THE WULFF THEOREM REVISITED

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1. INTRODUCTION.

The study of phase transitions leads in a natural way to the study of variational problems where the total energy functional involves bulk and surface energies terms (see FONSECA [17], [18], GURTIN [23], [24], KINDERLEHRER & VERGARA- CAFFARELLI [26]). It may also happen that the total energy reduces essentially to its interfacial component. As an example, for solid crystals with sufficiently small grains HERRING [25] assumes that interfaces are sharp and shows that the surface free energy plays a definite role in determining the shape of the crystal approaching an equilibrium configuration of minimum energy. The surface tension considered by HERRING was of the type

$$\int_{\partial E} F(n_E(x))dH_{N-1}(x) \tag{1.1}$$

where E is a smooth subset of $\mathbb{R}^N$, n_E is the outward unit normal to its boundary and F denotes the anisotropic free energy density per unit area. Clearly, when F is constant the problem

(P) to minimize (1.1) subject to meas(E) = constant,

reduces to the classical isoperimetric inequality. For anisotropic F, one of the first attempts to solve (P) is due to WULFF [38] in the early 1900's. His work was followed by that of DINGHAS [14], who proved that, among convex polyhedra, the *Wulff set* (or *crystal of F*)

$$W_F := \{x \in \mathbb{R}^N \mid x.n \leq F(n), \text{ for all } n \in S^{N-1}\}$$

is the shape having the least surface integral for the volume it contains. Later, TAYLOR [35], [36] and [37] obtained existence and uniqueness of solution using geometric measure theory tools. All the above mentioned results rely essentially in the application of the Brunn-Minkowski Theorem. Recently, DACOROGNA & PFISTER [12] presented a completely different proof in $\mathbb{R}^2$ which does not involve the Brunn-Minkowski inequality. Their arguments are purely analytical but, unfortunately, they cannot be extended to higher dimensions. Also, they obtain existence and uniqueness within a class of sets strictly contained in the class C of all measurable sets (bounded or unbounded) with finite perimeter. This is, of course, the natural class of sets for which (1.1) is defined.

In this paper we obtain existence of solution for (P) in C. The proof follows that of DINGHAS [14] and TAYLOR [37], although we work within the BV theory instead of using geometric measure theory terminology and tools. Hopefully, this proof will be more accessible to analysts. Its key ingredients are the Brunn-Minkowski Theorem and the parametrized indicator measures (see FONSECA [19]). These measures enable us to handling oscillating weakly converging sequences of surfaces and continuity and lower semicontinuity properties of functional of the type (1.1). The indicator measures (see RESHETNYAK [32]) are a refined version of the generalized surfaces of YOUNG [39] and they were studied by ALMGREM [2] (see also ALLARD [1]) under the name of varifolds.

We characterize the support of indicator parametrized measures associated to minimizing sequences for (P). This information allows us to conclude that if the Wulff set is polyhedral then minimizing sequences cannot oscillate.

In Section 2 we review briefly some concepts of the theory of functions of bounded variation and in Section 3 we provide a detailed analysis of the Wulff set and we study Γ and its biconjugate function Γ^{**}.

In Section 4 we recall some of the properties of indicator measures and we show that the Wulff set is a solution in $\mathcal{C}$ for the geometrical variational problem (P), generalizing TAYLOR's [37] result to unbounded sets. Also, and for completeness, we prove in Section 5 that the Wulff set is the only solution in the class $\mathcal{C}$, up to a translation and a set of measure zero. This result was obtained first by TAYLOR [36], using geometric measure theory techniques that involve approximation of a set by polyhedra. Recently, an analytical proof of was given by FONSECA & MÜLLER [20].

In Section 6 we provide a characterization of the support of indicator measures associated to minimizing sequences and we use this result to conclude that in some cases (e. g. if W_Γ is a polyhedron) minimizing sequences cannot develop oscillations.

Finally, in Section 7 we discuss a variational problem in nonlinear elasticity involving bulk and interfacial energy. Lately, there has been great progress in the analysis of minimization problems of this type. BOUCHITTE [8] and OWEN & STERNBERG [30], using singular perturbations and the Gamma - convergence approach (see De GIORGI [13]), obtained the Wulff set as the selected shape for a scalar-valued two-phases transition problem with infinitely many solutions. Recently, this result was generalized to the vector-valued case by AMBROSIO, MORTOLA & TORTORELLI [4] and AVILES & GIGA [5][1].

2. PRELIMINARIES.

We recal briefly some results of the theory of functions of bounded variation (see EVANS & GARIEPY [15], FEDERER [16], GIUSTI [22]). Let $\Omega \subset \mathbb{R}^N$ be an open and connected set and define $S^{N-1} := \{x \in \mathbb{R}^N | \, \|x\| = 1\}$.

Definition 2.1.

A function $u \in L^1(\Omega)$ is said to be a *function of bounded variation* ($u \in BV(\Omega)$) if
$$\int_\Omega |\nabla u(x)|\, dx := \sup\left\{ \int_\Omega u(x) \,.\, \operatorname{div} \varphi(x)\, dx \mid \varphi \in C_0^1(\Omega; \mathbb{R}^N), \|\varphi\|_\infty \leq 1 \right\} < +\infty.$$

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[1]. See also AMBROSIO & De GIORGI [3] for a related problem.

It follows immediatly from Definition (2.1) that if $u_£$ converges to u_0 in $L^!(Q)$ then

$$f \ |Vu(x)|dx \leq \liminf \ f \ |Vu_e(x)| \ dx. \qquad (2.2)$$
$$\bullet to \qquad e \to o \quad J\&$$

It turns out that bounded sets in BV are compact in L^1, precisely

Proposition 2.3.

Let QC $[R^N$ be an open bounded strongly Lipschitz domain and let C be a positive constant. Then, the set

$$\mathbf{JUG\ L^1\wedge)!\ f\ |u(x)|dx + f\ |Vu(x)|dx < cl}$$

is compact in $L^!(Q)$.

A particular case of a function of bounded variation is the characteristic function of a set of finite perimeter.

Definition 2.4.

If A is a subset of $[R^N$ then *the perimeter ofA in Q* is defined by

$$\mathbf{Fter_Q(A) := Jk\ |VX_A(x)|dx = \sup\{J_A\ div\ cp(x)\ dx\ |\ cp\ e\ cJ(Q;\ IR^N),\ \|cp\|L \leq 1\},}$$

where X A denotes the characteristic function of A.

If A has finite perimeter in IR^N then for any borel set E

$$\|\nabla \chi_A\|(E) = H_{N-1}(\partial^* A \cap E),$$

where H^ denotes the N-l dimensional Hausdorff measure, 3*A is the reduced boundary of A and $||V\%_A||$ is the total variation measure of the vector-valued measure V%A- Often, we use the notation

$$H_{N\_}!\ L3^*A\ (E) := H_{N\_1}O^*A\ n\ E).$$

Also, there exists a ||VXAII -measurable map n_A : 3*A -> $S^{N\sim l}$ such that $n_A(x)$ is the outward normal to 9*A at x,

$$-n_A||VX_AH = VX_A \quad \text{in} \quad £'(iR^N)$$

and the *generalized Green-Gauss theorem* holds, namely

$$\mathbf{J_{-A}\ div <p(x)\ dx = J_{RN}\ cp(x).\ n_A(x)\ \ d||VX_A ll}$$

$$\mathbf{= f_{\bullet\wedge 3^*A}\ \ cp(x).\ n_A(x)\ \ dH\wedge\wedge x)} \qquad \mathbf{(2.5)}$$

for all 9e $C\wedge(1R^N;\ IR^N)$.

Proposition 2.6.

If E has finite perimeter in £2, then for almost all R > 0

Per$_\Omega$(E n B(0, R)) ≤ Per$_{B(}$o,R)(E) + H_{NA}(*E* n 9B(0,R)).

We will use the ***Fleming-Rishel co-areaformula***

f |Vu(x)| dx = f $^{\infty}$ **Per$_Q$ {x € IR^N| u(x) > t} dt** **(2.7)**

Ja J-oo

for u e BV(£2), and also the ***change ofvariablesformula***

f $_N$ u(x) |det Vf(x)| dx = f *(* f , u(z) dH^fe)! dy **(2.8)**

where N ≥ p, f:IR^N-»[RPisa Lipschitz function and u : IR^N -> IR is measurable.

The next result shows that a bounded set of finite perimeter can be approached in BV by a sequence of C^∞ sets with the same volume. The proof can be found in the Appendix.

Lemma 2.9.

Let EC IR^N be a bounded set of finite perimeter. There exists a sequence of open, bounded setsEuC IR^N such that

(i)3E$_n$ **e** C~andEn,E C B(0, R) for some R > 0;

(ii)X$_{En}$ ->X$_E$ inL^IR*);

(iii)Per(E$_n$) ->Per(E);

(iv) meas(E$_n$) = meas(E).

3. THE WULLF SET.

In what follows F: SN"^{1} —> [0, +°o) denotes the surface free energy of a solid. For crystalline materials, HERRING [25] proposes some constitutive hypotheses for F based on molecular considerations where surface energies arise from interatomic interactions of finite range. It turns out that for ordered materials (i. e. materials with a lattice structure) F is not differentiate with respect to certain crystallographically simple directions. In this case, if we plot F radially as a function of the direction n, this plot will present cusped minima in certain directions corresponding to surfaces of particular simple structure with respect to the lattice (see FONSECA [17], PARRY [31]). At each point of this polar plot construct a plane perpendicular to the radius vector at that point. Then the volume Wp which can be reached from the origin without crossing any of the planes is the *Wulffset.* Precisely, assuming that F is continuous and bounded away from zero, i. e.

F(n) ≥ **a** for some **a** > 0 and for all n e S$^{1*"1}$ (3.1) we have

Definition 3.2.

The *Wulff set* (or *crystal of Γ*) is the set $W_\Gamma := \{x \in \mathbb{R}^N \mid x.n \le \Gamma(n) \text{ for all } n \in S^{N-1}\}$.

Clearly, if $\Gamma \equiv 1$ then W_Γ is the closed unit ball. Also, using HERRING's [25] idea it is easy to show that for solid crystals the lack of differentiability of Γ implies that its crystal is a polyhedron. This motivates the following definition.

Definition 3.3.

Γ is said to be *crystalline* if W_Γ is polyhedral. Moreover we say that Γ is *strictly crystalline* if it is crystalline and if $\Gamma(n) > \Gamma^{**}(n)$ unless n is normal to ∂W_Γ.

In order to relax the energy (1.1), we recall some concepts of the theory of convex functions (see ROCKAFELLAR [33]).

Definition 3.4.

Let $f : \mathbb{R}^N \to [-\infty, +\infty]$.

(i) The *Fenchel Transform of f* (or *polar of f*, or *conjugate of f*) is the function $f^* : \mathbb{R}^N \to [-\infty, +\infty]$ given by

$$f^*(y) := \sup_{x \in \mathbb{R}^N} \{x.y - f(x)\}.$$

(ii) The *bipolar* (or *biconjugate*) *of f* is the function $f^{**} : \mathbb{R}^N \to [-\infty, +\infty]$ defined by

$$f^{**}(x) := \sup_{y \in \mathbb{R}^N} \{x.y - f^*(y)\}.$$

(iii) If $x \in \mathbb{R}^N$, the *subgradient of f at x* is the set

$$\partial f(x) := \{y \in \mathbb{R}^N \mid f(x') \ge f(x) + y.(x' - x) \text{ for all } x' \in \mathbb{R}^N\}.$$

It is easy to verify that f^{**} is the lower convex envelope of f, i. e. $f^{**} = \sup\{g \mid g \text{ is convex and } g \le f\}$. The *indicator function of a set $C \subset \mathbb{R}^N$* is defined by

$$I_C(x) := \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{if } x \notin C. \end{cases}$$

The following results are well known.

Proposition 3.5.

(i) I_C is convex if and only if C is convex;

(ii) I_C^* is called the *support function of C* and $I_C^*(y) = \sup\{y.x \mid x \in C\}$;
(iii) if C is convex and if $x \in \partial C$ then

$$y \text{ is normal to } C \text{ at } x \Leftrightarrow y \in \partial I_C(x) \Leftrightarrow y.(x' - x) \leq 0 \text{ for all } x' \in C;$$

(iv) $\partial I_C(x)$ is a closed convex cone and $\partial I_C(x) = \{0\}$ if $x \in \text{int}(C)$;
(v) if C is closed and convex then $\partial I_C^*(y) = \{x \in \partial C \mid y \text{ is normal to } C \text{ at } x\}$.

Here Γ^* and Γ^{**} denote, respectively, the polar and bipolar functions of Γ_0 obtained by extending Γ as a function homogeneous of degree one :

$$\Gamma_0(x) := \begin{cases} \|x\|\,\Gamma\left(\frac{x}{\|x\|}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Proposition 3.6.
(i) W_Γ is convex, closed and bounded;
(ii) $\Gamma^{**} = I_{W_\Gamma}$;
(iii) $\Gamma^{**}(x) = \sup\{y.x \mid y \in W_\Gamma\}$ = support function of W_Γ;
(iv) if $x \in \partial W_\Gamma$ and if n is normal to W_Γ at x then $x.n = \Gamma(n) = \Gamma^{**}(n)$;
(v) the crystal of Γ^{**} is the crystal of Γ, i. e. $W_\Gamma = W_{\Gamma^{**}}$;
(vi) $0 \in \text{int}(W_\Gamma)$.

Proof. (i) It is clear that W_Γ is convex and closed. Also, if $x \in W_\Gamma$ then $\|x\| = \sup\{x.n \mid \|n\| = 1\} \leq \sup\{\Gamma(n) \mid \|n\| = 1\} =: M$, i. e. $W_\Gamma \subset B(0,M)$.
(ii) If $x \in W_\Gamma$, then for all $y \neq 0$

$$x.y - \Gamma_0(y) = \|y\|\left(x.\frac{y}{\|y\|} - \Gamma\left(\frac{y}{\|y\|}\right)\right) \leq 0$$

and as $x.0 - \Gamma_0(0) = 0$, $\Gamma^*(x) = 0$. On the other hand, if $x \notin W_\Gamma$ then there exists $n \in S^{N-1}$ such that $x.n - \Gamma(n) > 0$. Therefore, for all $k \in \mathbb{N}$

$$\Gamma^*(x) \geq x.(kn) - \Gamma_0(kn) = k\,(x.n - \Gamma(n))$$

and so $\Gamma^*(x) = +\infty$. We conclude that $\Gamma^{**} = I_{W_\Gamma}$.
(iii) By (ii) and by Proposition 3.5 (ii), Γ^{**} is the support function of W_Γ and $\Gamma^{**}(x) = \sup\{y.x \mid y \in W_\Gamma\}$.
(iv) As W_Γ is convex and bounded, the normal n(x) to ∂W_Γ at $x \in \partial W_\Gamma$ is uniquely defined for H_{N-1} a. e. x. We want to show that

$$x.n(x) = \Gamma(n(x)). \tag{3.7}$$

As $x \in W_\Gamma$, we know that $x.n(x) \leq \Gamma(n(x))$. Suppose that

$x.n(x) < \Gamma(n(x))$. (3.8)

By (ii) and by Proposition 3.5 (iii) we have

$\partial \Gamma^*(x) = \{n(x)\}$. (3.9)

We claim that there exists $\beta > 0$ such that

$\Gamma(m) - x.m \geq \beta$ for all $||m|| = 1$. (3.10)

Indeed, if $\Gamma(m) - x.m = 0$ for some unit vector m, then by (3.8) and (3.9) $m \neq n(x)$ and $m \notin \partial\Gamma^*(x)$. Thus, there exists $x' \in W_\Gamma$ such that

$0 < m.(x' - x)$ i. e. $\Gamma(m) < m.x'$

which is impossible by definition of the set W_Γ. Therefore 93.10) holds and if $x' \in B(x, \beta)$ then

$x'.m = (x'-x).m + x.m \leq ||x'-x|| + \Gamma(m) - \beta \leq \Gamma(m)$

for all $m \in S^{N-1}$ and so $B(x, \beta) \subset W_\Gamma$ contradicting the assumption $x \in \partial W_\Gamma$. We conclude that (3.7) holds. Hence

$\Gamma(n(x)) \geq \Gamma^{**}(n(x)) = \sup \{y.n(x) \mid y \in W_\Gamma\} \geq x.n(x) = \Gamma(n(x))$

which implies that $x.n = \Gamma(n) = \Gamma^{**}(n)$.

(v) Let $W_{\Gamma^{**}}$ be the crystal of Γ^{**}. As $\Gamma^{**} \leq \Gamma$ we have $W_{\Gamma^{**}} \subset W_\Gamma$. Also, by (iii) if $x \in W_\Gamma$ then $x.n \leq \sup \{y.n \mid y \in W_\Gamma\} = \Gamma^{**}(n)$, i. e. $x \in W_{\Gamma^{**}}$.

(vi) Suppose that the origin is not interior to W_Γ. Then there exist sequences $\{y_k\}$ and $\{n_k\}$ on S^{N-1} such that $n_k.y_k/k > \Gamma(n_k)$. Without loss of generality, we can assume that $y_k \to y$ and $n_k \to n$, where $||y|| = 1 = ||n||$. We conclude that $\Gamma(n) = 0$ which contradicts (3.1).

4. THE WULFF THEOREM: EXISTENCE.

Here we prove the isoperimetric inequality

$C\, \text{meas}(E)^{(N-1)/N} \leq J(\partial E)$ (4.1)

for all measurable sets E of bounded perimeter, where J(.) is the surface energy[2]

$$J(\partial E) := \int_{\partial^* E} \Gamma(n_E(x))\; dH_{N-1}(x)$$

and $C = J(\partial^* W_\Gamma)\, \text{meas}(W_\Gamma)^{(1-N)/N}$. The key idea of the proof is the use of the Brunn-Minkowski inequality. This was exploited formally by DINGHAS [14] and later made precise in the context of geometric measure theory by TAYLOR[3] [37]. Here we presente the proof in a BV framework that avoids the terminology and concepts of geometric measure theory, thus making it more accessible to analysts.

[2]Here n_E is the outward unit normal to the reduced boundary ∂^*E.

[3].We generalize TAYLOR's [37] result to unbounded sets.

Brunn-Minkowski Theorem 4.2.
If A and B are nonempty sets of 1R^N then
meas(A + B) ≥ (meas(A)$^{1/N}$ + meas(B)$^{1/N}$)^N.

We refer the reader to DACOROGNA & PFISTER [12], where (4.1) is obtained for a certain class of sets in IR^2 whithout using the Brunn-Minkowski Theorem. However, their argument is essentially two dimensional and it cannot be extended to higher dimensions.

It is clear that (4.1) follows from

Theorem 4.3.
Let E C [R^N be a set with finite perimeter and such that meas(E) = meas(Wp). Then

f$_{JZ*E}$ r(n$_E$(x)) dH^U) ≥ f$_{Ja*w_r}$ r(n$_{w_r}$(x)) dH$_N$_!(x).

Indeed, assume that meas(E) = A,^{N}meas(Wr). We recall that if *£1* CIR^N is an open bounded strongly Lipschitz domain, if *§* is a diffeomorphism and if f: S^{N-1} -» CR^N is a continuous function,

then n,$_{(Q)}$«D(x)) $\frac{\text{adjV<1)(x)nn(x)}}{\|\text{adj } \nabla\phi(x)n_\Omega(x)\|}$

and

$$\int_{\partial\phi(\Omega)} f(n_{\phi(\Omega)}(y))\, dH_{N-1}(y) \bullet L\, f\left(\frac{\text{adj } \nabla\phi(x)n_\Omega(x)}{\|\text{adj } \nabla\phi(x)n_\Omega(x)\|}\right) \|\text{adj } \nabla\phi(x)\, n_\Omega(x)\|\, dH_{N-1}(x).$$

Thus, setting <|>(x) := A,x, by Theorem 4.3 we have

$$\begin{aligned}\int_{\partial^* E} \Gamma(n_E(y))\ dH_{N-1}(y) &= \int_{\partial^*\phi(E/\lambda)} \Gamma(n_{\phi(E/\lambda)}(y))\ dH_{N-1}(y)\\ &= \lambda^{N-1}\int_{\partial^*(E/\lambda)} {}^r(n_{(E}/w(x))\ dH_{N-1}(x)\\ &\geq \lambda^{N-1}\int_{\partial^* W_\Gamma} Hn_{w\_}(x))\ dH\text{^}Cx)\\ &= \int_{\partial^*(\lambda W_\Gamma)} \Gamma(n_{\lambda W_\Gamma}(y))\ dH_{N-1}(y)\end{aligned}$$

i. e.

Corollary 4.4.
The dilation XWp minimizes the surface energy functional (1.1) among all sets of finite perimeter with volume equal to A,^{N}meas(Wr)-

Finally, (4.1) follows from Corollary 4.4 since

J(3E) ≥ *J(d(XW_T)) = X^{N~l}* JO(W_r)) = **[meas(E)/meas(WΓ)]^{(N-1)/N} J(∂(WΓ))**

= Cmeas(E)(^N-^1)^/N.

The proof of Theorem 4.3 will be divided into three parts. The first one deals with bounded open sets E having smooth (e. g. C^2) boundary. In the second step we prove the result for bounded sets of finite perimeter and on the third step we remove the boundedness restriction. The second and third parts rely heavily on the lower semicontinuity properties of surface energies. These were studied in detail in FONSECA [19]. For convenience, we summarize some of the relevant results for this work.

Let E C IR^N be a bounded set with finite perimeter. Associated to V $\%_E$ we have the nonnegative and finite Radon *indicator measure* A_E on $K^N x S^{N1}$ defined by

< A_E, F > = ∫ F(x, -n_E(x)) dH^Cx)
*Jd*E*

for all F e Co(IR^N x K^N). The slicing measures determine the factorization A_E = ¯*X_x*<8> rc_E, where $7U_E$ is the projection of A_E into IR^N and *$A^E_{,x}$} is a family of probability measures such that

<A_E,F> = 1 ^ (1 ^ F(x,y) dASy)) d$7U_E$(x).

In what follows, v denotes the *center ofmass,* i. e.

$$\nu(x) := \int_{S^{N-1}} y \, d\lambda_x^E(y).$$

Proposition 4.5.

Let {E_e} be a sequence of bounded sets of finite perimeter, with indicator measures {$A_€$}. If A_e-** A^ = X^°®7Coo weakly * in the sense of measures and if ^ -> *XE* strongly in L^1. Then E is a set of finite perimeter with indicator measure A_E =¯)*JiS) TC_E* ancf

(i) J_{a+E} F(x, n_{Ee}(x)) d W x) -» J ^ ^ , F(x,y) dAJx,y) = $£_N$ *[j*^ F(x,y) $d\lambda_x^\infty(y)$)$d\pi_\infty(x)$

forallFeC$_0$(IR^NxlR^N);

(ii) dH^iL 3*E = ||vj| dn^. and ||vjx)|| ≤ 1 for ^ a. e. x;

(ii) v_M = - ||vj| n_E, for dH^I. *d*E* a. e. x;

(iii)sup||vJ| = 8*E.

Using the indicator measures, *it* follows easily that (see FONSECA [19], Corollary 4.6 (ii))

Theorem 4.6.

Let $E_£$ C IR^N be a sequence of bounded sets with finite perimeter in K^N. If (meas($E_£$) + Per($E_£$)} is bounded and if $^\wedge_\varepsilon$ -> X_E in L^1(|R^N) then

$$\int_{Jd^*E} F(x, n_E(x))\, d H \wedge x \;) \leq \liminf_{e \to 0} \int_{J3^*E_e} F(x, n_{E_e}(x))\, dH_{N-1}(x)$$

for all nonnegative, continuous function F such that F(x,.) is convex and homogeneous of degree one for all x e IR^N.

If in addition there exists R > 0 such that E_e C B(0,R) for all e > 0, then (see FONSECA [17], Theorem 5.1 and Corollary 5.9) the following result provides necessary and sufficient conditions for continuity.

Proposition 4.7[4]**.**

The following assertions are equivalent:

(i) $Per(E_e) \to Per(E)$;

(ii) A_x^{∞} is a Dirac mass;

(iii) f F(x, n_E (x)) $dH_{N-1}(x)$ -> f F(x, $n_E(x)$) efflux) for all F e $C_0(IR^N x IR^N)$.

In order to prove Theorem 4.6, we obtain a lower bound for the relaxed energy.

Lemma 4.8.

Let E be a C°°, open, bounded domain. Then

$$\int_{JdE} T^{**}(n_E(x))^{xV}\, dH_{N-1}(x) \geq \liminf_{e \to o} \frac{\text{meas}(E + \pounds W_r) - \text{meas}(E)}{e}.$$

Proof. Let x e 3E, e > 0 and let

X := sup {t > 0 | x + e t $n_E(x)$ e E + £W_r and t$n_E(x)$ e W_r}.

By Proposition 3.6 (vi), there exists p > 0 such that B(0, p) C W_r and so, E C C E + eW_r. Choose 0 < 8 < ep/2. We claim that

B(x, 8) C E + eW_r.

Indeed, if y = x + z, with ||z|| < 8, and if x' e E is such that ||x' - x|| < 8 then we have

$$y = x' + \varepsilon\left(\frac{x - x'}{\varepsilon} + \frac{z}{\varepsilon}\right) \quad \in \quad E + eW_r.$$

Thus, x€ int (E + eWp) and as E + eWpis a bounded set, 0 < X < +<*>. Also, since Wp is closed then X$n_E(x)$ e Wp and, clearly, x + e X$n_E(x)$ e 3(E + eWp). Therefore

dist(x, d(E + eW_r)) ≤ ||x - (x + e A$a_E(x)$)|| = e (X$a_E(x)$). $n_E(x)$

[4]. **(ii) and (iii) are equivalent even when the sequence {Eg} is not uniformly bounded.**

$$\leq \varepsilon \max\{y.\, n_E(x) \mid y \in W_\Gamma\}$$

and so, by Proposition 3.6 (iii)

$$\int_{\partial E} \Gamma^{**}(n_E(x))\ dH_{N-1}(x) = \int_{\partial E} \max_{y \in W_\Gamma} y.n_E(x)\ dH_{N-1}(x)$$

$$\geq \limsup_{\varepsilon \to 0} \int_{\partial E} \frac{\operatorname{dist}(x, \partial(E + \varepsilon W_\Gamma\ ))}{\varepsilon} dH_{N-1}(x)$$

$$\geq \liminf_{\varepsilon \to 0} \frac{\operatorname{meas}(E + \varepsilon W_\Gamma\ ) - \operatorname{meas}(E)}{\varepsilon}.$$

Proof of Theorem 4.3. (i) Let E be a C^∞, open, bounded domain of $\mathbb{R}^N$ such that $\operatorname{meas}(E) = \operatorname{meas}(W_\Gamma)$. Then by Lemma 4.8 and by the Brunn - Minkowski inequality (see Theorem 4.2) we have

$$\int_{\partial E} \Gamma(n_E(x))\ dH_{N-1}(x) \geq \int_{\partial E} \Gamma^{**}(n_E(x))\ dH_{N-1}(x)$$

$$\geq \liminf_{\varepsilon \to 0} \frac{\operatorname{meas}(E + \varepsilon W_\Gamma) - \operatorname{meas}(E)}{\varepsilon}$$

$$\geq \liminf_{\varepsilon \to 0} \frac{(\operatorname{meas}(E)^{1/N} + \varepsilon \operatorname{meas}(W_\Gamma\ )^{1/N})^N - \operatorname{meas}(E)}{\varepsilon}.$$

Therefore, as $\operatorname{meas}(E) = \operatorname{meas}(W_\Gamma)$, by the generalized Green-Gauss Theorem (2.5) and by Proposition 3.6 (iv) we conclude that

$$\int_{\partial E} \Gamma(n_E(x))\ dH_{N-1}(x) \geq \operatorname{meas}(W_\Gamma) \lim_{\varepsilon \to 0} \frac{(1+\varepsilon)^N - 1}{\varepsilon}$$

$$= N \operatorname{meas}(W_\Gamma) = \int_{W_\Gamma} \operatorname{div} x\ dx = \int_{\partial^* W_\Gamma} x.n_{W_\Gamma}(x)\ dH_{N-1}(x)$$

$$= \int_{\partial^* W_\Gamma} \Gamma(n_{W_\Gamma}(x))\ dH_{N-1}(x).$$

(ii) Let E be a bounded set of $\mathbb{R}^N$ with finite perimeter and such that $\operatorname{meas}(E) = \operatorname{meas}(W_\Gamma)$. By Lemma 2.9 there exists a sequence of C^∞, open, bounded sets $E_n \subset \mathbb{R}^N$ and $R > 0$ such that

$$E_n, E \subset B(0,R) \text{ for all } n \in \mathbb{N} > 0,\ \operatorname{meas}(E_n) = \operatorname{meas}(E) = \operatorname{meas}(W_\Gamma)$$

and

$$\chi_{E_n} \to \chi_E \quad \text{in } L^1(\Omega) \quad \text{and} \quad \operatorname{Per}(E_n) \to \operatorname{Per}(E).$$

Thus, by Proposition 4.7 and by (i) we deduce that

$$\int_{\partial E} \Gamma(n_E(x))\ dH_{N-1}(x) = \lim_{n \to \infty} \int_{\partial E_n} \Gamma(n_{E_n}(x))\ dH_{N-1}(x)$$

$$\geq \int_{\partial^* W_\Gamma} \Gamma(n_{W_\Gamma}(x))\ dH_{N-1}(x).$$

(iii) Suppose that E is an unbounded set of finite perimeter with $\operatorname{meas}(E) = \operatorname{meas}(W_\Gamma)$. Define $f(R) := H_{N-1}(E \cap \partial B(0,R))$. Then, by Proposition 2.6 and as meas(E) = j f(R) dR < +~,

there exists **a** sequence Rn —> +°° such that

$f(R_n)$ -» 0 and En := EnB(O,R_n) is **a** set of finite perimeter. (4.9)

Using part (ii), **the argument** of **Corollary** 4.4, **and** (4.1)

f r(n_E(x)) dH^OO ≥ f Hnc (x)) dH^OO - f l{$_1$£-W_{N-1}(x)

J3*E ***Jd*E$_a$*** ^ **JaB(O,R_n)nE** \\$^K$***nJ***

≥ f T($n_{x\,w}$ 00) dH^OO - a f(R_n)

$$= \lambda_n^{N-1} \int_{\partial^* W_r} r(n_{W_r}(x))dH_{N-1}(x) - af(R_n), \quad (4.10)$$

where meas(E_n) = $A_{,n}^N$ meas(Wr)- As meas meas(E_n) -> meas(E) = meas(Wr), we have 7^ -> 1 and so (4.9) and (4.10) yield

f T(n_E(x)) d H ^ x) ≥ f T(n_{w_r}(x)) dH^^x).

^a*E Ja*w_r

Next, we obtain an apriori estimate on the diameter of a set E satisfying the following connection condition: for all £ e S^{1*-1} there exist -<*> < so(E,£) < si(E,£) < +oo such that GE,^(S) = 0 for all s ≤ so(E,^), GE,^(S) = 1 for all s ≤ Si(E,^) and GE,^ is strictly increasing in the interval (s_o(E£), si(E£)), where

$G_{E,\xi}$(s) :=$\dfrac{\text{meas(\{xe E|x.\%<s»}}{\text{meas(E)}}$ ***T-r***. **(4.11)**

Proposition 4.12.

There exist continuous functions ci:[0, +«>) ->[0, +<») and C2:[0, +oo) —>[0, +©<>) with ci(0) = 0, ci is decreasing, such that if E C $[R^N$ is **a** measurable set of finite perimeter with meas(E) = meas(Wp) satisfying (4.11) and if

f T(n_{W_r}(x)) dH_{N-}!(x) ≤ f T(n_E(x)) dH_{N-}!(x) ≤ (1 + *r*\\) f r(n_{w_r}(x)) dH^^x) (4.13)

*Jd*w$_r$* Ja*E Ja*w_r

then there exists a e $[R^N$ such that meas(E \ B(a, C2(ri))) ≤ **c_1(η).**

Proof, (i) Suppose that E is open, *dE* is smooth, meas(E) = meas(Wp) and E satisfies (4.11). Consider a unit vector e € $[R^N$ and for simplicity of notation set

f(t) := meas(E(t)) = $G_{E,e}$(t) meas(E) where E(t) :={x *e* E | x.e < t}.

Clearly 0 ≤ f(t) ≤ ^i :=meas(W_r), and **f**(t) = H_N_i({x e E | x.e = t}). As

meas$\left(\dfrac{u^{1/N}}{lf(0^{1/N}}E(t)\right)$ = **μ,** changing variables and by Theorem 4.3 we have

$$\int_{\partial^* E(t)} \Gamma(n_{E(t)})\, dH_{N-1} = \left(\frac{f(t)}{\mu}\right)^{\frac{N-1}{N}} \int_{\partial^*\left(\left(\frac{\mu}{f(t)}\right)^{1/N} E(t)\right)} \Gamma(n)\, dH_{N-1}$$

$$\geq \left(\frac{f(t)}{\mu}\right)^{\frac{N-1}{N}} \int_{\partial^* W_\Gamma} \Gamma(n_{W_\Gamma})\, dH_{N-1}.$$

In a similar way, with $E(t)' := \{x \in E \mid x.e \geq t\} = E \setminus E(t)$,

$$\int_{\partial^* E(t)'} \Gamma(n_{E(t)'})\, dH_{N-1} \geq \left(\frac{\mu - f(t)}{\mu}\right)^{(N-1)/N} \int_{\partial^* W_\Gamma} \Gamma(n_{W_\Gamma})\, dH_{N-1}.$$

Adding up these two inequalities yields

$$\int_{\partial^* E} \Gamma(n_E)\, dH_{N-1} + 2\int_{\{x\in E \mid x.e = t\}} \Gamma(e)\, dH_{N-1} \geq$$

$$\geq \int_{\partial^* W_\Gamma} \Gamma(n_{W_\Gamma})\, dH_{N-1}\left[\left(\frac{f(t)}{\mu}\right)^{\frac{N-1}{N}} + \left(\frac{\mu - f(t)}{\mu}\right)^{\frac{N-1}{N}} \right\},$$

and so, by (4.13)

$$g'(t) \geq C^* \left[g(t)^{(N-1)/N} + (1-g(t))^{(N-1)/N} - (1+\eta)\right] \tag{4.14}$$

where $g(t) := \dfrac{f(t)}{\mu}$ and $C^* := \dfrac{1}{2\Gamma(e)\mu} \displaystyle\int_{\partial^* W_\Gamma} \Gamma(n_{W_\Gamma})\, dH_{N-1}$. Let $F(s) := s^{(N-1)/N} + (1-s)^{(N-1)/N} - (1+\eta)$ for $0 \leq s \leq 1$ and let $0 < s(\eta) < 1/2$ be such that $F(s) > 0$ if and only if $s(\eta) < s < 1 - s(\eta)$. It is clear that

$$s(\eta) \to 0 \text{ as } \eta \to 0^+. \tag{4.15}$$

Let $t_i = t_i(E, \eta, e)$, $i = 1, 2$, be such that $g(t_1) = s(\eta)$ and $g(t_2) = 1 - s(\eta)$. We obtain

$$\operatorname{meas}(E \setminus \{x \in \mathbb{R}^N \mid t_1 \leq x.e \leq t_2\}) = \operatorname{meas}(E) - [f(t_2) - f(t_1)]$$

$$= \mu - \mu[1 - 2s(\eta)] = 2\mu\, s(\eta). \tag{4.16}$$

On the other hand, by (4.14)

$$C^*(t_2 - t_1) \leq \int_{t_1}^{t_2} \frac{g'(t)}{g(t)^{(N-1)/N} + (1-g(t))^{(N-1)/N} - (1+\eta)}\, dt$$

$$= 2\int_{s(\eta)}^{1/2} \frac{ds}{s^{(N-1)/N} + (1-s)^{(N-1)/N} - (1+\eta)}. \tag{4.17}$$

Finally, let $\{e_1, e_2, \ldots, e_N\}$ be the canonical orthonormal basis of $\mathbb{R}^N$ and consider $a \in \mathbb{R}^N$ and $R>0$ large enough so that

$$B(a, R) \supset \{x \in \mathbb{R}^N \mid t_1(E, \eta, e_i) \leq x.e_i \leq t_2(E, \eta, e_i),\ i = 1, \ldots, N\}.$$

By (4.17) we can obtain R as a continuous function $C_2(\eta)$, independent of the set E, and by (4.16) we deduce that

$$\operatorname{meas}(E \setminus B(a, R)) \leq \operatorname{meas}(E \setminus \{x \in \mathbb{R}^N \mid t_1(\eta, e_i) \leq x.e_i \leq t_2(\eta, e_i),\ i = 1, \ldots, N\})$$

$$\leq \sum_{i=1}^{N} \text{meas}(E \setminus \{x \in \mathbb{R}^N \mid t_1(\eta, e_i) \leq x.e_i \leq t_2(\eta, e_i)\})$$
$$\leq 2N\mu\, s(\eta) =: c_1(\eta)$$

which, together with (4.15) concludes the proof.

(ii) Suppose that $E \subset \mathbb{R}^N$ is a bounded, measurable set of finite perimeter with meas(E) = meas(W_Γ), satisfying (4.11). If $c_1(\eta) \geq$ meas(E) then choose any $a \in \mathbb{R}^N$. Assume that $c_1(\eta) <$ meas(E). By Lemma 2.9 and Proposition 4.7 there exists a sequence of smooth, open, bounded sets E_n with meas(E_n) = meas(W_Γ), such that $E_n, E \subset B(0, R)$ for some $R > 0$, meas($E_n \setminus E$) + meas($E \setminus E_n$) $\to 0$ and

$$\int_{\partial^* E} \Gamma(n_E(x))\, dH_{N-1}(x) = \lim_{n \to +\infty} \int_{\partial^* E_n} \Gamma(n_{E_n}(x))\, dH_{N-1}(x).$$

Moreover, it follows from the construction of the sets E_n that they verify the condition (4.11) (see Appendix). Let $\eta' > \eta$ be such that $c_1(\eta') <$ meas(E). For n large enough we have (4.13) for E_n and η' and so, by (i) there exist a_n such that

$$\text{meas}(E_n \setminus B(a_n, C_2(\eta'))) \leq c_1(\eta'). \tag{4.18}$$

As $E_n \subset B(0, R)$, it follows that $B(0,R) \cap B(a_n, C_2(\eta')) \neq \emptyset$ and so $\|a_n\| \leq R + C_2(\eta') \leq C = C(\eta)$. Thus, there exists a subsequence of $\{a_n\}$, $\{a'_n\}$ such that $a'_n \to a(\eta')$ with $a(\eta') \in \bar{B}(0, R + C_2(\eta'))$ and by (4.18)

$$\text{meas}(E \setminus B(a(\eta'), C_2(\eta'))) \leq c_1(\eta').$$

Letting $\eta' \to \eta^+$ and due to the continuity of c_1 and C_2, we have that for some subsequence $\{a(\eta')\}$, $a(\eta') \to a$ with meas($E \setminus B(a, C_2(\eta))$) $\leq c_1(\eta)$.

(iii) Assume that $E \subset \mathbb{R}^N$ is an unbounded, measurable set of finite perimeter with meas(E) = meas(W_Γ), satisfying (4.11). As in (ii), if $c_1(\eta) \geq$ meas(E) then choose any $a \in \mathbb{R}^N$. Suppose that $c_1(\eta) <$ meas(E) and as in the proof of Theorem 4.3 (iii), let $E_n := E \cap B(0,R_n)$, where $R_n \to +\infty$ and

$H_{N-1}(E \cap \partial B(0,R_n)) \to 0$. By (4.1) and Theorem 4.6 we have

$$\int_{\partial^* E} \Gamma^{**}(n_E(x))\, dH_{N-1}(x) \leq \lim_{n \to \infty} \int_{\partial^* E_n} \Gamma^{**}(n_{E_n}(x))\, dH_{N-1}(x)$$
$$\leq \int_{\partial^* E} \Gamma^{**}(n_E(x))\, dH_{N-1}(x) + \alpha \lim_{n \to \infty} H_{N-1}(E \cap \partial B(0,R_n))$$

and so,

$$\int_{\partial^* E} \Gamma^{**}(n_E(x))\, dH_{N-1}(x) = \lim_{n \to \infty} \int_{\partial^* E_n} \Gamma^{**}(n_{E_n}(x))\, dH_{N-1}(x).$$

Setting $E'_n = E_n/\lambda_n$, where meas(E_n) = λ_n^N meas(W_Γ), then $\lambda_n \to 1$ and

$$\lim_{n\to\infty}\int_{\partial^*E_n'}\Gamma^{**}(n_{E_n'}(x))\,dH_{N-1}(x) = \lim_{n\to\infty}\frac{1}{\lambda^{N-1}}\int_{\partial^*E_n}\Gamma^{**}(n_{E_n}(x))\,dH_{N-1}(x)$$
$$= \int_{\partial^*E}\Gamma^{**}(n_E(x))\,dH_{N-1}(x).$$

By Proposition 3.6 (iv) and by Theorem 4.3, (4.13) still holds for E, η and Γ^{**}. Hence, and as in (ii), for all $\eta' > \eta$ such that $c_1(\eta') < \text{meas}(E)$ there exist a_n such that

$$\text{meas}(E_n \setminus B(a_n, C_2(\eta'))) \leq c_1(\eta') \text{ for all n large enough.}$$

It remains to show that $||a_n||$ is bounded independently of η'. Indeed, let $c_1(\eta') \leq k$, $C_2(\eta') \leq K$ and choose $R > 0$ such that

$$\text{meas}(E \setminus B(0, R)) < \text{meas}(E) - k. \tag{4.19}$$

We claim that $||a_n|| \leq R + K$ for n large enough. Indeed, if for some subsequence $||a_n|| > R + K$ then $B(a_n, C_2(\eta')) \cap B(0, R) = \varnothing$ and so

$$\text{meas}(E \setminus B(0, R)) = \lim \text{meas}(E_n \setminus B(0, R)) \geq \limsup \text{meas}(E_n \cap B(a_n, C_2(\eta')) =$$
$$= \text{meas}(E) - \liminf \text{meas}(E_n \setminus B(a_n, C_2(\eta')) \geq \text{meas}(E) - k$$

which contradicts (4.19).

5. THE WULFF THEOREM: UNIQUENESS.

For completeness, we prove uniqueness for the Wulff Theorem, i. e. we show that the Wulff set or translations of it are the only solutions of the geometric variational problem

(P) minimize

$$\int_{\partial^*E}\Gamma(n_E(x))\,dH_{N-1}(x)$$

among all measurable sets $E \subset \mathbb{R}^N$ of finite perimeter with $\text{meas}(E) = \text{meas}(W_\Gamma)$. DINGHAS [14] obtained uniqueness within the class of polehedra and TAYLOR [36] extended this result using geometrical arguments. DACOROGNA & PFISTER [12] provided a proof that is entirely analytical but it cannot be extended to higher dimensions. Moreover, uniqueness is obtained for a class strictly contained in the one considered in this work. The proof presented here is due to FONSECA & MÜLLER [20] and, as in TAYLOR [36] is based on the Brunn-Minkowski Theorem and on the existence of an inverse for the Radon transform (see GEL'FAND, GRAEV & VILENKIN [21]).

Proposition 5.1.

Let $E \subset \mathbb{R}^N$ be a measurable set of finite perimeter. If E is a solution of (P) then $E = E_1 \cup E_2$ where $E_1 \cap E_2 = \varnothing$, $\text{meas}(E_2) = 0$, $H_{N-1}(\partial^*E_2) = 0$, E_1 is bounded and satisfies (4.11).

We divide the proof of this proposition into two lemmas.

Lemma 5.2.

If E is a solution of (P) then E = Ei u E_2 where EinE$_2$ = 0, meas(E_2) = 0, H_N_i(9*E_2) = 0 and Ei satisfies (4.11).

Proof, If E verifies (4.11) set Ei = E and E_2 = 0. Otherwise E admits a partition E = Ei u E_2, where EinE$_2$ = 0 and H_N-i(9*Ein3*E_2) = 0. We claim that either meas(Ei) = 0 or meas(E_2) = 0. Indeed, if meas(Ei) = 0 meas(E) and meas(E_2) = (1 - 0) meas(E) for some 0 e (0, l)then

f r(n_E(x)) dH^iCx) = f r(n_E(x)) dH^Cx) + f IWGO) dH^Cx) (5.3)

and as E is a solution of (P), by Corollary 4.4 we have

f Ja*w_r H n_w Wr GO) $dH_{N\_}$!(x) N l ≥ f 1/N Ja*($e^{1/N}$$w_r$) T(n) d H N ^ + f Ja*«i-e)$^{1/N}$$w_r$) * T(n) $dH_{N\_}$!

$$= (\theta^{(N-1)/N} + (1-\theta)^{(N-1)/N}) \int_{\bullet'a+Wp} \Gamma(n_{W_r}(x))\, dH_{N-1}(x)$$

which is impossible since 0 ^ 0,1. Therefore, we may suppose that meas(E_2) = 0 and so, by (5.3) and (4.1) we conclude that H_N_i(3*E_2) = 0.

Next, we show that E is bounded, up to a set of measure zero.

Lemma 5.4.

If E is a solution of (P) then meas (E \ B(0, R)) = 0 for R sufficiently large.

Proof. Setting Ti = 1/n on (4.13), by Proposition 4.12 it follows that

meas(E \ B(a_n, C_2(l/n)) -> 0. (5.5)

for some bounded sequence {a_n}. Indeed, let Ro be such that

meas(E \ B(0, Ro)) < meas(E)/2 (5.6)

and assume that, for some subsequence, ||a_n|| > Ro + K, where K is an upper bound for {C_2(l/n)}. Then

meas(E \ B(0, R_o)) ^ lim sup meas(E n B(a_n, C_2(l/n)) =

= meas(E) - liminf meas(E \ B(a_n, C_2(l/n)) = meas(E),

contradicting (5.6). Therefore, by (5.5) we have that for some a ***e*** R^N meas(E \ B(a, K)) = 0.

Proof of Proposition 5.1. Assume that E is a solution of (P). Then, by Lemmas 5.2 and 5.4 we can suppose that E satisfies (4.11) and meas(E \ B(0, R)) = 0 for R large enough. Therefore, given £ ***e*** S^{NA} by the Fleming-Rishel formula (2.7) we have

$$\text{meas}(E) = \int_{-\infty}^{+\infty} H_{N-1}(\{x \in E \mid x.\xi = s\})\, ds = \text{meas}(E \cap B(0,R))$$

$$= \int_{-R} H_{N-1}(\{x \in E \mid x.\xi = s\})ds$$

which implies that HN-I({X € E | x.£ = s}) = 0 for a. e. |s| > R. Choose Ro > R such that

H_N-i({xeE|x.ei = ±Ro})=O (5.7)

where {ei,..., c^} is the canonical basis of R^N. Then we can write E as a disjoint union of Ei and E2, where Ei := E n [-Ro, Ro]^N is bounded and connected and meas(E2) = 0. Moreover, by (5.7) it follows that H_N_i(3*Ei n d*E2) = 0 and so

f T(n_E(x)) dH^x) = f r(n_E(x)) dH^x) + f · IW00) dH^x).

Since meas(Ei) = meas(E) and E is a solution of (P), by (4.1) we conclude that HN-I(9*E2) = 0

Let A C $\mathbb{R}^N$ be a bounded, connected set and let £ e S^{N}~^{l}. As in (4.11) we define

$$G_{A,\xi}(s) := \frac{\text{meas}(\{x \in A \mid x.\xi < s\})}{\text{meas}(A)} \quad \text{and} \quad g_{A,\xi}(s) := \frac{H_{N-1}(\{x \in A \mid x.\xi = s\})}{\text{meas}(A)}.$$

Lemma 5,8.

If A and B are strongly Lipschitz, bounded sets satisfying (4.11) and if meas(A) = meas(B) then for all e > 0

meas(A + εB) ≥ meas(A) rr. u f ^ r r r + ε $\frac{\gamma_{A,\xi}(t)}{\gamma_{B,\xi}(t)}$) dt

Jb v 1tóJ J {

where $\gamma_{A,\xi}(t) := g_{A,\xi}(G^{-1}_{A,\xi}(t))$.

Proof. By the Fleming-Rishel formula (2.7) we have

$$G_{A,\xi}(s) = \int_{-\infty}^{s} g_{A,\xi}(\lambda)\, d\lambda$$

and so, 0 ≤ $G_{A>}$£ ≤ 1 and $G'_{A,\xi}$(s) = g_A,^(s). **Let** so := sup {s | $G_{A(}$^(s) = 0} and si := sup {s | GA,^(S) = 1}. As A is bounded, open and connected, -°o < so < s\ < +«> and gA,^(s) > 0 in (so, si). Therefore, G_A£ admits an inverse G_A^{-1}^ : (0, 1) -> (so, si) and, setting YA,^(t) := gA^CG^$_\xi$(t)) we obtain

$$\frac{dG^{-1}_{A,\xi}}{dt}(t) = \frac{1}{\gamma_{A,\xi}(t)} \tag{5.9}$$

We can assume, without loss of generality, that \ = ei and write x = (xi, x^1). Let

A_t := {x^1 e RN-i | (x_1, x') e A and x_2 = G'.^Ct)}, for t e (0, 1).

As

{G'^(t) + G^Ct)} x (A_t + BO C A + B,

setting z(t) := G^(t) + Gg^t) by (5.9) we have

meas(A + B) ≥ I fZ(l) Jz(O) H_{N-1}(A C *) + B -1 Z ^ d s

$$= \int_0^1 H_{N-1}(A_t + B_t)\, z'(t)\, dt$$

= I ^1 ^{H}N-l<^{A}t + ^{B}t) f 1 TT + 1 FT 1 dt.

By the Brunn-Minkowski Theorem (see Theorem 4.2)

H_{N}_i(A_t + BON-^{1} ≥ H_N.!(A_t)N-i + H_N-i(B_t)N-i

= (YA£(t) measCA))^{1}^"^{1} + (YB^CO meas(B))$^{N"!}$

and so

meas(A + B) ≥ f [yi/^-$^{1)}$(t)meas(A) + ^/>(|J-$^{1)}$(t)meas(B)]$^{N"1}$(1 / v. -(f) + — ^ J dt (5.10)

It is easy to verify that

g£B,^(s) = g£B,£(s/£)/£, G£B,^(S) = GB,^(S/£) and $\gamma_{\varepsilon B,\xi}(t) = \gamma_{B,\xi}(t)/\varepsilon$

which, together with (5.10) imply

meas(A + eB) ≥ meas(A) [bfA^it) + e ^T"^{1}^^""""1, " " ^ + " ^ ^ dt

$$= \text{meas}(A) \int_0^1 \left(1 + \varepsilon \left(\frac{\gamma_{B,\xi}(t)}{\gamma_{A,\xi}(t)}\right)^{\frac{1}{N-1}}\right)^{N-1} \left(1 + \varepsilon \frac{\gamma_{A,\xi}(t)}{\gamma_{B,\xi}(t)}\right) dt.$$

Theorem 5.11.
If E is a solution of (P) then || % E + c - XWJJIL^1 = °»where

c:= 1 / meas(W_r)^Jw_r ATTTI (∫ x d x - ∫ JE x d x l)

Proof. Let E be a solution of (P) and consider the translated sets E^1 := E - a and W := W - b, where

a : = I f / meas(W_r)JE T^TT xdxandb:= I f / meas(W_r)Jw_r 7^T x d x - (5.12)

By Proposition 5.1 we can suppose that E is bounded and satisfies (4.11). Hence, by Lemma 2.9 there exists a sequence of smooth, open, bounded sets E_nC $\mathbb{R}^N$ verifying (4.11) such that E_n, E'

$\subset B(0, R)$ for some $R > 0$, $meas(E_n) = meas(E')$, $Per(E_n) \to Per(E')$ and $meas(E_n \setminus E') + meas(E' \setminus E_n) \to 0$. By Lemma 4.8 and Lemma 5.8 we have

$$\int_{\partial^* E_n} \Gamma(n_{E_n}(x))\, dH_{N-1}(x) \ge \liminf_{\varepsilon \to 0} \frac{meas(E_n + \varepsilon W) - meas(W)}{\varepsilon}$$

$$\ge meas(W_\Gamma) \liminf_{\varepsilon \to 0} \int_0^1 \frac{\left(1 + \varepsilon \left(\frac{\gamma_{W',\xi}(t)}{\gamma_{E_n,\xi}(t)}\right)^{\frac{1}{N-1}}\right)^{N-1} \left(1 + \varepsilon \frac{\gamma_{E_n,\xi}(t)}{\gamma_{W',\xi}(t)}\right) - 1}{\varepsilon}\, dt$$

$$= meas(W_\Gamma) \int_0^1 \left[(N-1) \left(\frac{\gamma_{W',\xi}(t)}{\gamma_{E_n,\xi}(t)}\right)^{1/(N-1)} + \frac{\gamma_{E_n,\xi}(t)}{\gamma_{W',\xi}(t)} \right] dt.$$

As $g_{E_n,\xi}(s) = 0$ if $|s| > R$, setting $t = G_{E_n,\xi}(s)$ we obtain

$$\int_{\partial^* E_n} \Gamma(n_{E_n}(x))\, dH_{N-1}(x) \ge$$

$$\ge meas(W_\Gamma) \int_{-R}^{R} \left[(N-1) \left(\frac{\gamma_{W',\xi}(G_{E_n,\xi}(s))}{g_{E_n,\xi}(s)}\right)^{\frac{1}{N-1}} + \frac{g_{E_n,\xi}(s)}{\gamma_{W',\xi}(G_{E_n,\xi}(s))} \right] g_{E_n,\xi}(s)\, ds. \qquad (5.13)$$

On the other hand, as $meas(E_n \setminus E') + meas(E' \setminus E_n) \to 0$ we have $\|g_{E_n,\xi} - g_{E',\xi}\|_{L^1} \to 0$ and $\|G_{E_n,\xi} - G_{E',\xi}\|_\infty \to 0$ and so, Proposition 4.7, (5.13) and Fatou's Lemma we conclude that

$$\int_{\partial^* E'} \Gamma(n_{E'}(x))\, dH_{N-1}(x) \ge$$

$$\ge meas(W_\Gamma) \int_{-R}^{R} \left[(N-1) \left(\frac{\gamma_{W',\xi}(G_{E',\xi}(s))}{g_{E',\xi}(s)}\right)^{\frac{1}{N-1}} + \frac{g_{E',\xi}(s)}{\gamma_{W',\xi}(G_{E',\xi}(s))} \right] g_{E',\xi}(s)\, ds.$$

As E' satisfies (4.11), $G_{E',\xi}$ is strictly increasing in $(s_0(E',\xi), s_1(E',\xi)) \subset (-R, R)$ and so, by the change of variables formula (2.8), by Theorem 4.3 and by Proposition 3.6 (iv) we have

$$N\, meas(W_\Gamma) = \int_{\partial^* W_\Gamma} \Gamma(n_{W_\Gamma}(x))\, dH_{N-1}(x) = \int_{\partial^* E'} \Gamma(n_{E'}(x))\, dH_{N-1}(x) \ge$$

$$\ge meas(W_\Gamma) \int_0^1 \left[(N-1) \left(\frac{\gamma_{W',\xi}(t)}{\gamma_{E',\xi}(t)}\right)^{1/(N-1)} + \frac{\gamma_{E',\xi}(t)}{\gamma_{W',\xi}(t)} \right] dt. \qquad (5.14)$$

However $(N-1)\, a^{1/(N-1)} + 1/a \ge N$ and equality holds only if $a = 1$. Thus (5.14) implies that

$\gamma_{E',\xi}(t) = \gamma_{W',\xi}(t)$ for almost all $t \in (0, 1)$

which, by (5.9) yields

$G^{-1}_{E',\xi}(t) = G^{-1}_{W',\xi}(t) + C$ for some constant C and for almost all t.

Hence

$G_{E',\xi}(s + C) = G_{W',\xi}(s)$ for almost all s

which, after differentiating, implies that

$$g_{E',\xi}(s + C) = g_{W',\xi}(s) \text{ for a. e. } s \in \mathbb{R}. \tag{5.15}$$

We claim that C = 0. Indeed, by (5.12) and (5.15)

$$0 = \int_{E'} x.\xi \, dx = \int_{-\infty}^{+\infty} s \, g_{E',\xi}(s) \, ds = \int_{-\infty}^{+\infty} (s + C) \, g_{E',\xi}(s + C) \, ds$$

$$= \int_{-\infty}^{+\infty} s \, g_{E',\xi}(s + C) \, ds + C\int_{-\infty}^{+\infty} g_{E',\xi}(s + C) \, ds$$

$$= \int_{W'} x.\xi \, dx + C \, \text{meas}(W_\Gamma) = C \, \text{meas}(W_\Gamma).$$

Thus,

$$g_{E',\xi}(s) = g_{W',\xi}(s) \text{ for a. e. } s \in \mathbb{R} \text{ and for all } \xi \in S^{N-1},$$

and so, due to the existence of the inverse of the Radon transform (see GEL'FAND, GRAEV & VILENKIN [21]) we conclude that $\|\chi_{E'} - \chi_{W'}\|_{L^1} = 0.$

6. CHARACTERIZATION OF MINIMIZING SEQUENCES.

Consider the problem

(P) minimize

$$\int_{\partial^* E} \Gamma(n_E(x)) \, dH_{N-1}(x)$$

in $\mathcal{Q} := \{E \subset \mathbb{R}^N \mid E$ is measurable, bounded set, $\text{Per}(E) < +\infty$, $\text{meas}(E) = \text{meas}(W_\Gamma)$ and E satisfies (4.11)$\}$[5]. In this section we characterize the minimizing sequences in terms of the support of the indicator measure[6]. This will allow us to determine if oscillations may be present and, in particular, we show that if the Wulff set is polyhedral then there are no oscillations[7].

Proposition 6.1.

If$\{E_\varepsilon\}$ is a minimizing sequence for (P), then there exist translations $E_\varepsilon - a_\varepsilon$ such that $\text{meas}((E_\varepsilon - a_\varepsilon)\setminus W_\Gamma) + \text{meas}(W_\Gamma \setminus (E_\varepsilon - a_\varepsilon)) \to 0.$

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[5]. By Proposition 3.6, Theorem 4.3 and Theorem 5.11 we know that the solutions of (P) are translations of the Wulff set and are also the solutions of the relaxed problem

(P**) To minimize in $\mathcal{Q}$ $\int_{\partial^* E} \Gamma^{**}(n_E(x)) \, dH_{N-1}(x).$

The class of admissible sets $\mathcal{Q}$ is chosen taking into account Proposition 5.1.

[6]. See Propositions 4.5 and 4.7.

[7]. Thus, if in Example 5.10 in FONSECA [19] we take E to be the Wulff set of Γ, where $\Gamma(n) := \sup \{x.n \mid x \in E\}$, then $\{E_k\}$ cannot be a minimizing sequence.

Proof. As E_ε are connected, by (4.12) there exist $R > 0$ and $b_\varepsilon \in \mathbb{R}^N$ such that $\text{meas}(E_\varepsilon \backslash B(b_\varepsilon,R)) \to 0$. Set

$$c_\varepsilon := \frac{1}{\text{meas}(W_\Gamma)}\left(\int_{(E_\varepsilon - b_\varepsilon)\cap B(0,R)} x\,dx - \int_{W_\Gamma} x\,dx\right). \tag{6.2}$$

Consider a subsequence of $\{E_\varepsilon\}$. As $\{c_\varepsilon\}$ is bounded, there exists a subsequence (which we still denote by $\{c_\varepsilon\}$) such that $c_\varepsilon \to c$ with

$$\text{meas}((E_\varepsilon - a_\varepsilon)\backslash B(c,R)) \to 0, \text{ where } a_\varepsilon := b_\varepsilon + c_\varepsilon. \tag{6.3}$$

On the other hand, by Proposition 2.6 and as $\{\text{Per}(E_\varepsilon)\}$ is bounded, the sequence $\{\chi_{E_\varepsilon - a_\varepsilon \cap B(c;\,R)}\}$ is bounded in $BV(B(c; R))$, and so by Proposition 2.3 we conclude that there is a subsequence (for convenience we use the same notation) such that

$$\chi_{E_\varepsilon - a_\varepsilon \cap B(c;\,R)} \to \chi_K \quad \text{strongly in } L^1, \text{ for some set } K. \tag{6.4}$$

By (6.3) we conclude that

$$\chi_{E_\varepsilon - a_\varepsilon} \to \chi_K \quad \text{strongly in } L^1 \tag{6.5}$$

which, together with Proposition 3.6 (iv) and Theorem 4.6 implies

$$\begin{aligned}\int_{\partial^* K} \Gamma^{**}(n_K(x))\, dH_{N-1}(x) &\le \liminf_{\varepsilon\to 0} \int_{\partial^* E_\varepsilon} \Gamma^{**}(n_\varepsilon(x))\, dH_{N-1}(x)\\ &\le \liminf_{\varepsilon\to 0} \int_{\partial^* E_\varepsilon} \Gamma(n_\varepsilon(x))\, dH_{N-1}(x)\\ &= \int_{W_\Gamma} \Gamma(n_{W_\Gamma}(x))\, dH_{N-1}(x) = \int_{W_\Gamma} \Gamma^{**}(n_{W_\Gamma}(x))\, dH_{N-1}(x).\end{aligned}$$

Finally, by Theorem 5.11 we have $\text{meas}((K + k)\backslash W_\Gamma) + \text{meas}(W_\Gamma \backslash (K + k)) = 0$, where

$$k := \frac{1}{\text{meas}(W_\Gamma)}\left(\int_{W_\Gamma} x\,dx - \int_K x\,dx\right).$$

By (6.2) and (6.4) k must be zero and the result follows from 6.5.

Using the same notation as in Proposition 6.1, we characterize the indicator measure of a minimizing sequence.

Theorem 6.6.

Let $\{E_\varepsilon\}$ is a minimizing sequence for (P) and let $\Lambda_\varepsilon \equiv \lambda_x^\varepsilon \otimes \pi_\varepsilon$ be the indicator measure associated to $E_\varepsilon - a_\varepsilon$. There exists a subsequence converging weakly * in the sense of measures to $\Lambda_\infty \equiv \lambda_x^\infty \otimes \pi_\infty$, where for π_∞ a. e. $x \in \mathbb{R}^N$

support $\lambda_x^\infty \subset \{y \in S^{N-1} \mid \Gamma^{**}(-y) = \Gamma(-y) = -x'.\,y$ for all $x' \in \partial^* W_\Gamma$ such that $n_{W_\Gamma}(x)$ and $\pi_\infty << dH_{N-1} \lfloor \partial^* W_\Gamma << \pi_\infty$.

Proof. By Proposition 4.5 (i) the total variation of Λ_ε is equal to $\mathrm{Per}(E_\varepsilon)$. Thus, as $\{\mathrm{Per}(E_\varepsilon)\}$ is bounded, $\{\Lambda_\varepsilon\}$ admits a weakly * convergent subsequence (still denoted by $\{\Lambda_\varepsilon\}$), $\Lambda_\varepsilon \to \Lambda_\infty \equiv \lambda_x^\infty \otimes \pi_\infty$ weakly *. Also, by Proposition 3.6 and as $\{E_\varepsilon\}$ is a minimizing sequence for (P) we obtain

$$\int_{\mathbb{R}^N x S^{N-1}} \Gamma^{**}(-y)\ d\Lambda_\infty(x, y) = \lim_{\varepsilon \to 0} \int_{\partial^* E_\varepsilon} \Gamma^{**}(n_\varepsilon(x))\, dH_{N-1}(x)$$

$$= \lim_{\varepsilon \to 0} \int_{\partial^* E_\varepsilon} \Gamma(n_\varepsilon(x))\, dH_{N-1}(x)$$

$$= \int_{\mathbb{R}^N x S^{N-1}} \Gamma(-y)\ d\Lambda_\infty(x, y)$$

i. e.

$$\int_{\mathbb{R}^N} \left(\int_{S^{N-1}} [\, \Gamma(-y) - \Gamma^{**}(-y)\,]\, d\lambda_x^\infty(y) \right) d\pi_\infty(x) = 0.$$

Since $\Gamma \geq \Gamma^{**}$, we deduce that for π_∞ a. e. $x \in \mathbb{R}^N$

$$\text{support } \lambda_x^\infty \subset \{y \in S^{N-1} \mid \Gamma^{**}(y) = \Gamma(y)\}. \tag{6.7}$$

Let us define the nonnegative Radon measures η_ε by

$$< \eta_\varepsilon , \varphi > = \int_{\partial^*(E_\varepsilon - a_\varepsilon)} \varphi(x)\, \Gamma^{**}(n_{E_\varepsilon - a_\varepsilon}(x))\ dH_{N-1}(x) \quad \text{for all } \varphi \in C_0(\mathbb{R}^N).$$

By (6.5) and by Theorem 4.6, we have

$$\eta_{W_\Gamma} \leq \liminf \eta_\varepsilon \text{ with } \eta_{W_\Gamma}(\mathbb{R}^N) = \lim \eta_\varepsilon(\mathbb{R}^N)$$

and so

$$\eta_{W_\Gamma} = \lim \eta_\varepsilon.$$

Therefore, if $\varphi \in C_0(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} \varphi(x)\, d\eta_{W_\Gamma}(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \varphi(x)\, d\eta_\varepsilon(x) = \lim_{\varepsilon \to 0} \int_{\partial^*(E_\varepsilon - a_\varepsilon)} \varphi(x)\, \Gamma^{**}(n_{E_\varepsilon - a_\varepsilon}(x))\ dH_{N-1}(x)$$

$$= \int_{\mathbb{R}^N} \varphi(x) \left(\int_{S^{N-1}} \Gamma^{**}(-y)\ d\lambda_x^\infty(y) \right) d\pi_\infty(x)$$

i. e.

$$d\eta_{W_\Gamma} = \Gamma^{**}(n_{W_\Gamma}(x))\ dH_{N-1} \llcorner \partial^* W_\Gamma = \left(\int_{S^{N-1}} \Gamma^{**}(-y)\ d\lambda_x^\infty(y) \right) d\pi_\infty.$$

On the other hand, by Proposition 4.5

$$dH_{N-1} \llcorner \partial^* W_\Gamma = \|\nu_\infty\|\, d\pi_\infty \text{ and } \nu_\infty = - \|\nu_\infty\|\, n_{W_\Gamma},$$

and since Γ^{**} is homogeneous of degree one, we have

$$\Gamma^{**}(-\nu_\infty(x))\ d\pi_\infty(x) = \Gamma^{**}(n_{W_\Gamma}(x))\ dH_{N-1} \llcorner \partial^* W_\Gamma = \left(\int_{S^{N-1}} \Gamma^{**}(-y)\ d\lambda_x^\infty(y) \right) d\pi_\infty(x)$$

or, for π_∞ a. e. $x \in \mathbb{R}^N$

$$\Gamma^{**}(-\nu_\infty(x)) = \int_{S^{N-1}} \Gamma^{**}(-y)\ d\lambda_x^\infty(y). \tag{6.8}$$

Let $x' \in \partial^* W_\Gamma$ be such that $-\nu_\infty(x)$ is normal to $\partial^* W$ at x'. By Proposition 3.5 we have $x' \in \partial\Gamma^{**}(-\nu_\infty(x))$ and so, for all $y \in S^{N-1}$ there exists $\theta(-y) \geq 0$ such that

$$\Gamma^{**}(-y) = \Gamma^{**}(-\nu_\infty(x)) + x'.(-y + \nu_\infty(x)) + \theta(-y). \tag{6.9}$$

As λ_x^∞ is a probability measure and as $\nu_\infty(x)$ is its center of mass, (6.8) and (6.9) imply that $\theta(-y) = 0$ for λ_x^∞ a. e. $y \in S^{N-1}$, i. e.

$$\Gamma^{**}(-y) = \Gamma^{**}(-\nu_\infty(x)) + x'.(-y + \nu_\infty(x)) \tag{6.10}$$

which, together with (6.7) and Proposition 3.6 (iv), yields

support $\lambda_x^\infty \subset \{y \in S^{N-1} \mid \Gamma^{**}(-y) = \Gamma(-y) = -x'.y$ for all $x' \in \partial^* W_\Gamma$ such that $\eta_{W_\Gamma}(x)$

Finally, by (4.1) we have $W_\Gamma \supset B(0,\alpha)$ and so, by Proposition 3.6 (iii)

$\Gamma^{**}(n) \geq \alpha$ for all $n \in S^{N-1}$

which, together with (6.8) and as λ_x^∞ is a probability measure, implies that

$\|\nu_\infty\| \neq 0$ for π_∞ a. e. $x \in \mathbb{R}^N$.

From Proposition 4.5 (ii) we conclude that

$$\pi_\infty << dH_{N-1} \llcorner \partial^* W_\Gamma << \pi_\infty.$$

Next, we provide some examples in which minimizing sequences cannot oscillate, i. e.

$$\chi_{E_\varepsilon} \to \chi_{W_\Gamma} \text{ strongly in } L^1 \text{ and } \mathrm{Per}(E_\varepsilon) \to \mathrm{Per}(W_\Gamma). \tag{6.11}$$

Proposition 6.12.

Let Γ^{**} be strictly convex except radially[8], i. e. $\Gamma^{**}(x) = \Gamma^{**}(x_0) + y.(x - x_0)$ for some $y \in \partial\Gamma^{**}(x_0)$ if and only if x is parallel to x_0. If $\{E_\varepsilon\}$ is an uniformly bounded minimizing sequence then (6.11) holds.

Proof. By Proposition 2.3, Proposition 3.6 (v), Theorem 4.6 and Theorem 5.11, given any subsequence there exists a subsequence (which we still denote by $\{E_\varepsilon\}$) and a translation of W_Γ (without loss of generality we may assume that it is W_Γ) such that

$$\chi_{E_\varepsilon} \to \chi_{W_\Gamma} \text{ strongly in } L^1.$$

By Proposition 4.7 and as λ_x^∞ is a probability measure, it suffices to show that supp λ_x^∞ cannot have two distinct points for π_∞ a. e. x. From (6.10) we have that if $y, y' \in$ support λ_x^∞, then

$$\Gamma^{**}(-y) = \Gamma^{**}(-\nu_\infty(x)) + x'.(-y + \nu_\infty(x))$$

which implies that y is parallel to $\nu_\infty(x)$ and, in a similar way, y' is parallel to $\nu_\infty(x)$. As supp $\lambda_x^\infty \subset S^{N-1}$, we conclude that either y = y' or y = -y'. We claim that y = y'. Indeed, if y' = -y, by Theorem 6.6 we have

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[8]. This is the case when Γ^{**} is identically equal to 1. Then Γ^{**} is called the *area integrand* and $W_\Gamma = B(0; 1)$.

$T^{**}(-y^1) = -x.y' = x.y = - T^{**}(y)$
contradicting (4.1).

Proposition 6.13.

If the Wulff set is polyhedral and if $\{E_e\}$ is an uniformly bounded minimizing sequence then (6.11) holds.

Proof. As in the proof of Proposition 6.12, it suffices to show that supp $A_{\tilde{x}}$ reduces to a point for %<*> a. e. x. Suppose that Wp has faces Fi,..., F_p with outward unit normal respectively ni,..., n_p. By Theorem 6.6 and as 7Go « CIHN-I L 3*Wp « TGo, we must prove that if x e Fi then supp X^° = {-ni}. Indeed, if y *e* supp A_x~ then y **e** S^{N-1} and by Theorem 6.6 we have

$T^{**}(-y) = -x^f.y$ for all x^1 **e** Fi

i. e. $x^f.y$ is constant on Fi which implies that y = ± n^ If y = ni then by Proposition 3.6 (iv)

$T^{**}(-y) = -xjy = F^{**}$ (nO

which is impossible since F^{**} is strictly positive on the unit sphere.

Remark 6.14.

The cases presented on the previous two propositions are actually distinct. In fact, if T(x) = $F^{**}(x) = ||x||$ satisfies the hypotheses of Proposition 6.12 although Wp = B(O,1) is not a polyhedron. Conversely, if in $\mathbb{R}^2$ we consider the square C := {(xi, X2) | |xi| ≤ 1 and |X2| ≤ 1} then by Proposition 3.6 (iii)

$F^{**}(a, b) = \max\{axi + bx_2 1\ (x_b\ x_2)$ **e** $C\} = |a| + |b|$

which is not strictly convex.

7. A VARIATIONAL PROBLEMS INVOLVING BULK AND INTERFACIAL ENERGIES.

The total energy for materials that can change phase involves bulk and interfacial contributions (see FONSECA [17], [18], GURTIN [23], [24], KINDERLEHRER & VERGARA-CAFFARELLI [26]). Recently, variational problems for functional of this type have been investigated by BOUCHITTE [8] and OWEN & STERNBERG [30] who, using singular perturbations and the F - convergence approach, show that the Wulff set is the selected shape for a scalar-valued two-phases transition problem with infinitely many solutions. Recently, this result was generalized to the vector-valued case by AMBROSIO, MORTOLA & TORTORELLI [4] and AVILHS & GIGA [5]. Here we will study a model where the Wulff set appears as the minimizing configuration. Associated to an elastic body immersed on a melt with zero bulk energy we consider the energy

$$E_w(u) := J_{\Omega} W(Vu(x))\ dx + iQu(Q)) \qquad (7.1)$$

where *Q,* C IR^N is the reference configuration, u : *£1* —» K^N is a deformation of the solid, W : M\S(NxN,+) —> [0, +©o) is the stored energy density, $M^{N\times N}$ denotes the set of all NxN real matrices, M_+^{NxN} := {F e $M^{N\times N}$ | det F > 0}, and I(.) is a nonlocal surface energy given by

$$IOA) := \inf 1\ \underset{lya(RA)}{f}\ H\, n \wedge y))\ dH_{N\_1}(y) \mid R \text{ is a rotation in } K^N 1. \qquad (7.2)$$

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As before, *T:* S^{N-1} —> [a, +<*>) is continuous, a > 0, and nA(y) denotes the outward unit normal to 3Aaty.

We will prove that (7.1) admits a minimizer on a suitable class of admissible deformations Q. Moreover, if $u_£$ is a minimizer for $E_£$wC) then the deformed configurations *n£(Q)* approach a set C geometrically similar to the Wulff set Wp, i. e. C = R(Wp - a) for some rotation R and some a e [R^N. We assume that *Q,* is an open bounded strongly Lipschitz domain and we define

Q := { u e $W^{1)p}$(Q; IR^N) | det Vu > 0 a.e. in *Q,* Per (u(Q)) <+oo, | $_u$(x) dx = m and

$$\int_{ft} \backslash\ \det Vu(x)\ dx \leq \text{meas } u(Q) = \text{meas}(W_r)l$$

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where *me* DR^N is fixed and p > N. It turns out that functions of Q are almost everywhere invertible[9]. Indeed,

Proposition 7.3 ([6], [27]).

Let *Q* C [R^N be an open bounded strongly Lipschitz domain, let p > N and let u e W ^ Q ; R^N) be such that det Vu > 0 a. e. Then

(i) u maps sets of measure zero into sets of measure zero;

(ii) u maps measurable sets to measurable sets;

(iii) meas u(Q) = meas *u(U);*

(iv) I det Vu(x) dx = f card{u~^{1}(y)} dy

whenever one of the two integrals exists;

(v) $\underset{MA)}{f}$ f(y) *caxdin^iy)}* dy = $\underset{JA}{f}$ f(u(x)) detVu(x) dx

for all A measurable set and whenever one of the two integrals exists.

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9. For results concerning invertibility of Sobolev functions, we refer the reader to BALL [6] and MARCUS & MIZEL [27].

Thus, if $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ and if $\det \nabla u > 0$ a. e. then

$$\text{meas}(u(\Omega)) = \int_{u(\Omega)} dy \le \int_{u(\Omega)} \text{card}\{u^{-1}(y)\}\, dy = \int_{\Omega} \det \nabla u(x)\, dx$$

and so, if $u \in \mathcal{Q}$ then

$$\text{meas}(W_r) = \text{meas}(u(\Omega)) = \int_{\Omega} \det \nabla u(x)\, dx \tag{7.4}$$

and

u is one-to-one a. e. in Ω, i. e. $\text{card}\{u^{-1}(y)\} = 1$ for almost all $y \in u(\Omega)$. (7.5)

This injectivity condition was used first by CIARLET & NECAS [9] (see also CIARLET [10]).

Definition 7.6. ([7])

$W : M_+^{N\times N} \to [0, +\infty)$ is said to be *polyconvex* [10] if there exists a convex function $G : M^{N\times N} \times M^{N\times N} \times (0, +\infty) \to \mathbb{R}$ such that $W(F) = G(F, \text{adj } F, \det F)$ for all $F \in M_+^{N\times N}$ [11].

The following lower semicontinuity result is well known (see BALL [7]).

Theorem 7.7.

Let $\Omega \subset \mathbb{R}^N$ be an open bounded strongly Lipschitz domain, let $p > N$ and let $u_j \in W^{1,p}(\Omega;\mathbb{R}^N)$. If W is a polyconvex function and if $u_j \rightharpoonup u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^N)$ then

$\det \nabla u_j \rightharpoonup \det \nabla u$ weakly in $L^{p/N}(\Omega)$, $\text{adj } \nabla u_j \rightharpoonup \text{adj } \nabla u$ weakly in $L^{p/(N-1)}(\Omega; M^{N\times N})$

and

$$\int_{\Omega} W(\nabla u(x))dx \le \liminf_{j\to\infty} \int_{\Omega} W(\nabla u_j(x))dx.$$

Assume that

(H1) $\Gamma(n) = \Gamma^{**}(n)$ for all $n \in S^{N-1}$;

(H2) the bulk energy density $W : M_+^{N\times N} \to [0, +\infty)$ is polyconvex and satisfies the growth condition $W(F) \ge C_1 \|F\|^p + C_2$ for some $C_1, C_2 \in \mathbb{R}$ with $C_1 > 0$. Moreover, $W(F) \to +\infty$ if $\det F \to 0+$ [12].

Theorem 7.8.

Under the hypotheses (H1) and (H2) the functional (7.1) admits a minimizer on $\mathcal{Q}$.

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[10]. For a detailed study of such functions we refer to BALL [7] and DACOROGNA [11]. See also MORREY [29].

[11]. Here adjF denotes the matrix of cofactors of F. If F is invertible then $F^{-1} = \frac{(\text{adj } F)^T}{\det F}$.

[12]. This condition is taken in order to prevent interpenetration of matter and change in orientation

Before proving this result, we study some properties of the surface energy I(.).

Proposition 7.9.

Let $E \subset \mathbb{R}^N$ be a bounded set of finite perimeter with meas(E) = meas(W_Γ). Then

(i) (*Frame indifference*) $I(\partial E) = I(\partial(RE))$ for all $R \in SO(N) := \{R \in M_+^{NxN} \mid R^T R = 1\!\!1\}$;

(ii) There exists $R_0 \in SO(N)$ such that

$$I(\partial E) := \int_{\partial^*(R_0E)} \Gamma(n_{R_0E}(y))\, dH_{N-1}(y);$$

(iii) $I(\partial E) \geq I(\partial W_\Gamma)$ and equality holds if and only if meas($E \setminus R(W_\Gamma - a)$) + meas($R(W_\Gamma - a)\setminus E$) =0 for some rotation R and some $a \in \mathbb{R}^N$;

(iv) $I(\partial E) \geq \alpha$ Per(E).

Proof. (i) Follows from the fact that SO(N) = SO(N)R for all $R \in SO(N)$.

(ii) Let $\{R_j\}$ be a minimizing sequence for $I(\partial E)$. There exists a subsequence R_j' such that $R_j' \to R_0$ $\in SO(N)$ and so, by (H1) and Theorem 4.6

$$\int_{\partial^*(R_0E)} \Gamma(n_{R_0E}(y))\, dH_{N-1}(y) \leq \liminf \int_{\partial^*(R_j'E)} \Gamma(n_{R_j'E}(y))\, dH_{N-1}(y).$$

(iii) As meas(RE) = meas(E) = meas(W_Γ) for all $R \in SO(N)$, we conclude that

$$\int_{\partial^* W_\Gamma} \Gamma(n_{W_\Gamma}(x))\, dH_{N-1}(x) = I(\partial W_\Gamma(x)) \leq I(\partial E).$$

(iv) Given $R \in SO(N)$ and as $\Gamma \geq \alpha$ on S^{N-1}

$$\int_{\partial^*(RE)} \Gamma(n_{RE}(y))\, dH_{N-1}(y) \geq \alpha\, \mathrm{Per}(RE) = \alpha\, \mathrm{Per}(E)$$

and so $I(\partial E) \geq \alpha$ Per(E).

Proof of Theorem 7.8. Let $\{u_i\}$be a minimizing sequence in $\mathcal{Q}$. Due to the growth condition of hypothesis (H2) and as the average of u_i is always equal to m, we have that $\{u_i\}$ is bounded in $W^{1,p}(\Omega; \mathbb{R}^N)$. By Theorem 7.7 there exists a subsequence (still denoted by $\{u_i\}$) and there exists $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ such that

$$u_i \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega; \mathbb{R}^N) \text{ and strongly in } L^\infty(\Omega; \mathbb{R}^N), \qquad (7.10)_1$$

$$\det \nabla u_i \rightharpoonup \det \nabla u \text{ weakly in } L^{p/N}(\Omega), \qquad (7.10)_2$$

and

$$\int_\Omega W(\nabla u(x))\, dx \leq \liminf_{i\to\infty} \int_\Omega W(\nabla u_i(x))\, dx. \qquad (7.10)_3$$

From $(7.10)_2$ it follows that

$$\det \nabla u \geq 0 \text{ a. e. and } \int_\Omega \det \nabla u(x)\, dx = \lim \int_\Omega \det \nabla u_i(x)\, dx, \qquad (7.11)$$

which, together with (H2) and $(7.10)_3$ implies that

$$\det \nabla u > 0 \text{ a. e. in } \Omega. \tag{7.12}$$

Also, as $\text{meas}(u_i(\Omega)) = \text{meas}(W_\Gamma)$, by Proposition 7.9 (iv) we deduce that the sequence of the characteristic functions of the sets $u_i(\Omega)$ is bounded in BV. As u is continuous, $u(\Omega)$ is a bounded set and so, since $u_i \to u$ strongly in L^∞, we can find an open bounded smooth domain U such that $u_i(\Omega), u(\Omega) \subset\subset U$ for all i. Therefore

$$\sup_i \|\chi_{u_i(\Omega)}\|_{BV(\mathbb{R}^N)} = \sup_i \|\chi_{u_i(\Omega)}\|_{BV(U)} < +\infty$$

and by (2.2) and Proposition 2.3 there exists a subsequence (still denoted by $\{u_i\}$) such that

$$\chi_{u_i(\Omega)} \to \chi_A \text{ strongly in } L^1 \text{ and } \text{Per}(A) \le \liminf_i \text{Per}(u_i(\Omega)) < +\infty. \tag{7.13}$$

We claim that[13] (see CIARLET & NECAS [9])

$$\int_\Omega \det \nabla u(x)\, dx \le \text{meas}(u(\Omega)). \tag{7.14}$$

Indeed, as $u(\overline{\Omega})$ is compact, given $\delta > 0$ there exists an open set V_δ such that $u(\overline{\Omega}) \subset V_\delta$ and $\text{meas}(V_\delta \backslash u(\overline{\Omega})) < \delta$. As u_i converges uniformly to u, for i large enough we have $u_i(\overline{\Omega}) \subset V_\delta$ and so by (7.4) and $(7.10)_2$ we deduce that

$$\int_\Omega \det \nabla u(x)\, dx = \lim \int_\Omega \det \nabla u_i(x)\, dx = \lim \text{meas}(u_i(\Omega))$$
$$\le \text{meas}(V_\delta) \le \text{meas}(u(\Omega)) + \delta.$$

Letting $\delta \to 0$ we obtain (7.14) which, together (7.12), (7.4) and (7.5) implies that u is one-to-one a.e. in Ω. Therefore, by Proposition 7.3 (v) if $\varphi \in C(\mathbb{R}^N)$ then

$$\int_{u_i(\Omega)} \varphi(y)\, dy = \int_\Omega \varphi(u_i(x)) \det \nabla u_i(x)\, dx$$
$$= \int_\Omega [\varphi(u_i(x)) - \varphi(u(x))] \det \nabla u_i(x)\, dx + \int_\Omega \varphi(u(x)) [\det \nabla u_i(x) - \det \nabla u(x)]\, dx$$
$$+ \int_{u(\Omega)} \varphi(y)\, dy. \tag{7.15}$$

On the other hand, by $(7.10)_{1,2}$

$$\lim_i \int_\Omega \varphi(u(x)) [\det \nabla u_i(x) - \det \nabla u(x)]\, dx = 0$$

and

$$\lim_i \left| \int_\Omega [\varphi(u_i(x)) - \varphi(u(x))] \det \nabla u_i(x)\, dx \right| \le \lim_i \|\det \nabla u_i\|_{L^{p/N}} \|\varphi(u_i) - \varphi(u)\|_{L^{p/(p-N)}}$$
$$\le \text{Const.} \lim_i \|\varphi(u_i) - \varphi(u)\|_{L^{p/(p-N)}}$$
$$= 0.$$

Hence, by (7.15) we conclude that

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13. This argument is used in CIARLET & NECAS [9].

$X_{u_i}(n)$ ~* X uto) weakly * in the sense of measures

which, together with (7.13), (7.4) and (7.11) implies that

X_{Ui} (Q) -» Xu(Q) strongly in L^1, Per (u(Q)) < H** and meas(u(Q)) =

= meas(W_r) = f det Vu(x) dx. (7.16)

Finally, u e Q and by (7.10)3 it remains to show that

I(3u(Q)) ≤ liminf I(3ui(Q)).

Consider a subsequence of {ui} (still denoted by {ui}) and by Proposition 7.9 (ii) for each i choose Ri G SO(N) such that

$$I(\partial u_i(\Omega)) = \int_{\partial^*(R_i u_i(\Omega))} r(n_{R u to)}(x))\, dH_{N-1}(x).$$

There exists a subsequence (still denoted by {Ri}) such that Ri —> Ri G SO(N) and so, by (7.16) and Theorem 4.6 we have

IOU(Q)) ≤ f T($n_{R\,u(fi)}$(x)) dHN^OO ≤

$$\leq \liminf \int_{\partial^*(R_i u_i(\Omega))} r(n_{R_i U_i(n)}(x))\, dH_{N-1}(x) = \liminf I(\partial u_i(\Omega)).$$

Now suppose that

(H3) *Cl* is an open, bounded, strongly Lipschitz subset of IR^N diffeomorphic to the unit ball B(0,l), i. e. there exists a Lipschitz function h : *Q*, -» B(0,l) such that h is one-to one, surjective and detVh(x) > 9 > 0 a. e. x G *Q*.

Consider the family of perturbed problems

($P_£$) minimize in Q

E_e(u) := e f W(Vu(x)) dx + IQU(Q)).

Theorem 7.17.

If (H2) and (H3) hold and if u_e is a solution for ($P_£$) then there exists a subsequence {u^} such that u^ -* u weakly in W ^ Q ; IR^N) and meas(u_{T1}(Q)\u(Q))+meas(u(Q)\u_{T1}(Q)) -> 0, where **u(Ω)** = RWp + a for some rotation R and some a G IR^N.

The proof of this result relies on the fact that we can find uo G Q such that detVuo is bounded away from zero and *uo(Q)* is a translation of the Wulff set. We construct this deformation using the Minkowski functional associated to Wp.

Definition 7.18.

Let C be a bounded, convex subset of $\mathbb{R}^N$ such that $0 \in \text{int } C$. The *Minkowski functional* μ_C *of C* is defined by $\mu_C(x) := \inf\{t > 0 | x \in tC\}$.

Proposition 7.19.

(i) $0 \le \mu_C(x) < +\infty$ and $\mu_C(x) = 0$ if and only if $x = 0$;

(ii) $\mu_C(tx) = t\mu_C(x)$ for $t \ge 0$;

(iii) $\text{int } C = \{x \in \mathbb{R}^N | \mu_C(x) < 1\}$ and $\bar{C} = \{x \in \mathbb{R}^N | \mu_C(x) \le 1\}$.

(iv) There exists $\sigma > 0$ such that $\sigma \|x\| \le \mu_C(x) \le \|x\|/\sigma$ for all x and $|\mu_C(x) - \mu_C(y)| \le \|x - y\|/\sigma$ for all x, y.

Proof. The proofs of parts (i), (ii) and (iii) can be found in RUDIN [34]. We prove (iv). Let $\sigma > 0$ be such that $B(0, \sigma) \subset C \subset B(0, 1/\sigma)$. We claim that μ_C is a Lipschitz function, precisely

$$|\mu_C(x) - \mu_C(y)| \le \|x - y\|/\sigma, \text{ for all } x, y. \tag{7.20}$$

Indeed, if $y \in tC$ for $t > 0$, then

$$\frac{x}{t + \frac{1}{\sigma}\|x - y\|} = (1 - \tau)\frac{y}{t} + \tau \frac{\sigma(x - y)}{\|x - y\|}$$

where

$$\sigma = \frac{\|x - y\|}{\sigma t + \|x - y\|} \in (0, 1).$$

As C is convex, we conclude that

$$\frac{x}{t + \frac{1}{\sigma}\|x - y\|} \in C$$

and so $\mu_C(x) \le t + \|x - y\|/\sigma$. Letting $t \to \mu_C(y)^+$ we obtain (7.20), and setting $y = 0$ in (7.20) yields $\mu_C(x) \le \|x\|/\sigma$. On the other hand, if $x \in tC$ then $\|x\| \le t/\sigma$ and so $\sigma \|x\| \le \mu_C(x)$.

Define

$$F_C(x) := \begin{cases} \dfrac{\|x\|}{\mu_C(x)} x & \text{if } x \ne 0 \\ \\ 0 & \text{if } x = 0. \end{cases}$$

Proposition 7.21.

F_C is a one-to-one Lipschitz function that maps B(0,1) onto intC and $F_C(\partial B(0,1)) = \partial C$. Moreover $\det \nabla F_C(x) > \sigma^N$ a. e. $x \in \Omega$.

Proof. We prove that F_C is injective. If $F_C(x) = F_C(x')$ then either $x = x' = 0$ or $x = \lambda x'$ for some $\lambda > 0$. By Proposition 7.19 (ii) we have

$$\frac{||x'||}{\mu_C(x')} x' = \frac{||\lambda x'||}{\mu_C(\lambda x')} \lambda x' = \lambda \frac{||x'||}{\mu_C(x')} x'$$

and so $\lambda = 1$. As $\mu_C(F_C(x)) = ||x||$, we conclude that $F_C(B(0,1)) \subset \text{int}\, C$ and $F_C(\partial B(0,1)) \subset \partial C$. Conversely, if $x \in \text{int}\, C \setminus \{0\}$ then, by Proposition 7.19 (iii) $\mu_C(x) < 1$ and

$$x = F_C\left(\frac{\mu_c(x)}{||x||} x\right) \in F_C(B(0,1)).$$

Thus $F_C(B(0,1)) = \text{int}\, C$ and similarly, $\partial C = F_C(\partial B(0,1))$. We show that

$$||F_C(x) - F_C(y)|| \le (2\sigma^{-1} + \sigma^{-3})\, ||x - y|| \text{ for all } x, y. \qquad (7.22)$$

If $x = 0$ or $y = 0$ then (7.22) follows from Proposition 7.19 (iv). Suppose that $x \neq 0$ and $y \neq 0$. Then, by Proposition 7.19 (iii)

$$\begin{aligned}
||F_C(x) - F_C(y)|| &\le ||(||x|| - ||y||)\frac{x}{\mu_C(x)}|| + ||y||\, ||\frac{x}{\mu_C(x)} - \frac{y}{\mu_C(y)}|| \\
&\le \frac{|\,||x|| - ||y||\,|}{\sigma} + \frac{\mu_C(y)}{\sigma}\, ||\frac{x}{\mu_C(x)} - \frac{y}{\mu_C(y)}|| \\
&\le 2\,\frac{||x - y||}{\sigma} + \frac{1}{\sigma}\, ||x\left(\frac{\mu_C(y)}{\mu_C(x)} - 1\right)|| \\
&\le 2\,\frac{||x - y||}{\sigma} + \frac{1}{\sigma^2}\, \mu_C(x)\, |\frac{\mu_C(y)}{\mu_C(x)} - 1| \\
&\le \left(\frac{2}{\sigma} + \frac{1}{\sigma^3}\right) ||x - y||.
\end{aligned}$$

As μ_C is a homogeneous of gegree one Lipschitz function,

$$\nabla \mu_C(x) \,.\, x = \mu_C(x) \text{ for almost all } x$$

which, together with Proposition 7.19 (iii) implies that

$$\begin{aligned}
\det \nabla F_C(x) &= \det\left[\frac{||x||}{\mu_C(x)}\mathbb{1} + x \otimes \left(\frac{x}{||x||\mu_C(x)} - \frac{||x||\nabla\mu_C(x)}{\mu_C(x)^2}\right)\right] \\
&= \det\left\{\frac{||x||}{\mu_C(x)}\left[\mathbb{1} + x \otimes \left(\frac{x}{||x||^2} - \frac{\nabla\mu_C(x)}{\mu_C(x)}\right)\right]\right\} \\
&= \left(\frac{||x||}{\mu_C(x)}\right)^N \left[1 + x \,.\left(\frac{x}{||x||^2} - \frac{\nabla\mu_C(x)}{\mu_C(x)}\right)\right] \\
&= \left(\frac{||x||}{\mu_C(x)}\right)^N \ge \sigma^N.
\end{aligned}$$

Proposition 7.23.

Under the hypothesis (H3) there exists a homeomorphism g : Q -»int Wp such that g is a Lipschitz function and det Vg is bounded away from zero.

Proof. By Proposition 3.6 the Wulff set is convex, bounded and it contains a neighborhood of the origin. Therefore, if we set C : = Wpand g := Fc ° h then g is a Lipschitz diffeomorphism of *Q*, into int Wp and det Vg ≥ $\mathbf{a^N 0}$ a. e. x e *£1*.

Proof of Theorem 7.17. Let g be as in Proposition 7.23 and define

$$U_0(x) := g(x) + \frac{m}{\text{meas}(Q)} \text{r-rr} \frac{1}{\text{meas}(Q)} r \int_{J\wedge} g(x)\, dx.$$

By Proposition 7.3 (iv) uo e Q and since ||Vuoll~ ^ M for some M > 0 and detVuo is bounded away from zero, from (H2) we conclude that

$$\mathbf{J_\Omega W(Vu_0(x))dx<+oo.}$$

Suppose that $u_£$ is a solution for $(P_£)$. Then

$$E_£(u_£) = e\int_{\Omega} W(Vu_£(x))dx + lOu_£(Q)) \le E_£(u_0) = 6\int_{\Omega} W(Vu_o(x))\, dx + lQW_r).$$

and so, by Proposition 7.9 (iii) we have for sufficiently small e

$$\mathbf{f\ W(Vu_£(x))\ dx \le f\ W(Vu_o(x))\ dx\ and\ I@u_£(Q)) \le e\, f\ W(Vu_o(x))\ dx + lOW_r).}$$

Thus, as in the proof of Theorcm 7.8, there exists a subsequence {u^} such that u^ -* u weakly in W^{1}^^; IR^N) and strongly in L°°(£2; IR^N), meas(uq(£2)\u(Q)) + measCuC^XoqCQ)) —> 0, Per(u(£2)) < +«> and meas(u(fli)) = meas(Wp). It remains to show that, up to a set of measure zero, Ru(£2) is a translation of the Wulff set for some rotation R. By Proposition 7.9 (ii), for each T| there exists R_n e SO(N) such that

$$\mathbf{I(\partial u_\eta(\Omega)) = \int_{\partial^*(R_\eta u_\eta(\Omega))} \Gamma(n_{R_\eta u_\eta(\Omega)}(x))\, dH_{N-1}(x)}$$

and so, with (for a subsequence) R^-* R e SO(N), by Proposition 3.6 (v), Theorem 4.6 and Proposition 7.9 (iii) we have

$$\int_{\partial^*(Ru(\Omega))} \Gamma^{**}(n_{Ru(\Omega)}(x))\, dH_{N-1}(x) \le \liminf_{\eta \to 0} \int_{\partial^*(R_\eta u_\eta(\Omega))} \Gamma^{**}(n_{R_\eta u_\eta(\Omega)}(x))\, dH_{N-1}(x)$$

$$\le \liminf_{\eta \to 0} IOU^\wedge\Omega)) = lOW_r)$$

$$= \mathbf{\int_{Jd^*w_r} r^{**}(n_{w_r}(x))\ dH^{\wedge\wedge}x).}$$

Thereforc, by Theorem 5.11 we conclude that u(Q) = RWp + a up to a set of measure zero.

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APPENDIX.

Here we prove Lemma 2.9. We recall the following regularity theorem for sets of finite perimeter (see EVANS & GARIEPY [15]).

Theorem A.I.

If $Per_a(E)$ < +00 then there exists a sequence $\{u_e\}$ *e* *C°°(Q)* such that

(i) u_e converges to *%$_E$* in *U(Q) ;*

(ii) supportUgC$\overline{E}$ + B(0; e);

(iii)0≤u_e≤l;

(iv)Per_n(E) = lim **f** |Vu_e(x)| dx.

The approximating sequence $\{u_e\}$ is of the form

$$u_\varepsilon(x) = \sum_{i=0}^{\infty} (\chi_E f_i) * \varphi,$$

where {fi} forms a partition of unity for *Q,* and <p is a mollifier. The first part of the proof of Lemma 2.9 follows the proof of Lemma 1 in MODICA [28].

Lemma 2.9.

Let EC IR^N be a bounded set of finite perimeter. There exists a sequence of open, bounded sets EnC $[R^N$ such that

(i)3E_n G C°°andEn,E C B(0,R) for some R > 0;

(ii)X_{En} ->%E inLto;

(iiiJPerCEn) ->Per(E);

(iv) meas(E_n) = meas(E).

Proof. *Step 1.* Here we prove the existence of a sequence A_n that satisfies (i), (ii) and (iii). Indeed, let $\{u_e\}$ e *Cr(Q)* be as in Theorem A.1 and let u := $\%_E$. As u_e converges to $\%_E$ in *V(Q),* for all **a** > 0

meas{xe R^N| | u_e(x) - u(x)|≥ **a** }-> 0

and so, let e(n) be such that

meas{x e $(R^N$ | | $u_{e(n)}$(x) - u(x)| ≥ 1/n } ≤ 1/n. (A.2)

Let

$\wedge_n$:= ess inf {Per{x G IR^N| U^ > t} | 1/n ≤ t ≤ (1-1/n)}.

It is clear that there exists a set Y_n C (1/n, 1-1/n) such that for all t e Y_n

$$\mathrm{Per}\,\{x \in \mathbb{R}^N \mid u_{\varepsilon(n)}(x) > t\} \le \xi_n + 1/n \tag{A.3}$$

and

$$\mathrm{meas}(Y_n) > 0. \tag{A.4}$$

Let $X_n := \{x \in \mathbb{R}^N \mid \nabla u_{\varepsilon(n)}(x) = 0\}$. By Sard's Lemma

$$\mathrm{meas}\,(u_{\varepsilon(n)}(X_n)) = 0$$

and so, due to (A.4), there exists $t_n \in Y_n$ such that

$$\nabla u_{\varepsilon(n)}(x) \neq 0 \tag{A.5}$$

for all x such that $u_{\varepsilon(n)}(x) = t_n$. Set

$$A_n := \{x \in \mathbb{R}^N \mid u_{\varepsilon(n)}(x) > t_n\}.$$

By (A.5) $\partial A_n \in C^\infty$ and by (A.2)

$$\begin{aligned}\mathrm{meas}\,\{x \in A_n \mid x \notin E\} + \mathrm{meas}\,\{x \in E \mid x \notin A_n\} &\le \mathrm{meas}\,\{x \in \mathbb{R}^N \mid u(x) = 0 \text{ and } u_{\varepsilon(n)}(x) > t_n\} + \\ &\quad + \mathrm{meas}\,\{x \in \mathbb{R}^N \mid u(x) = 1 \text{ and } u_{\varepsilon(n)}(x) \le t_n\} \\ &\le 2\,\mathrm{meas}\{x \in \mathbb{R}^N \mid |u_{\varepsilon(n)}(x) - u(x)| \ge 1/n\} \\ &\le 2/n.\end{aligned}$$

Thus

$$\chi_{A_n} \to \chi_E \quad \text{in } L^1(\Omega) \tag{A.6}$$

and so, by (2.5)

$$\mathrm{Per}(E) \le \liminf \mathrm{Per}(A_n). \tag{A.7}$$

Finally, by (A.3)

$$\mathrm{Per}(A_n) \le \xi_n + 1/n \le 1/n + \mathrm{Per}\,\{x \in \mathbb{R}^N \mid u_{\varepsilon(n)}(x) > t\}$$

for all $t \in (1/n, 1\text{-}1/n)$, which together with the Fleming-Rishel formula (2.7) implies that

$$\begin{aligned}\mathrm{Per}(A_n)\left(1 - \frac{2}{n}\right) &\le \frac{1}{n}\left(1 - \frac{2}{n}\right) + \int_{1/n}^{1-1/n} \mathrm{Per}\{x \in \mathbb{R}^N \mid u_{\varepsilon(n)} > t\}\,dt \\ &\le \frac{1}{n}\left(1 - \frac{2}{n}\right) + \int_\Omega |\nabla u_{\varepsilon(n)}(x)|\,dx\end{aligned}$$

and so, by Theorem A.1 (iv)

$$\limsup \mathrm{Per}(A_n) \le \mathrm{Per}(E). \tag{A.8}$$

By (A.7) and (A.8) we conclude that

$$\lim \mathrm{Per}(A_n) = \mathrm{Per}(E).$$

Step 2. Define

$$E_n := \lambda_n A_n \quad \text{where} \quad \lambda_n = \left(\frac{\mathrm{meas}\,E}{\mathrm{meas}\,A_n}\right)^{\frac{1}{N}}.$$

Clearly $\{E_n\}$ satisfy (i) and (iv), and by (A.6)

$$\lambda_n \to 1. \tag{A.9}$$

Thus, as $\mathrm{Per}(E_n) = \lambda_n^{N-1}\mathrm{Per}(A_n)$ with $\lim \mathrm{Per}(A_n) = \mathrm{Per}(E)$, we deduce (iii), namely

$$\lim \mathrm{Per}(E_n) = \mathrm{Per}(E).$$

It remains to show (ii). Since E is bounded, by (iii), (iv) and Theorem A.1 (ii), there exists an open, bounded, strongly Lipschitz domain $\Omega \subset \mathbb{R}^N$ such that

$E_n, E \subset\subset \Omega$ and $\{ \|\chi_{E_n}\|_{BV(\Omega)} \}$ is bounded.

By Proposition 2.3, there exists a subsequence (that for convenience we still denote by χ_{E_n}) such that

$$\chi_{E_n} \to \chi_A \quad \text{in } L^1(\Omega), \text{ for some subset } A \subset \Omega. \tag{A.10}$$

We claim that $\chi_E = \chi_A$ a. e. Indeed, let $\varphi \in C_0(\mathbb{R}^N)$. Then by (A.6) and (A.9)

$$\lim_n \int_{E_n} \varphi(x)\,dx = \lim_n \lambda_n^N \int_{A_n} \varphi(\lambda_n y)\,dy = \lim_n \int_{A_n} \varphi(y)\,dy + \lim_n \int_{A_n} [\varphi(\lambda_n y) - \varphi(y)]\,dy$$

$$= \int_E \varphi(y)\,dy \tag{A.11}$$

since, by Lebesgue's dominated convergence theorem and by (A.6),

$$\lim_n \left| \int_{A_n} [\varphi(\lambda_n y) - \varphi(y)]\,dy \right| \le \lim_n \left[\int_E |\varphi(\lambda_n y) - \varphi(y)|\,dy + 2\,\|\varphi\|_\infty\,[\text{meas}(A_n \setminus E) + \text{meas}(E \setminus A_n)] \right]$$

$$= 0.$$

Therefore, by (A.11)

$\chi_{E_n} \overset{*}{\rightharpoonup} \chi_E$ weakly * in the sense of measures

which, together with (A.10) yields (ii).