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GEOMETRIC SINGULARITIES FOR SOLUTIONS OF SINGLE CONSERVATION LAWS*

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0. INTRODUCTION

In this paper we describe the geometric framework for the study of generation and propagation of shock waves in $\mathbb{R}^n$ appearing in weak solutions of single conservation laws

$$
\text{(P)} \qquad \begin{cases} \dfrac{\partial y}{\partial t} + \sum \dfrac{\partial f_i(y)}{\partial x_i} = 0 \\ y(0, x_1, \cdots, x_n) = \phi(x_1, \cdots, x_n), \end{cases}
$$

where f_i's and ϕ are C^∞-functions. Single conservation laws play an important role in various fields, e.g., gas dynamics (see e.g., [10]) and oil reservoir problems (see, e.g., [4]).

The *geometric solution* y of (P) is defined in the framework of *one-parameter unfoldings of immersions* and it is constructed by the method of characteristics. Although y is initially smooth there is, in general, a critical time beyond which characteristics cross. The geometric solution past the critical time is multi-valued, that is singularities appear.

The notion of *entropy solutions* (see [7]) has provided the right weak setting for the study of (P). Existence and uniqueness of the entropy solution of (P) have been established in [7]. The single-valued entropy solution is in general discontinuous and coincides with the smooth geometric solution until the first critical time. After the characteristics cross, the entropy solution develops *shock waves,* i.e., surfaces across which the entropy solution is discontinuous.

The entropy solution in a neighborhood of the critical time when characteristics cross for the first time has been constructed by Nakane in [9] in the case of $\mathbb{R}^n$. The entropy solution is constructed by selecting the proper discontinuous branch of the geometric solution so that the *entropy condition* is satisfied across the discontinuity. The case $n = 1$ has been studied e.g., by Ballou [2], Chen [3], Guckenheimer [5] and Jennings [6].

The framework introduced in [9] describes only the way that shocks are generated. In this paper, we present the geometric framework to study the global

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evolution of shock waves. The entropy solution just after the first critical time is a discontinuous single-valued branch of the geometric solution. In order to study the evolution of the shock surface we need to study how the different branches defining the entropy solution evolve in time. As the two branches evolve, the appearance of singularities in a branch affects the form of the discontinuity. Whenever a new singularity appears a new single-valued discontinuous branch has to be chosen so that the entropy condition is satisfied across the discontinuity.

In the present paper, we classify the bifurcations of singularities appearing in a single branch of the multi-valued geometric solution in Sections 3,4. This classification predicts the generic way that shocks are generated and how two shocks interact at their end points. In order to further study the evolution of a shock surface we also classify how various singularities appearing in different branches bifurcate in time in Sections 5,6. This classification describes how one shock interacts with another at a middle point, how different branches evolve from an initially smooth shock surface and how more than two shocks interact.

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1. GEOMETRY OF FIRST ORDER QUASI-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Since single conservation laws are the special case of first order quasi-linear partial differential equations, we start to study the gometric framework of these equations. We consider a first order quasi-linear partial differential equation:

$$\text{(Q)} \qquad \sum_{i=1}^{n} a_i(x,y) \frac{\partial y}{\partial x_i} - b(x,y) = 0,$$

where a^x,*y)* *(i* = l,...,n) and *b(x,y)* are smooth functions. We construct the geometric framework of the above equations in the projective cotangent bundle 7T: PT*(R^n x R) -> R^n x E. We now review geometric properties of this space. We consider the tangent bundle r : TPT*(R^n x E) ^ *PT*(R^n* xR) and the differential map *d*K : TPT*(R^n* xR) -4 T(R^n x R) of ?r.

For any *X* € TPT*(R^n x R), there exists an element *a e* $T^*_{(x,y)}$(R^n x R) such that *T(X)* = [a]. For an element *V* € $T_{(x4,)}$(R^n x R), the property *a(V)* = 0 does not depend on the choice of representative of the class [a]. Thus we can define the canonical contact structure on PT*(R^n x R) by

$$K = \{X \in TPT^*(\mathbb{R}^n \times R) \mid r(X)(d\pi(X)) = 0\}.$$

Because of the trivialization PT*(R^n x R) S (R^n x R) x P(R^n x R)*, we can choose ((#i,..., x_n, j/), [£i;... ; $£_n$; *rj\)* as the homogeneous coordinate system, where [&;... ; $£_n$; *v]* is *the* homogeneous coordinate of the projective space P(R^n x ℝ)*.

It is easy to show that *X* € J^(x,y),[£;ti]) if an(l on^*Y* if]C?=i A^i + Ar/ = 0, where *d*(X)* = *Y,7=i* ^ ^ 7 + A^- An immersion *i : L* -> PT*(R^n x R) is said to be *a Legendrian immersion* if dimL = *n* and *$di_q(T_qL)$* C -Ki(q) for any *q* € *L.*

A first order quasi-linear differential equation (briefly, *an equation)* is defined to **be a hypersurface**

$$E(a_1,. \qquad \in PT^*(\mathbb{R}^n \times \mathbb{R})| \sum_{i=1}^{n} a_i(x,y)\xi_i + b(x,y)\eta = 0\}.$$

***A geometric solution of**E(a\,...,a_n, 6)* **is a Legendrian submanifold** *L* **of PT*(R^n x R) lying in** *E(a\,...,* **a_n, 6) such that TT|L is an embedding. We consider the meaning of the notion of geometric solutions. Let** *S* **be a smooth hypersurface in R^n x R, then we have a unique Legendrian submanifold 5̂ in PT*(R^n x R) such that TT(£̂) =** *S* **which is given as follows :**

Ŝ = {((x, j/), [£; r/])|the vector $\sum_{i=1}^{n}$ & 2) ← + 17 ft ← is a normal vector of 5 in R^n x R}.

It follows that if *L* **is a geometric solution of** *E{a\,...,* **a_n, 6), then we have** *L* **= $\widehat{\text{TT(L)}}$.**

We consider the condition of a smooth hypersurface 5 that *S* **is contained in** *E(ai,...* **,a_n,6). For any (xo,j/o) € 5, there exists a smooth submersion germ / : (R^n x R,(x_o,2/o)) -+ (R,0) such that (/^(O^^cyo)) = (5, (x_o,j/_o)). A vector $\sum_{i=1}^{n} \mu_i$ ∂! + ^F ∂ ^s tan 6 ent *o ^ at (x??/) e (^ (xo? J/o)) ^ a»d only if it satisfies $\sum_{i=1}^{n} \mu_i \frac{\partial f}{\partial x_i}$ + ^$\frac{\partial f}{y}$ = 0 at (x,** *y).* **Then we have the following representation of 5̂ :**

$$(\hat{S}, ((x_0, y_0); [\xi_0; \eta_0])) = \{((x,y), [\frac{\partial f}{\partial x}; \frac{\partial f}{\partial y}])|(x,y) \in (S, (x_0, y_0))\}.$$

By the classical theory of first order partial differential equations, the characteristic vector field of *E(a\,...,* **a_n, 6) is defined to be**

$$X(ai,...,a_n,6) = \sum_{i=1}^{n} a i(x,j/)\frac{\partial}{\partial x_i} + 6(x,z/)\frac{\partial}{\partial y}.$$

Then we have the following characterization theorem of geometric solutions.

Proposition 1*1. *Let S be* ***a*** *smooth hypersurface in* R^n xR. *Then* Ŝ *is a geometric solution ofE(a\,...,* a_n, 6) *if and only if the characteristic vector field X(a\,...,* a_n, 6) *is tangent to S.*

Proof **We have a local representation**

$$(\hat{\ } ((x_0, y_0); [\xi_0; \eta_0])) = \{((x,y), [\frac{\partial f}{\partial x}; \quad])|(x,y) \in (S, (x_0, y_0))\},$$

where / : (R[* x *R,(x_o,y_o))* **-> (R,0) is a submersion germ with /^(O) = 5. It follows that 5 C** *E(ai,. ..,a_n,b)* **if and only if**

$$\sum_{i=1}^{n} a_i(x,y)\frac{\partial f}{\partial x_i} + b(x,y)\frac{\partial f}{\partial y} = 0$$

for (x,y) € (£, (XQ, J/O))« The last condition is equivalent to the condition that X(ai,..., a_n, 6) is tangent to 5 at (x, j/).

The Cauchy problem can be solved by the method of characteristic. We say that *a geometric Cauchy problem (GCP)* is given for an equation *E(a\\,...,* a_n, *b)* if there is given an (n — l)-dimensionai submanifold *V* : 5' C R^n x R such that the characteristic vector field *X(a\\,...,* a_n, 6) is not tangent to S^1. The following result is well-known (cf. [1]), however we clarify the geometric situation in this paper.

Theorem 1.2 (Existence theorem). *A GCP i' : S'* C R^n x R *has a unique geometric solution, that is there exists a smooth hypersurface S* C R^n x R, 5' C 5 *such that* $\hat{S}$ C *E{a\\^...,*a_n, 6), *and any two such smooth hypersurfaces coincide in a neighborhood of* S^f.

Proof. Consider the embedding *i : S* C R^n x R, where 5 C (-e,s)x5'isa neighborhood of 0 x S', *e* > 0; here *i(t,q)* = $T_t(q)$, *q* € 5", *(t,q)* € 5, and T_t is an one-parameter group of translations along -X"(ai,..., a_n, 6). By the construction, we have X(ai,...,a_n, 6) € TS,so that 5 C *E(a\\,...,*a_n, *b)* by Proposition 1.1.

On the other hand, for any geometric solution *L* of *E{a\\,...,* a_n, 6), TT(L) must be invariant under -X"(ai,..., a_n, 6), so that it is coincide with 5 in some neighborhood.

We consider the canonical projection TII : R^n xR —• R^n given by 7Ti(:r, *y)* = x. Let *L* be a geometric solution of *E(a\\,...,* a_n, 6). We say that g € L is a *singular point* if rankd(7Ti o *ir\\L)* ^ n. We remark that *q e L is* not a singular point if and only if there exists a smooth function germ / : (R^n,xo) —* R such that (5, (XQ,J/O)) == (graph/,(x_o,j/o))$_5$ where z_0 = TIX o 7r(g), y_0 = $f(x_0)$ and graph/ = *{(x,f(x))\\x* G $(\mathbb{R}^n, x_0)\}$.

2. EVOLUTION EQUATIONS

From now on we consider the following equations:

$$\text{(E)} \qquad \wedge + \sum_{i=1}^{n} a_i(t,x, \S \pounds_i - b(t,x,y) = 0,$$

where aj(t,x,j/) (i = l,...,n) and 6(t,x,y) are smooth functions. In this case we can construct the geometric framework in the projective cotangent bundle II : *PT*(R* x R^n x R) -• R x R^n x R with the trivialization PT*(R x R^n x E) ≃ (E x R^n x R) x P(R x R^n x R)*. and the natural homogeneous coordinate ((t, *x\\,...,* x_n, y), [a; Ci5 • • • 5 ^n? *v])-* We also adopt the canonical contact structure *K* on PT*(R x R^n x R).

The equation is a hypersurface

$$E(1, a_1, \ldots, a_n, b) = \{((t,x,y), [\sigma; \xi; \eta]) \in PT^*(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}) | \quad cr + \sum^{n} ai(t, x, j/) \wedge + b(t,x,y)rj = 0\}$$

and the characteristic vector field is given by

$$X(1, a_1, \ldots, a_n, b) = \frac{\partial}{\partial t} + \sum_{i=1}^{n} a_i(t,x,y) \frac{\partial}{\partial x_i} + b(t,x,y)\frac{\partial}{\partial y}.$$

We remark that the dimension of geometric solutions is $n + 1$.

We usually distinguish the time and the space, however GCP is not enough to serve to consider this case. We need a more restricted framework.

We say that *a geometric Cauchy problem associated with the time parameter* (GCPT) is given for an equation 15(1, a_t,..., a_n, 6) if a GCP *%': S'* C R x R^n x R with *i'(S')* C c x R^n x R for some c € R is given. Since *X(l,a_u... ,*a_n,6) £ *T(c* x R^n x R), the condition J£(1,ai,...,a_n,6) £ *TS^f* is automatically satisfied when *i'(S')* C c x R^n x R. It follows that GCPT is given if and only if there exists an n-dimensional submanifold ť': S^f C c x R^n x R for some c € R^n.

Remark. The Cauchy problem j/(0, *x)* = $(x) is a GCPT. The initial submanifold is given by

$$S_{\phi,0} = \{(0, x, \phi(x)) | x \in \mathbb{R}^n\}.$$

Let *i : S'* C 0 x R^n x R be an initial submanifold of GCPT. By Theorem 1.2, we can construct a unique geometric solution $\hat{S}$ such that S|0 x R^n x R = S". Since *X(l, a_1,...,* a_n, 6) G *TS, S* is transverse to c x R^n x R for any *c* G (K, 0). It follows that 5_C = 5 fl (c x R^n x R) is also an n-dimensional submanifold. The problem of studying the singularities of the geometric solution is formulated as follows:

Geometric problem. Classify the generic bifurcations of singularities of the one-parameter family of smooth mappings

$$\tilde{\pi}_t| : S_t \to \mathbb{R}^n$$

with respect to the time parameter t, where TT* : *t* x R^n x R —• R^n is given by *(t,a:,y)=x.

In order to study the singularities of the geometric solution we identify geometric solutions with one-parameter unfoldings of immersions. Such a characterization, which is given in Section 4 permits the use of the available singularity theory of one-parameter unfoldings of immersions. In the next section we present the necessary background material that we use in Section 4.

3. ONE PARAMETER UNFOLDINGS OF IMMERSIONS

We now describe the notion of unfoldings of immersions. Let *R* be an (n + 1)-dimensionaJ smooth manifold and /x(i?,uo) -* (R?*o) be a submersion germ. We call a map germ *I: (R,UQ)* -> R x R^n x R of the form I(t,z) = *(n(u),x(u),y(u)) an unfolding of immersions* if J is an immersion germ and J|/x""^{1}(t) is also an immersion for each *t* € (R,*o)«

In order to study bifurcations of singularities of unfoldings of immersions, we introduce the following equivalence relation. Let J*· : (J?, *Ui)* —» (R x R^n x R, *{t^x^* y») (i = 0,1) be unfoldings of immersions. We say that *XQ* and *X* are *t-P-A-equivalent* if there exist a diffeomorphism germ

* : (R x (R^n x R),(to,*o,Vo)) - (R x (R^n x *R),($t_u x_{uyi}$))*

of the form

$$\Phi(t, x, y) = (\phi_1(t), \phi_2(t, x), \phi_3(t, x, y))$$

and a diffeomorphism germ $: (Jf2, tio) —* (-R, *u\)* such that $ o 2o = Ji o *.

In an analogous way to that of ArnoPd-Zakalyukin's theory for Legendrian singularities ([1],[12]), we can construct generating families of unfoldings of immersions. Let *F* : (R x R^n x R,0) -• (R,0) be a function germ such that F|{0} x R^n x R is submersive. Then *F"^1^)* is a smooth (n + l)-manifold and TT_F : (F^O),*)) -• R is a submersion, where TT(£,X, *y) = t.* Let *X*: (i?, *uo)* —> (R x R^n x R,) be an unfolding of immersions. Since the image of X is a codimension one submanifold germ of R x R^n x R such that each Image I(/x""^1(t)) is also a codimension one submanifold of *{t}* x R^n x R, We can choose a function germ $\tilde{F}$: (R x R^n x R, (t_0, z_0, j/o)) -> (R, 0) such that *F\{to}* x R^n x R is submersive and F""^1^) = Image J. We define a function germ *F* : (R x R^n x R,(0)) -• (R,0) by F(t,*x,y)* = *F(t + t_o,x + x_o,y + y_0).* We call *F a generating family ofX.* We remark that if *Fi (i* = 0,1) are generating families of the same unfolding of immersions, then there exists a function germ A: (RxR^nxR,0) -> (R, 0) with A(0) ^ 0 such that A(t,x,y)-F_o(t,x,y) = Fi(t,x,y). We also consider an equivalence relation among generating families of regular Legendrian unfoldings. Let

Fi:(RxR^n xR,0)-+(R,0)

be generating families of unfoldings of immersions *Xi* (t = 0,1). We say that *FQ* and *F* are *t-P-K-equivalent* if there exists a diffeomorphism germ

$: (E x E ^n x R,0) -• (R x R^n x R,0)

of the form

$$\Phi(t,x,y) = (\phi_1(t), \phi_2(t,x), \phi_3(t,x,y))$$

such that

$$\langle F_1 \circ \Phi \rangle \mathcal{E}_{(t,x,y)} = \langle F_0 \rangle \mathcal{E}_{(t,x,y)},$$

where *{Fo)s(t ,xy)* denotes the ideal generated by Fo in the ring *£(t,x,y) of* function germs of (t,x,j/) -variables at the origin.

For a generating family F of I, we define

$$T_e(P\text{-}\mathcal{K})(f) = \langle \frac{\partial f}{\partial y}, f \rangle \mathcal{E}_{(x,y)} + \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle \mathcal{E}_x$$

and *P-IC-codf* = dim$_R$f($_{X)y}$)/T_e(P-/C)(/), where / = F|0 x R^n x R. We also say that *F* is *a P-IC-versal deformation off* if

S(x»y) = <^|0 xR-x R)$_R$ + *T_e(P-tC)(f).*

By definition, we have the following simple proposition.

Proposition 3.2. *Let Fi (i = 0,1) be generating families of unfoldings of immersions Xi. Then XQ and X\ are t-P-A~equivalent if and only if FQ and F\ are t -P -K-equivalent.*

We can classify generic unfoldings of immersions under the t-P-^4-equivalence for any fixed n. In [12], Zakalyukin has given a generic classification of function germs F : (R x R^n x R^{fc},0) -• (R,0) under the t-P-/C-equivalence for $n \leq 5$. In our case, $k = 1$, so that we can detect generic normal form for any n by the same method as that of in Zakalyukin ([12], Part 2.2). Since the set of function germs F(t,x, *y)* with submersive function F|{0} x R^n x R is open, we have the following theorem as a corollary of Zakalyukin's theorem.

Theorem 3.3. *For the generic unfolding of immersions 1, the generating family F is t-P-K-equivalent to one of the germs in the following list:*

$$({}^{0}A_k) \qquad y^{k+1} + \sum_{i=1}^{k} x_i y^{i-1} \ (1 \le k \le n)$$

$$({}^{1}A_k) \qquad y^{k+1} + y^{k-1}(t \pm x_k^2 \pm \cdots \pm x_n^2) + \sum_{i=1}^{k-1} x_i y^{i-1} \ (2 \le k \le n+1).$$

Remarks. Each germ F in the above theorem is a P-/C-versal deformation of $f = F|_{\{0\}\times \mathbf{R}^n \times \mathbf{R}}$.

On the other hand, we can prove that F is "structurally stable" relative to the t-P-/C-equivalence if and only if F is a P-JC-versal deformation of / by using the ordinary method of singularity theory (cf. [8]). By this fact, we can assert that the above theorem gives a classification of structural stable one-parameter unfoldings of immersions, so that we call the corresponding unfolding of immersions in the above list *a stable unfolding of immersions.* We argue the stability of these in Appendix. We only give the list of these unfoldings of immersions:

$$({}^{0}A_k) \qquad (*, uf^1 + \coprod_{i=1}^{k-1} {}_{Ui}ul_{Ul}, \ldots, u_n) \ (1 <\_k <\_n)$$

$$({}^{1}A_k) \ (t, u_n^{k+1} + u_n^{k-1}(t \pm u_{k-1}^2 \pm \cdots \pm u_{n-1}^2) + \sum^{k} u_i u_n^i, u_1, \ldots, u_n) \ (2 \le k \le n+1).$$

We also remark that the germ of type IA2 describes the situation that how the singularity of the geometric solution appears or vanishes.

4. REALIZATION THEOREMS

In this section we identify the geometric solution of a (GCPT) introduced in Section 2 with the notion of unfoldings of immersions.

Let $i : S'$ C 0 x R^n x R be an initial submanifold of GCPT. By Theorem 1.2, we can construct a unique geometric solution $\hat{S}$ such that S|0 x R^n x R = 5". Since X(l,ai,... ,a_n,6) € TS, S is transverse to c x R^n x R for any c € (R,0). It follows that 5_C = 5 n (c x R^n x R) is also an n-dimensional submanifold. If we consider the local parametrization of 5, we may assume that S is the image of an immersion germ J : (R x R^n,0) -> R x R^n x R such that $I\backslash c$ x R^n is also an immersion for any c € (R, 0). Hence we can choose the coordinate representation of I as J(t, x) = (t, $x(t, u)$, $y(t, u)$). This completes the proof of first part of Theorem 4.1.

Theorem 4.1. (1) *The local geometric solution of the geometric Cauchy problem associated with the time parameter for the quasi-linear first order partial differential equation* 25(1, a_x,..., a_n, 6) *is* $\widehat{\text{Image} J}$, *where* I : (R x R^n, 0) -• R x R^n x R *is an unfolding of immersions.*
(2) *Let X:* (R x R^n, 0) -> R x R^n x R *be an unfolding of immersions. Then there exist function germs* a»(t, x, *y)* (t = 1,..., n *and* &(£, x, *y)* *such that* $\widehat{\text{Image} X}$ *is a local geometric solution of the geometric Cauchy problem associated with the time parameter for the quasi-linear first order partial differential equation* 2£(1, ai,..., a_n, 6), *where the initial submanifold is* J(0 x R^n).

Proof. The assertion (1) is already proved by the previous arguments. We prove the assertion (2). By the definition, we have J(i,x) = (<,x(£,ii), *y(t,u)).* Since J is an immersion germ, we have an epimorphism

* ^X(0)v^ X JK X 1KJ —• UQ ^Jft X IK j,

defined *by l*(h) = hoi,* where J* : C|^0)(R x R^n x R) is the ring of function germs at *1(0).* It follows that there exist a<, 6 € I* : Cf^0)(R x E^n x R) such that

$$a.i(t, x(t, u), y\{t, u)) = \frac{dy}{dt}(t,u) \qquad (t = 1,\ldots,n)$$

and

$$b(t, x(t,u), y(t,u)) = \frac{dy}{dt}(t,u).$$

These relations denote that each curve X(t, n) from 1(0, *u)* is a characteristic flows, so that X(l, ai,..., a_n, 6) € TJ(RxR^n). By Proposition 1.1, $\widehat{\text{Image} X}$ is a geometric solution of 2£(1,ai,...,a_n, 6).

Remark. Let *F :* (R x R^n x R, 0) -• (R, 0) be a generating family of an unfolding of immersions J. If $\widehat{\text{Image} J}$ is a geometric solution of 2?(l,ai,... ,a_n, 6), then we have

$$\frac{\partial F}{\partial t} + \sum_{i=1}^{n} a_i(t,x,y)\wedge- + b(t,x,y)\wedge=0 \quad \text{on } F^{-1}(0).$$

The above theorem guarantees that the class of unfoldings of immersions supplies the correct class to describe the geometric solutions of (GCPT) for first order quasi linear partial differential equations of evolution type. Thus, generic results for the singularities of unfoldings of immersions can be translated to generic results in the class of all first order quasi linear partial differential equations 2?(l$_9$ai,...,a_n,6) and all initial conditions. However, our purpose is to find generic normal forms of singularities for geometric solutions of single conservation laws, so that we must also concern ourselves with what are the types of singularities that the geometric solution to a given first order quasi linear partial differential equation might exhibit. Representation Theorems 4.2-4.4 address this question.

Let *i :* (S',uo) —• R^n x R be an immersion germ, where dim 5' = n. We define an immersion germ $\tilde{t}$: (R x 5', (O,uo)) -> R x R^n x R by $\tilde{\%}$*(t,u) = (t,i(u)).* We call $\tilde{i}$ *a trivial unfolding of immersions induced by i.* Let *I: (R,uo)* -* R x R^n x R be an unfolding of immersion germs. We say that *X has a trivial bifurcation* if I is t-P-^4-equivalent to a trivial unfolding of immersions induced by J|//""^{1}(to). Then we have the following theorem

Theorem 4.2. *Let E(l, a\,..., a_n.b) be a quasi-linear first order partial differential equation and I : (R,uo) -+ R x R^n x R be a stable unfolding of immersions. If I has a trivial bifurcation, then there exists an unfolding of immersions V such that Image!' is a local geometric solution of* J£(l,ai,... ,a_n.6) *and* J, *V are t-P-A-equivalent.*

Proof **Let *G :* (RxR^n xR, 0) —> (R, 0) be a generating family of the unfolding of immersions X. Since *G\{0} x R^n x R* is submersive, the set <*f*>$_9$ = {(0, x, j/)|/(x, *y)* = 0} is an initial condition for the (GCPT), where *g* = G|{0} xR^{n}xR. By the argument of the proof of the first part of we can construct an unfolding of immersions J' which is a locally unique geometric solution around <*f*>$_g$.**

We now choose a generating family *F : (R x R^n x* R,0) -> (R,0) of *T.* By definition, F""^{1}(0)|{0} x R^n x *R* = <*f*>$_9$, so we may assume that / and *g* are P-/C-equivalent, where / = F|{0} x R^n x R. Since *F* is a P-/C-versal deformation of / and J has a trivial bifurcation, ^|{0} x R^n x R € *T$_e${P-K)(f),* and hence £(*,v) = *T$_e$(P-K)(f).* Therefore, it follows that £$_{(a\cdot,y)}$ = *T$_e$(P-)C)(g)* and *G* is also a P-/C-versal deformation of *g.* By the uniqueness theorem of *P-fC*-versa! deformations (see [ ]), *F* and *G* are t-P-AC-equivalent. This completes the proof.

We remark that °Afc-type germs in Theorem 3.3 can always be realized as geometric solutions for any first order quasi-linear partial differential equation. However, if an unfolding of immersions has a non-trivial bifurcation, the situation is different as follows:

Example 4.3. Consider the equation : *f£ + y^2-§%:* = 0.

We now consider the function germ /(x, *y)* = j/^{3} + x, then <*f*>*f* = {(—j/^{3},y)|j/ G (R, 0)} is a curve in (R x R, 0). We adopt <*f*>$_f$ as an initial condition for the (GCPT), then we can get an unfolding of immersion J by the method of characteristics. Let G(t, x, *y)* be a generating family of X, then p(x, *y)* = G(0, x, *y)* satisfies that <*f*>*f* = p~^{1}(0). It follows that there exists A G ^(x,y) ^{8UC}h that A(0) ^ 0 and *f(x,y)* = *X(x,y)g(x,y).* Define a function germ *F :* (R x R x R,0) -> (R,0) by *F(tjX$_9$y)* = A(x,j/)G(t,x,t/), then we have F^O) = G^J(0) = Image J. and F(0,x,j/) = /(x,t/). It follows that ^ + j/^{2}|f = 0 on F-^O). Then we have §£ + *y^2*^ = 0 mod(F>$_{£(< x y)}$. Since F(0,x, *y)* = /(0,x,j/), we have ^|$_t$=o + J/^{2} € *(y^3 + x)s$_{iXtyy}$* We can easily calculate that *T$_e$(P-!C)(f)* = (j/^{2},x)$_{(£(x y)}$ +5^, so that ^JT € *T$_e$(P-)C)(f).* This formula show that F cannot be a P-/C-versai deformation of /. However the generating family of type IA2 : *y^z* + *yt* + *x* is a P-JC-versal deformation of /. Then *F* is not £-P-£-equivalent to germ of type IA2. Since *F* and *G* are t-P-/C-equivalent, G is not £-P-/C-equivalent to germ of type IA2. Thus the unfolding of immersions of type ^{x}*Ai* cannot be realized as a geometric solution of

$$\frac{y}{z_i} = 0.$$

Hence, we assume a kind of non-degeneracy condition on the first order quasi-linear partial differential equation. We say that a first order quasi-linear partial differential equation *E(l,ai,.. ,a_n,b)* is *k-non-degenerated at (t_o,x_o,y_0)* if

$$\frac{\partial^k a}{dy^k_i}(t_0, x_0, y_0) \neq 0,$$

where *ffi(to,xo,yo)* = *(^^(to,xo,yo),...,* ^TK*O,ŽO,2/O)). We simply say that 2?(l,ai,...,an,6) is *non-degenerated at* (to>xo>2/o) if it is 1-non-degenerated at $(t_0, x_0, y$/o)- Then we have the following realization theorem.

Theorem 4,4. *Let* 2£(1, ai,...,an, 6) 6e a /irsi order guasi-Jinear *partial differential equation and X* : (i?,tto) —» R X R^n x R *be a stable unfolding of immersions. If the equation is non-degenerated at* (to^ChJ/o)* *then there eonsts an unfolding of immersions* X^1 *such that* $\widehat{ImageV}$ *is a local geometric solution of* J£(l, ai,. •., *an.b) and T,* X^1 *are t-P-A-equivalent*

Proof. Without loss of generality, we assume that (to>#o5j/o) = (0,0,0). Let *G :* (R x R^n x R, 0) -> (R, 0) be a generating family of the unfolding of immersion *X*. Since *g* = G|0 x R^n x R is submersive, the set <7^{-1}(0) is the initial condition for the corresponding (GCPT). By the arguments of the characteristic method, we can construct a unfolding of immersions X^1 which is the local unique geometric solution around p""^{1}(0).

We now choose a generating family *F* : (R x R^n x R,0) -+ (R,0) of J'. By definition, F~^{1}(0)|{0} x R^n x R is equal to p""^{a}(0), then we may assume that /, *g* are P-/C-equivalent, where / = *F\{0}* x R^n x R.

If *P-fC-codg* = 0, then *P-IC-codf* = 0, so that / is already a *P-K*-versal deformation of itself. Hence, for the same reason as in the proof of Theorem 4.2, *F* is £-P-/C-equivalent to *G.*

We now assume that *P-tC-codg* = 1, so that *P-K-codf* = 1. If ^-$|_{t=0}$ ∉ T_e(P-/C)(/), then we can get the required assertion by the uniqueness of the *P-K*-versal deformation as in the previous case.

Suppose that ^-|t=o € *T_e(P-K)(f)* for any generating family *F* of X^1. Since F'"^{1}(0) is a geometric solution of J5(l,ai,...,an,6), we have a relation

$$\frac{\partial F}{\partial t} + \sum_{i=1}^{n} a_i(t,*,j,)!£+^h\wedge^x\wedge\% = \circ \text{ on } F^{-1}(0),$$

so that

$$-\frac{dF}{} \equiv \sum_{i=1}^{n} a_i(t>^x>v) for^{QW}_{+h}(t>^x>v)\text{-}ej;^{Qp} \mod \langle F\rangle_{\mathcal{E}_{(t,x,y)}}.$$

It follows that that

$$-\wedge \;|_{t=0} \equiv \sum_{i=1}^{n} > (0,x,y)£ + b(0 \; X \wedge \mod \langle f\rangle_{\mathcal{E}_{(x,y)}}.$$

Therefore

$$-\wedge \;|_{t=x=0} \equiv \sum_{i=1} \phi_i(0,0,y) \quad (0,y) + b(0,0 \gg)\wedge \mod \langle f_0\rangle_{\mathcal{E}_y},$$

where *fo(y)* = /(0,y). We may assume that /o € 9Jl^ for *k* ≥ 3, where $9Jt_y$ is the unique maximal ideal of S_y.

We now consider the Taylor polynomial of a;(t,z, *y*) for sufficiently higher order at (t, x, 0) with respect to *y*-variables as follows :

$$Oi(t,x,y) = m(t,x,0) + \frac{\partial a}{y*}(t,x,0)y + \frac{1}{2} \wedge (t \wedge \wedge y^{2} + \text{higher terms.}$$

On the other hand

$$-\frac{\partial F}{\partial t}|_{t=0} \in T_e(P\text{-}\mathcal{K})(f) = \langle f(x,y), \frac{\partial f}{\partial y}(x,y)\rangle \mathcal{E}_{(x,y)} + \langle \frac{\partial f}{\partial x_1}(x,y), \ldots, \frac{\partial f}{\partial x_n}(x,y)\rangle \mathcal{E}_x .$$

It follows that

$$\sum_{i=1}^{n} a_i(0,0,y)\frac{\partial f}{\partial x_i}(0,y) + b(0,0,y)\frac{\partial f_0}{\partial y}(y)$$
$$\in \langle f_0(y), \frac{\partial f_0}{\partial y}(y)\rangle \mathcal{E}_y + \langle \frac{\partial f}{\partial x_1}(0,y), \ldots, \frac{\partial f}{\partial x_n}(0,y)\rangle_{\mathbf{R}},$$

and

$$\sum_{i=1} \wedge(\wedge(0,0,0) + \text{-}\wedge(0,0,0)y, + \wedge\text{-}(0,0,0)y^2 + \text{higher terms})\wedge(0,y)$$
$$\in \langle f_0(y), \frac{\partial f_0}{\partial y}(y)\rangle \mathcal{E}_y + \langle \frac{\partial f}{\partial x_1}(0,y), \ldots, \frac{\partial f}{\partial x_n}(0,y)\rangle_{\mathbf{R}}.$$

For any linear isomorphism $A : \mathbf{R}^n \longrightarrow \mathbf{R}^n$, we have a relation

$$\frac{\partial f(Ax,y)}{\partial x_i} = \sum_{j=1}^{n} A_{ij}\frac{\partial f}{\partial x_j}(Ax,y).$$

We remark that the vector space

$$\frac{\partial f_0}{\partial y} \ldots \langle \frac{\partial f}{\partial x_1}(0,y), \ldots, \frac{\partial f}{\partial x_n}(0,y)\rangle_{\mathbf{R}}$$

is an invariant under the action of the linear isomorphism *A*. Since

$$\sum_{i=1} \ldots (0,0,0) | \ldots (0,y) \in \langle \ldots (0,y), \ldots, \frac{\partial f}{\partial x_n}(0,y)\rangle_{\mathbf{R}},$$

we have

$$(\sum_{i=1}^{n} \wedge (0,0,0) \wedge (\ldots)) + \text{higher terms}$$
$$\in \langle f_0, \frac{\partial f_0}{\partial y}\rangle \mathcal{E}_y + \langle \frac{\partial f}{\partial x_1}(0,y), \ldots, \frac{\partial f}{\partial x_n}(0,y)\rangle_{\mathbf{R}}.$$

By assumptions, ^-(0,0,0) ^ 0 and ^£(0,0) ^ 0 for some *i,j* = l,...,n. If necessary, by applying some linear isomorphism *A,* we have

$$\sum_{i=1}^{n} \frac{\partial a_i}{\partial y}(0,0,0)\frac{\partial f}{\partial x_i}(0,0) \neq 0.$$

It follows that M_y C </o, ^)*e,* + <J£(0, *y),...,* ^ (0 , j/))R mod 2Kj.

Since JC-cod(/o) is finite (for the definition of JC-finiteness, see [11]), then there exists *k* € N such that 2ttJ C *(fo^)s*$_v$ C 9KJ-^{1}. If y *e {fo^)e*$_y$*,* we have 2Jl$_y$ = (y05 ^fe.)^. Then this case is corresponding to the case *k* = 1. We may assume that ?/ € (J^(0,y),..., ^"(0,2/))R- It follows that there exist real numbers *Xi (i* = l,...,n) such that *y* = $2^=1 Ai^(0,y). If necessary, applying a linear isomorphism *A,* we may assume that *y* = ^-(0,y) for some *i* = l,...,n. By the same arguments as those of previous paragraphs, we can asserts that y^1 6 </o> *y^)e*$_v$+(^£"(°> #)> • • •» ^£-(°'^))K- We can continue this procedure up to degree *k — 1.* Eventually, every polynomial of degree *k* — 1 is contained in the vector space

$$\mathfrak{M}_y \subset \langle f_0, \frac{\partial f_0}{\partial y}\rangle_{\mathcal{E}_y} + \langle \frac{\partial f}{\partial x_1}(0,y), \ldots, \frac{\partial f}{\partial x_n}(0,y)\rangle_{\mathbb{R}} \bmod m_y^r.$$

Since m^k_y C </o, ^)f$_v$, we have

$$\mathcal{E}_y = \langle f_0, \frac{\partial f_0}{\partial y}\rangle_{\mathcal{E}_y} + \langle \frac{\partial f}{\partial x_1}(0,y), \ldots, \frac{\partial f}{\partial x_n}(0,y)\rangle_{\mathbb{R}}.$$

It follows that

$$\mathcal{E}_{(x,y)} = T_e(P\text{-}\mathcal{K})(f) + \mathfrak{M}_y\mathcal{E}_{(x,y)},$$

thus we have £"(x,y) = T_e*(P-)C)(f)* by the Malgrange preparation theorem. This contradicts to the fact that P-/C-cod (/) = 1.

By the classification in Theorem 3.3, the generic nontrivial bifurcations of singularities are given by the germs lA* *(k* = 2,..., *n*+1). Especially the first singularity appears in the form of IA2, if the initial condition is smooth. We show that the *^{l}Ak* singularity appear at an *(k* — l)-non-degenerated point of the equation as follows :

Theorem 4.5. *If an* lA* *singularity appears at a point* (to^o^J/o)? *then the equation* 2?(l,ai,...,On,6) *is s-non-degenerated at* (£0,£0,2/0) *f*$^{or\ some}$ ^ = 1,... ,fc —1.

Proof. The method of the proof is analogous to those of Example 4.3 and Theorem 4.4.

Without loss of generality, we assume that (<o, £0? J/o) = (0? 0> 0). We consider the function germ *f(x*$_u$... ,x$_n$,y) = $y^{km¥1}$+y^k~^{1} *(±xl±- • -±Xň)+lL,7Ě7*^{1} x* J/*-^{1}^then we adopt the hypersurface <*f*>$_f$ C (R^n x R,0). as an initial condition for the (GCPT). We can get an unfolding of immersions J by the method of characteristics. Let *G*(t, £1,... x$_{n5}$ *y)* be a generating family of J, then *g(x\,..., x*$_n$*)* = G(0, *x*\,... x$_n$, y) satisfies that $\phi_f = \phi^{-1}(0)$. It follows that there exists $A \in \mathcal{E}(x,y)$ such that $A(0) \neq 0$ and $f(x_1,\dots,x_n,y) = A(x_1,\dots,x_n,y)g(x_1,\dots,x_n,y)$. Define a function germ

$$F:(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R},0)\to(\mathbb{R},0)$$

by $F(t,x_1\dots x_n,y) = A(x_1,\dots,x_n,y)G(t,x_1,\dots,x_n,y)$, then we have $F^{-1}(0) = G^{-1}(0) = \text{Image}\,I$ and $F(0,x_1,\dots,x_n,y) = f(x_1,\dots,x_n,y)$. It follows that

$$\frac{\partial F}{\partial t}(t,x,y) + \sum_{k} a_k(t,x,y)\frac{\partial F}{\partial x_k}(t,x,y) + b(t,x,y)\frac{\partial F}{\partial y}(t,x,y) = 0 \quad \text{on } F^{-1}(0),$$

so that

$$-\frac{\partial F}{\partial t} \equiv \sum_{i\in I} a_i(t,x,y)\frac{\partial F}{\partial x_i} + b(t,x,y)\frac{\partial F}{\partial y}(t,x,y) \bmod \langle F\rangle\mathcal{E}_{(t,x,y)}.$$

It follows that

$$-\frac{\partial F}{\partial t}|_{t=0} \equiv \sum_{i=1}^{n} a_i(0,x,y)\frac{\partial f}{\partial x_i} + b(0,x,y)\frac{\partial f}{\partial y} \bmod \langle f\rangle\mathcal{E}_{(x,y)}.$$

We now consider the Taylor polynomial of $a_i(t,x,y)$ of sufficiently higher order at $(t,x,0)$ with respect to y-variables as follows :

$$a_i(t,x,y) = a_i(t,x,0) + \frac{\partial a_i}{\partial y}(t,x,0)y + \frac{1}{2}\frac{\partial^2 a_i}{\partial y^2}(t,x,0)y^2 + \text{higher terms}.$$

On the other hand, we have

$$\frac{\partial f}{\partial x_i} = \begin{cases} 1 & i = 1 \\ y^{i-1} & 2 \le i \le k-1 \\ \pm 2x_i y & k \le i \le n \end{cases}$$

and

$$\frac{\partial f}{\partial y} = (k+1)y^k + (k-1)y^{k-2}(\pm x_k^2 \pm \cdots \pm x_n^2) + \sum (i-1)x_i y^{i-2}.$$

It follows that

$$\begin{aligned}
-\frac{\partial F}{\partial t}|_{t=0} \equiv{}& a_1(0,x,0) + \frac{\partial a_1}{\partial y}(0,x,0)y + \frac{1}{2}\frac{\partial^2 a_1}{\partial y^2}(0,x,0)y^2 + \text{higher terms}\\
&+\sum_{i=2}^{k-1}\Big(a_i(0,x,0) + \frac{\partial a_i}{\partial y}(0,x,0)y + \frac{1}{2}\frac{\partial^2 a_i}{\partial y^2}(0,x,0)y^2\Big)y^{i-1} + \text{higher terms}\\
&+\sum_{i=k}^{n}\Big(a_i(0,x,0) + \frac{\partial a_i}{\partial y}(0,x,0)y + \frac{1}{2}\frac{\partial^2 a_i}{\partial y^2}(0,x,0)y^2\Big)(\pm 2x_i y) + \text{higher terms}\\
&+ b(0,x,y)\frac{\partial f}{\partial y}(x,y)\\
&\qquad \bmod \langle f, \frac{\partial f}{\partial y}\rangle\mathcal{E}_{(x,y)} + \mathfrak{M}^{k+2}_{(x,y)}.
\end{aligned}$$

We can show that all elements of £($_x$,y) except $cy^{k\sim l}$ are contained in T_e(P-£)(/), where c is a constant real number.

If the equation 2?(l,ai,...,a_n,6) is not *(k* — l)-non-degenerated at the point (0,0,0), then we have f^-(0,0,0) = 0 for any *i* = l,...,n and *s* = l,...,fc - 1. Then we have

$$\sum_{i=1} \frac{1}{(k-i)!} \frac{\partial^{(k-i)} a_i}{\partial y^{(k-i)}}(0, x, 0) \in \mathfrak{M}_x.$$

This means that ^|*=o € *T_e{P-K){f)* by the above calculations. This formula show that *F* cannot be a P-JC-versal deformation of /. However the generating family of type *^{l}Ak* is a P-/C-versal deformation of /. Then *F* is not t-P-/C-equivalent to germ of type lA*. Since *F* and G are t-P-/C-equivalent, *G* is not t-P-/C-equivalent to germ of type *^{l}Ak**

On the other hand, let *H* be a generating family of a geometric solution of 2?(l,ai,... ,a_n,6). Suppose that if is t-P-/C-equivalent to the germ of *^{1}Aj$_c$,* then there exist a diffeomorphism germ

$: (E x R ^{n} x R,0) -^ (R x R ^{n} x R,0)

of the form

$$\Phi(t,x,y) = (\phi_1(t), \phi_2(t,x), \phi_3(t,x,y))$$

and an element A(£,x, *y)* € *£(t,x,y)* ^{suc}^ that A(0) ^ 0 and A(t,x, *y)H o* $(t,x, j/) = F(t,x,j/), where

$$F(t,x,y) = y^{k+1} + y^{k-1}(t \pm x_k^2 \pm \cdots \pm x_n^2) + \sum^{Jk-l} x_i y^{i-1} \ (2 \le k \le n+1).$$

Thus we have A(0,x,y)if(0,<^2(0,x),^3(0,x,j/)) = /(x,j/). Let G be a generating family of the geometric solution which is constructed by the characteristic method from the initial data *<f>f.* By the uniqueness of the geometric solution, we can assert that *H* is t-P-/C-equivalent to *G.* However, by the previous arguments, *G* cannot be £-P-/C-equivalent to *F.* This contradicts to the assumption that *H* is t-P-/C-equivalent to *F.* Thus the unfolding of immersions of type lA* cannot be realized as a geometric solution of J£(l,ai,.. •,a_n, 6) at the point (to,£o, Jto)-

The following corollary of the above theorem shows that the first singularity appears at a non-degenerated point of the equation, if the initial condition is smooth.

Corollary 4.6. *If an* IA2 *singularity appears at a point* (<O5^o$_5$yo)? *then the equation* 2?(1, *a*\,..., a_n, 6) *is non-degenerated at* (to, x_0, y_0).

5. SINGLE CONSERVATION LAWS

In this section we apply the previous results on first order quasi-linear partial differential equations to single conservation laws.

Here, we consider the equation of the form

$$\text{(C)} \qquad \frac{\partial y}{\partial t} + \sum_{i=1}^{n} \frac{\partial f_i(y)}{\partial x_i} = 0,$$

where /*'s are smooth functions. This equation is rewritten as

$$\text{(C}'\text{)} \qquad \frac{\partial y}{\partial t} + \sum_{i=1}^{n} \frac{df_i}{dy}(y)\frac{\partial y}{\partial x_i} = 0,$$

so that it is considered as *E(l,dfi/dy(y),... ,df$_n$/dy(y),0)*. Set *a^y)* = $\wedge(y)$, $1 \leq i \leq n$. Then the characteristic equation associated with (P) through $(0,x_0)$ is given as follows:

$$\begin{cases} \frac{dXi}{dt}(t) = ai(y(t,x(t)), & xi(0) = x_{ifi} \\ \frac{dy}{dt}(t,x(t)), & y(0,x(0)) = \phi(x_0). \end{cases}$$

The solution of the characteristic equation can be expressed by

x(t) = x_0 +ta(<f>(x_0)) and *y(t,x(t))* = y(0,x(0)) = *<f>(x_o).*

Therefore we define the corresponding immersion

$$J[a,<f>] : \mathbf{R}\times\mathbf{R}^n \text{-}\wedge \mathbf{R}\times\mathbf{R}^n\times\mathbf{R}$$

by

I[a,<f>]{t,u) = (t,u + ta(<f>(u)),<l>(u)),

where a = *(a\,...,* a_n). We also define mappings

$$F[a,<t>) : \mathrm{R}\times\mathrm{R}^n \text{->} \mathrm{R}^n$$

by

$$a,\phi](t,u) = u + ta(\phi(u))$$

and

$$f[a,\phi;t_0] : \mathbb{R}^n \to \mathbb{R}^n$$

by

$$f[a,\phi;t_0](u) = F[a,\phi](t_0,u).$$

By the classical theory of characteristics, it follows that the classical solution of (C) is expressed by *y(to,x) = 4>{f[a,<f>;to]~*(x))* at *x* which is not critical value of /[a*, 4>;to*. Then our problem is to study the singularities of *f[a, <f>;to*. We can calculate the Jacobian matrix *J(f[a,<f>;t_o])* of *f[a,<t>;to* at «o as follows:

$$\begin{pmatrix} 1 + C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & 1 + C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & 1 + C_{nn} \end{pmatrix},$$

where *aj* = to^(0K))|J·(to).

By definition (C) is called *k-non-degenerated &t* (to^o^J/o) if ^*k±((y)* 7^ 0 at (<o,^o, J/o), where *%£(y) = (^r(y),...,* ^(y)). Especially (C) is said to be non-*degenerated* if it is 1-non-degenerated. Then we can summarize the corresponding results for general first order quasi-linear partial differential equations of evolution type to single conservation laws.

Theorem 5.1. *Let X:* (i?, *UQ)* —> (E x E^n x E, (to, *xo, Vo)) be a stable unfolding of immersions. Then*
(1) *If X has a trivial bifurcation, then there exists an unfolding of immersions X^1 such that $\widehat{ImageX^1}$ is a local geometric solution* o/.E(l,ai,...,a_n.6) *andX, X' are t-P-A-equivalent.*
(2) *If the equation (C) is nondegenerate at* (to,£o> 3/0), *then there exists an unfolding of immersions X' such that $\widehat{ImageX^1}$ is a local geometric solution of (C) and X, X^1 are t-P-A-equivalent.*
(3) *If an ^lAk singularity appears at a point* (to, £o,Jto)$_5$ *then the equation (C) is non-degenerated at* (to,£o,2/o)-

Proof. **The statements of (1), (2) are the same as those of Theorems 4.2 and 4.4. By Theorem 4.5, if an ^xAk singularity appears at a point (to,xo,2/o) = *(to)Uo+toa(<f>(uo))i <j>(uo)),* then the equation (C) is s-non-degenerated at (to, #o, j/o) for some s = 1,... ,fc — 1. However, if (C) is s-non-degenerated at (to,#o,2/o) for some *s* ≥ 2, then ^*((f>(uo))* = 0. It follows that the Jacobian matrix of /[a, </>;to] is the unit matrix at *UQ.* SO the point (to,£o,2/o)'** n o t a singular point of the geometric solution. This contradicts to the fact that the geometric solution has ^lAk singularities at (to,xo,j/o)«**

In order to deal with generic properties for the Cauchy problem of single conservation laws (P), we need a kind of the transversality theorems which is given in the Appendix. For notions and elementary properties we use hereafter, see Appendix. It follows from Theorems 3.3 and A.? that the list of germs of map germs in the remark after Thoerem 3.3 gives a classification of map germs / : (E^n,0) —> (E^n,0) with .4-cod (/) ≤ 1 and dimKerd/ ≤ 1 :

$$({}^0A_k) \qquad \left(u_n^{k+1} + \sum_{i=1}^{k-1} u_i u_n^i, u_1, \ldots, u_n\right) \ (1 \le k \le n)$$

$$({}^1A_k) \ \left(u_n^{k+1} + u_n^{k-1}(\pm u_{k-1}^2 \pm \cdots \pm u_{n-1}^2) + \sum^{k} u_i u_n^i, u_1, \ldots, u_n\right) \ (2 \le k \le n+1).$$

Let $J^\ell(E^n,E^n)$ be the £-jet space of mappings E^n -> E^n, where *I* ≥ *n* + 2. We consider a stratified subset 5 = *{Sj}j£i* of $J^*(E^n,E^n)$ each strata is given by the .A-orbit of the £-jet of the germ in the above list. By Theorem A.?, we have the following theorem.

Theorem 5.2. *There exists a residual subset O* C C°°(E^n,E) *with the following properties: For any <f> € O and* (to,t*o) € E x E^n, *the germ* /[a,<f>] *at* (to,tto) *is t-P-A-equivalent to one of germs in the list of Remarks after Theorem 3.3.*

6. MULTI-UNFOLDINGS OF IMMERSIONS

As we discussed in the introduction, in order to study how the shock waves for the the entropy solution of (P) evolve, we study the bifurcation of singularities appearing in different branches of the corresponding geometric solution. We classify

the bifurcations of singularities of different branches by classifying the bifurcations of singularities of multi-unfoldings of immersions which are expressed in terms of multi-germs.

Let *Xi* : *(R,u0)* -^ (R x R^n x R,(0,0,j/;) *(i* = l,...,r) be unfoldings of immersions, where t/i,...,y$_r$ are distinct. We call (Ji,...,X$_r$) *a multi-unfolding of immersions.* Let (Ti,...,J$_r$) and (J{,...,*X'$_r$)* be multi-unfoldings of immersions. We say that these are *t-(P-A)(,y equivalent* if there exist diffeomorphism germs

$i: (R x R^n x R, (0,0, y<) - (R x R^n x R), (0,0,yj) (i = 1,... ,r)

of the form

$$\Phi_i(t,x,y) = (\phi_1(t), \phi_2(t,x,y), \phi_3^i(t,x,y))$$

and a diffeomorphism germ \t : *(R,uo)* —> (-R, UQ) such that $; o 2* = Xi' o $ for any *i* = l,...,r. This equivalence describes how bifurcations of singularities of geometric solutions interact.

By the arguments in Section 3, there exist generating families *F{* : (R x R^n x R, 0) -> (R, 0) of Z;, *i* = 1,..., r. We call *F* = (Fi,..., *F$_r$) a multi-generating family of* the multi-unfolding of immersions (Ji,... ,I$_r$). We also consider an equivalence relation among multi-generating families of multi-unfoldings of immersions. Multi-generating families *F* = (F$_x$,...,F$_r$) and *F'* = *(F{,...,F^)* are t-(P-/C)$_{(r)}$-*equivalent* if there exists a diffeomorphism germ

*i : (RxR^n xR,0) -> (R x R^n xR,0) (z = l,...,r)

of the form

*i(t,x,y) = *{<t>i{t),<t>$_2${t,x,y),<t>\(t,x,y))*

such that

$$\langle F_i' \circ \Phi\rangle_{\mathcal{E}_{(t,x,y)}} = \langle F_i\rangle_{\mathcal{E}_{(t,x,y)}}.$$

We have the following simple proposition.

Proposition 6.1. *Let F* = (i*i,..., *F$_r$) and F^1* = (F{,..., *F'$_r$) be multi-generating families of multi-unfoldings* (Ji,... ,I$_r$) *and* (JJ,... ,2£) *respectively.*
Then (Ji,..., J$_r$) and (J{,... ,2£) are t-(P-^4)($_r$)-egmva/ent t/ and onto/ t/F and F' are t-(P-/C)($_r$)-*equivalent.*

According to the above proposition it is enough to give a classification of generic multi-generating families under the (P-/C)($_r$)-equivalence. For this we need to extend the results of the previous section to multi-generating families.

For generating families *Fi* of $/?n 1'1 = 1,. • •, r, we define a subspace of £T$_{(x,y)}$ by

$$T_e(\mathrm{P}_{(r)}\text{-}\mathcal{K})(f) = <\wedge, h)e\wedge_{y)} \ \times \ \ldots \ \times \frac{}{\partial y}, /_r>\wedge_{ff)} + \langle \frac{}{\partial x_1}, \ldots, \frac{}{\partial x_n}\rangle_{Sm}$$

and (P-^)$_{(r)}$-cod / = dim$_R$^$_{x\ y)}$/T$_c$((P-X:)$_{(r)}$)(/), where /< = F<|{0} x R^w x R. We also say that F is a (P-/C)($_r$)-ver5a/ *deformation of f* if

$$\mathcal{E}_{(x,y)} = \langle \frac{\partial F}{\wedge} |\{0\} \times \mathrm{R}^n \times \mathrm{R}\rangle_{\mathrm{R}} + T_e((P\text{-}\mathcal{K})_{(r)})(f).$$

By the versality theorem in [10], we have the following uniqueness result.

Theorem 6.2. *Let F and G be {P-K)(,yversal deformations of f and g respectively. Then F and G are t~(P-K)(,yequivalent if and only if f and g are* (P-/C)($_r$)-*equivalent.*

Our objective is to classify multi-generating families F = (Fi,..., F_r) which are (P-/C)($_r$)-versal deformations of / = F|{0} x R^n x R. Then we need a classification of multi-germs / of (P-/C)($_r$)-cod/ $\leq$ 1. The following estimate of codimensions is useful for such a classification.

Lemma 6.3. *£ i = i £-cod(f_0,i) _< (P-JC)(,ycod(f) + n.*
Here, £-cod(f_0,i) = $\mathfrak{m}_{\mathbb{R}}\mathcal{E}_y/\langle j_y^{0,i}, f_{0,i}\rangle_{\mathcal{E}_y}$.

Proof **We have /C-cod(/$_0$,;) = dim$_R$£(Xiy)/(|k,/;)£(x,f) +9tt$_x$£(x4/), so that**

$$\sum_{i=1}^{r}\mathcal{K}\text{-cod}\,(f_{0,i})$$
$$=\dim_{\mathbb{R}}\mathcal{E}^r_{(x,y)}/\langle\frac{\partial f_1}{\partial y},f_1\rangle_{\mathcal{E}_{(x,y)}}\times\cdots\times\langle\frac{\partial f_r}{\partial y},f_r\rangle_{\mathcal{E}_{(x,y)}}+\mathfrak{M}_x\mathcal{E}^r_{(x,y)}$$
$$\leq\dim_{\mathbb{R}}\mathcal{E}^r_{(x,y)}/T_e(\mathrm{P}_{(r)}\text{-}\mathcal{K})(f)$$
$$+\dim_{\mathrm{R}}\frac{T_e((P\text{-}\mathcal{K})_{(r)})(f)}{\langle\frac{\partial f_1}{\partial y},f_1\rangle_{\mathcal{E}_{(x,y)}}\times\cdots\times\langle\frac{\partial f_r}{\partial y},f_r\rangle_{\mathcal{E}_{(x,y)}}+\mathfrak{M}_x\mathcal{E}^r_{(x,y)}}$$
$$\leq\dim_{\mathbb{R}}\mathcal{E}^r_{(x,y)}/T_e((P\text{-}\mathcal{K})_{(r)})(f)+\dim_{\mathbb{R}}\langle\frac{\partial \tilde f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}\rangle_{\mathbb{R}}$$
$$\leq(\text{P-K})_{(r)}\text{-cod}\,(/)+n.$$

We say that multi-function germs /$_0$ = (/o,i,... ,/o,r) and g_0 = (po,i,---,f/o,r) are *K.(,yequivalent\i* /o,i and po,» ^{a}r^e AC-equivalent for i = 1,..., r. For the definition and properties of the AC-equivalence, see [11]. We now define

$$\mathrm{T_e(AC)_{(r)}(/_0)} = (\wedge Jo,i)e_v \;\; \mathrm{x} - \mathrm{x} \;\; (\wedge,Me_v\text{-}$$

We also define the notion of /C($_r$)-versal deformation of /o as follows : Let $\tilde F$ = ($\tilde F$i,... ,$\tilde F_r$) be a 5-parameter deformation of /$_0$ (i.e. F» : (R^J x R,0) -• (E,0) is a function germ such that $\tilde F$j|{0} x R = /o,» for any i = l,...,r.) We say that $\tilde F$ = ($\tilde F$i,..., $\tilde F_r$) is a /C($_r$)-versa/ *deformation* of a multi-germ /o if

$$\mathcal{E}_y=\langle\frac{\partial\tilde F}{\partial u_1}|\{0\}\ \mathrm{x\ R},\ldots,\ \wedge|\{0\}\ \mathrm{x\ R}\rangle_{\mathrm{R}}+\mathrm{T_c(AC)_{(r)}(/_0)}.$$

We also define *the discriminant set of* $\tilde F$ as follows :

$$D_{\tilde F}=\cup_{i=1}^r D_{\tilde F_i},$$

where

$$Dp. = \{ue\ R^s\backslash Fi\{u,_y) = \frac{9\tilde F\text{-}}{Q}\wedge\text{-}fay) = \cdots = \frac{9\tilde F\text{-}}{}of(^u>y) = 0 \text{ for some } y\in(\mathbb{R},0)\}.$$

We have the following lemma.

Lemma 6.4. *// a multi-generating family F is an one-parameter (P-)C)(r)-versal deformation of f* = F|{0} x R^n x R, *then it is an* (n + *l)-parameter (K,)(ryversal deformation of* f_0 = *F\{0} x* {0} x R.

We summarize the strategy for the classification, which will be presented in the next section, as follows :

Step 1. We classify multi-germs /o = (/o,i» • • • > /o,r) with (/C)(r)-cod (/o) $\leq$ n + 1 under the (/C)(r)-equivalence, where (JC)(r)-cod (/o) = *Y%=i* £-^{c}°d(/o,t)-

Step 2. We construct (n+l)-parameter (/C)(r)-versal deformations $\bar{F}$ of the normal forms /o = (/o,i> • • • > /o,r) obtained by Step 1. We fix each germ $\bar{F}$ and we consider a smooth function germ *t* : (R^{n+1},0) -• (R,0). By the theorem in Zakalyukin [36, part 2.2], we can classify *t* under coordinate changes of R^{n+2} which preserve the discriminant set *Dp*. This classification generically corresponds to the classification of *F* under the (P-/C)(r)-equivalence which preserves the projection on the t-space (i.e. the t-(P-/C)(r)-equivalence).

Step 3. We can check that each normal form obtained by Step 2 is a (P-/C)(r)-versal deformation when we consider *t* as a parameter. Thus we can detect normal forms of generic multi-Legendrian unfoldings by the P(r)-Legendrian equivalence.

7. CLASSIFICATIONS

In this section we pursue the strategy we referred to at the end of the last section. Since we consider how singularities interact, we may assume that /C-cod (/o,t) $\geq$ 1 for a multi-germ /o = (/o,i> • • • >/o,r)« It follows from Lemma 6.3 that r $\leq$ *n* + 1. By Theorem 3.3, we have the following classification.

Lemma 7.1. *For generic multi-generating families F = (F\^...^F$_r$) of multi-unfoldings of immersions, the corresponding multi-germs* /o = (/o,i... ,/o,r) *are* (/C)(r)*-equivalent to one of the multi-germs in the following list :*

$$(y^{k_1+1},\ldots,y^{k_r+1}) \qquad r \leq \sum^{r} k_i \leq n+1.$$

We can easily construct a £(r)-versal deformation for each multi-germ by the usual method. Then the corresponding list is as follows :

$$\Big(y^{k_1+1} + \sum^{k_1} u_{1,i}y^{i-1}, \ldots, y^{k_r+1} + \sum^{k_r} u_{r,i}y^{i-1}\Big) \quad r \leq \sum^{r} k_i \leq n+1.$$

Let

£(^1,1,..., tii$_{fMl}$,..., u$_r$,i,..., *u$_r$^$_r$, y)*

be a /C(r)-versal deformation of a multi-germ *g* = (</i,..., p$_r$), where /x» = *K-codgi*) for *i* = 1,..., r. Define a multi-germ $\bar{G}$ by

$$\bar{G}(u_{1,1},\ldots,u_{1,\mu_1},\ldots,u_{r,1},\ldots,u_{r,\mu_r},\text{in},\ldots,\text{tx}_M,y)$$
$$=G(u_{1,1},\ldots,u_{1,\mu_1},\ldots,u_{r,1},\ldots,u_{r,\mu_r},y)$$

for // = n +1 - ^ $_{=1}$ *fa*. We now consider a function germ *t*: (R^n+^{a}, 0) -+ (R, 0) on **the (t*i,i,... ,ui$_{>fll}$,... ,tt$_r$,i,... ,^r,/A$_r$,^i$_9$---,W/i)-space. We can apply the theorem** of part 2.2 in [12], so we get the following :

Proposition 7.2. (Zakalyukin [12]) *Suppose that*

$$\frac{dt}{\partial u_{1,\mu_1}} \neq 0 \qquad \frac{dt}{\quad\;_{r,\mu_r}} \neq 0$$

and

*Ui,i=—=ttrfnr=0 t5 a *Morse function germ.*

Then there exists a diffeomorphism germ <f>: $(\mathrm{R}^{n+1},0)$ -» $(\mathrm{R}^{n+1},0)$ *preserving the discriminant set DQ such that to<f> is equal to u\ or ±u* $_{\mathrm{Ml}}$ ± • • • ± $u_{r\,M}$ ±$(ui)^2$ ±

We notice that a submersion germ t : $(\mathrm{R}^{n+1},0)$ —> (R,0) which satisfies the assumption of the proposition is generic. We can detect generic normal forms of multi-unfoldings of immersions as follows.

Theorem 7.3. *The multi-generating family of a generic multi-unfolding of immersions is t-(P-)C)(ry equivalent to one of the multi-germs in the following list:*
$({}^0A_{k_1 \cdots k_r})$

$$(y^{k_1+1} + \sum_{i=1}^{k_1} x_i y^{i-1}, \ldots, y^{k_r+1} + \sum_{i=1}^{k_r} x_{\sum_{j=1}^{r-1} k_j + i} y^{i-1}),$$

$$\textit{where } 1 \leq r \leq \sum_{j=1}^{r} k_j \leq n.$$

$({}^1A_k)$

$$y^{k+1} + y^{k-1}(t \pm x_k^2 \pm \cdots \pm x_n^2) + \sum^{k-1} x_i y^{i-1},$$

$$\textit{where } 2 \leq k \leq n+1.$$

$({}^1A^0_{k_1}A_{k_2,\ldots,k_r})$

$$(y^{k_1+1} + y^{k_1-1}(t + x_{k_1+k_2} + \cdots + x_{\sum k} + \cdots + x_\ell$$

$$\pm x_{\ell+1}^2 \pm \cdots \pm x_n^2) + \sum_{i=1}^{k_1-1} x_i y^{i-1},$$

$$y^{k_2+1} + \sum^{k_2} x_{k_1+1} y^{i-1}, \ldots, y^{k_r+1} + \sum^{k_1} x_{\sum_{j=1}^{r-1} k_j + i} y^{i-1}),$$

$$\textit{where } t = \sum_{j=1}^{r} k_d, \quad 2 \leq r \leq l \leq n+1.$$

Proof. Let $G(u_{1,1},\ldots,u_{1,\mu_1},\ldots,u_{r,1},\ldots,u_{r,\mu_r},y)$ be a $\mathcal{K}_{(r)}$-versal deformation of a multi-germ $g = (g_1,\ldots,g_r)$, where $\mu_i = \mathcal{K}\text{-}cod\,g_i$ for i = 1,...,r. Define a multi-germ $\bar{G}$ by

$$\bar{G}(u_{1,1},\ldots,u_{1,\mu_1},\ldots,u_{r,1},\ldots,u_{r,\mu_r},t,x_1,\ldots,x_n,y)$$
$$=G(u_{1,1},\ldots,u_{1,\mu_1},\ldots,u_{r,1},\ldots,u_{r,\mu_r},y)$$

for /z = n + 1 -]£i=i Mi-

Let F(t,z,y) be a multi-generating family of a t-(P-.4)$_{(r)}$-stable multi-unfolding of immersions. Then *F* is a (P-/C)$_{(r)}$-versal deformation of a multi-germ / = F|{0} x R^n x R, so that *F* is a (JC)$_{(r)}$-versal deformation of a multi-germ $/_0$ = /|{0} x R. If / is (JC)$_{(r)}$-equivalent to *g*, then *F* and $\bar{G}$ are (P-JC)$_{(r)}$-equivalent (i.e., there exist diffeomorphism germs

$$*i: (R^{n+1} \text{ x } R,0) \text{ -> } ((R \text{ x } R^n) \text{ x } R,0) \quad (i = 1,\dots,r)$$

of the form

$$\Psi(u,q) = (\psi_1(u), \psi_2(u), \psi_3(u,y))$$

such that \t*(F)$£_{(u,y)}$ = $(\bar{G})s_{(uy)}$). By the remark after Proposition 7.2, we may assume that *fa* satisfies the assumption of Proposition 7.2, so that there exists a diffeomorphism germ <*f*> *:* (R^{n+1},0) —» (R^{n+1},0) preserving the discriminant set UjLj-Dg$_i$ such that *%l)\o<f>\s* equal to *U* or dbtii^ ±- • -±u_{ryflr} db(iti)^2 ±- • •db(i6$_{/1}$)^2. Here, the discriminant set *U^$_{=1}$DQ:* is the wave front set of a multi-germ of a Legendrian submanifold in PT*R^{n+1}. Then we can construct the unique contact lift <*j*> of <*f*> preserving the multi-germ of the Legendrian submanifold. That is, (<f> x *lR)*$\bar{G}$(u,y)* = *$\bar{G}$(cf)(u),y)* gives the same multi-germ of the Legendrian submanifold given by $\bar{G}$. We can choose the coordinates of (R x R^n,0) as follows

$$\begin{cases} t = \psi_1 \circ \phi(u) \\ X = -02 \; O <f)(u), \end{cases}$$

where *u* = (txi,i,..., ui$_{>Ml}$,..., *u$_r$\,..., u$_r$^$_r$, u\,..., u^).* By the above equality we can represent the coordinates *u* by the coordinates (t,x). This procedure gives the normal forms as follows : Let *G* be a germ of the form

$$(y^{k_1+1} + \sum_{i=1}^{k_1} u_{1,i}y^{i-1}, \dots, y^{k_r+1} + \sum^{k_r} u_{r,i}y^{i-1}) \quad r \le \sum^{r} ki \le n+1.$$

It follows from Porposition 7.2 that *t* = *u* or *t* = ±ni$_{i/Xl}$ ± • • • ± $u_{r?Mr}$ ± (ixi)^2 ± • • • ± $(u_M)^2$, where ^ = *kj* and // = n + 1 - 5^J$_{=1}$fc^. If 1 ≤ r ≤ J^jjj fcj ≤ n and *t* = txi, then we can use the coordinates

$$t^*i_{l}i = \text{^»}, \; (1 \le i \le h) \; u_{2yi} = x_{k_1+i}, \; (1 \le i \le k_2), \dots, u_{r,i} = x_{\ell+i}, \; (1 \le i \le k_r),$$

where *I* = X^^=i^{-1} ^{fc}i^ *s°* that we get the normal forms *(°A$_{kb}$,...,k$_r$)>* If r = 1 and 2 ≤ fc ≤ n + 1, we already got the nirmal forms in Theorem 3.3. If 2 ≤ r < EJ=i fej ≤ n + 1 and t = ±^i,$_{Ml}$ ± • • • ± *u$_r$^$_r$* ± *(u^^{2}* ± •. • ± $(u_M)^2$, then we can detect the normal forms *CA^A^^kr)* by the appropriate coordinate changes.

Finally, we can easily check that each normal form is (P-/C)$_{(r)}$-versal.

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13. **ABSTRACT** *(Maximum 200 words)*

In this paper we describe the geometric framework for the study of generation and propagation of shock waves in Rn appearing in weak solutions of scalar conservation laws. We first define the notion of geometric solutions for scalar conservation laws in the framework of one-parameter unfoldings of immersions. The geometric solutions are, in general, multi-valued and they are constructed by the method of characteristics. We use singularity theory techniques to classify the generic types of multi-valuedness of the geometric solutions. Such a classification is used to construct the unique entropy solution of the scalar conservation law by selecting the proper single-valued discontinuous branch of the geometric solution satisfying the entropy condition across the discontinuity.

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