_n\}$ admits on strongly convergent subsequence.

□

Remark: From Corollary 1 it is easy to obtain a well known compactness result introduced by Th.Aubin [A] for the Yamabe problem and used by Brezis–Nirenberg [B–N] in our contest (cf. [St]). Namely,

let $\{u_n\}$ satisfy $(1)_{\lambda,n}$ and (ii)*; if $c < S^{N/2}$ then $\{u_n\}$ admits a convergent subsequence.

Indeed, for $c > 0$ (for $c = 0$ there is nothing to prove) from $(1)_{\lambda,n}$ one easly derives:

$$\Gamma(\{u_n\}) \cap \left\{\| \nabla u \|_2^2 < c\right\} \subset \left\{u = 0\right\}.$$

In addition, $(1)_{\lambda,n}$ also allows to take $m \geq 1$ in (i). Thus, condition (0.3) is obviously satisfied.

Since every $u \in \Gamma(\{u_n\})$ satisfies $(1)_\lambda$, we also have:

Corollary 2: Let $\{u_n\} \subset H_0^1(\Omega)$ satisfy $(1)_{\lambda,n}$, (i) and (ii)*.

If every solution u of $(1)_\lambda$ with $\|\nabla u\|_2^2 < c$ satisfy:

(1) $c < \|\nabla u \|_2^2 + (m - \nu(u))\, b_{N,m - \nu(u)}$

Then u_n admits a convergent subsequence.

□

It is not difficult to exhibit examples showing the optimality of conditions (1) for $m = 1, 2$. However, for $m \geq 3$ the values $b_{N,m}$ are unknown and condition (1) becomes difficult to check in concrete cases. But, if we resort to the Lieb-Thirring conjecture we can obtain a more useful version of Corollary 1.

For $A \neq A_k$ $k = 1, 2, \ldots,$ we have:

Corollary 3: Let $N = 3, \ldots, 6$ and assume that the Lieb-Thirring conjecture holds. If the sequence $\{u_n\}$ satisfies (l)x/»,n' (i) and (ii)* and for every « € r $(\{u_n\})$ n $\{\| \nabla u \|_2^2 < c\}$ we have,

$$(I)' \quad \|\nabla(u_n - u)\|_2^2 < (m - Ku)) S^{N/2}$$

then $\{u_n\}$ admits a convergent subsequence.

□

Remark (0.1): As already pointed out Corollary 3 always holds for $m = 1, 2$. On the other hand, for $m \geq 3$ a counterexample to the statement of Corollary 3 would disprove the Lieb—Thirring conjecture.

Let also observe that situations where the hypothesis of Corollary 1 are verified, naturally occur when studying multiplicity question for $(1)_\lambda$, $\lambda > 0$.

We shall present an application in the last section.

We conclude with another compactness result. It relies on the comparison between the Morse index of the "approximating" sequence and that of the solutions of (*).

To this purpose, given a nontrivial solution u of (*) define:

$m_\infty(u)$ = number of eigenvalue $\mu \in (0, p-1]$, counted with multiplicity, of $(*)_\mu$.

We have,

Theorem 2: Let $\{u_n\} \subset H_0^1(\Omega)$ satisfy $(1)_{\lambda,n}(\lambda \neq \lambda_k)$ and $(ii)^*$. Assume that there exists $m \in \mathbb{N}$ and $\{s_n\} \subset \mathbb{R} : \nu(u_n, s_n) \geq m$ and $s_n \longrightarrow (p-1)$. If for every solution $u_i \neq 0$ of $(*)$ $i = 1,..,k$, and every solution u_0 of $(1)_\lambda$ with,

$$\sum_{i=1}^{k} \|\nabla u_i\|_2^2 + \|\nabla u_0\|_2^2 = c$$

we have,

$$\sum_{i=1}^{k} m_\infty(u_i) + m^*(u_0) < m \tag{0.3}$$

then $\{u_n\}$ admits a convergent subsequence.

□

Remark (0.2): This result could be used to find nontrivial solutions of $(1)_\lambda$ with $\lambda = 0$. As well known problem $(1)_{\lambda=0}$ is very delicate and existence or nonexistence situations can occur according to the topological property of Ω (see [P] and [Ba–C]). By theorem 2, nontrivial solutions of $(1)_{\lambda=0}$ would exist as long as one could construct an "approximating" sequence with "large" Morse index but "small" energy.

So, it is an interesting problem to see for which domain Ω such construction is possible.

The Proof of Theorem 1 and 2:

The proof of theorem 1 and 2 rely on a blow up argument. We have collected useful inequalities as well as regularity properties for solutions of $(*)$ and $(*)_\mu$ in Appendix I and II.

The Proof of Theorem 1:

Write $u_n = u_0 + w_n$ with $w_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$.

Using $(1)_{\lambda,n}$ and the fact that u_0 satisfy $(1)_\lambda$, one easily derives the following for w_n:

$$-\Delta w_n = |w_n|^{p-2} w_n + g_n \quad \text{in } H^{-1} \tag{1.1}$$

where

$$g_n = \left[|u_0 + w_n|^{p-2} (u_0 + w_n) - |u_0|^{p-2} u_0 - |w_n|^{p-2} w_n \right] + \lambda w_n + f_n.$$

Hence, using the Calculus Lemma of Appendix I, the fact that $w_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ and $u_0 \in L^q(\Omega)$ for every q, we derive:

$$g_n \longmapsto 0 \text{ strongly in } H^{-1}.$$

Furthermore from (i) we obtain $v_{l,n} \in H_0^1(\Omega)$ satisfying:

$$\begin{aligned} & -(\Delta v_{l,n} + \lambda v_{l,n}) = \mu_{l,n} |u_n|^{p-2} v_{l,n} \quad \text{in } H_0^1(\Omega) \\ & \int_\Omega |u_n|^{p-2} v_{l,n} v_{p,n} = 0 \text{ if } l \neq p \\ & \int_\Omega |\nabla v_{l,n}|^2 = 1 \end{aligned} \tag{1.2}$$

with $\mu_{l,n} \in (0, s_n]$ and $l = 1,\ldots,m$.

Hence, taking a subsequence if necessary, we find $v_l \in H_0^1(\Omega)$ and $\mu_l \in [0, 1]$ such that

$$v_{l,n} \rightharpoonup v_l \text{ weakly in } H_0^1(\Omega)$$

and

$$- (A\ vj + A\ vj) = {}_h\ I\ u_0 1\ *\sim\backslash \quad \text{in } Hj(fi) \tag{1.3}$$

Notice that in particular, Vj € C"(ft) n C(I7), (6ee [B-K]).

Write,

l,n 1 i,n

as above we derive that $w_{1,n}$ satisfies the following,

$$-\Delta w_{1,n} = \mu_{1,n} |w_n|^{p-2} w_{1,n} + g_{1,n} \text{ in } H^{-1} \tag{1.4}$$

with $gj_{,n}$ i—> 0 strongly in H"~^{1}

From now on it is understood that we are taking subsequences when needed. Combining results of Struwe [St.] and Brezis—Coron [Br—C], from (l)x _ we obtain an integer r, sequences $\{R_{j,n}\}$ C R^+ and $\{x_{j,n}\}$ c fi, solutions u_j of (*), j = 1, ..., r satisfying:

(a) $\| \wedge (w_n - \sum_{j=1}^{i} u_{j,n}) \|_2 \longrightarrow 0$ as $n \longrightarrow +\infty$

where,

$$u_{j,n} = R_{j,n}^{\frac{N-2}{2}} u_j (R_{j,n} (x - x_{j,n}));$$

(b) $\| \nabla w_n \|_2^2 \longrightarrow \sum^{r} \| \nabla u_j \|_2^2$ and $R_{j,n}$ dist $(x_{j,n}, \partial\Omega) \longrightarrow +\infty$ as $n \longrightarrow +\infty$;

(c) H j * k the- $\max \left\{ \frac{R_{j,n}}{R_{k,n}}, \frac{R_{k,n}}{R_{j,n}}, (R_{j,n} + R_{k,n}) |x_{j,n} - x_{k,n}| \right\} \longrightarrow +\infty$

as $n \longrightarrow +\infty$

Let us mention that (c) is a consequence of the regularity properties of the solutions of (*) as derived in Appendix II, and the obvious modification of an argument of Brezis—Coron ([Br-C, theorem 2.]).

In $\Omega_{j,n} = \left\{ x \in R^n : x_{j,n} + \frac{x}{R_{j,n}} \in \Omega \right\}$ define:

$$w^*_{j,n} = R_{j,n}^{\frac{2-N}{2}} w_n\left(x_j + \frac{x}{R_{j,n}}\right);$$

$$w^*_{j,l,n} = R_{j,l,n}^{\frac{2-N}{2}} w_n\left(x_j + \frac{x}{R_{j,n}}\right)$$

$j = 1,\dots,r$ and $l = 1,\dots,m$.

We consider $w^*_{j,n}$ and $w^*_{j,l,n}$ defined in the whole R^n by setting them equal zero outside $\Omega_{j,n}$.

Since $\|\nabla w^*_{j,n}\|_2 = \|\nabla w_n\|_2$ is uniformly bounded, we can find a subsequence, which we still call $w^*_{j,n}$, and $w^*_j \in D^{1,2}(IR^n)$:

$$w^*_{j,n} \rightharpoonup w^*_j \text{ weakly in } D^{1,2}(R^N).$$

Similarly, for (a subsequence of) $w^*_{j,l,n}$ we find $w^*_{j,l} \in D^{1,2}(R^N)$ such that,

$$w^*_{j,l,n} \rightharpoonup w^*_{j,l}, \text{ weakly in } D^{1,2}(R^N).$$

Moreover from (1.1) and (1.4) one easily derives:

$$-\Delta w^*_{j,n} = |w^*_{j,n}|^{p-2} w^*_{j,n} + g^*_{j,n} \text{ in } \Omega_{j,n} \qquad (1.1)^*$$

and $g^*_{j,n} \longrightarrow 0$ strongly in $(D^{1,2})'$;

$$-\Delta w^*_{j,l,n} = \mu_{l,n} \mid w^*_{j,n} \mid^{p-2} w^*_{j,l,n} + g^*_{j,l,n} \text{ in } \Omega_{j,n} \qquad (1.4)^*$$

and $g^*_{j,l,n} \longrightarrow 0$ strongly in $(D^{1,2}(\mathbb{R}^N))^*$.

Now set,

$$u^*_{j,k,n}(x) = \left[R_{j,n}\right]^{\frac{2-N}{2}} u_{k,n}\left[x_{j,n} + \frac{x}{R_{j,n}}\right] = \left[\frac{R_{k,n}}{R_{j,n}}\right]^{\frac{N-2}{2}} u_k\left[R_{k,n}(x_{j,n} - x_{k,n}) + \frac{R_{k,n}}{R_{j,n}} x\right].$$

Using the results of Appendix II and property (c) we shall derive the following:

$$\text{if } j \neq k \Longrightarrow u^*_{j,k,n} \rightharpoonup 0 \text{ weakly in } D^{1,2}(\mathbb{R}^N). \qquad (1.5)$$

On the other hand it is easy to check that for $j = k$ we have, $u^*_{j,k,n} = u_k$ for every n.

To establish (1.5) notice that $\mid \nabla u_k \mid \in L^\infty(\mathbb{R}^N)$ (see Appendix II).

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, if $\frac{R_{k,n}}{R_{j,n}} \longrightarrow 0$ as $n \longrightarrow +\infty$, then

$$\left| \int \nabla u^*_{j,k,n} \cdot \nabla \varphi \right| \leq \left[\frac{R_{k,n}}{R_{j,n}}\right]^{\frac{N}{2}} \int \left| \nabla u_k \left[R_{k,n}(x_{j,n} - x_{k,n}) + \frac{R_{k,n}}{R_{j,n}} x\right] \right| \, \left| \nabla \varphi \right|$$

$$\leq \left[\frac{R_{k,n}}{R_{j,n}}\right]^{N/2} \| \nabla u_k \|_\infty \int \mid \nabla \varphi \mid \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty.$$

On the other hand, if $\frac{R_{k,n}}{R_{j,n}} \longrightarrow +\infty$ as $n \longrightarrow +\infty$ and $R_{j,n}(x_{j,n} - x_{k,n}) \longrightarrow x_{j,k}$

(take a subsequence if necessary), then for arbitrary e > 0, let *6* > 0 small enough to guarantee:

$$\int_{\{|x - x_{j,k}| < 2\delta\}} |\nabla \varphi|^2 \leq \frac{\epsilon^2}{\| H_{k} \|_2^2}$$

Thus, for n large we have:

I J ' »j*,k,n - ' *> I ≤ J I V u*j,k,n | | ∇ φ | + $\int_{\{|x - x_{j,k}| \geq 2\delta\}} |\nabla u^*_{j,k,n}| \, |\nabla \varphi|$

$\{| x - x_{j,k} | < 2\delta\}$

≤ ‖ '«k ‖ (j I v *> I²)^1/2 + (j I v \ I²) ‖ v v ‖₂ ≤ *i* + 0(1).

$\{|x - x_{j,k}| < 2\delta\}$ $\{|x - x_{j,k}| \geq \delta\}$

Finally, if | R.j,n (x.j,n — x^ ,n) |———> + ∞ then we can find suitable constants R_n (depending on supp <*p*): R_n———> + ∞ as n——> + ∞ and

$$\left| \int_{\{|y| \geq R_n\}} \nabla u^*_{j,k,n} \nabla \varphi \right| \leq \left(\int_{\{|y| \geq R_n\}} |\nabla u_k|^2 \right)^{1/2} \|\nabla \varphi\|_2 \longrightarrow 0 \text{ as } n \longrightarrow +\infty.$$

Since,

$\| \nabla (w^*_{j,n} - u_j - \sum_{j \neq k} u^*_{j,k,n}) \|_2 = \| \nabla$ (wn " i V } *h*—> ○ as n ——> + ∞

from (1.5) we conclude,

$$w^*_j = u_j \qquad j = 1, \ldots, r. \tag{1.6}$$

This together with $(1.4)^*$ yields:

$$\int \nabla w^*_{j,l} \cdot \nabla \varphi = \mu_l \int |u_j|^{p-2} w^*_{j,l}\varphi \qquad \forall \varphi \in D^{1,2}(\mathbb{R}^N).$$

That is $w^*_{j,l}$ is a (classical) solution for,

$$\begin{cases} -\Delta w^*_{j,l} = \mu_l \, |u_j|^{p-2} w^*_{j,l} & \text{in } \mathbb{R}^N \\ w^*_{j,l} \in D^{1,2}(\mathbb{R}^N) \end{cases} \tag{1.7}$$

with $0 \le \mu_l \le 1$, $l = 1,\ldots,m$ and $j = 1,\ldots,r$

(See Appendix II for the regularity properties of $w^*_{j,l}$).

Now, notice that $\mu_l > 0$ for every $l = 1,\ldots,m$. Indeed if $\mu_l = 0$,

then from (1.3) it follows that necessarily $v_l = 0$ (only here we use the fact that $\lambda \neq \lambda_k$ $\forall k$).

Furthermore,

$$1 = \| \nabla v_{l,n} \|_2^2 = \| \nabla w_{l,n} \|_2^2 = \mu_{l,n} \int |w_n|^{p-2} w_{l,n}^2 + o(1)$$

with $\int |w_n|^{p-2} w_{l,n}^2$ uniformly bounded, and $\mu_{l,n} \longrightarrow 0$. This is clearly impossible.

Thus, we can conclude $\mu_l \in (0, 1]$ $\forall l = 1,\ldots,m$.

Assume $\nu_0 = k_0 < m$, otherwise (0.2) is trivially satisfied. This implies that $v_l = 0$ for $l = k + 1,\ldots,m$.

So, for $l > k_0$ we get,

$$1 = \| \nabla w_{l,n} \|_2^2 = \mu_{l,n} \int_\Omega |w_n|^{p-2} w_{l,n}^2 + 0(1) = \mu_{l,n} \int_{\Omega_{1,n}} |w^*_{1,n}|^{p-2} (w^*_{1,l,n})^2 + o(1)$$

$$= \mu_{l,n} \int |u_1 + \sum_{k \ge 2} u^*_{1,k,n}|^{p-2} (w_{1,l^*,n})^2 + o(1) = \mu_l \int |u_1|^{p-2} (w^*_{1,l})^2 +$$

$$+ \mu_{1,n}\int | \sum_{k\geq 2} u^*_{1,k,n}|^{p-2}(w^*_{1,l,n})^2 + o(1) = \|\nabla w^*_{1,l}\|_2^2 + \mu_{2,l}\int |u_2 + \sum_{k\geq 3} u^*_{2,k,n}|^{p-2}(w^*_{2,l,n})^2$$

$$+ o(1) = \dots = \sum_{j=1}^{I} \| \nabla w^*_{j,l} \|_2^2 + o(1).$$

Hence, going to the limit as $n \to +\infty$, for $l > k_0$ we find,

$$\sum_{j=1}^{I} \| \nabla w^*_{j,l} \|_2^2 = 1; \qquad (1.8)$$

so the $w^*_{j,l}$ cannot all vanish identically.

Furthermore, if $k_0 < l$, $h \leq m$ and $l \neq h$ then,

$$0 = \int |u_n|^{p-2} v_{l,n}\, v_{h,n} = \int_{\Omega} |w_n|^{p-2} w_{l,n}\, w_{h,n} + o(1) =$$

$$= \int_{\Omega_{1,n}} |u_1 + \sum_{k\geq 2} u^*_{1,k,n}|^{p-2}\, w^*_{1,l,n}\, w^*_{1,h,n} + o(1) = \int |u_1|^{p-2}\, w^*_{1,l}\, w^*_{1,h} +$$

+ J I k \2 •!*, I"" ∧,l,n-!*, $+ o(1) = \sum_{j=1}^{I} \int |u_j|^{p-2} w^*_{j,l}\, w^*_{j,h} + o(1).$

That is,

$$\sum_{j=1}^{I} \int |u_j|^{p-2}\, w^*_{j,l}\, w^*_{j,h} = 0 \qquad (1.9)$$

for $k_0 < l$, $h \leq m$ and $l \neq h$.

Notice that to derive (1.8) and (1.9) we have repeatly used the calculus lemma of Appendix I.

Thus, (1.7), (1.8) and (1.9) yield,

$$\sum_{j=1}^{r} \nu_\infty(u_j) \geq m - k_0.$$

In conclusion,

$$\lim_{n\to\infty} \| \nabla (u_n - u_o) \|_2^2 = \lim_{n\to\infty} \| \nabla w_n \|_2^2 = \sum_{j=1}^{r} \| u_j \|_p^p ,$$

and

$$\sum_{j=1}^{r} \| u_j \|_p^p \geq (m - k_0)\, b_{N,m-k_0}.$$

This concludes the proof.

Corollary 1, is now an immediate consequence of theorem 1. Indeed let $\{ u_{n_k} \}$ be a subsequence of $\{ u_n \}$ and $u \in H_0^1(\Omega)$ such that $u_{n_k} \rightharpoonup u$ weakly in $H_0^1(\Omega)$.

Since u satisfies,

$$c = \| \nabla u \|_2^2 + \lim_{k\to+\infty} \| \nabla (u_{n_k} - u) \|_2^2,$$

by theorem 1. we have that necessarily $\| \nabla u \|_2^2 \geq c$, that is $u_{n_k} \to u$ strongly.

The Proof of Theorem 2.

Take a subsequence if necessary and assume that $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$, for suitable $u_0 \in H_0^1(\Omega)$.

The blow up technique seen above gives solutions $u_1,\ldots,u_r$ for $(*)$,sequences $\{ R_{j,n} \} \subset \mathbb{R}^+$

and

$\{x_{j,n}\} \subset \Omega$ such that,

$$\| \nabla (u_n - u_0 - \sum_{j=1}^{r} u_{j,n}) \|_2 \longrightarrow 0 \text{ with } u_{j,n} = R_{j,n}^{\frac{N-2}{2}} u_j (R_{j,n}(x-x_{j,n}));$$

and

$$\| \nabla u_0 \|_2^2 + \sum_{j=1}^{r} \| \nabla u_j \| = c.$$

We are done once we show that, necessarily $u_j = 0$, $\forall\, j = 1, \ldots, r$. Arguing by contradiction, assume that $u_j \neq 0$ for $j = 1,\ldots, r_0$ and some $r_0 \in \{1,\ldots,r\}$.
From (0.3) it follows that,

$$m^*(u_0) = k_0 < m \tag{1.10}$$

In virtue of the given assumptions we can find eigenfunctions $v_{j,n} \in H_0^1(\Omega)$ satisfying:

$$-(\Delta v_{j,n} + \lambda v_{j,n}) = (p-1)\, \mu_{j,n} \, |u_n|^{p-2} v_{j,n} \text{ in } H_0^1(\Omega);$$

$$\| \nabla v_{j,n} \|_2^2 = 1 \text{ and } \int_\Omega |u_n|^{p-2} v_{j,n} v_{k,n} = 0 \text{ if } k \neq j \text{ and } n \in \mathbb{N}$$

with $0 < \mu_{j,n} \leq 1$ and $n \in \mathbb{N}$.

Let $v_j \in H_0^1(\Omega)$ and $\mu_j \in [0, 1]$ such that,

$$v_{j,n} \longrightarrow v_j \text{ weakly in } H_0^1(\Omega) \text{ and } \mu_{j,n} \longrightarrow \mu_j$$

(take a subsequence if necessary).

Thus v_j must satisfy:

$$-(\Delta v_j + \lambda v_j) = (p-1)\,\mu_j\,|u_0|^{p-2}\,v_j \quad \text{in} \quad H_0^1(\Omega)$$

As above we rule out the possibility that $\mu_j = 0$. So $\mu_j \in (0, 1]$. Thus, from (1.10) it must result: $v_j = 0$ for $k_0 < j \le m$. A blow up argument similar to the one given above, will give functions

$w^*_{j,l} \in D^{1,2}(\mathbb{R}^N)$ with $j = 1,\dots, r_0$ and $l = k_0 + 1,\dots, m$ such that,

$$-\Delta w^*_{j,l} = (p-1)\,\mu_l\,|u_j|^{p-2}\,w^*_{j,l} \quad \text{in} \quad \mathbb{R}^N$$

$$\sum_{j=1}^{r_0} |\nabla w^*_{j,l}|^2 = 1 \quad \text{and} \quad \sum_{j=1}^{r_0} \int |u_j|^{p-2}\, w^*_{j,l}\, w^*_{j,h} = 0$$

for $k_0 < l, h \le m$ and $l \ne h$.

That is,

$$\sum_{j=1}^{r_0} m_\infty(u_j) \ge m - k_0.$$

This clearly contradicts (0.2).

□

An Application:

We investigate changing sign solutions for $(1)_\lambda$ with $0 < \lambda < \lambda_1$. Our goal is to give a rather simple proof of a result already established in [C–S–S], [Z] and [T].

Let

$$I(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - \frac{\lambda}{2}u^2 - \frac{1}{p}|u|^p \qquad u \in H_0^1(\Omega)$$

be the "action" functional corresponding to $(1)_\lambda$.

That is, critical points of I give solutions of $(1)_\lambda$. Since I is even, we can use the theory of Ljusternik-Schnirelman to seek critical points for I.

To this end, define:

$$E = \left\{ A \subset H_0^1(\Omega) : A \text{ is closed and symmetric (i.e. } u \in A \Rightarrow -u \in A) \right\}.$$

For every $A \in E$, denote by $i(A) \in \mathbb{N}$ the Krosnoselski genus of A (cf [St.]) and let,

$$\mathcal{H} = \left\{ h : H_0^1(\Omega) \to H_0^1(\Omega) \text{ odd homeomorphism} \right\}.$$

Given $k \in \mathbb{N}$, set

$$\Sigma_k = \left\{ A \in E : i(A \cap h(S)) \ge k \quad \forall h \in \mathcal{H} \right\}$$

where

$$S = \{u \in H_0^1(\Omega) : \|u\| = 1\}.$$

Following Ljusternik—Schnirelman, we define,

$$c_k = \inf_{A \in \Sigma_k} \sup_A I$$

So $-\infty < c_1 \le c_2 \le c_3 \le \dots$, are the natural candidates when seeking critical values for I.
However, since the functional I lacks compactness, this will be established only for $k = 1, 2$.
The case $k = 1$ has been obtained in [B—N]. They show that there exists $u_1 > 0$ satisfying:

$$I'(u_1) = 0 \quad \text{and} \quad I(u_1) = c_1.$$

This has been achieved by providing the estimate,

$$0 < c_1 < \frac{1}{N} S^{N/2},$$

(cf. [St.]).

Notice that Cj is the smallest positive critical value for I.

That is,

c: = min | c > 0 : 3 u 6 Hj(fi) with I(u) = c and F(u) = o|.

Also notice that if u is a critical point for I and c, ≤ I(u) < c« then u cannot change sign in n.

Here, we use Theorem 1. to handle the case k = 2.

Let

A = ju € Hj(n) u *t* 0 and (I'(u), u) = o}

((•,•) is the standard scalar product on Hj(ft)). Given u e LP(fl) u *i* 0, denote by (/i,(u), v,(u)) the first eigenpair for the eigenvalue problem:

$$-(\Delta v + \lambda v) = \mu\, |u|^{p-2} v \quad \text{in } H_0^1(\Omega) \qquad (\star)$$

Under the normalization: v,(u) > 0 on ft and || V Vj(u) {(= 1, the map:

$$L^p(\Omega) \longrightarrow H_0^1(\Omega)$$
$$u \longrightarrow v_1(u)$$

is continuous and even.

This implies that if we set,

$$F = (u \in A: \int_\Omega |u|P-^2 u_{V1}(u) = 0\}$$

then,

$$A \cap F \, O \qquad \forall A \in E_2. \tag{2.1}$$

Furthermore, it is not difficult to check that,

$$\inf_F I > c_0 \tag{2.2}$$

(see [T] for details).

Using a deformation lemma proved by Brezis-Nirenberg (see [B-N,1] corollary 4) from (2.1) and (2.2) one derives a sequence $\{u_n\} \subset H_0^1(\Omega)$ satisfying:

(1) $I(u_n) \to c_2$,

(2) $\|I'(u_n)\| \to 0$

(3) $u_n \in F$

(see [T] for details).

To be more precise, we should notice that although in general I does not satisfy the P.S. condition; it does however satisfy property (27) of [B-N,1]. So the statement of corollary 4. in [B-N,1] is still valid for I.

Alternatively, from (2.1) and (2.2) one could use a result of Ghoussoub [G] to obtain a sequence $\{u_n\} \subset H_0^1(\Omega)$ satisfying (1), (2) and dist $(u_n, F) \to 0$ is $n \to +\infty$.

In addition if $N \geq 6$, then the following estimates hold:

$c_2 \wedge c_1$ '*' $\frac{I}{TT}$ M/O   v∧#o/

In particular,

$$c_1 < c_2 < \frac{2}{N} S^{N/2} \tag{2.3'}$$

Notice that (2) is equivalent to $(1)^\wedge_{,n}$. Furthermore $u_n \in F$ implies

$$\| \nabla u_n \|_2^2 - \lambda \| u_n \|_2^2 = \| u_n \|_p^p \text{ and } \int_\Omega |u_n|^{p-2} v_1(u_n)\, u_n = 0$$

That is,

$$\| \nabla u_n \|_2^2 - \lambda \| u_n \|_2^2 = \| u_n \|_p^p = N\, I(u_n) \longrightarrow N\, c_2, \tag{2.4}$$

and

$$\nu(u_n, 1) \geq 2.$$

If necessary, take a subsequence to find $u_0 \in H_0^1(\Omega)$ such that

$$u_n \rightharpoonup u_0 \text{ weakly in } H_0^1(\Omega).$$

Assume that $\| u_0 \|_p^p < N\, c_2$. If $u_0 \neq 0$, then u_0 is a critical point for I and $c_1 \leq I(u_0) < c_2$. So necessarily,

$$\nu_0 = \nu(u_0, 1) = 1,$$

since u_0 cannot change sign in Ω.

Thus,

$$N\, I(u_0) + \lim_{n \to +\infty} \| \nabla (u_n - u_0) \|_2^2 = N\, c_2 < N\, c_1 + S^{N/2} = N\, c_1 + (2 - \nu_0)\, b_{N,\, 2-\nu_0}$$

(we have used the fact that $b_{N,1} = S^{N/2}$), which contradicts theorem 1.

Similarly if $u_0 = 0$, then,

$$\lim_{n \to +\infty} \| \nabla (u_n - u_0 \|_2^2 = N\, c_2 < 2\, S^{N/2} = 2\, b_{N,2}$$

which also contradict theorem 1 since $\nu(u_0 \equiv 0, 1) = 0$.

In conclusion, $\| u_0 \|_p^p \geq N\, c_2$, Hence from (2.4) it follows that u_n converges strongly to u_0.

Thus, for $0 < \lambda < \lambda_1$ and $N \geq 6$, problem $(1)_\lambda$ admits a solution u satisfying

$$\int_{\Omega} |u|^{p-2} v_1(u)u = 0$$

where $v_1(u)$ is the first eigenfunction for $(\star)$

In principle, one could apply this idea for every c_k's, $k \geq 3$. But, in order to establish stronger multiplicity results for $(1)_\lambda$ one faces two types of difficulties. The first one is to obtain sharp estimates on the values $b_{N,m}$ for $m \geq 3$. Secondly, it is not clear whether or not one can construct eigenfunctions $v_k(u)$ of $(\star)$ which are *even* and *continuous* in u, for every $k \geq 2$.

APPENDIX I:

This first appendix is devoted to derive a useful calculus inequality.

Calculus Lemma: Let $p > 2$.

There exists a constant $C_1 > 0$ (depending on p only) such that for every $0 < \alpha < \min \{p-2, 1\}$ we have:

$$\text{(I)} \qquad \big| |a+b|^{p-2} - |a|^{p-2} - |b|^{p-2} \big| \leq C_1 \left(|a|^\alpha |b|^{p-2-\alpha} + |b|^\alpha |a|^{p-2-\alpha} \right);$$

for every $a, b \in \mathbb{R}$.

Proof:

Use the homogeneity to see that (I) is equivalent to,

$$\big| |t+1|^{p-2} - 1 - |t|^{p-2} \big| \leq C_1 \left(|t|^\alpha + |t|^{p-2-\alpha} \right) \qquad \text{(I)}'$$

for $|t| \leq 1$.

We have,

$$\big| |t+1|^{p-2} - 1 \big| = \big| (t+1)^{p-2} - 1 \big| = (p-2) \left| \int_0^1 (1+st)^{p-3}\, t ds \right| \qquad \text{(A.1)}$$

To estimate the right hand side of (A.1) we distinguish two cases.

Case 1: $2 < p < 3$.

$$\left| \int_0^1 \frac{t}{(1 + st)^{3-p}} ds \right| \le |t| \int_0^1 \frac{ds}{(1 - s)^{3-p}} = \frac{1}{(p-2)} |t| \le \frac{1}{(p-2)} |t|^{\alpha}$$

since $0 < \alpha \le 1$ and $|t| \le 1$.

Case 2: $p \ge 3$.

$$\left| \int_0^1 t (1 + st)^{p-3} ds \right| \le 2^{p-3} |t| \le 2^{p-3} |t|^{\alpha}$$

In conclusion,

$$\left| |t + 1|^{p-2} - 1 \right| \le C_1 |t|^{\alpha} \quad \forall |t| \le 1. \tag{A.2}$$

Furthermore, for $|t| \le 1$ we have,

$$|t|^{p-2} \le |t|^{p-2-\alpha} \tag{A.3}$$

So (I) immediately follows from (A.2) and (A.3).

□

APPENDIX II.

In this appendix we collect some regularity results for solutions of $(*)$

Proposition A: Let $a \in L^{N/2}(\mathbb{R}^N)$ and $u \in D^{1,2}(\mathbb{R}^N)$ satisfy:

$$\int \nabla u \cdot \nabla \varphi = \int a(x) u \varphi \qquad \forall \varphi \in D^{1,2}(\mathbb{R}^N).$$

Then u € $L^q(R^N)$ V q ≥ p.

Furthermore if a e $L^N/^2(R^N)$ n L^R^1*) is locally Hőlder continuous, then

u e $C^2(R^N)$, | V n| e L*(R^N) 2 ≤ t ≤ + *m* and u satisfy:

$$\begin{cases} - A\ u = a(x)u & \text{in } R^N \\ u(x) - 0 & \text{as } |x| \to +m \\ |Vu(x)| - 0 & \text{as } |x| \to +en. \end{cases}$$

H, in addition a e ^(R^1^), then u e ^(R^N).

Corollary: If u € $D^{1>2}(R^N)$ satisfies:

$$\int \nabla u - V <p-\ |u|^{p\ "\ 2} u \wedge = 0 \qquad \forall \varphi \in D^{1,2}(R^N)$$

then u € *C^w(ti®)* n $L^q(R^N)$ for p ≤ q ≤ + m, |V u| 6 L*(R^N) for 2 ≤ t ≤ + x and u satisfies:

$$\begin{cases} - A\ u = |u|^{p-2} u & \text{in } R^N \\ u(x) \to 0 & \text{as } |x| \to +\infty \\ |U(x)| - .\ 0 & \text{as } |x| \to +\infty. \end{cases}$$

Proof of Proposition:

The following argument is due to Brezis-Kato [B—K].

2 f 2s 21

Let s = jj^jj and L > 0. Define *(p* = min ⌈|u| , L ⌉ u.

Observe that *<p* € $D^{1,2}$ ^) , u *<p* € *L^l{vF)* and | J u *ip*\ < $||u||^p_p$.

We have:

$$\int a(x)u\varphi = \int \nabla u . \nabla \wedge = \int |\nabla u|^2 \min\{|u|^{2s}, L^2\} + 2s \int_{\{|u|^s \le L\}} |u|^{2s} |\nabla u|^2 =$$

$$= \int |\nabla (u \min\{|u|^s, L\})|^2 + s \int_{\{|u|^s \le L\}} |u|^{2s} |\nabla u|^2.$$

That is,

$$\int |\nabla (u \min\{|u|^s, L\})|^2 \le \int a(x) u \varphi = \int a(x) u^2 \min\{|u|^{2s}, L^2\} \le$$

$$\le K \int_{\{|a| \le K\}} u^2 \min\{|u|^{2s}, L^2\} + \int_{\{|a| \ge K\}} |a|\, u^2 \min\{|u|^{2s}, L^2\} \le$$

$$\le K \|u\|_p^p + \Big(\int_{\{|a| \ge K\}} |a|^{N/2}\Big)^{2/N} \Big(\int |u \min\{|u|^s, L\}|^p\Big)^{\frac{N-2}{N}} \le$$

$$\le K \|u\|_p^p + S \Big(\int_{\{|a| \ge K\}} |a|^{N/2}\Big)^{2/N} \|\nabla (u \min\{|u|^s, L\})\|_2^2 \tag{A.4}$$

Since $a \in L^{N/2}(\mathbb{R}^N)$, we have:

$$\int_{\{|a| \ge K\}} |a|^{N/2} \longrightarrow 0 \quad \text{as } K \longrightarrow +\infty.$$

Choose K large enough to guarantee

$$\Big(\int_{\{|a| \ge K\}} |a|^{N/2}\Big)^{2/N} \le \frac{1}{2S}$$

From (A.4) we derive, $\|\nabla (u \min\{|u|^s, L\})\|_2^2 \le 2K \|u\|_p^p$ and in particular,

$$\int_{\{|u|^s \leq L\}} |\nabla(|u|^s u)|^2 \leq 2K\|u\|_p^p \quad \forall L > 0.$$

Hence letting $L \to +\infty$ we conclude:

$$|\nabla(|u|^s u)| \in L^2(\mathbb{R}^N).$$

By the Sobolev embedding theorem (see [Fr]) it follows that $u \in L^{p(s+1)}(\mathbb{R}^N)$ and by interpolation, $u \in L^q(\mathbb{R}^N)$ for $p \leq q \leq p(s+1)$

Iterate this procedure by choosing s_1 with $s_1 + 1 = (s+1)\frac{N-2}{N}$, and s_k:

$s_k + 1 = (s_{k-1} + 1)\frac{N}{N-2}$ for $k \geq 2$.

This yields, $u \in L^q(\mathbb{R}^N)$ for $q \geq p$.

Next we show that,

$$u(x) = \int \Gamma(x-y)\, a(y)\, u(y)\, dy \quad \text{a.e. } x \in \mathbb{R}^N \tag{A.5}$$

where $\Gamma(x-y) = a_N \frac{1}{|x-y|^{N-2}}$, and $a_N = \left[N(N-2)\, \omega_N\right]^{-1}$.

From (A.5) the rest of our statement will follow immediately. In fact by the Calderon-Zygmund inequality (see [G-T th. 9.9]) we know that $au \in L^t(\mathbb{R}^N) \Rightarrow D_{ij}u \in L^t(\mathbb{R}^N)$ $\forall\, i,j = 1,\dots,N$.

Our assumptions allow to take $t > N$. Thus by the Sobolev inequality (see [Fr]) we get in particular that u and $|\nabla u| \in L^\infty(\mathbb{R}^N)$; (hence $|\nabla u| \in L^t(\mathbb{R}^N)$ $2 \leq t \leq +\infty$).

While, by the Morrey's estimates (see [Fr]) we have that u and $|\nabla u|$ are locally Hölder.

Thus, from the analysis of Newtonian potentials (see [GT, lemma 4.2]) we finally conclude $u \in C^2(\mathbb{R}^N)$. Furthermore, for t large there exists a constant C (depending on N and t

only) such that:

$$\|\nabla u\|_{L^\infty(\mathbb{R}^N\setminus B_R)} \le C\,\|\nabla^2 u\|_{L^t(\mathbb{R}^N\setminus B_R)}$$

and

$$\|u\|_{L^{t^*}(\mathbb{R}^N\setminus B_R)} \le C\,\|\nabla u\|_{L^t(\mathbb{R}^N\setminus B_R)}$$

where B_R is the ball of radius R.

Hence, letting $R \to +\infty$, we conclude:

$$u(x) \to 0 \text{ and } |\nabla u(x)| \to 0 \text{ as } |x| \to +\infty.$$

A boot strap argument finally gives $u \in C^\infty(\mathbb{R}^N)$ if $a \in C^\infty(\mathbb{R}^N)$.

So we are left to establish (A.5).

To this purpose take,

$$\varphi_\epsilon(x) = c_N \frac{\epsilon^{\frac{N-2}{2}}}{(\epsilon^2 + |x-y|^2)^{\frac{N-2}{2}}} \in D^{1,2}(\mathbb{R}^N)$$

with $\epsilon > 0$ and the constant c_N is adjusted so that

$$-\Delta\,\varphi_\epsilon = \varphi_\epsilon^{p-1} \text{ in } \mathbb{R}^N.$$

Notice that for every $R > 0$ we have:

$$\int_{\{|x-y|\le R\}} \nabla u \cdot \nabla \varphi_\epsilon = \int_{\{|x-y|\le R\}} \varphi_\epsilon^{p-1} u(y) + \int_{\{|x-y|=R\}} u(y)\, \nabla \varphi_\epsilon(y)\cdot(x-y)\,dS =$$

$$\int_{\{|x-y|\le R\}} \varphi_\epsilon^{p-1}\, u + c\, \frac{\epsilon^{\frac{N-2}{2}}}{(\epsilon^2+R^2)^{N/2}} \int_{\{|x-y|=R\}} u(y)\,ds \qquad \text{(A.6)}$$

Since, $\int_0^{+\infty} dR \int_{\{|x-y|=R\}} |u(y)|^p dS < +\infty$, for a sequence $R_n \to +\infty$

we have,

$$\int_{\{|x-y|=R_n\}} |u(y)|^p \to 0 \quad \text{as} \quad n \to +\infty$$

Thus,

$$\frac{R_n}{(\epsilon^2+R_n^2)^{N/2}} \left| \int_{\{|x-y|=R_n\}} u(y)\, dS \right| \le \frac{1}{R_n^{N-1}} (\sigma_N R_n^{N-1})^{1-\frac{1}{p}} \left(\int_{\{|x-y|=R_n\}} |u(y)|^p\, dS\right)^{1/p} \to 0$$

as $n \to +\infty$ (σ_N = surface of the unit sphere).

This yields,

$$\int a(y)\,u(y)\,\varphi_\epsilon(y)\,dy = \int \nabla u \cdot \nabla \varphi_\epsilon = \lim \left(\int_{\{|x-y|\le R_n\}} \nabla u \cdot \nabla \varphi_\epsilon\right) =$$

$$= \lim_{n\to+\infty} \left(\int_{\{|x-y|\le R_n\}} \varphi_\epsilon^{p-1} u + o(1)\right) = \int \varphi_\epsilon^{p-1} u.$$

In other words,

$$\int \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}} u(x + \epsilon\, y)dy = \left[\frac{1}{c_N}\right]^{p-2} \int \frac{a(y)\ u(y)}{(\epsilon^2+|x-y|^2)^{\frac{N-2}{2}}}\, dy.$$

Thus, letting $\epsilon \longrightarrow 0$, we obtain (A.5).

□

The corollary can be easily derived by a bootstrap argument from the fact that $u \in L^q(\mathbb{R}^N)$ for every $q \geq p$ and

$$u(x) = \int \frac{|u(y)|^{p-2} u(y)\ dy}{|x-y|^{N-2}} \qquad \text{a.e. } x \in \mathbb{R}^N.$$

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