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Local Invertibility of Sobolev Functions

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1 Introduction.

The aim of this paper is to give a simple proof of local invertibility of functions $v \in W^{1,N}(\Omega, \mathbf{R}^N)$, where $\Omega \subset \mathbf{R}^N$ is an open set and $det\nabla v(x) > 0$ *a.e.* $x \in \Omega$ (Theorem 3.1). We show that the local inverse function w is $W^{1,1}$ and under suitable hypotheses we improve regularity of w to $W^{1,s}$ for some $s > 1$. Precisely, it is shown that v is locally invertible almost everywhere in the sense that, for almost every $x \in \Omega$ there is an open neighborhood D of x and there is a function $w \in W^{1,1}(v(D), D)$ such that $v(D)$ is an open set,

$$v \circ w(y) = y \;\; a.e. \;\; y \in v(D), \tag{1}$$

$$w \circ v(x) = x \;\; a.e. \;\; x \in D \tag{2}$$

and

$$\nabla w(y) = (\nabla v)^{-1}(w(y)) \;\; a.e. \;\; x \in D, \tag{3}$$

where $(\nabla v)^{-1}(w(y))$ is the inverse matrix of $\nabla v(w(y))$. Moreover, if we assume that $|\frac{adj(\nabla v)}{det\nabla v}|^s det\nabla v \in L^1(\Omega)$ for some $1 \leq s < +\infty$ then as in [Sv], we prove that $w \in W^{1,s}(v(D), D)$. One can then deduce easily that if $det\nabla v(x) \geq \gamma > 0$ *a.e.* $x \in \Omega$, $v \in W^{1,q}(\Omega)^N$ and $q \geq N(N-1)$, then $v : D \to v(D)$ and $w : v(D) \to D$ are homeomorphisms, (1) holds for every $y \in v(D)$, (2) holds for every $x \in D$, $w \in W^{1,N}(v(D), D)$ and v is an open mapping on $\Omega \setminus L$ for a suitable $L \subset \mathbf{R}^N$ which has zero measure (see Corollary 3.3). In particular, we conclude that if $N = 2$, $v \in W^{1,2}(\Omega)^2$ and $det\nabla v(x) \geq \gamma > 0$ *a.e.* $x \in \Omega$ then $w \in W^{1,2}(v(D), D)$ and there is a set of measure zero $L \subset \mathbf{R}^N$ such that v is an open mapping on $\Omega \setminus L$ and we obtain a weaker version of [IS].

Conversely if $v \in W^{1,q}(\Omega)^N$, for some $q > N$, $det\nabla v(x) \neq 0$ *a.e.* $x \in \Omega$ and if for almost every $x_0 \in \Omega$ v is locally almost invertible in a neighborhood of x_0 in the sense of (1) - (3), then there are open sets $\Omega_1, \Omega_2 \subset \mathbf{R}^N$ and a set of measure zero $N \subset \mathbf{R}^N$ such that $\Omega = \Omega_1 \cup \Omega_1 \cup N$, $det\nabla v(x) > 0$ *a.e.*$x \in \Omega_1$ and $det\nabla v(x) < 0$ *a.e.* $x \in \Omega_2$ (see Corollary 3.2).

Note that a homeomorphism $v \in W^{1,\infty}(\Omega)^N$ need not to satisfy $det\nabla v(x) \neq 0$ *a.e.*$x \in \Omega$. Such an example is provided in [MZ] (see Remarks 3.4).

The result in this paper is in the same spirit as the work in [Ba] (1981), [CN] (1987), [Sv] (1988) and [TQ] (1988). As far as we know, the existence and the regularity of the local inverse function w is not an immediate consequence of these earlier results where assumptions are placed either on the trace $v|_{\partial\Omega}$ or on $|v(\Omega)|$. By an elementary lemma (Lemma 3.5) and the invertibility result found in [TQ], one can obtain the existence of the local inverse function w and then deduce its regularity. Due to his relaxed assumption $q > N-1$ (here we have $q \geq N$), Q. Tang used an elaborated method to obtain the existence of an inverse $w \in W^{1,1}_{loc}$ under the condition introduced by [CN], $\int_\Omega det\nabla v(x)dx \leq |v(\Omega)|$.

The proof that we present here concerning the local invertibility of v is independent of the work by [Ba], [CN], [Sv], [TQ], and the method employed relies on basic properties of the degree theory.

In the sequel of this paper, we fix a bounded, open set $\Omega \subset \mathbf{R}^N$ and we consider a function $v \in W^{1,q}(\Omega)^N$. We denote by ∇v the gradient of v i.e. the $N \times N$ matrix of the partial derivatives of v and by $adj\nabla v$ the adjugate [1] of ∇v.

As an application of the local invertibility property, we study the weak lower semicontinuity of functionals E of the form

$$E(u,v) = \int_\Omega W\Big(\nabla u(x)(\nabla v(x))^{-1}\Big)dx,$$

defined on the set

$$B_{p,q} = \{(u,v) \in W^{1,p}(\Omega,\mathbf{R}^N) \times W^{1,q}(\Omega,\mathbf{R}^N)|\ det\nabla v(x) = 1\ a.e\ \ x \in \Omega\}$$

where $1 \leq p < +\infty$, $N \leq p \leq +\infty$, $\frac{1}{p} + \frac{N-1}{q} = \frac{1}{r} \leq 1$. When $N = 3$, $\nabla u(x) \cdot (\nabla v(x))^{-1}$ represents the lattice of a neutral elasto-plastic change of state of a perfect cubic crystal, u is the elastic deformation and v corresponds to the slip or plastic deformation. (for details, see Ericksen [Er], Davini & Parry [DP], Fonseca & Parry [FP] and Dacorogna & Fonseca [DF]). We prove that, under some convexity and growth assumptions on the function W, E is weakly lower semicontinous on $B_{p,q}$. If $r > 1$ and $q \neq +\infty$ we rely on the div-curl lemma (see Tartar [Ta]) to prove that

$$\nabla u_\epsilon \cdot (\nabla v_\epsilon)^{-1} \rightharpoonup \nabla u \cdot (\nabla v)^{-1} \text{weakly in } L^r$$

whenever

$$(u_\epsilon, v_\epsilon) \rightharpoonup (u,v) \text{ weakly in } B_{p,q}.$$

We notice, however, that the growth condition of W and the weak lower semicontinuity of E on $B_{p,q}$ do not always imply the existence of the minimum of E on $B_{p,q}$. Indeed, $\{(\nabla u_\epsilon \cdot (\nabla v_\epsilon)^{-1}\}$ being relatively compact in L^r does not imply that $\{(\nabla u_\epsilon\}$ or $\{(\nabla v_\epsilon)^{-1}\}$ are relatively compact in, respectively, $L^p, L^{\frac{q}{N-1}}$ (see [DF] Proposition 4.1).

The paper is organized as follows: in the second section we fix notations and recall some definitions and well known properties related to Brouwer degree. In the third section we prove the local invertibility property of the mappings $v \in W^{1,q}(\Omega,\mathbf{R}^N), q \geq N$, under the condition $det\nabla v(x) > 0$ *a.e.* $x \in \Omega$. In view of our applications, in addition we prove that if $v_\epsilon \rightharpoonup v$ weakly in $W^{1,q}$ $q \geq N$ and $det\nabla v_\epsilon(x) > 0$ then, up to a subsequence, v_ϵ and v are respectively locally invertible on open sets $D_\epsilon(x)$ and $D(x)$ for almost every $x \in \Omega$, where $D_\epsilon(x)$ and $D(x)$ are neighbourhoods of x, such that $v_\epsilon(D_\epsilon(x)) = v(D(x))$ does not depend on ϵ. The last section is devoted to the applications where we obtain the weak lower semicontinuity for a class of functionals E on $B_{p,q}$.

[1] we recall that if A is a $N \times N$ matrix, then $adjA$ is a matrix such that $A \cdot adjA = I_N detA$.

2 Preliminaries.

In the sequel we will use the following notations.
For $x = (x_1, \cdots, x_N) \in \mathbf{R}^N$, $|x|$ stands for $(|x_1|^2 + \cdots + |x_N|^2)^{\frac{1}{2}}$ and $|x|_\infty := \max\{\, |x_1|, \cdots, |x_N|\, \}$.
If $A \subset \mathbf{R}^N$ $|A|$ denotes the Lebesgue measure of A, A^c denotes its complement, $dist(x, A)$ is defined by $\inf\{|x-y| : \ y \in A\ \}$ and $\rho(x, A)$ is given by $\inf\{|x-y|_\infty : \ y \in A\ \}$.
If $\Omega \subset \mathbf{R}^N$ is an open set, $v \in L^1(\Omega)^N$, then ∇v is the $N \times N$ matrix of the distributional derivatives $\frac{\partial v_i}{\partial x_j}$ and $det \nabla v$ is the determinant of ∇v.

We recall some properties of mappings.

Lemma 2.1 *Let Ω be a bounded, open set in $\mathbf{R}^N$ and $v \in (W^{1,N}_{loc}(\Omega))^N$ such that $det\nabla v(x) > 0$ a.e.$x \in \Omega$. Then v is a continuous mapping on Ω. Futhermore, if K is a compact set and V is an open set such that $K \subset V \subset\subset \Omega$, then there is a constant C_N depending only on N, such that*

$$|v(x) - v(y)| \leq M^{\frac{1}{N}} C_N \theta(|x-y|)$$

for every $x, y \in K$ verifying $|x-y| \leq \delta$, where

$$M = \int_V |\nabla v(x)|^N dx,$$

$$\theta(t) = \left(\frac{2}{log(\frac{2}{t})}\right)^{\frac{1}{N}},$$

and

$$\delta = \min\{2, \frac{1}{2}(dist(K, \mathbf{R}^N \setminus V))^2\}.$$

Proof. This lemma is an immediate consequence of Theorem 3.5, p 294, Proposition 3.3, p 292 in [GR] and Theorem 4.4 p 339 in [Re] (see also [Man]). It can also be shown that, under the above hypotheses, v is a monotonic mapping (see the definition of monotonic mapping below).

Definition 2.2 ([GR]) *Let Ω be a bounded, connected, open set in $\mathbf{R}^N$ and $v \in W^{1,N}(\Omega)^N$. We say that v* **is monotonic at the point** *$x \in \Omega$ if there is a number $0 < r(x) \leq d(x, \partial\Omega)$ such that for almost every $r \in (0, r(x))$ the pre-image of the intersection of the set $v(B(x,r))$ with the unbounded connected component of $\mathbf{R}^N \setminus v(\partial B(x,r))$ is of measure 0 in $B(x,r)$. We say that v is a monotonic mapping in Ω if v is monotonic at every point $x \in \Omega$.*

We make some remarks on the Brouwer degree theory. For details we refer the reader to [Ll].

Let $\Omega \subset \mathbf{R}^N$ be a bounded, open set and $v : \bar{\Omega} \to \mathbf{R}^N$, a continuous function. For every $p \in \mathbf{R}^N \setminus v(\partial\Omega)$ the *Brouwer degree* $d(v, \Omega, p)$ *of* v *with respect to* Ω *at* p is a well defined integer depending only on the boundary values of v. In particular if $v \in C^1(\bar{\Omega})^N$ and $p \in \mathbf{R}^N \setminus (v(\partial\Omega) \cup v(Z_v))$, we have

$$d(v, \Omega, p) = \sum_{x \in v^{-1}(p)} sign\, det \nabla v(x),$$

where

$$sign\, t = \begin{cases} 1 & \text{if } t > 0 \\ -1 & \text{if } t < 0 \end{cases}$$

and $v(Z_v)$ denotes the image of the set $\{x \in \Omega \mid det \nabla v(x) = 0\}$.

We give some additional properties on the degree.

Proposition 2.3 ([GR]) *Let* $\Omega \subset \mathbf{R}^N$ *be an open, bounded set,* $v \in C^0(\bar{\Omega})^N$ *and let* $p \in \mathbf{R}^N \setminus v(\partial\Omega)$. *Let* C_p *be the connected component of* $\mathbf{R}^N \setminus v(\partial\Omega)$ *containing* p. *Then we have the following properties:*

$$d(v, \Omega, p) = d(u, \Omega, p) \text{ if } u \in C^0(\bar{\Omega})^N \text{ and } |u - v|_\infty < dist(p, v(\partial\Omega)), \tag{4}$$

$$d(v, \Omega, p) \neq 0 \Rightarrow \exists x \in \Omega \text{ such that } v(x) = p, \tag{5}$$

$$d(v, \Omega, p) = d(v, \Omega, q) \ \forall \ q \in C_p, \tag{6}$$

$$d(v, \Omega, p) = d(\phi, \Omega, p) \text{ if } \phi \in C^0(\bar{\Omega})^N \text{ and } \phi = v \text{ on } \partial\Omega. \tag{7}$$

Moreover, the degree is invariant under homotopy, i.e.

$$d(H(\cdot, t), \Omega, p) = d(H(\cdot, 0), \Omega, p), \tag{8}$$

for every homotopy $H \in C^0(\bar{\Omega} \times [0,1])^N$ *such that* $p \notin H(\partial\Omega, t)$, *for every* $t \in [0,1]$. *Finally, if* $K \subset \bar{\Omega}$ *is a compact set and* $p \notin v(K)$ *then (excision property)*

$$d(v, \Omega, p) = d(v, \Omega \setminus K, p) \tag{9}$$

and if $\Omega = \cup_{i=1}^{+\infty} \Omega_i$, Ω_i *mutually disjoint open sets then (decomposition property)*

$$\sum_i d(v, \Omega_i, p) = d(v, \Omega, p). \tag{10}$$

Proof. We refer the reader to [Ll].

Lemma 2.4 *Let* $\Omega \subset \mathbf{R}^N$ *be a bounded, connected, open set and* $v \in W^{1,N}(\Omega)^N$ *such that* $det \nabla v(x) > 0$ *a.e.* $x \in \Omega$. *Let* $f : \mathbf{R}^N \to \mathbf{R}$ *be a measurable function. Then*
(i) for every measurable set $E \subset \Omega$, $x \to f \circ v(x)$ *and* $y \to N(v, E, y)$ *are measurable and*

$$\int_E f \circ v(x) |det \nabla v(x)| dx = \int_{\mathbf{R}^N} N(v, E, y) f(y) dy, \tag{11}$$

where $N(v,E,y)$ is the cardinality of the elements of the set $\{x \in E|\ v(x) = y\}$.
(ii) If, in addition, f is a continuous, bounded function, then for every connected, open set $V \subset\subset \Omega$ such that $|\partial V| = 0$

$$\int_V f \circ v(x) det \nabla v(x) dx = \int_{\mathbf{R}^N \setminus v(\partial V)} d(v, V, y) f(y) dy. \tag{12}$$

(iii) If $D \subset\subset \Omega$ is an open such that $|\partial D| = 0$ and $p \in \mathbf{R}^N \setminus v(\partial D)$, then

$$d(v, D, p) = \int_D f(v(x)) det \nabla v(x) dx, \tag{13}$$

where V is the connected component of $\mathbf{R}^N \setminus v(\partial D)$ containing p and f is any nonnegative, continuous real-valued function with compact support in V and satisfying $\int_{\mathbf{R}^N} f(x)dx = 1$.

Remarks 2.5 A function $v : \Omega \to \mathbf{R}^N$ is said to satisfy the N-property (Lusin's property) if

$$|v(E)| = 0$$

whenever $E \subset \Omega$ is a measurable set such that $|E| = 0$ and v is said to satisfy the N^{-1}-property if

$$|v^{-1}(A)| = 0$$

whenever $A \subset \mathbf{R}^N$ is a measurable set such that $|A| = 0$.
a) It is known that if $v \in W^{1,N}(\Omega)^N$, $det \nabla v(x) > 0$ *a.e.* $x \in \Omega$ then v satisfies the N and the N^{-1}-property (see [GR], p. 296-297).
b) Also, if $v \in W^{1,q}(\Omega)^N$ with $q > N$ then v satisfies the N-property (For details we refer the reader to [MM]).

Proof of Lemma 2.4. We refer the reader to [GR], Theorem 1.8, p. 280, Theorem 2.6, p. 288 or also to [Sv] for the proof of (11) and (12) in the case where D is a domain.
First we prove that (12) is still valid even if D is not connected and (13) is a by-product of this fact. To achieve this, let us remark that by Vitali's covering theorem there are $\{D_i\}$ a countable family of open balls mutually disjoint and a set N of measure zero such that $(\cup_i D_i) \cap N = \emptyset$ and

$$(\cup_i D_i) \cup N = D.$$

Setting $B = \cup_i D_i$, we have $\cup_i \partial D_i \subset \partial B$. If $y \in \mathbf{R}^N \setminus (v(\partial B) \cup v(\partial D))$, then by the decomposition formula (10)

$$\sum_i \chi_{v(D_i)} d(v, D_i, y) = \sum_i d(v, D_i, y) = d(v, B, y). \tag{14}$$

Let $K = \bar{D} \setminus B$. As K is a compact set and $K \subset \partial D \cup N$, if $y \notin v(K)$ then, by the excision property of the degree (9), we obtain

$$d(v, D, y) = d(v, D \setminus K, y) = d(v, B, y). \tag{15}$$

Using the fact that v has the N-property (see Remark 2.5), $D_i \subset\subset \Omega$, $|\partial D| = |N| = |\partial D_i| = 0$, by (12), (14) and (15) we obtain

$$\begin{aligned}\int_D f \circ v(x) det\nabla v(x) dx &= \int_B f \circ v(x) det\nabla v(x) dx \\ &= \sum_i \int_{D_i} f \circ v(x) det\nabla v(x) dx \\ &= \sum_i \int_{v(D_i)\setminus v(\partial D_i)} d(v, D_i, y) f(y) dy \\ &= \int_{v(B)\setminus v(\partial B)} d(v, B, y) f(y) dy \\ &= \int_{v(D)\setminus v(\partial D)} d(v, D, y) f(y) dy.\end{aligned}$$

Since $\int_{\mathbf{R}^N} f(x)dx = 1$ and as the compact support of f is included V, we conclude that

$$\int_D f \circ v(x) det\nabla v(x) dx = d(v, D, p).$$

∎

Definition 2.6 *Let $\Omega \subset \mathbf{R}^N$ be an open set, let $v : \Omega \to \mathbf{R}^N$ be a function and $x_0 \in \Omega$.*
1.- *We say that v is* completely differentiable *at x_0 if there is a number $R_0 > 0$, a function $\epsilon : \mathbf{R} \to \mathbf{R}$ and a $N \times N$ matrix $\nabla v(x_0)$ such that*

$$v(x_0 + h) = v(x_0) + \nabla v(x_0) h + |h|\epsilon(|h|)$$

for every $h \in B(0, R_0)$ and $\lim_{t\to 0} \epsilon(t) = 0$. In this case we call $det\nabla v(x_0)$ the Jacobian of v at x_0.
2.- *We say that v is* weakly differentiable *at x_0 if there is a set $A \subset \mathbf{R}$ and a $N \times N$ matrix $\nabla v(x_0)$ such that $\lim_{r\to 0} \frac{|A\cap[0,r]|}{r} = 1$ and*

$$\liminf_{t\to 0, t\in A} \gamma_{x_0}(t) = 0,$$

where

$$\gamma_{x_0}(t) = \sup\{|\frac{v(x_0 + tz) - v(x_0)}{t} - \nabla v(x_0) z| \mid |z| = 1\}.$$

In this case we call $det\nabla v(x_0)$ the weak Jacobian of v at x_0.

Lemma 2.7 *Let Ω be a bounded open set in $\mathbf{R}^N$.*
i) If $v \in W^{1,N}(\Omega)^N$ is a monotonic mapping, then v is almost everywhere in Ω completely differentiable.
ii) If $v \in W^{1,q}(\Omega)^N, q > N$, then v is almost everywhere in Ω completely differentiable.
iii) If $v \in W^{1,q}(\Omega)^N, q > N - 1$, then v is almost everywhere in Ω weakly differentiable.

Proof. We refer the reader to [GR] Theorem 5.4, p. 175, to [Re] and to [MZ].

3 Local invertibility in $W^{1,9}$.

We first state the main result of this section (Theorem 3.1) and some of its corollaries.

Theorem 3.1 *Let* ft C R^N *be a bounded, open set and let* v € $W^{hN}(Q)^N$ *be a function such that* $detVv(x) > 0$ *a.e.* x € ft. *Then for almost every* x_0 G ft, t; *is locally almost invertible in a neighborhood of* x^ *in the sense that there exists* $r = r(x_0) > 0$, *an open set* $D = D(x_0)$ CC ft *and a function* $w : B(yo,r)$ —• /?, *with* j/o = t>(x$_0$), *such that*

$$w \in W^{1,1}(B(y_0,r))^N$$

$$w \text{ o } v(x) = x \quad a.e. \quad x \text{ € } D$$

$$v \text{ o } w(y) = y \quad a.e. \quad y \text{ € } B(y_0, r),$$

$$Vw(y) = (Vv)\sim^l(w(y)) \quad a.e. \quad y \text{ € } B(y_o,r).$$

If, in addition, \^^\$^9$*detVv* € $L^l(Q)$ *for some* $1 <\_s < +oo$ *then* w € W^l**$(B(y_0,r), D)$.

Before proving Theorem 3.1 we list some of its consequences.

Corollary 3.2 *Let* ft C R^N *be a bounded, open set,* $q \geq N$, *and* v e $W^{hq}(Q,)^N$ *be a function such that* $detVv(x)$ ^ 0 *a.e.* x £ ft.

a) Assume that fti,ft2 C R^N *are two open sets and* N C R^N *is a set of measure zero such that* ft = fti U fti U N, $detVv(x) > 0$ *a.e.* x € ftj, *and* $detVv(x) < 0$ a.e. x € fi$_2$. *Then for almost every* XQ € ft v *is locally almost invertible in a neighborhood of* x_0 *in the sense above.*

b) Conversely, if $q > N$, v 6 W^ft)^ *and if for almost every* x_0 6 ft v *is locally almost invertible in a neighborhood of* x_0, *then there are open sets* ft$_x$,ft$_2$ C R^N *and a null set* N C R^N *such that* ft = fti U ft$_2$ U JV, $detVv(x) > 0$ a.e. x € fti, *and* $detVv(x) < 0$ a.e. x € ft2-

Corollary 3.3 *Let* $q \geq N$, *let* ft CR^N *be a bounded, open set and let* v € W^ft)* *be a function such that* $detVv(x) = 1$ a.e. x € ft. *Then the inverse function* w *of Theorem S.I is such that*

$$w \text{ € } W^l>?\&(v(D))^N.$$

If, in addition, $q \geq N(N - 1)$ *then* w *o* $v(x) = x$ *for every* x 6 2?, v *o* $w(y) = y$ *for every* y € $B(yo,r)$, v *is a local homeomorphism and* v *is an open mapping on* ft \ X *for some set* L C ft *of zero measure. In particular, if* $N = 2$ *then* $N(N - 1) = N = 2$ *and* v *i* a local homeomorphism at* XQ.

We make some remarks and state some lemmas needeed for the proofs of Corollaries 3.2 and 3.3, which will appear at the end of this section.

Remarks 3.4

1.- Recall that $v \in W^{1,N}(\Omega)^N$ is said to be a mapping of bounded distortion if $|\nabla v(x)|^N \leq K(det\nabla v(x))$ for almost every $x \in \Omega$ and for some constant K. It is well known that every mapping of bounded distortion $v \in W^{1,N}(\Omega)^N$ is locally a homeomorphism at almost every point $x_0 \in \Omega$ (see [Re] Theorem 6.6, pp. 187). Moreover mappings of bounded distortion are open mappings or constant in Ω (see [Re] Theorem 6.4, pp. 184).

2.- Note that, even if $v \in C^1(\bar{\Omega})^N$ is such that $det\nabla v(x) \geq \gamma > 0 \quad \forall x \in \Omega$, we cannot expect a global invertibility of v without any regularity assumptions on the trace of v. As an example consider

$$\Omega = \{(x,y) \in \mathbf{R}^2 : 1 < x^2 + y^2 < 2\}, \quad v(x,y) = (x^2 - y^2, 2xy).$$

For every $(x,y) \in \Omega$ we have $det\nabla v(x,y) = 4(x^2+y^2) \geq 4$ although $v(x,y) = v(-x,-y)$ (see also [Ba]).

3.- Under the assumptions of Theorem 3.1, we cannot expect v to be locally invertible everywhere. The following example is provided in [Ba]: let $N \geq 3$ and consider the cylinder

$$\Omega = \{x \in \mathbf{R}^N : 0 \leq R < 1, |x_N| < 2\}$$

where $x = (x_1, \cdots, x_N), R = \sqrt{x_1^2 + \cdots + x_{N-1}^2}$ and $v = (v_1, \cdots, v_N)$ is defined by

$$\begin{aligned} &v_i(x) = R^{-\alpha} x_i, \quad i = 1, \cdots, N-1, \\ &v_N(x) = R^{\beta} x_N, \quad |x_N| \leq 1, \\ &v_N(x) = [2(|x_N| - 1) + (2 - |x_N|)R^{\beta}] sign x_N, \quad 1 \leq |x_N| \leq 2, \end{aligned}$$

where $N < q < N(N-1)$, $\frac{1}{N-1} - \frac{1}{q} < \alpha < \frac{N-1}{q}$, $\beta = \alpha(N-1)$. It is shown in [Ba] that $v \in W^{1,q}(\Omega)^N$, $det\nabla v(x) > 1 - \alpha > 0$ almost everywhere. However, one can easily see that

$$v^{-1}(0) = \{(0, \cdots, 0, \lambda) : \quad |\lambda| \leq 1\}$$

and so v cannot be locally invertible at any point $(0, \cdots, 0, \lambda)$ for $|\lambda| \leq 1$.

4.- An example of a mapping $v \in W^{1,\infty}(\Omega)^2$, $(\Omega \subset \mathbf{R}^2)$ is exhibited in [Ba], with $det\nabla v(x) = 1 \quad a.e. \quad x \in \Omega$, for which there is no sequence $v_r \in C^1(\bar{\Omega})^2$ such that $v_r \to v$ uniformly and $J_{v_r}(x) > 0 \quad a.e. \quad x \in \Omega$. Therefore, to prove Theorem 3.1 one cannot approximate the function v by a sequence of smooth functions v_r, expecting the functions v_r to be locally invertible.

5.- Note that for every bounded, open set $\Omega \subset \mathbf{R}^N$, there exist a measurable set $E \subset \Omega$ of non-zero measure and a homeomorphism $v \in W^{1,\infty}(\Omega)^N$ such that $det\nabla v(x) = 0$ for every $x \in E$. The following example is provided in [MZ], Remarks 3.7. Let $E \subset [0,1]$ be a Cantor set of positive one dimensional measure $0 < 1 - \alpha < 1$ and write

$$v_1(x) = v_1(x_1, \cdots, x_N) = \int_0^{x_1} \chi_{E^c}(t)dt$$

where χ_{E^c} is the characteristic function of the complement of E in $[0,1]$. Then $v(x) = (v_1(x), x_2, \cdots, x_N)$ is such that $v \in W^{1,\infty}(\Omega)^N$, has $det\nabla v(x) = 0$ *a.e.* $x \in E \times [0,1]^{N-1}$ and v is a homeomorphism of $[0,1]^N$ onto $[0,\alpha] \times [0,1]^{N-1}$.

6.- Due to the previous remark, the assumption $det\nabla v(x) \neq 0$ *a.e.* $x \in \Omega$ in Corollary 3.2 is essential.

Lemma 3.5 *Let $\Omega \subset \mathbf{R}^N$ be an open set, let $v \in C^0(\Omega)^N$ and $x_0 \in \Omega$ such that v is completely differentiable at x_0. Assume that $det(\nabla v(x_0)) \neq 0$. Then there is $r_0 > 0$ such that for every $0 < r \leq r_0$ the following assertions hold:*

$$v(x_0 + h) \neq v(x_0) \ \textit{for every}\ h \in \bar{B}(0,r) \setminus \{0\}. \tag{16}$$

$$d(v, B(x_0,r), v(x_0)) = sign(det\nabla v(x_0)). \tag{17}$$

Proof. Since v is completely differentiable at x_0,

$$v(x_0 + h) = v(x_0) + \nabla v(x_0)h + |h|\epsilon(|h|),$$

for $h \in \bar{B}(0, R_0)$, some $R_0 > 0$ and $\lim_{t\to 0}\epsilon(t) = 0$. Let

$$a = \inf\{|\nabla v(x_0)h|, \ \ h \in \mathbf{R}^N, |h| = 1\}.$$

Since $det(\nabla v(x_0)) \neq 0$ we obtain that $a > 0$. Using the fact that $\lim_{t\to 0}\epsilon(t) = 0$, we may find $0 < r_0 \leq R_0$ such that $|\epsilon(t)| \leq \frac{a}{2}$ for every $|t| \leq r_0$.

Claim 1: $v(x_0 + h) \neq v(x_0)$ for every $h \in \bar{B}(0, r_0)$ such that $h \neq 0$.
Indeed, if $0 < |h| \leq r_0$ then $|\epsilon(|h|)| \leq \frac{a}{2}$ and so

$$\left|\frac{v(x_0 + h) - v(x_0)}{|h|}\right| \geq \left|\nabla v(x_0)\frac{h}{|h|}\right| - |\epsilon(|h|)| \geq \frac{a}{2} > 0$$

and we obtain (16).

Claim 2: $d(v, B(x_0, r), v(x_0)) = sign(det\nabla v(x_0))$ for every $0 < r \leq r_0$.
Let us first notice that from (16), $d(v, B(0,r), v(x_0))$ is well defined for every $0 < r \leq r_0$. Fix $0 < r \leq r_0$ and set

$$u(x_0 + h) = v(x_0) + \nabla v(x_0)h, \ \ h \in \bar{B}(0,r).$$

The function u is a linear mapping defined on J3(x_o,r). Since *det(Vv(x_0))* ^ 0 we know that u is a one-to-one mapping and so *u(x_0)* ^ *u(x)* for every *x* G *dB(x_o,r)* i.e. *v(x_0)* ^ u(x) for every *x* G 0J3(x_o,r). Therefore *d(u,B(x_o,r),v(x_o))* is well defined and ***d(u,B(x_o,r),v(xo)) = sign(det(Vv(x_0))).* The application *H* defined by**

$$H(x,t) = tv(x) + (l\text{-}t)u(x) \qquad xGB(x_o,r), \quad <G[O,1]$$

is a homotopy between t; and u. Moreover, for every *x* G *dB(x_o,r)* we have

$$|H(x,<) - v(x_o)\backslash = r\{|Vt;(xo)£\wedge^{+ t€\{lx \ "" \ *o|)l\}} \gtrsim \frac{ra}{T} > o .$$

|X X0| I

Therefore *v{x_0)* £ *H(dB(x_o,r),t)* for every *t* G [0,1] and <*(#(., t),£(x_o,r),v(x_o)) is well defined. By (8) we obtain that d(i/(-,t),i?(xo,r),t;(xo)) is independent of i, hence, taking *t* = 0, t = 1 we obtain

$$d(v,B(x_0,r),v(x_0)) = d(u,B(x_0,r),v(x_0)) = sign(det(\nabla v(x_0)))$$

and (17) is proved. I

Remark 3.6 The relation between complete differentiability and topological degree was first observed by Reshetnyak ([Re]).

Lemma 3.7 ***Let Cl CH^N be an open*** *set,* ***let v* G *W^{hN}(il)^N be*** *such that detVv{x)* > 0 **a.e. x G ft.** ***Then for every* Xo G ft *such that v is*** *completely differentiable at x_0 and* ***detVv(xo)* > 0 *there is RQ* = JRO(XO) *such that for every* 0 < *R*** *< Ro the following holds:*

N(v,B(x_o,R),y)) = 1 for almost every y* G *C_R, **(18)**

d(v,B(x_o,R),y) = 1 for every y E C_R, **(19)**

d(v,B,y) = 1 for every y* G *v{B)\v(dB), **(20)**

for every non empty, open set B* C *V~^{X}(CR)* fl *B(xo,R) such that\dB\ = 0,

where CR is the connected component ofR^N \v(dB(x_o,R)) containing y_0 := v(x_0) *and* ***N(v,E,y) is the cardinality of the set {x* G *E\ v{x) = y}.***

Proof. By lemmas 2.2 and 2.7, t; is continuous and monotonic on ft and is completely differentiable at almost every point x G ft. Fix x_0 G ft such that t; is completely differentiable **at xo and *detVv(xo)* > 0.**

Proof of (19). By lemma 3.5 there is *Ro* > 0 such that *B{x_0, RQ)* CC ft and *d(v, B(z_o<R), y_0)* = 1 for every *0* < *R* < *RQ* and (19) follows from (6).

Proof of (18). Using the fact that *detVv(x)* > 0 a.e. x G ft, (11), (12) and (19) yield (18).

Proof of (20). As v satisfies the N-property (see Remark 2.5) $|v(\partial B(x_0, R_0))| = 0$ and $|v(\partial B)| = 0$. Since B is a non empty open set, by (11) we have that $|v(B)| \neq 0$ and so $|v(B) \setminus v(\partial B)| \neq 0$. Let $y \in v(B) \setminus v(\partial B)$ and C be the connected component of $\mathbf{R}^N \setminus v(\partial B)$ containing y. As $|v(\partial B(x_0, R_0))| = 0$ and since $d(v, B, \cdot)$ is a constant on C, we may assume without loss of generality that $y \notin v(\partial B(x_0, R_0))$. Let $\rho_\epsilon \in C^\infty(\mathbf{R}^N)$ be such that

$$\begin{aligned} & 0 \le \rho_\epsilon(y), \ \forall y \in \mathbf{R}^N, \ \ \forall \epsilon > 0 \\ & \frac{1}{2} \le \rho_\epsilon(y), \ \forall y \in B(0, \frac{\epsilon}{2}) \\ & supp\rho_\epsilon \subset B(0,\epsilon), \ \ \forall \epsilon > 0 \\ & \int_{\mathbf{R}^N} \rho_\epsilon(y)dy = 1 \ \ \forall \epsilon > 0. \end{aligned} \tag{21}$$

Since $y \in v(B)$ there is $x \in B$ such that $y = v(x)$. By (6) we have

$$\lim_{\epsilon \to 0} \int_B \rho_\epsilon(v(z) - y) det\nabla v(z) dz = d(v, B, y) \tag{22}$$

and using the continuity of v at x, we deduce that for every $\epsilon > 0$ there is $\delta > 0$ such that $|v(z) - y| \le \frac{\epsilon}{2}$ for every $z \in B(x, \delta)$. Recalling that $det\nabla v(z) > 0 \ \ a.e. \ \ z \in B(x, \delta)$, by (21) and (22) we obtain

$$d(v, B, y) > 0. \tag{23}$$

Finally as the degree $d(v, \cdot, y)$ is a non decreasing function of the set, using (19) and the fact that $B \subset v^{-1}(C_R) \cap B(x_0, R)$ we obtain

$$d(v, B, y) \le d(v, B(x_0, R), y) = 1 \tag{24}$$

which together with (23) and the fact that the degree is an integer number, yields (20). ∎

Lemma 3.8 *Let Ω, v, R_0 and x_0 be as in Lemma 3.7, (18), (19). Let C_{R_0} be the connected component of $\mathbf{R}^N \setminus v(\partial B(x_0, R_0))$ containing $y_0 := v(x_0)$. Then for every $r > 0$ such that $B(y_0, r) \subset\subset C_{R_0}$, if $O := v^{-1}(B(y_0, r)) \cap B(x_0, R_0) \subset\subset B(x_0, R_0)$ then*

$$v(O) = B(y_0, r), \ \ v(\partial O) \subset \partial v(O) = \partial B(y_0, r). \tag{25}$$

Proof. It is clear that $v(O) \subset B(y_0, r)$. Conversely, if $y \in B(y_0, r)$, by (19) $d(v, B(x_0, R_0), y) = 1$ and so by (5) there exists $x \in B(x_0, R_0)$ such that $y = v(x)$, implying $y \in v(O)$. Let $x \in \partial O$ and let $\{a_n\} \subset O$, $\{b_n\} \subset B(x_0, R_0) \setminus O$ be such that

$$\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} b_n = x.$$

We have $v(a_n) \in v(O) = v(v^{-1}(B(y_0, r))) = B(y_0, r)$ and $v(b_n) \notin v(O) = B(y_0, r)$. Using the continuity of v at x, we have

$$v(x) = \lim_{n \to +\infty} v(a_n) = \lim_{n \to +\infty} v(b_n)$$

and this gives $x \in \partial v(O)$. ∎

Lemma 3.9 *Let $v \in W^{1,N}(\Omega)^N$, $det\nabla v(x) > 0$ a.e. $x \in \Omega$ and let $x_0 \in D$ be such that $v(x) \neq v(x_0)$ for every $x \in \bar{B}(x_0, R_0) \setminus \{x_0\}$. Let $0 < R < R_0$ and let C be an open set containing $y_0 = v(x_0)$. Then there is $r > 0$ such that $v^{-1}(B(y_0, r)) \cap B(x_0, R) \subset\subset B(x_0, R)$.*

Proof. Define

$$d(\delta) = \sup\{|x - x_0| : \ x \in \bar{B}(x_0, R), \ |v(x) - v(x_0)| \leq \delta\}.$$

Since $v(x) \neq v(x_0)$ for every $x \in \bar{B}(x_0, R) \setminus \{x_0\}$ and v is uniformly continuous on $\bar{B}(x_0, R)$ we have

$$\lim_{\delta \to 0} d(\delta) = 0.$$

Take now $r > 0$ such that $d(r) < \frac{R}{2}$. We have

$$v^{-1}(B(y_0, r)) \cap B(x_0, R) \subset B(x_0, \frac{R}{2}) \subset\subset B(x_0, R).$$

∎

Proof of Theorem 3.1 Let Ω' be the set of points $x_0 \in \Omega$ such that v is completly differentiable at x_0 and $det\nabla v(x_0) > 0$. By Lemmas 2.7 and Lemma 2.2 we obtain $|\Omega \setminus \Omega'| = 0$. In the sequel, we fix $x_0 \in \Omega'$, we set $y_0 = v(x_0)$ and show that v is locally invertible at x_0. By Lemma 3.5 and Lemma 3.7 there is $R_0 > 0$ such that $B(x_0, R_0) \subset\subset \Omega$,

$$N(v, B(x_0, R_0), y) = 1 \ \ a.e. \ \ y \in C_{R_0}, \tag{26}$$

where C_{R_0} is the connected component of $\mathbf{R}^N \setminus v(\partial B(x_0, R_0))$ containing y_0, with $N(v, B(x_0, R_0), y_0) = 1$. By Lemma 3.9 we deduce that there is $r > 0$ such that

$$v^{-1}(B(y_0, r)) \cap B(x_0, R_0) \subset\subset B(x_0, R_0) \tag{27}$$

and

$$B(y_0, r) \subset\subset C_{R_0}. \tag{28}$$

Setting $D = v^{-1}(B(y_0, r)) \cap B(x_0, R_0)$, by (27) and (28) we have $D \subset v^{-1}(C_{R_0}) \cap B(x_0, R_0)$ and by Lemma 3.8

$$v(D) = B(y_0, r), \ \ v(\partial D) \subset \partial v(D) = \partial B(y_0, r). \tag{29}$$

By the N^{-1}-property of v (see Remark 2.5 and (29)), we have $|\partial D| = 0$ which together with (20) yields

$$d(v, D, y) = 1 \ \ \forall\, y \in v(D) \setminus v(\partial D). \tag{30}$$

Using the definition of D, the fact that $D \subset B(x_0, R_0)$, (26) and (28), we obtain

$$N(v, D, y) = 1 \ \ a.e. \ \ y \in v(D). \tag{31}$$

Let $N := \{y \in v(D) \equiv B(y_0, r) | \ d(v, D, y) \neq 1\}$ and define the candidate for local inverse function, w, by

$$w(y) = x \ \text{ if } \ y \in v(D) \setminus N, \ \text{ and } \ v(x) = y, x \in D \tag{32}$$
$$w(y) = x, \ \text{ if } \ y \in N, \ \ v(x) = y \tag{33}$$

$x \in D$ being chosen by the axiom of choice.

Claim1: $w \in L^\infty(B(y_0, r))^N$.
We have $w(y) \in D \subset \Omega$ for every $y \in v(D)$ and so w is uniformly bounded in $v(D)$. To prove that w is Lebesgue measurable, fix $\alpha \in \mathbf{R}$ and show that the set

$$A := \{y \in v(D) : \ \ w_i(y) \geq \alpha\}$$

is measurable. We obtain $A = A_1 \cup A_2$ where

$$A_1 := \{y \in v(D) \setminus N : \ \ w_i(y) \geq \alpha\},$$

$$A_2 := \{y \in N : \ \ w_i(y) \geq \alpha\}.$$

Since $|A_2| = 0$ we deduce that A_2 is measurable. Using the fact that the restriction of v to $v^{-1}(v(D) \setminus N)$ is one-to-one, one can see that

$$\begin{aligned} A_1 \quad &= \{v(x) : \ \ x \in v^{-1}(v(D) \setminus N), \ \ x_i \geq \alpha\} \\ &= (v(D) \setminus N) \cap \left(\cup_{n=0}^{+\infty} v\{x \in \bar{B}(x_0, R_0), \alpha + n \leq x_i \leq \alpha + n + 1\}\right). \end{aligned}$$

Using the fact that for every $n \in \mathbf{N}$, $\{x \in \bar{B}(x_0, R_0), \ \ \alpha + n \leq x_i \leq \alpha + n + 1\}$ is a compact set, v is a continuous function and $v(D) \setminus N$ is measurable we obtain that A_1 is measurable and we conclude that $w \in L^\infty(B(y_0, r))^N$.

Claim 2:

$$v \circ w(y) = y \ \text{ for every } \ y \in v(D) \equiv B(y_0, r), \tag{34}$$
$$w \circ v(x) = x \ \text{ for every } \ x \in D \setminus v^{-1}(N). \tag{35}$$

This follows immediatly from (32) and (33). One notice that, due to (30) and Remark 2.5, $|v^{-1}(N)| = 0$.

Claim 3: $f \circ w$ is measurable for every $f : D \to \mathbf{R}$ measurable.
Knowing that every Lebesgue measurable set is a union of a Borel measurable set and a set of measure zero, to show that $f \circ w$ is measurable, by claim 1 it suffices to show that $w^{-1}(R)$ is measurable for every $R \subset D$ so that $|R| = 0$. Let R be a subset of D such that $|R| = 0$. We have by (34)

$$w^{-1}(R) \subset v(R),$$

and since $|R| = 0$, by the N-property of v, we obtain that $|w^{-1}(R)| = 0$. Thus $w^{-1}(R)$ is measurable.

Let $g : v(D) = B(y_0, r) \to \mathbf{R}$ be defined by

$$g(y) = \frac{|adj \nabla v(w(y))|}{det \nabla v(w(y))}.$$

Claim 4: $g \in L^1(v(D))$.
By claim 3 g is measurable. By Lemma 2.4 (11) where we set $f = \chi_{v(D)}$ the indicator of the set $v(D)$, by claim 2, and (31) we obtain

$$\int_{v(D)} |g(y)|dy = \int_D |g \circ v(x)| det \nabla v(x) dx = \int_D |adj \nabla v(x)| dx.$$

Therefore $g \in L^1(v(D))$.

Claim 5: $w \in W^{1,1}(v(D))^N$ and $\nabla w(y) = (\frac{adj \nabla v(w(y))}{det \nabla v(w(y))})^T$.
To prove claim 5, we fix $\phi \in C_0^\infty(v(D))$ and set $K = supp\phi$. We show that

$$\int_{v(D)} w_\alpha(y) \frac{\partial \phi}{\partial y_j}(y) dy = -\int_{v(D)} \frac{\left(adj \nabla v(w(y))\right)_\alpha^j}{det \nabla v(w(y))} \phi(y) dy.$$

Set $\delta = dist(K, \partial v(D)) > 0$. Using the uniform continuity of v on $\bar{D} \subset B(x_0, R_0)$ we choose $\epsilon > 0$ such that

$$|v(x) - v(x')| \leq \frac{\delta}{4} \text{ for every } x, x' \in \bar{D}, \ |x - x'| \leq \epsilon. \tag{36}$$

Let $\{v_n\} \subset C^\infty(\bar{D})^N$ be such that

$$v_n \to v \text{ in } C^0(\bar{D})^N \tag{37}$$

and

$$v_n \rightharpoonup v \text{ in } W^{1,N}(D)^N.$$

By (37) we can assume without loss of generality that

$$|v - v_n|_\infty \leq \frac{\delta}{4} \text{ for every } n \in \mathbf{N}. \tag{38}$$

By the fact that $v(\partial D) \subset \partial v(D)$ (see (29)), by (36) and (38) we have that

$$dist(x, \partial D) < \epsilon \text{ implies } \phi(v_n(x)) = 0$$

and so

$$\phi \circ v_n \in C_0^\infty(D).$$

In the sequel we denote by A^j_α the component of the j-row and the α-column of the $N\times N$ matrix A. By (11), (31) and (35) and the fact that for every $n\in\mathbf{N}$, $\sum_{\alpha=1}^N \frac{\partial(adj\nabla v_n)^j_\alpha}{\partial x_\alpha}=0$ for every $j=1,\cdots,N$ we have

$$\begin{aligned}
\int_{v(D)} w_\alpha(y)\frac{\partial\phi}{\partial y_j}(y)dy &= \int_D w_\alpha(v(x))\frac{\partial\phi}{\partial y_j}(v(x))det\nabla v(x)dx\\
&= \lim_{n\to+\infty}\int_D x_\alpha\frac{\partial\phi}{\partial y_j}(v_n(x))det\nabla v_n(x)dx\\
&= \lim_{n\to+\infty}\int_D x_\alpha\sum_{k=1}^N\Big(adj\nabla v_n(x)\Big)^j_k\frac{\partial}{\partial x_k}\phi(v_n(x))dx\\
&= -\lim_{n\to+\infty}\int_D\Big(adj\nabla v_n(x)\Big)^j_\alpha\phi(v_n(x))dx\\
&= -\int_D\Big(adj\nabla v(x)\Big)^j_\alpha\phi(v(x))dx\\
&= -\int_D\frac{\Big(adj\nabla v(w\circ v(x))\Big)^j_\alpha}{det\nabla v(w\circ v(x))}\phi(\circ v(x))det\nabla v(x)dx\\
&= -\int_{v(D)}\frac{\Big(adj\nabla v(w(y))\Big)^j_\alpha}{det\nabla v(w(y))}\phi(y)dy.
\end{aligned}$$

This equality together with claim 4 yields claim 5.

Claim 6: $\nabla w\in W^{1,s}(v(D))$ if and only if $|g\circ v|^s det\nabla v\in L^1(v(D))$ for $1\le s<+\infty$. Recall that $g(y)=\frac{|adj\nabla v(w(y))|}{det\nabla v(w(y))}$ and that $w\in L^\infty(v(D))^N$. Thus $\nabla w\in W^{1,s}(v(D))$ if and only if $\nabla w\in L^s(v(D))$. The result now follows from claim 5 and (11). ∎

Remarks 3.10 It is possible to show that if $v\in W^{1,q}(\Omega)^N$, $q>N-1$, $adj\nabla v\in L^{\frac{N}{N-1}}(\Omega)$, $det\nabla v(x)>0$ a.e. in Ω and if v is continuous, then there is local invertibility a.e. in Ω, i.e. for a.e. $x_0\in\Omega$ there exists $r>0$ such that $v|_{B(x_0,r)}$ is almost everywhere injective with the inverse $w\in BV_{loc}(v|_{v(B(x_0,r))},\mathbf{R}^N)$ and there exists a set $E\subset v(B(x_0,r))$ such that

$$\begin{aligned}
&E \textit{ is an open set of } v(B(x_0,r)),\\
&|v(B(x_0,r)\setminus E|=0,\\
&w\in W^{1,1}(E,\mathbf{R}^N),\\
&v\circ w(y)=y \ \textit{ a.e. } \ y\in v(B(x_0,r)),\\
&w\circ v(x)=x \ \textit{ a.e. } \ x\in B(x_0,r).
\end{aligned}$$

To see this, we recall that by Lemma 2.7 iii) v is weakly differentiable a.e. in Ω and by adapting the proof of Lemma 3.5 accordingly, it is possible to show that

$$d(v,B(x_0,r),v(x_0))=1$$

for some $r > 0$. Let C_0 be the connected component of $\mathbf{R}^N \setminus v(\partial B(x_0, r))$ which contains $v(x_0)$. Then

$$d(v, B(x_0, r), y) = 1 \tag{39}$$

for every $y \in C_0$ and so, if we choose $0 < r' < r$ such that

$$B(x_0, r') \subset B(x_0, r) \cap v^{-1}(C_0)$$

then by (39) and since $det\nabla v > 0$ a.e. we have

$$d(v, B(x_0, r'), y) \leq 1$$

for every $y \in \mathbf{R}^N \setminus v(\partial B(x_0, r'))$. It suffices now to use the results in [TQ], (1.3) - (1.5), (2.26) and Theorem 3.7 (i). Note, however, that in [TQ], it is assume that $adj\nabla v \in L^r$, $r \geq \frac{q}{q-1}$ and if $N - 1 < q < N$, then $\frac{q}{q-1} > \frac{N}{N-1}$.

As it turns out, [TQ]'s results still hold for $r = \frac{N}{N-1}$ as remarked by [MTY] (see Theorem 5.3 in [MTY]).

Proof of Corollary 3.2.
Proof of a) We have

$$v \in W^{1,N}(\Omega_1)^N, \quad det\nabla v(x) > 0 \;\; a.e. \;\; x \in \Omega_1$$

and

$$v \in W^{1,N}(\Omega_2)^N, \quad det\nabla v(x) < 0 \;\; a.e. \;\; x \in \Omega_2.$$

It suffices to apply Theorem 3.1 to v and to $R_0 v$ in Ω_2, where R_0 is a constant rotation with $detR_0 = -1$.
Proof of b) We now assume that $v \in W^{1,q}(\Omega)^N$, $q > N$ $det\nabla v(x) \neq 0$ $a.e.$ $x \in \Omega$ and for almost every $x_0 \in \Omega$, v is locally almost injective in a neighborhood of x_0 in the sense that there is an open set $D \equiv D(x_0) \subset\subset \Omega$ and there is a function $w : v(D) \to D$ such that

$$w \circ v(x) = x \;\; a.e. \;\; x \in D. \tag{40}$$

By Vitali's covering theorem there is a countable family of non-empty, open, mutually disjoint balls $\{B_i, \; i \in \mathbf{N}\}$ and there is a sequence of functions $w_i : v(\bar{B}_i) \to \Omega$ such that $\bar{B}_i \subset \Omega$ and

$$\begin{aligned} &|\Omega \setminus \cup_{i=1}^{+\infty} B_i| = 0, \\ &w \circ v(x) = x \;\; a.e. \;\; x \in B_i. \end{aligned} \tag{41}$$

The task ahead will be to partition B_i into three subsets B_i^1, B_i^2 and N_i such that B_i^1, B_i^2 are two open sets, N_i is a set of measure zero,

$$\begin{aligned} &det\nabla v(x) > 0 \;\; a.e. \;\; x \in B_i^1, \\ &det\nabla v(x) < 0 \;\; a.e. \;\; x \in B_i^2. \end{aligned}$$

Using the fact that $v \in W^{1,q}(B_i)^N$, $q > N$, by Lemma 2.7 and (41) we deduce that there is a set $A_i \subset \bar{B}_i$ of measure zero such that v is completely differentiable at every $x \in B_i \setminus A_i$,

$$\begin{aligned} w \circ v(x) = x \;\text{ for every }\; x \in \bar{B}_i \setminus A_i \\ det \nabla v(x) \neq 0 \;\text{ for every }\; x \in \bar{B}_i \setminus A_i. \end{aligned} \tag{42}$$

Let $\{C^j\}$ be the countable collection of the (open) connected components of $\mathbf{R}^N \setminus v(\partial B_i)$. By Remark 2.5 a), c) we have

$$|v^{-1}(v(\partial B_i \cup A_i))| = 0. \tag{43}$$

We claim that

Claim1 $d(v, B_i, v(x)) = sign\, det \nabla v(x)$ for every $x \in B_i \setminus v^{-1}(v(\partial B_i \cup A_i))$.
Fix $x \in B_i \setminus v^{-1}(v(\partial B_i \cup A_i))$.
step1 We prove that $d(v, B(x, r_0), v(x)) = sign\, det \nabla v(x)$ for r_0 small enough. Using the fact that v is completely differentiable and $det \nabla v(x) \neq 0$, by Lemma 3.5 we deduce that there is $r_0 > 0$ such that for every $0 < r \leq r_0$ we have

$$d(v, B(x, r), v(x)) = sign\, det \nabla v(x).$$

step2 We show that $d(v, B_i, v(x)) = sign\, det \nabla v(x)$. Indeed, setting $K = \bar{B}_i \setminus B(x, r_0)$, K is a compact set included in $\bar{B}_i$ and by (42) $v(x) \notin v(K)$ because $v(x) \notin v(A_i)$. By the excision property of the degree (see Proposition 2.3) we obtain

$$d(v, B_i, v(x)) = d(v, B(x, r_0), v(x)) = sign\, det \nabla v(x).$$

Claim2 $sign\, det \nabla v(x) = sign\, det \nabla v(x')$ for every $x, x' \in v^{-1}(C^j) \setminus v^{-1}(v(\partial B_i \cup A_i))$.
Assume that $x, x' \in v^{-1}(C^j) \setminus v^{-1}(v(\partial B_i \cup A_i))$. Using claim 1 and the fact that the degree $d(v, B_i, \cdot)$ is constant on each C^j, we obtain that $sign\, det \nabla v(x) = sign\, det \nabla v(x')$.

We now conclude the proof of b). Let $I = \{j \in \mathbf{N} | det \nabla v(x) > 0 \;\; a.e. \;\; x \in v^{-1}(C^j)\}$ and $J = \{j \in \mathbf{N}, det \nabla v(x) < 0 \;\; a.e. \;\; x \in v^{-1}(C^j)\}$. Set

$$B_i^1 = \cup_{j \in I} v^{-1}(C^j) \cap B_i,$$

$$B_i^2 = \cup_{j \in J} v^{-1}(C^j) \cap B_i,$$

and

$$N_i = B_i \setminus (B_i^1 \cup B_i^2).$$

Then $B_i = B_i^1 \cup B_i^2 \cup N_i$ and setting $\Omega_1 = \cup_i B_i^1$, $\Omega_2 = \cup_i B_i^2$ and $N = \Omega \setminus (\Omega_1 \cup \Omega_2)$, then $|N| = 0$ and Ω_1, Ω_2 have the required properties. ∎

Proof of Corollary 3.3: To obtain that $w \in W^{1,\frac{q}{N-1}}(v(D), D)$ we take $s = \frac{q}{N-1}$ in Theorem 3.1. If $q \geq N(N-1)$ then $w \in W^{1,N}$,

$$det\nabla w(y) = \frac{1}{det\nabla v(w(y))} > 0 \ \ a.e. \ \ y \in v(D)$$

and so, by Lemma 2.2 we deduce that w is continuous. Hence v and w are homeomorphisms and v is an open mappings in Ω' for some $\Omega' \subset \Omega$ open, where $|\Omega \setminus \Omega'| = 0$. ∎

4 Semicontinuity involving variation of the domain.

The variational treatment of crystals with defects leads to the study of functionals of the type

$$E(u, v) = \int_\Omega W(\nabla u(x)(\nabla v(x))^{-1})dx$$

where $\Omega \subset \mathbf{R}^N$ is a reference domain, W is the strain energy density, u is the elastic deformation and v represents the slip (rearrangement) or plastic deformation with $det(\nabla v(x)) = 1 \ \ a.e. \ \ x \in \Omega$. The underlying kinematical mode for slightly defective crystals was introduced by Davini [Dav] and later developed by Davini and Parry [DP]. As it turns out, matrices of the form

$$\nabla u(x)(\nabla v(x))^{-1}$$

represent lattice matrices of defect-preserving deformations (neutral deformations) and taking the viewpoint that equlibria correspond to a variational principle, Fonseca & Parry [FP] studied the structure of some kind of generalized minimizers (Young measure solutions) for the energy $E(\cdot, \cdot)$ (related variational problems were also investigated in [DP]).

Using the Div-Curl Lemma it follows that if $u_n \rightharpoonup u$ in $W^{1,\infty}$ $w*$ and $v_n \rightharpoonup v$ in $W^{1,\infty}$ $w*$ then

$$\nabla u_n(\nabla v_n)^{-1} \rightharpoonup \nabla u(\nabla v)^{-1} \text{ in } L^\infty \ w*.$$

Lower semicontinuity and relaxation properties of $E(\cdot, \cdot)$ were adressed only under additional material symmetry assumptions on W. Existence and regularity properties for minimizers of $E(\cdot, \cdot)$ were obtained in [DF]. Following this work, we stress the fact that the direct methods of the calculus of variations fail to apply to this problem, as sequential weak lower semicontinuity of $E(\cdot, \cdot)$ is not sufficient to guarantee the existence of minimizers. Indeed with $W(F) = ||F||^r$, it is shown in [DF] that there are no minimizers in $\{(u, v) \in W^{1,\infty} \times W^{1,\infty} | \ u(x) = x \text{ on } \partial\Omega, \ det(\nabla v(x)) = 1 \ a.e.\}$ if $0 < r < N = 2$, while for $r > N$ existence is obtained for smooth (u, v) (Theorem 2.3 [DF]).

It is clear that if $\{(u_n, v_n)\}$ is a minimizing sequence and if, $||\nabla u_n(\nabla v_n)^{-1}||^r$ is bounded in L^1 then

$$\nabla u_n(\nabla v_n)^{-1} \rightharpoonup L \text{ in } L^r, u_n|_{\partial\Omega} = u_0, det(\nabla v_n) = 1 \ \ a.e.$$

and so if some type of lower semicontinuity prevails, then

$$\int_\Omega W(L)dx \le \liminf \int_\Omega W(\nabla u_n(\nabla v_n)^{-1})dx. \tag{44}$$

It would remain to show that L would still have the same structure, precisely

$$L = \nabla u(\nabla v)^{-1}$$

where $u|_{\partial\Omega} = u_0$, $det(\nabla v) = 1$ $a.e.$ Note that (44) is always satisfied if W is a convex function. On the other hand, formally, as $det(\nabla v) = 1$ $a.e.$ and setting $w = u(v^{-1})$ then the energy becomes

$$\int_{v(\Omega)} W(\nabla w(y))dy$$

which is now an energy functional involving variations of the domain. Hence, under this new formulation, quasiconvexity seems to be more appropriate than convexity (see [AF], [Ba] and [Da]).
Suppose that W is a quasiconvex function, i.e.

$$W(F) \le \frac{1}{|Q|}\int_Q W(F + \nabla\phi(x))dx$$

where $Q = (0,1)^N, \phi \in W_0^{1,\infty}(Q)^N$ and let $\nabla u_n(\nabla v_n)^{-1} \rightharpoonup L$ in L^r. Can we say that

$$\int_\Omega W(L) \le \liminf \int_\Omega W(\nabla u_n(\nabla v_n)^{-1})?$$

As an example, consider

$$W(F) = ||F||^2 + |det(F)|.$$

Although we are unable to answer this question, we prove the following result which is the main theorem of this section.

Theorem 4.1 *Let $W : M^{N\times N} \to \mathbf{R}$ be a quasiconvex function such that*

$$-C_1(1 + ||A||^s) \le W(A) \le C_2(1 + ||A||^r)$$

for some constants $C_1, C_2 > 0, r > s \ge 1$, $p \ge 1$, $q \ge N$, $\frac{1}{p} + \frac{N-1}{q} = \frac{1}{r}$ ($W \ge 0$ if $r = s = 1$). If $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)^N$, $v_n \rightharpoonup v$ in $W^{1,q}(\Omega)^N$, and $det(\nabla v_n) = 1$ a.e. in Ω then

$$\int_\Omega W(\nabla u(\nabla v)^{-1})dx \le \liminf \int_\Omega W(\nabla u_n(\nabla v_n)^{-1})dx.$$

Before proving Theorem 4.1 we make some remarks.

Remarks 4.2 1. It is dear that if it € $W^{1,*}$, *v* € W^{l}« and *detVv* = 1 a.e. then Vu(Vv)-^{1} € *V*.

2. If r > 1 then 5 < r is a necessary condition as the counterexample by Murat and Tartar shows (see [BM]). Here r = s = 2 = Ar,ft = $(0,l)^2$, *W(F)* = *det{F)*, u_n - *u* in $H^l(Sl)$, $v_n(x)$ = *x* and

$$/_{Jo} \text{de*Vu j£ liminf} /_{Jo} detVu_n.$$

3.The growth condition cannot be dropped even if *W* is polyconvex and nonegative. Precisely if the relation between p, g, r and *s* does not occur, the conclusion of Theorem 4.1 may be false. Indeed, using the example by Malý ([Ma]) with *q* = +oo, *p* < TV—1, *W(F)* = *detF*, *N* = r = 5, we may find u_n —* u in W^{liP}, tz(x) = *x* with $r_n(z)$ = x and

$$/_{Jo} \backslash det(Vu)\backslash > \text{liminf} /_{Jo} \backslash det(Vu_n)\backslash.$$

Moreover the growth condition prescribed in Theorem 4.1 is the well known growth condition ensuring weak lower semicontinuity of *E(u,id)* in $W^{l}>^{p}$ (see [AF] and [Da]).

4. We may ask if these results can be extended to the case ^-j- < *q* < *N*, since, due to Müller's result ([Mu]), if we assume that *DetVv* = 1 a.e. then *DetVv* = *detVv* a.e. in n.

5.Having obtained lower semicontinuity of the energy in Theorem 4.1, the question now amounts to showing that one can find a minimizing sequence {$Vu_n(Vi;_n)^{""1}$} where {u_n} is bounded in W^{l}* and *{v_n}* is bounded in W^{l*q}. Actually, one only needs to show that there exists a sequence {$/_n$} C $W^{l,oo}(fl;il)$ such that $v_n o f_n$ is bounded in W^{l}* and

$$\begin{array}{ccc} \text{f } detVf_n(x) & = & 1 \text{ a.e. } x6ft \\ \text{t } /»(*) & = & \text{i } x € an. \end{array}$$

Due to the examples provided in [DF], we know that this may not be possible since the infimum of *E* may be zero, which prevents the existence of minimizing sequence bounded in W^{l}* *x* W^{l}«.

As usual in variational problems for which existence of minimizers is not guaranteed (such as variational problems for material that change phase and, here, for slightly defective materials), rather then studying the macroscopic limit of $Vu_n(Vz;_n)^{\sim 1}$ we focus on the properties of the minimizing sequences.

The following may help to understand better why boundedness of {$Vu_n(Vv_n)^{\sim l}$} may not entail the boundedness of {Vu_n} and {Vv_n}. Using Theorem 4.1 we show that we may construct a minimizing sequence {$Vu^VrJ^{\wedge 1}$} with $||Vu_c||_p$ = 0(^r), $||Vr_c||_9$ = 0(^). for any a, *f3* > 0.

Consider the "perturbed" family of variational problems

$$E_\epsilon(u,v) = \int_\Omega W(\nabla u(\nabla v)^{-1})dx + \epsilon^{\alpha p}||\nabla u_\epsilon||_p^p + \epsilon^{\beta q}||\nabla v_\epsilon||_q^q,$$

where $u|_{\partial\Omega} = u_0$, $det\nabla v = 1$ *a.e.*, $\frac{1}{|\Omega|}\int_\Omega v(x)dx = 0$. Using the direct method of the Calculus of Variations, Poincaré's inequality and Theorem 4.1, it follows immediatly that there exists $(u_\epsilon, v_\epsilon) \in W^{1,p} \times W^{1,q}$ such that

$$E_\epsilon(u_\epsilon, v_\epsilon) = \inf\{E_\epsilon(u,v) : \ (u,v) \in W^{1,p} \times W^{1,q}, \ det\nabla v = 1 \ a.e.\}.$$

Then, given an admissible pair (u,v)

$$\begin{aligned} E(u,v) &= \lim_{\epsilon\to 0+} E_\epsilon(u,v) \\ &\geq \limsup_{\epsilon\to 0+} E_\epsilon(u_\epsilon, v_\epsilon) \\ &\geq \limsup_{\epsilon\to 0+} E(u_\epsilon, v_\epsilon), \\ &\geq \inf E. \end{aligned}$$

Doing the same with $\liminf_{\epsilon\to 0+} E(u_\epsilon, v_\epsilon)$ and taking the infimum in (u,v) we conclude that

$$\inf E = \lim_{\epsilon\to 0+} E(u_\epsilon, v_\epsilon)$$

and $||\nabla u_\epsilon||_p = 0(\frac{1}{\epsilon^\alpha})$, $||\nabla v_\epsilon||_q = 0(\frac{1}{\epsilon^\beta})$.

The following two lemmas will be useful to prove Theorem 4.1.

Lemma 4.3 *Let Ω', Ω be two open sets of $\mathbf{R}^N$ such that $\Omega' \subset\subset \Omega$, let $q \geq N$ and $v, v_n \in W^{1,q}(\Omega)^N$ be such that $det\nabla v(x) = det\nabla v_n(x) = 1$ a.e. $x \in \Omega$. Assume that $v_n \rightharpoonup v$ in $W^{1,q}(\Omega)^N$. Then there exists a subsequence of $\{v_n\}$ (not relabelled) such that for almost every $x_0 \in \Omega'$ there exist open sets $D, D_n \subset \Omega'$ containing x_0, there exist $n_0 \in \mathbb{N}$, $r_0 \equiv r(x_0) > 0$, $w : B(y_0, r_0) \to D$, $w_n : B(y_0, r_0) \to D_n$, with $y_0 = v(x_0)$ such that for $n \geq n_0$*

$$\begin{aligned} &w_n \circ v_n(x) = x \ \ a.e. \ \ x \in \bar{D}_n, \\ &v_n \circ w_n(y) = y \ \ for\ every \ \ y \in \bar{B}(y_0, r_0) \ \ and \ \ v_n(D_n) = B(y_0, r_0) \\ &w \circ v(x) = x \ \ a.e. \ \ x \in \bar{D} \ \ and \ \ v(x_0) \neq v(x) \ \ for \ \ x \in D, x \neq x_0 \\ &v \circ w(y) = y \ \ for\ every \ \ y \in \bar{B}(y_0, r_0) \ \ and \ \ v(D) = B(y_0, r_0), \\ &w_n, w \in W^{1,\frac{q}{N-1}}. \end{aligned}$$

Proof. Using Lemma 2.2 and the Ascoli-Arzela Theorem we obtain that, up to a subsequence, v_n converges to v uniformly in $\bar{\Omega}'$. By Lemmas 3.7 and 2.7 for almost every

$x_0 \in \Omega'$, there is $R_0 > 0$ such that

$$\begin{aligned}
&B(x_0, R_0) \subset\subset \Omega', \\
&N(v, B(x_0, R_0), y)) = 1 \text{ for almost every } y \in C_{R_0}, \\
&d(v, B(x_0, R_0), y) = 1 \text{ for every } y \in C_{R_0}, \\
&d(v, B, y) = 1 \text{ for every } y \in B \setminus v(\partial B), \\
&\text{for every non empty open set } B \subset v^{-1}(C_{R_0}) \cap B(x_0, R_0) \text{ such that } |v(\partial B)| = 0,
\end{aligned}$$

where C_{R_0} is the connected component of $\mathbf{R}^N \setminus v(\partial B(x_0, R_0))$ containing $y_0 := v(x_0)$. Since v is completely differentiable at x_0 and $det \nabla v(x_0) \neq 0$ we may assume without loss of generality that $N(v, B(x_0, R_0), y_0)) = 1$. Fix $0 < \epsilon < d(y_0, v(\partial B(x_0, R_0)))$ and choose $n_0 \in \mathbf{N}$ such that $|v_n - v|_\infty < \epsilon$. Set

$$A_\epsilon := \{y \in C_{R_0} : \ dist(y, v(\partial B(x_0, R_0))) > \epsilon\}.$$

It is obvious that A_ϵ is a non-empty open set.
Claim 1 $d(v_n, B(x_0, R_0), y)$ exists and is equal to 1 for every $y \in A_\epsilon$ and every $n \geq n_0$.
By Proposition 2.3 (4), together with the fact that $d(v, B(x_0, R_0), y) = 1$ for every $y \in C_{R_0}$, we have

$$d(v_n, B(x_0, R_0), y) = 1 \tag{45}$$

for every $y \in A_\epsilon$ and every $n \geq n_0$.

By Lemma 3.9 there is $0 < r_0 < R_0$ such that

$$B(y_0, r_0) \subset\subset A_\epsilon \quad \text{and} \quad v^{-1}(B(y_0, r_0)) \cap B(x_0, R_0) \subset\subset B(x_0, R_0)). \tag{46}$$

Claim 2 We claim that

$$B(y_0, r_0) \subset\subset C^n_{R_0} \tag{47}$$

where $C^n_{R_0}$ is the connected component of $\mathbf{R}^N \setminus v_n(\partial B(x_0, R_0))$ containing y_0.
We prove first that $A_\epsilon \subset \mathbf{R}^N \setminus v_n(\partial B(x_0, R_0))$. Assume on the contrary that there is $y \in A_\epsilon \cap v_n(\partial B(x_0, R_0))$ and choose $x \in \partial B(x_0, R_0)$ such that $y = v_n(x)$. We would have $|v_n(x) - v(x)| = |y - v(x)| > \epsilon > |v_n - v|_\infty$ which yields a contradiction. Fix $r' > r_0$ such that $\bar{B}(y_0, r') \subset A_\epsilon$. We have $B(y_0, r')$ is a connected set included in $\mathbf{R}^N \setminus v_n(\partial B(x_0, R_0))$ and containing y_0. We deduce that $B(y_0, r') \subset C^n_{R_0}$ and $B(y_0, r_0) \subset\subset C^n_{R_0}$.

Set $D = v^{-1}(B(y_0, r_0)) \cap B(x_0, R_0) \subset\subset \Omega'$ and $D_n = v_n^{-1}(B(y_0, r_0)) \cap B(x_0, R_0) \subset\subset \Omega'$. By (45), (46), (47) and using arguments similar to the ones of the proof of Theorem 3.1, together with Corollary 3.3 we deduce that for $n \geq n_0$ there is $w_n : \bar{B}(y_0, r_0) \to \bar{D}_n$, there is $w : \bar{B}(y_0, r_0) \to \bar{D}$ such that

$$w_n, w \in W^{1, \frac{2}{N-1}}(B(y_0, r_0))^N,$$

$$
\begin{aligned}
&w_n \circ v_n(x) = x \ \text{ a.e. } \ x \in \bar{D}_n, \\
&v_n \circ w_n(y) = y \ \text{ a.e. } \ y \in \bar{B}(y_0, r_0), \\
&w \circ v(x) = x \ \text{ a.e. } \ x \in \bar{D} \ \text{ and } \ v(x_0) \neq v(x) \ \text{ for } \ x \in \bar{D}, x \neq x_0, \\
&v \circ w(y) = y \ \text{ a.e. } \ y \in \bar{B}(y_0, r_0).
\end{aligned}
$$

Finally by Lemma 3.8, $v_n(D_n) = v(D) = B(y_0, r_0)$.

Remarks 4.4

1. It follows from the proof above that if the conclusion of Lemma 4.3 holds for $r \equiv r(x_0) > 0$ then it holds also for $0 < r' < r$. Thus, as v is continuous on $\bar{D}$, $v(x) \neq v(x_0)$ for $x \in D$ and $x \neq x_0$, we deduce that

$$\lim_{r \to 0} \max\{|x - x_0|, \ \ x \in \bar{D}, v(x) \in B(y_0, r_0)\} = 0.$$

2. It is possible to show that $\lim_{n\to+\infty} |D \Delta D_n| = 0$. We divide the proof into two cases.
Claim 1 $\lim_{n\to+\infty} |D \setminus D_n| = 0$.
Let $F_\epsilon = B(y_0, r_0 - \epsilon)$ and $O_\epsilon = v^{-1}(F_\epsilon) \cap D$. We prove first that for each ϵ fixed there exists $n_0 \equiv n_0(\epsilon) \in \mathbb{N}$ such that $n \geq n_0$ implies $O_\epsilon \subset D_n$. Indeed, since $\{v_n\}$ converge to v uniformly, there exists $n_0 \equiv n_0(\epsilon) \in \mathbb{N}$ such that $|v - v_n|_\infty \leq \frac{\epsilon}{2}$ for every $n \geq n_0$. If $x \in O_\epsilon$, we obtain

$$|v_n(x) - y_0| \leq |v(x) - y_0| + |v(x) - v_n(x)| < r_0$$

and so $x \in D_n$. As $\cup_\epsilon O_\epsilon = D$ and the sequence (O_ϵ) is non-increasing, we have

$$\lim_{\epsilon \to 0} |D \setminus O_\epsilon| = 0$$

which, together with the fact that $|D \setminus D_n| \leq |D \setminus O_\epsilon|$ for $n \geq n_0$, yields claim 1.
Claim 2 $\lim_{n\to+\infty} |D_n \setminus D| = 0$.
For $\epsilon > 0$ take $n_0 \equiv n_0(\epsilon) \in \mathbb{N}$ such that $|v - v_n|_\infty \leq \frac{\epsilon}{2}$ for every $n \geq n_0$. For $n \geq n_0$, we have

$$\{x \in B(x_0, R_0) : r - \frac{\epsilon}{2} \leq |v_n(x) - y_0| < r\} \subset \{x \in B(x_0, R_0) : r - \epsilon \leq |v(x) - y_0| < r + \epsilon\}$$

and since v has the N^{-1} property (see Remark 2.5) we obtain

$$|\cap_\epsilon \{x \in B(x_0, R_0) : r - \epsilon \leq |v(x) - y_0| < r + \epsilon\}| = |\{x \in B(x_0, R_0) : |v(x) - y_0| = r\}| = 0.$$

To conclude the proof of claim 2 it suffices to remark that for $n \geq n_0$ we obtain

$$D_n \setminus D \subset \{x \in B(x_0, R_0) : r - \epsilon \leq |v(x) - y_0| < r + \epsilon\}.$$

Lemma 4.5 *Let $p \geq 1$, $q \geq N$, $r \geq 1$ be such that $\frac{1}{p} + \frac{N-1}{q} = \frac{1}{r}$. Assume that $\Omega \subset \mathbf{R}^N$ is an open, bounded set, $u_n, u \in W^{1,q}(\Omega)^N$, $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)^N$, $v_n, v \in W^{1,q}\Omega)^N$, $det\nabla v_n = det\nabla v = 1$ a.e. in Ω and $v_n \rightharpoonup v$ in $W^{1,q}(\Omega)^N$. Let $x_0 \in \Omega$, and w_n, w be, respectively the local inverse function of v_n, v, in the open neighborhoods D_n, D of x_0, let $y_0 = v(x_0)$ and $B(y_0, r_0)$ be as in Lemma 4.3 and Remark 4.4. Then the following hold:*

i) $u_n \circ w_n \in W^{1,r}(B(y_0,r_0))^N$ *and* $\nabla(u_n \circ w_n)(y) = \nabla u_n(w_n(y))(\nabla v_n(w_n(y)))^{-1}$ *a.e.*

ii) $u_n \circ w_n \rightharpoonup u \circ w$ *in* $W^{1,r}(B(y_0,r_0))^N$ *if* $r > 1$,

iii) $u_n \circ w_n \to u \circ w$ *in* $L^1(B(y_0,r_0))^N$
and $\{u_n \circ w_n\}$ *is bounded in* $W^{1,1}(B(y_0,r_0))^N$ *if* $r = 1$.

Proof. We remind that by Lemma 4.3 we have

$$w_n, w \in W^{1,\frac{q}{N-1}}(B(y_0,r_0))^N, \quad v(D) = B(y_0,r_0), v_n(D_n) = B(y_0,r_0), \tag{48}$$

$$\nabla w(y) = (\nabla v(w(y)))^{-1}, \nabla w_n(y) = (\nabla v_n(w_n(y)))^{-1} \text{ a.e. } \; y \in B(y_0,r_0). \tag{49}$$

$$N(v,D,y) = N(v_n,D_n,y) = 1 \text{ a.e. } \; y \in B(y_0,r_0), \tag{50}$$

$$w \circ v(x) = x \;\; a.e. \;\; x \in D, \;\; w_n \circ v_n(x) = x \;\; a.e. \;\; x \in D_n. \tag{51}$$

First step. We prove that $u \circ w, u_n \circ w_n \in W^{1,r}(B(y_0,r_0))^N$.
In fact by the change of variables formula (11), (48), (49), (50) and (51) we have

$$\begin{aligned}\int_{B(y_0,r_0)} |u \circ w(y)|^r dy &= \int_{v(D)} |u \circ w(y)|^r N(v,D,y) dy \\ &= \int_D |u(x)|^r dx < +\infty.\end{aligned}$$

Thus

$$u \circ w, \; u_n \circ w_n \in L^r(B(y_0,r_0))^N.$$

Let $\phi \in C_0^\infty(B(y_0,r_0))$. By (11), (48), (49), (50), (51) and using the fact that each vector row of $adj\nabla v$ is divergence free, we have

$$\begin{aligned}\int_{B(y_0,r_0)} u_i \circ w(y) \frac{\partial \phi}{\partial y_j} dy &= \int_D u_i(x) \frac{\partial \phi}{\partial y_j} \circ v(x) dx \\ &= -\int_D \sum_{l=1}^N \frac{\partial u_i}{\partial x_l}(x)((\nabla v(x))^{-1})_j^l \phi \circ v(x) dx \\ &= -\int_{B(y_0,r_0)} \sum_{l=1}^N \frac{\partial u_i}{\partial x_l}(w(y))((\nabla v)^{-1} \circ w(y))_j^l \phi(y) dy.\end{aligned}$$

Thus

$$u \circ w \in W^{1,r}(B(y_0,r_0))^N$$

and

$$\nabla u \circ w(y) = \nabla u(w(y))(\nabla v(w(y)))^{-1} \text{ a.e. in } \; B(y_0,r_0).$$

We have a similar result for $u_n \circ w_n$.
Second step. We conclude that $\{u_n \circ w_n\}$ is bounded in $W^{1,r}(B(y_0, r_0))^N$.
Indeed
$$\int_{B(y_0,r_0)} |u_n \circ w_n(y)|^r dy = \int_{D_n} |u_n(x)|^r dx \leq \int_{\Omega} |u_n(x)|^r dx.$$
Since $r \leq p$ and $\{u_n\}$ is bounded in $W^{1,p}(\Omega)^N$ we deduce that $\{u_n \circ w_n\}$ is bounded in $L^r(B(y_0, r_0))^N$.
Also
$$\begin{aligned}\int_{B(y_0,r_0)} |\nabla u_n \circ w_n(y)|^r dy \quad &= \int_D |\nabla u_n(x)(\nabla w_n(x))^{-1}|^r dx \\ &\leq C'[\int_{\Omega} |\nabla u_n(x)|^p dx]^{\frac{r}{p}} [\int_{\Omega} |\nabla v_n(x)|^{\frac{q}{N-1}} dx]^{\frac{r(N-1)}{p}} \leq C\end{aligned}$$
for some constant C which does not depend on y_0, r and n. Thus $\{u_n \circ w_n\}$ is bounded in $W^{1,r}(B(y_0, r_0))^N$.
Third step. We prove that, up to a subsequence, $u_n \circ w_n$ converges strongly in $L^1(B(y_0, r_0))$ to $u \circ w$. Let $f \in C(\bar{B}(y_0, r_0))$. By Remark 4.4 $\lim_{n \to +\infty} |D \Delta D_n| = 0$ and so
$$\chi_{D_n}(x) \to \chi_D(x) \ \ a.e. x \in \Omega.$$
Using the fact that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)^N$, $v_n \rightharpoonup v$ in $W^{1,q}(\Omega)^N$ and assuming, without loss of generality, that $u_n \to u$ a.e., $v_n \to v$ a.e. we obtain by (11) and the Lebesgue Dominated Convergence Theorem that
$$\begin{aligned}\lim_{n \to +\infty} \int_{B(y_0,r_0)} u_n \circ w_n(y) f(y) dy \quad &= \lim_{n \to +\infty} \int_{D_n} u_n(x) f(v_n(x)) dx \\ &= \int_D u(x) f(v(x)) dx \\ &= \int_{B(y_0,r_0)} u \circ w(y) f(y) dy.\end{aligned}$$
Therefore $u_n \circ w_n$ converges strongly to $u \circ w$ in measure and applying the Sobolev Imbedding Theorem to the bounded sequence $\{u_n \circ w_n\}$ in $W^{1,r}(\Omega)$, we conclude that, up to a subsequence, $u_n \circ w_n$ converges strongly in $L^1(B(y_0, r_0))$ to $u \circ w$.
Fourth step. Using the second and the third step we conclude that $\{\nabla u_n \circ w_n\}$ is bounded in $W^{1,r}(\Omega)^N$,
$$u_n \circ w_n \rightharpoonup u \circ w \ \text{ in } \ W^{1,r}(\Omega)^N \ \text{ if } \ r > 1$$
and
$$u_n \circ w_n \to u \circ w \ \text{ in } \ L^1(\Omega)^N \ \text{ if } \ r = 1.$$
∎

We now give the proof of Theorem 4.1.

Proof of Theorem 4.1. Without loss of generality (and, if necessary, after extracting a **subsequence of {(u_n,v_n)}), we assume that**

liminf / *W(Vu$_n$(x){Vv$_n${x))~^{l})dx* = lim / M^^{r}(Vu$_n$(x)(Vi;$_n$(x))-^{1})dx < +00.

Fix e > 0 and let Q$_c$ CC ft be an open set such that |ft \ ft$_c$| < c. By Lemma 2.2 and **the Ascoli-Arzela Theorem, without loss of generality, we assume that** *v_n* converges to *v* **uniformly in ft$_c$. Set**

***C* = {*x* € ft$_c$ *v* is completely differentiate and almost invertible at x},**

***A* = {J9(x)|x 6 C,D(x) is an open set of ft$_e$, v(D(x)) is an open ball}**

and

ft'e = *U$_{D€A}$D.*

As in the proof of Lemma 3.9, it is easy to see that

inf*{diamD(x) \ D(x)* G *A}* = 0,

for every x G *C.* By Lemma 4.3 and Vitali's covering theorem (see [Fe] Theorem 2.8.17, **p. 151) there exist {x^j, *j* € N} C ft$_o$ {-D^J, *j* € N} a family of** mutually disjoint, open **neighborhoods of, respectively, Xj, *N* a set of measure zero such that,**

$$\Omega = N \cup_N D^i,$$

and *v :* D^j —> J5(y^J,W) admits an inverse ti^^{7} 6 VV$^{rl?}$^"(J5(j/^{J},r^j),$_J$D^j), in the sense of **Theorem 3.1, for some rj > 0 and with y^J = v(x^J). Recall that**

V(uot^)(y) = VtxK(y))(Vt;K(j/))r^1 a.e. *y* G JB(y$^{J'}$,H),

ti^ o i?(x) = **x a.e** x **6 D^J,**

voxv^i(y)-y **— a.e y€B(t/^{J},H)**

and D^j = t;-^{1}(B(y^j,r^i))nJ5(x^J, JR^') **for** some **i#** > **0.** Fix *k* G **N.** By Lemma 4.5 we obtain **for each *j* = 1, • • •, *k* and up to a subsequence, the existence of *w^3_n* G *W^{li}^(B(y\ r^J))^A* which is the inverse function of Vnl^ where *D^3_n* = *v~^{l}(B(y^J,r^3))* *D B(x^J,R^J).* Recall that $\frac{1}{r} = \frac{1}{p} +$ ^ **and also**

u^ o v$_n$(x) = x a.e. x G D^j_n,

$u_n \circ w^j_n \in W^{1,r}(B(y^j, \quad))^N$,

$\nabla(u_n \circ w^j_n)(y)$ = V«$_n$K(y))(Vv$_n$K(y)))-^{1} a.e.,

U$_n$ o ^ - u o t ^ in *W^{lfr}(B(y^i_yr^i))^N* if r > 1,

u$_n$ow>$_n$-*uoxv> **in *L^1(B{y^i_tr^i)f$_t$***

{u$_n$ou^} is bounded in *W^{l,l}(B(y^i,r^i))^{ff}* if r = 1,

$$\lim_{n\to+\infty} |D^j_n \Delta D^j| = 0.$$

Fix

$$0 < \eta < \min\{r^j : j = 1, \cdots, k\}.$$

There exists $n(\eta) \in \mathbb{N}$ such that for every $n \geq n(\eta)$ we obtain

$$\max\{|v_n(x) - v(x)| \mid x \in \Omega_\epsilon\} < \eta.$$

Since $D^j = v^{-1}(B(y^j, r^j)) \cap B(x^j, R^j)$, we deduce that for every $n \geq n(\eta)$

$$D^j_n(\eta) := D^j_n \cap v_n^{-1}(B(y^j, r^j - \eta)) \subset D^j$$

and so $D^i_n \cap D^j_n = \emptyset$ if $i \neq j$. Set

$$D^j(\eta) := D^j \cap v^{-1}(B(y^j, r^j - \eta)).$$

We divide the rest of the proof of Theorem 4.1 into two cases.

First case We assume that $1 = r = \frac{1}{p} + \frac{N-1}{q}$ and there is a constant C such that $0 \leq W(F) \leq C(1 + |F|)$ for every $F \in M^{N\times N}$. Since $W \geq 0$ and $\{D^j(\eta)\}, \{D^j_n(\eta)\}$ are mutually disjoint for every $n \in \mathbb{N}$, we have by [FM]

$$\begin{aligned}
\int_{\cup_{j=1}^k D^j(\eta)} W(\nabla u(x)(\nabla v)^{-1}(x))dx &= \sum_{j=1}^k \int_{D^j(\eta)} W(\nabla u(x)(\nabla v)^{-1}(x))dx \\
&= \sum_{j=1}^k \int_{B(y^j, r^j - \eta)} W((\nabla u \circ w^j)(y))dy \\
&\leq \sum_{j=1}^k \liminf_{n\to+\infty} \int_{B(y^j, r^j-\eta)} W((\nabla u_n \circ w^j_n)(y))dy \quad (52) \\
&= \sum_{j=1}^k \liminf_{n\to+\infty} \int_{D^j_n(\eta)} W(\nabla u_n(x)(\nabla v_n)^{-1}(x))dx \\
&\leq \liminf_{n\to+\infty} \sum_{j=1}^k \int_{D^j} W(\nabla u_n(x)(\nabla v_n)^{-1}(x))dx \\
&\leq \liminf_{n\to+\infty} \int_\Omega W(\nabla u_n(x)(\nabla v_n)^{-1}(x))dx.
\end{aligned}$$

Letting η go to zero, k go to infinity and then ϵ go to zero we have

$$E(u, v) \leq \liminf_{n\to+\infty} E(u_n, v_n).$$

Second case We assume that $1 < r = \frac{1}{p} + \frac{N-1}{q}$ and there are some constants $C_1, C_2 > 0, 1 \leq s \leq r$ such that $-C_1(1 + |F|^s) \leq W(F) \leq C_2(1 + |F|^r)$ for every $F \in M^{N\times N}$. The proof follows as in the first case, where on step (52) we use the lower semicontinuity results of [Da] instead of [FM]. Since $\{\nabla u_n(x)(\nabla v_n)^{-1}(x))\}$ is weakly relatively compact in Ω, we have

$$\begin{aligned}
\int_{\cup_{j=1}^k D^j(\eta)} W(\nabla u(x)(\nabla v)^{-1}(x))dx &= \sum_{j=1}^k \int_{D^j(\eta)} W(\nabla u(x)(\nabla v)^{-1}(x))dx \\
&= \sum_{j=1}^k \int_{B(y^j,r^j-\eta)} W((\nabla u \circ w^j)(y))dy \\
&\leq \sum_{j=1}^k \liminf_{n\to+\infty} \int_{B(y^j,r^j-\eta)} W((\nabla u_n \circ w_n^j)(y))dy \\
&= \sum_{j=1}^k \liminf_{n\to+\infty} \int_{D_n^j(\eta)} W(\nabla u_n(x)(\nabla v_n)^{-1}(x))dx \\
&= \sum_{j=1}^k \liminf_{n\to+\infty} [\int_{D^j} W(\nabla u_n(x)(\nabla v_n)^{-1}(x))dx \\
&+ \int_{D_n^j(\eta)\setminus D^j} W(\nabla u_n(x)(\nabla v_n)^{-1}(x))dx \\
&- \int_{D^j\setminus D_n^j(\eta)} W(\nabla u_n(x)(\nabla v_n)^{-1}(x))dx] \\
&\leq \sum_{j=1}^k \liminf_{n\to+\infty} \int_{D^j} W(\nabla u_n(x)(\nabla v_n)^{-1}(x))dx \\
&+ C_1 \int_{D^j \Delta D_n^j} (1 + |\nabla u_n(x)(\nabla v_n)^{-1}(x))|^s)dx \\
&\leq \liminf_{n\to+\infty} \sum_{j=1}^k \int_{D^j} W(\nabla u_n(x)(\nabla v_n)^{-1}(x))dx \\
&\leq \liminf_{n\to+\infty} \int_\Omega W(\nabla u_n(x)(\nabla v_n)^{-1}(x))dx.
\end{aligned}$$

Letting η go to zero, k go to infinity and then ϵ go to zero we conclude that

$$E(u,v) \leq \liminf_{n\to+\infty} E(u_n, v_n).$$

∎

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References :

[AF] Acerbi E. & N. Fusco, Semicontinuity and relaxation for integrals depending on vector valued functions, J. Math. Pures Appl. 62 (1983), p. 371-387.

[Ba] Ball J.M., Global invertibility of Sobolev functions and the interpenetration of the matter, Proc. Roy. Soc. Edinburgh 88A (1988), p. 315-328.

[BM] Ball J.M. & F. Murat, $W^{1,p}$ quasiconvexity and variational problems for multiple integrals, J. Funct. Anal. 58 (1984), p. 225-253.

[CN] Ciarlet P.G. & J. Nečas, Injectivity and self contact in non linear elasticity, Arch. Rat. Mech. Anal. 97 (1987), p. 171-188.

[Da] Dacorogna B., *Direct Methods in the Calculus of Variations*, vol. 78, 1987, Springer.

[DF] Dacorogna B. & I. Fonseca, A minimization problem involving variation of the domain, Comm. Pure Appl. Math., vol XLV (1992), p. 871-897.

[Dav] Davini C. , A proposal for a continuum theory of defective crystals, Arch. Rat. Mech. Anal. 96 (1986), p. 295-317.

[DP] Davini C. & G. Parry, On the defect-preserving deformations in crystals, Int. J. of Plasticity 5 (1989), p. 295-317.

[Er] Ericksen J.L., Twinning of crystals I, p. 77-96 in : *Metastability and Incompletely Posed Problems*, S. Antman et al., eds., IMA Vol in Appl. Math, 3, 1987.

[Fe] Federer H., *Geometric Measure Theory*, 1969, Springer.

[FP] Fonseca I. & G. Parry, Equilibrium configurations of defective crystals, Arch. Rat. Mech. Anal. 97 (1987), p. 189-223.

[FM] Fonseca I. & S. Müller, Quasiconvex integrands and lower semicontinuity in L^1, to appear.

[GR] Gol'dshtein V.M. & Yu. G. Reshetnyak, *Quasiconformal Mappings and Sobolev spaces*, vol. 54, 1990, Kluwer Academic Publishers.

[IS] Iwaniec T. & V. Šverák, On mappings with integrable dilatation, to appear.

[Ll] Lloyd N.G., *Degree Theory*, 1978, Cambridge University Press.

[Ma] Malý J., Weak lower semicontinuity of polyconvex integrals, to appear.

[Man] Manfredi J., Weakly monotone functions, to appear.

[MZ] Martio O. & W.P. Ziemer, Lusin's condition (N) and mappings with non-negative jacobians, to appear.

[MM] Marcus M. & V. J. Mizel, Transformations by functions in Sobolev spaces and lower semicontinuity for parametic variational problems, Bul. Amer. Math. Soc. vol. 79, no. 4 (1973), p. 709-795.

[Mo] Morrey C.B., *Multiple Integrals in the Calculus of Variations*, 1966, Springer.

[Mu] Müller S., A remark on the distributional determinant, C.R.A.S Paris 311 (1990), 13-17.

[MTY] Müller S., Tang Q. & B.S. Yan, On a new class of elastic deformations not allowing for cavitation, (preprint).

[Re] Reshetnyak Yu. G., *Space Mappings with Bounded Distortion*, Translation of Mathematical Monographs, AMS, Vol. 73, 1989.

[Sc] Schwartz J.T., *Nonlinear Functional Analysis*, New York: Courant Institute Lecture Notes, 1974.
[Sv] Šverák V., Regularity properties of deformations with finite energy, Arch. Rat. Mech. Ana. 100 (1988), p. 105-127.
[TQ] Tang Q, Almost-everywhere injectivity in nonlinear elasticity, Proc. Roy. Soc. Edinburgh 109 (1988), p. 79-95.
[Ta] Tartar L. , Compensated compactness and applications to partial differential equations, p 136-212, *Nonlinear Analysis and Mechanics*, Heriot-Watt Symposium, vol. IV, R. Knops, ed., Res, Notes in Math. 39, 1979 Pitman, San Francisco.

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