NWAT

93-025

Maxwell Equations in a Nonlinear Kerr Medium

Oscar P. Bruno
Georgia Institute of Technology

Fernando Reitich
Carnegie Mellon University

Research Report No. 93-NA-025

August 1993

Sponsors

U.S. Army Research Office
Research Triangle Park
NC 27709

National Science Foundation
1800 G Street, N.W.
Washington, DC 20550

Maxwell Equations in a Nonlinear Kerr Medium

Oscar P. Bruno* Fernando Reitich*

Abstract

In this paper we present an exact calculation of the transfer function associated with the nonlinear Fabry-Perot resonator. While our exact result cannot be evaluated in terms of elementary functions, it does permit us to obtain a number of simple approximate expressions of various orders of accuracy. In addition, our derivation yields criteria of validity for the approximate formulae. Our approach is to be compared with others in which approximations are introduced in the model itself, either through the equations or through the boundary conditions.

Our lowest order approximate formula turns out to be identical, interestingly, with the result obtained from the slowly varying envelope approximation (SVEA). Thus, our validity criteria apply to the SVEA result, and predict well its domain of validity and its breakdown for short wavelengths and for very high intensities and nonlinearities. The simple higher order formulae we present provide improved estimations in such regimes.

•School of Mathematics, Georgia of Technology, Atlanta, Georgia 30332-0160
department of Mathematics, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213-3890

1 Introduction

The interest in the nonlinear optical properties of materials, which are only observable at high field intensities, has grown steadily since such high intensities were made available with the invention of the laser in 1960. Since then, a number of effects associated with various types of nonlinearities exhibited by materials have been observed, and a number of practical uses of these effects have been found [2, 7, 3].

Here we deal with the so-called *optical Kerr effect.* Optical Kerr media are characterized by an intensity dependent refractive index of the form

$$n^2 = nl + non_2|E|^2$$

where E denotes the complex electric field. It is known [7] that Kerr media can be used to construct bistable optical systems. These are systems in which a certain input field can produce two (or more) different output states, or, in other words, systems for which the transfer function is multivalued.

The simplest and best known of the bistable devices is the nonlinear Fabry-Perot interferometer, first used by Szőke et al [8], in which a non-linear material is placed in a cavity between two partially reflecting mirrors. The importance of bistable devices lies on their potential applicability as optical switches in all-optical computers: they make it possible to use a light pulse to have the input intensity exceed threshold values, and therefore, to have the device switch between two output intensity levels [7].

The basic nonlinear mechanisms underlying non-dissipative Fabry-Perot resonators are well understood, and several approximate theories that predict their bistable behavior have been given [5, 6, 7]. One of the most accurate of these theories, due to Felber and Marburger [6], is based on the well known slowly varying envelope approximation (SVEA). A different, semi-exact theory was also presented in [6]. This calculation, which incorporates approximations only through certain boundary conditions, leads to results which are of the same order of accuracy as the SVEA expression, see §4.

In this paper we present an exact calculation of the optical properties of these devices. While our exact result cannot be evaluated in terms of elementary functions, it does permit us to obtain a number of simple approximate expressions of various orders of accuracy. In addition, our derivation yields criteria of validity for the approximate formulae. Our approach is to be compared with those mentioned above, in which approximations are introduced in the model itself, either through the equations or through the boundary conditions.

Our lowest order approximate formula turns out to be identical, interestingly, with the result obtained from the SVEA. Thus, our validity criteria apply to the SVEA expression, and predict well its domain of validity and its breakdown for short wavelengths and for very high intensities and nonlinearities. The simple higher order formulae we present provide improved estimations in such regimes.

2 The nonlinear Fabry-Perot interferometer

The simplest bistable optical device is the Fabry-Perot interferometer, first used in nonlinear optics by Szöke et al [8], see Figure 1. The middle region in the figure is occupied by a nonlinear material; on both sides of the nonlinear material we find the mirrors, i.e., plates of glass coated with a reflective material. The device, consisting of the mirrors and the nonlinear cavity, is placed in air.

In either of the seven regions of Figure 1, the electric field $\vec{E}$, the displacement vector $\vec{D}$ and the magnetic field $\vec{H}$ must satisfy the time harmonic Maxwell equations

$$\begin{aligned} \mathrm{V} \times \vec{E} &= iufi_0\vec{H} \\ \mathrm{V} \times \vec{H} &= -iu\vec{D}. \end{aligned} \qquad (1)$$

In any of these regions the displacement vector is given by

$$\vec{D} = €_0\vec{E} + \vec{P},$$

where €o is the dielectric constant of vacuum and $\vec{P}$ is the polarization vector. The polarization contains, in general, linear and nonlinear contributions

$$\vec{P} = \vec{P}^L + \vec{P}^{NL};$$

the linear part equals

$$\vec{P}^L = \epsilon_0\chi^{(1)}\vec{E}$$

where $\chi^{(1)}$ is the linear susceptibility of the medium. In the linear materials (glass, coating) $\vec{P}^{NL} \approx 0$ and so

The cavity, on the other hand, is assumed to be occupied by a nonlinear Kerr medium. In this case the vector $\vec{P}^{NL}$ is cubic in $\vec{E}$, and we have

$$\vec{P}^{NL} = \chi^{(3)}\vec{E}\vec{E}\vec{E}$$

where $\chi^{(3)}$ is the third-order susceptibility tensor.

We deal with normally incident TE-polarized light, with the electric field directed along the (vertical) z axis

$$\vec{E} = E\vec{e}_z.$$

Therefore, and since $d/dz = 0$, $\nabla \cdot \vec{E} = 0$, equations (1) reduce, in the nonlinear region, to

$$E'' + \frac{\omega^2}{c^2}n_0^2E + \omega^2\mu_0 P_z^{NL} = 0. \tag{2}$$

Here $n_0 = \sqrt{1 + \chi^{(1)}}$ is the linear part of the nonlinear refractive index and P_z^{NL} is the z component of $\vec{P}^{NL}$. Since

$$P_z^{NL} = 3\chi^{(3)}_{3333}|E|^2E$$

equation (2) can be rewritten as

$$E'' + \frac{\omega^2}{c^2}(n_0^2 + n_0n_2|E|^2)E = 0 \tag{3}$$

where

$$n_2 = \frac{3\chi^{(3)}_{3333}}{\epsilon_0 n_0}.$$

Thus, the intensity dependent refractive index of the Kerr medium is

$$n = \sqrt{n_0^2 + n_0n_2|E|^2},$$

or, accurate to first order in n_2,

$$n = n_0 + \frac{n_2}{2}|E|^2 = n_0 + n_2\langle E^2\rangle \quad \text{(see e.g. [2]).}$$

In our geometry, the nonlinear medium is placed between two partially reflective mirrors. The mirrors consist of dielectric layers of widths d_1 and d_1' ($d_1 + d_1' = d$) and refractive indices n_1 and n_1', representing the glass and coating respectively (see Figure 2). The problem of determining the optical response of this device can be reduced to a problem in the cavity. Indeed, equations (27) and (28) in the Appendix (see also the discussion below equation (28)) relate the incident and transmited intensities to the electric field at the boundaries of the cavity. The cavity equations can then be written

$$E'' + (k_0^2 + k_0k_2|E|^2)E = 0, \qquad 0 < z < L, \tag{4}$$
$$E(L) = be^{ik(L+d)}E_T, \quad E'(L) = ikae^{ik(L+d)}E_T,$$

where E_T is the complex amplitude of the transmitted electric field. Here a and b are the mirror parameters defined in equation (30) of the Appendix, and we have put

$$k = \frac{\omega}{c},\ k_0 = \frac{\omega}{c}n_0 \text{ and } k_2 = \frac{\omega}{c}n_2.$$

Now, from the Appendix we have

$$E_I = \frac{1}{2}e^{ikd}\left(aE(0) + \frac{1}{ik}E'(0)\right), \tag{5}$$

and, therefore, the solution of the cavity problem permits us to relate E_T to E_I, i.e., to obtain the transfer function.

Equations (4) and (5) can be simplified by introducing the variables

$$u(z) = \sqrt{\frac{k_2}{k_0}}E(L - \frac{z}{k_0}),\ u_T = \sqrt{\frac{k_2}{k_0}}e^{ik(L+d)}E_T b \text{ and } u_I = e^{-ikd}\sqrt{\frac{k_2}{k_0}}\frac{2ki}{k_0 b}E_I.$$

In these variables, equations (4) and (5) translate into

$$u'' + (1 + |u|^2)u = 0, \quad 0 < x < k_0L, \tag{6}$$
$$u(0) = u_T, \quad u'(0) = -iKu_T$$

and

$$«/ = iKu(k_0L) - u'(k_0L) \tag{7}$$

where $K = Q = a + i/3$, see also (32).

3 Exact solution and explicit approximations

3.1 Exact solution

In order to solve equation (6) let us put

$$u(z) = p(z)e^{i0}M$$

so that

$$u'(z) = (p'(z) + ip(z)6'\{z)) \text{ e'«(*)} \quad \text{and}$$
$$u''\{z) = [(p''(z) - p\{z)e'\{zf) + i \{2p'\{z)e'(z) + p\{z)d''(z))) \text{ e''(*>.}$$

Notice that equation (6) is invariant under the transformation u —• $e^{tc}u$ for any real constant c. Thus, without loss of generality we may assume that tij > 0; from (6) we then get

$$(p''(z) - p\{z)6'\{zf + (1 + p\{zf) p(z)) + i (2p'(z)d'(z) + p\{z)6''\{z)) = 0$$
$$\rho(0) = u_T, \quad 6(0) = 0$$
$$\rho'(0) = \beta u_T, \quad \theta'(0) = -a \tag{8}$$

Taking imaginary parts in (8) and multiplying by p it follows that

$$\theta'(z) = -\frac{u_T^2\alpha}{\rho(z)^2}.$$

We can therefore write the real part of (8) as

$$\rho''(z) - \frac{u_T^4\alpha^2}{\rho(z)^3} + (1 + \rho(z)^2)\,\rho(z) = 0$$
$$\rho(0) = u_T, \quad \rho'(0) = \beta u_T, \tag{9}$$

or, integrating once, as

$$\frac{\rho'(z)^2}{2} + \frac{u_T^4\alpha^2}{2\rho(z)^2} + \frac{\rho(z)^2}{2} + \frac{\rho(z)^4}{4} = \frac{\beta^2 u_T^2}{2} + \frac{u_T^4\alpha^2}{2u_T^2} + \frac{u_T^2}{2} + \frac{u_T^4}{4}. \tag{10}$$

We note here that oscillatory solutions of equation (10) are in one to one correspondence with the solutions of the second order equation (9) from which (10) was derived.

Multiplying (10) by $4\rho^2$ and setting $\hat{I} = \rho^2$ we get the equation

$$\frac{\hat{I}'(z)^2}{2} + \hat{I}(z)^3 + 2\hat{I}(z)^2 - K\hat{I}(z) + c = 0$$
$$\hat{I}(0) = q, \quad \hat{I}'(0) = 2\beta q \tag{11}$$

where

$$q = |u_T|^2, \quad c = 2q^2\alpha^2 \text{ and } K = 2(1 + \alpha^2 + \beta^2)q + q^2.$$

Once the solution to (11) is known, the normalized incident intensity $\hat{I}_0 = |u_I|^2$ can be obtained from (7). Indeed,

$$\hat{I}_0 \equiv |u_I|^2 = |u'(k_0L) - i\kappa u(k_0L)|^2 = |\rho'(k_0L) + i\rho(k_0L)\phi'(k_0L) - i(\alpha + i\beta)\rho(k_0L)|^2$$

$$= \frac{q^2}{2} - \frac{\hat{I}(k_0L)^2}{2} + (1 + 3\alpha^2 + \beta^2)q - (1 - \alpha^2 - \beta^2)\hat{I}(k_0L) + \beta\hat{I}'(k_0L).$$

Finally, the transmissivity τ is given by

$$\frac{1}{\tau} \equiv \frac{|E_I|^2}{|E_T|^2} = \frac{n_0^2|b|^4}{4}\frac{|u_I|^2}{|u_T|^2} = \frac{1}{4\alpha^2}\frac{\hat{I}_0}{q} \tag{12}$$

$$= \frac{1}{4\alpha^2}\left\{\frac{q}{2}\left(1 - P(k_0L)^2\right) + (1 + 3\alpha^2 + \beta^2) - (1 - \alpha^2 - \beta^2)P(k_0L) + \beta P'(k_0L)\right\}$$

where $P \equiv P_q = \frac{\hat{I}}{q}$ is the oscillatory solution of the equation

$$\frac{P'(z)^2}{2} + qP(z)^3 + 2P(z)^2 - KP(z) + \hat{c} = 0\,, \tag{13}$$

$$P(O) = 1, \quad P'(0) = 2/?, \tag{14}$$

and where

$$q = \quad s(\delta))$$

$$\hat{c} = 2a^2 \text{ and } \hat{\kappa} = 2(1 + a^2 + /?^2) + q.$$

3.2 Asymptotic formula for q small

When $ri2$ = 0 equation (13) reduces to a second order linear equation whose solution is

$$P_0(z) = 14£L \quad + A\cos(2z + \phi)$$

with A and <i> given by

$$\begin{cases} A\cos(\phi) = \frac{(1-\alpha^2-\beta^2)}{2} \\ A\sin(\phi) = -/?. \end{cases};$$

using equation (34) in the Appendix, we then find

$$P_Q(z) = \frac{(1 + Q + \wedge)}{2} + y\wedge + (1 - a^2 - /?^2)^2 \cos(2z + (?) \tag{15}$$

$$= \frac{(1+R)}{l + R + 2y/\bar{R}\cos(6)} + \frac{8R}{(l + R + 2\sqrt{R}\cos(\delta))^2} \cos(2z + \delta).$$

In this case our expression (12) for the transmissivity gives

$$\frac{1}{7} = \frac{1}{4\alpha^2}\left\{(1 + 3\alpha^2 + \beta^2) - (1 - \alpha^2 - \beta^2)P(k_0L) + \beta P'(k_0L)\right\}$$

$$= 1 + \frac{4R}{(1 - Rf)} \sin^2(k_0L + \delta)$$

which is the classical formula for the transmissivity of a linear Fabry-Perot resonator (see e.g. [4, p. 325]).

In the nonlinear device we consider, however, the coefficient n_2 and therefore q are not zero, though we have $0 < n_2 \ll 1$. The oscillatory solution P_q of (13)–(14) is still a periodic function, but in this case we do not have a simple formula such as (15). Of course, for small values of q, the solution P_q is close to P_0. In the expression (12), however, we need values of P at the point $z = k_0 L \gg 1$. Clearly, then, we cannot use P_0 as an approximation to P_q in (12) since small differences in the periods may yield widely different values of the solutions at the large value $z = k_0 L$.

A good approximation can be obtained from P_0, however, by simply adjusting its period, as we show now. Let us denote by T_q the period of P_q ($T_0 = \pi$). From (13), (14) we see that

$$T_q = \frac{1}{\sqrt{2}} \int_{p_2}^{p_1} \frac{dP}{\sqrt{-qP^3 - 2P^2 + \hat{\kappa} P - \hat{c}}}$$

where $p_3 < 0 < p_2 < p_1$ are the roots of the polynomial

$$F_q(P) = qP^3 + 2P^2 - \hat{\kappa} P + \hat{c}.$$

(It is not hard to show that for all $q \geq 0$ the roots of F_q are indeed ordered as indicated above.) We note now that $P_q(z)$ solves the equation (13) subject to the conditions

$$P(z_M) = p_1\,, \quad P'(z_M) = 0 \tag{16}$$

where $z_M \leq 0$ is a point where P attains its maximum

$$z_M = -\int_1^{p_1} \frac{dP}{\sqrt{-F_q(P)}}.$$

Then, it is easily checked that the oscillatory solution $\tilde{P}$ of the equation

$$\frac{P'(z)^2}{2} + 2\left(\frac{\pi}{T_q}\right)^2 P(z)^2 - \hat{\kappa} P(z) + \hat{c} = 2\left(\frac{\pi}{T_q}\right)^2 p_1^2 - \hat{\kappa} p_1 + \hat{c} \tag{17}$$

subject to (16) satisfies

$$\backslash \tilde{P}(z) - P_q(z) \backslash = O(q) \tag{18}$$

for $z_M < z < z_M + T_q/2$ and therefore, by periodicity, for all real z. Indeed, since $F_q(pi) = 0$ it follows that the right hand side of (17) vanishes for $q = 0$, which implies that, for $q = 0$, $P_q = \tilde{P}$. In particular, $\tilde{P}(0) = P_q(0) + O(q) = 1 + O(q)$ and since

$$\tilde{P} = \left(\frac{T_q}{2\pi}\right)^2 \hat{\kappa} + \left(p_1 - \left(\frac{T_q}{2\pi}\right)^2 \hat{\kappa}\right) \cos(2\pi/T_q z - 2\pi/T_q z_M) \tag{19}$$

we conclude from (15) that

$$z_M = -\frac{\delta}{2} + O(q).$$

Now, from (19) we see that the amplitude and phase of $\tilde{P}$ can be perturbed by $O(q)$ and the resulting function will still satisfy (18). More precisely, if we let

$$\hat{B} = \left(\frac{T_q}{2\pi}\right)^2 \hat{\kappa} + O(q) = \frac{\hat{\kappa}}{4} + O(q)$$

$$\hat{A} = \left(p_1 - \left(\frac{T_q}{2\pi}\right)^2 *\right) + 0\{q) = W\hat{T}^2 \, '' 2Z + 0(q)$$

$$\$ = -2Tr/T_q z_M + O(q) = 6 + O(q)$$

then

$$\backslash P_q(z) - (\& + \hat{A}\cos(2n/T_q z + \$)) \backslash = O(?) \quad \textbf{for all real } *.$$

The simplest expressions we can take for these constants are those that correspond to $q = 0$, i.e.,

$$\hat{B} = \frac{\hat{\kappa}}{4}$$

$$\hat{A} = \frac{1}{2}\sqrt{4\beta^2 + (1 - \alpha^2 - \beta^2)^2}$$

$$\hat{\phi} = \delta;$$

and we have

$$P_q(z) = \frac{(1+\alpha^2 + P2)}{2} + \backslash yJAp + (1 - a* - py \cos(2*z/T_q + 6) + O(q) \quad (20)$$

$$= \frac{(1+R)}{1+R+2\sqrt{R}\cos(\delta)} + \frac{8R}{(1+R+2\sqrt{R}\cos(\delta))^2} \cos(27rz/r, + 6) + 0(q)$$

Furthermore, if the conditions

$$q < 1 \text{ and } k_0 L q^n = 0(1), \quad (21)$$

are satisfied, the error in the approximation will be of order q at $z = k\$L$ even if we replace T_q in (20) by its Taylor polynomial in q of degree n

$$T_q \ll t_0 + t_1 q + t_2 q^2 + \cdots + t_n q^n + O(q^{n+1}).$$

A simple calculation shows, for example, that

$$\begin{cases} to = \pi \\ t_1 = -\frac{3\pi}{8}(1 + \alpha^2 + 3?) \\ t_2 = \frac{3\pi}{256}[35(a^2 + /?^2)^2 + 50a^2 + 70/?^2 + 19] \cdot \end{cases}$$

Thus, for n = 2, the approximate formula for P_q reads

$$P_q(z) \approx \frac{(1+\alpha^2+\beta^2)}{2} + \frac{1}{2}\sqrt{4\beta^2 + (1-\alpha^2-\beta^2)^2} \cos\left[2(1+\mu_1 q + \mu_2 q^2)z + \delta\right]$$

$$= \frac{(1+R)}{1+R+2\text{-/R}\cos(<5)} \quad \frac{8R}{(1+R+2\sqrt{R}\cos(\delta))^2} \cos\left[2(1 + fiiq + H2q^2)z + \delta\right]$$

where

$$\begin{cases} \mu_1 = \frac{3}{8}(1+\alpha^2+\beta^2) \\ \mu_2 = -\frac{3}{256}\left[23(\alpha^2+\beta^2)^2 + 26\alpha^2 + 46\beta^2 + 7\right]. \end{cases}$$

Prom (12) we then find that the transmissivity r is given by

$$r = \{1 + \frac{4R}{(1-R)^2}\sin^2\left[k_0L(1+\mu_1 q+\mu_2 q^2)+\delta\right]\}^{-1} + O(q).$$

Alternatively, setting

$$\nu_1 = \frac{n_2\mu_1}{n_0\alpha} = \frac{3n_2(1+R)}{4n_0(1-R)}$$

$$\nu_2 = \frac{n_2^2\mu_2}{n_0^4\alpha^2} = -\frac{3n_2^2}{32n_0^4}\,\frac{\left[7+24R+7R^2-8\sqrt{R}\cos(\delta)\left(1+\sqrt{R}\cos(\delta)+R\right)\right]}{(1-R)^2}.$$

the approximate formula for r reads

$$\tau \approx \left\{1+\frac{4R}{(1-R)^2}\sin^2\left[k_0L(1+\nu_1|E_T|^2+\nu_2|E_T|^4)+\delta\right]\right\} \qquad (22)$$

This is our new approximate expression for the transmissivity. It yields very accurate results provided the validity criteria

$$q = n_2|E_T|^2\frac{(1+R+2\sqrt{R}\cos(\delta))}{n_0^2(1-R)} \ll 1 \qquad (23)$$

and

$$W = O(1) \qquad (24)$$

are verified, as is usually the case in practice. Of course, formulae which incorporate phase terms of order higher than $n = 2$ can be obtained easily. The $n = 1$ approximation, on the other hand, is easily seen to coincide with the SVEA result of [6]; it gives good approximations provided the more restrictive conditions $q < 1$ and

$$k_0Lq = O(1)$$

are satisfied.

4 Discussion

Approximate formulae for the transmissivity, similar to the ones we present here, were given by Felber and Marburger [5, 6]. In the first of these papers an expression for the transmissivity was obtained under the assumption that the nonlinear refractive index is constant throughout the cavity. This expression is of the form (22) except for the phase of the sine function, which is incorrect even to first order in the nonlinearity ri2. As acknowledged by the authors, the corresponding values for the transmissivity differ substantially from the true values. The results in [5] are more interesting for the insight involved in their derivation and for the light they shed on the mechanisms at work in optical bistability than for their quantitative accuracy.

In [6] the authors present two different approximate formulae for the transmissivity of the nonlinear Fabry-Perot interferometer: one of them is obtained by means of the slowly varying envelope approximation (SVEA); the other, which involves approximations only through the boundary conditions, is given in terms of a certain elliptic function.

The SVEA expression again coincides with (22) except for the phase, though this time the phase is correct to first order in ri2. Our validity criterion (21) tell us then that the SVEA result is accurate as long as the conditions (23) and *k^Lq* = 0(1) are verified, as is often the case in the applications. In Figure 3 we present plots of $\backslash E j \backslash^2$ vs. $\backslash E T \backslash^2$ as given by the exact solution, by SVEA and by the approximate formula (22). Here we focus on the high nonlinearity and field intensity range, where the SVEA begins to break down; similar plots can be obtained in the short wavelength regime. This figure shows us the beneficial effect of incorporating the second order term in the phase of the transmissivity.

Finally, let us discuss the semi-exact calculation given in [6]. The authors only present expressions corresponding to mirrors with vanishing phase change 6 = 0, but their methods apply also to the general case. The corresponding general result is

$$\tau = j1 + \wedge \wedge)\ 2^{\sin 2} [\cos\text{-}HawK*)) + 6/2]\ \mathbf{I}\ ' \qquad (25)$$

where $\mathrm{cn}_m(;z)$ is a Jacobian elliptic function (see e.g. [1]),

$$\nu^2 = n_0^2 + \frac{n_2}{2}\left(\frac{3(1+R)+2\sqrt{R}}{(1-R)}\right)|E_T|^2, \quad m = \frac{n_2}{\nu^2}\left(\frac{2\sqrt{R}}{(1-R)}\right)|IM^2$$

and C+ is determined by the approximate phase-change condition at the back mirror $(z = L)$

$$\nu\left(\frac{k_0 L}{n_0} - \zeta_+\right) = \mathrm{cn}_m^{-1}\left(\cos(-\delta/2)\right).$$

The only approximations in this semi-exact derivation occur in the cavity boundary conditions, see also [6, Eqns. (18)-(22)]. As it happens, however, these errors affect the transmissivity phase to second order in the nonlinearity. In other words, the result (25) contains errors of the same order as those occurring in the SVEA expression.

A Appendix: Mirrors and equivalent jump conditions

Here we derive certain relations between the values of the fields at the two surfaces of a mirror in a nonlinear Fabry-Perot resonator. We consider first the case in which the mirror is substituted by an uncoated piece of glass; that is to say, we take *d[* = 0 in Figure 2. In the case of coated mirrors (with *d[* / 0) the calculation is similar and we will only present the final results.

Consider, then, an arrangement like the one in Figure 2 with *d[* = 0, where a nonlinear medium is placed between two dielectric layers of width *d* and refractive index *n*. Assume an electromagnetic field, with electric field in the plane of the figure, is normally incident on the left end of this device, i.e. at *z* = —*d*. To obtain relations for the values of the fields on the surfaces of the *linear* dielectric layers, we need to consider the *characteristic matrix* of the dielectric (see e.g. [4, p. 61]), which relates the values of the electric and magnetic fields at different points in the layer. For a dielectric of index *n* and for points with abscisae differing by di, the characteristic matrix is given by

$$M = \begin{pmatrix} \cos(k_1 d_1) & -\frac{i\mu_0}{n_1}\sin(k_1 d_1) \\ -i\frac{n_1}{\mu_0}\sin(k_1 d_1) & \cos(k_1 d_1) \end{pmatrix}. \tag{26}$$

Taking into account the continuity of the tangential component of the electromagnetic field, it then follows (see [4, §1.6]) that the amplitudes of the incident and reflected fields are related to the fields at $z = 0$ by

$$\begin{pmatrix} e^{-ikd_1} & e^{ikd_1} \\ \frac{e^{-ikd_1}}{\mu_0} & -\frac{e^{ikd_1}}{\mu_0} \end{pmatrix} \begin{pmatrix} E_I \\ E_R \end{pmatrix} = M \begin{pmatrix} E(0) \\ \frac{1}{ik\mu_0}E'(0) \end{pmatrix}.$$

Note that this relation does not depend on the refractive index of the medium to the right of $z = 0$. Analogously, we have

$$\begin{pmatrix} E(L) \\ '(L) \end{pmatrix} = M \begin{pmatrix} E_T e^{ik(L+d_1)} \\ \frac{1}{\mu_0} E_T e^{ik(L+d_1)} \end{pmatrix}.$$

In particular we can write

$$E_I = | \ast e \, '' \, ' \, (aE(0) + | \;\; E'(0) \tag{27}$$

and

$$E(L) \;=\; be^{ik}(^{L}+^{dl} >\pounds r_f \qquad E\backslash L) \;=\; ikae^{ik\{L}+^{dl} \wedge_T \tag{28}$$

where

$$a = (\cos(\text{fcidi}) - \text{ini} \sin(\text{fcirfi})) \text{ and } 6 = (\cos(\text{fcidi}) - \frac{i}{n_i}\sin(\text{fcidi})). \tag{29}$$

These equations provide relations between the values of the electric field at the boundary of the cavity and the incident and transmitted amplitudes.

If an additional dielectric layer with index of refraction $n_1' = ck_1'/\omega$ and width $d_1' \neq 0$ is added so as to model a glass plate coated with a reflective material (see Figure 2), a similar calculation to the one carried out above shows that equations (27)-(28) continue to hold as long as we replace d_1 by $d = d_1 + d_1'$ and (29) by

$$
\begin{aligned}
a &= \cos(k_1 d_1)\cos(k_1' d_1') - \frac{n_1'}{n_1}\sin(k_1 d_1)\sin(k_1' d_1') \\
&\quad - i\left(n_1' \cos(k_1 d_1)\sin(k_1' d_1') + n_1 \sin(k_1 d_1)\cos(k_1' d_1')\right) \quad \text{and} \\
b &= \cos(k_1 d_1)\cos(k_1' d_1') - \frac{n_1}{n_1'}\sin(k_1 d_1)\sin(k_1' d_1') \\
&\quad - i\left(\frac{1}{n_1}\sin(k_1 d_1)\cos(k_1' d_1') + \frac{1}{n_1'}\cos(k_1 d_1)\sin(k_1' d_1')\right).
\end{aligned}
\tag{30}
$$

The quantities a and b introduced above characterize the transmission properties of the mirrors. Usually, however, two different numbers, the reflectivity R and the phase change on reflection δ, are used for this purpose. To find expressions for R and δ in terms of a and b we again consider first the case of an uncoated glass. Assume the incident electromagnetic wave, of amplitude E_I, propagates in a material of refractive index n_0 (see Figure 4). The electric field is then given by

$$
E(z) = \begin{cases} E_R e^{-ik_0 z} + E_I e^{ik_0 z}, & -\infty < z < 0 \\ E_B e^{-ik_1 z} + E_F e^{ik_1 z}, & 0 < z < d_1, \\ E_T e^{ikz}, & d_1 < z < \infty \end{cases}
$$

where

$$
k = \frac{\omega}{c}, \quad k_1 = \frac{\omega}{c} n_1 \quad \text{and} \quad k_0 = \frac{\omega}{c} n_0.
$$

Using the characteristic matrix (26) and the continuity of the tangential components of the electromagnetic field we obtain

$$
\begin{pmatrix} E_I + E_R \\ \frac{k_0}{k\mu_0}(E_I - E_R) \end{pmatrix} = M \begin{pmatrix} E_T e^{ikd_1} \\ \frac{1}{\mu_0} E_T e^{ikd_1} \end{pmatrix}
$$

Thus,

$$\begin{pmatrix} E_I \\ E_R \end{pmatrix} = \frac{e^{ikd_1}E_T}{2} \begin{pmatrix} 1 & k/k_0 \\ 1 & -k/k_0 \end{pmatrix} \begin{pmatrix} \cos(k_1 d_1) - \frac{i}{n_1}\sin(k_1 d_1) \\ \cos(k_1 d_1) - i n_1 \sin(k_1 d_1) \end{pmatrix}$$

that is

$$E_I = \frac{1}{2}\left(b + \frac{ka}{k_0}\right)E_T e^{ikd_1}$$

$$E_R = \frac{1}{2}\left(b - \frac{ka}{k_0}\right)E_T e^{ikd_1}. \qquad (31)$$

Letting

$$K = \frac{ka}{k_0 b} \qquad (32)$$

equations (31) give us

$$r = \sqrt{R}\, e^{i\delta} = \frac{E_R}{E_I} = \frac{b - ka/k_0}{b + ka/k_0} = \frac{1-K}{1+K}$$

where R and δ are the reflectivity and phase change on reflection of the dielectric layer. In particular, if α and β denote the real and imaginary parts of K we have

$$K = \frac{ka}{k_0 b} = \alpha + i\beta = \frac{1-R}{1+R+2\sqrt{R}\cos(\delta)} + i\frac{(-2\sqrt{R}\sin(\delta))}{1+R+2\sqrt{R}\cos(\delta)}. \qquad (33)$$

and, therefore,

$$R = \frac{(1-\alpha)^2+\beta^2}{(1+\alpha)^2+\beta^2}, \quad \tan(\delta) = \frac{2\beta}{1-\alpha^2-\beta^2}. \qquad (34)$$

In the case of actual mirrors with a reflective coating of width $d_1 \neq 0$ (on the incident side), a calculation similar to the one above shows that equations (32)-(34) give the correct reflectivity and phase change provided a and b are defined as in (30).

Acknowledgments. OB gratefully acknowledges support from NSF through grant No. DMS-9200002. This work was partially supported by the Army Research Office and the National Science Foundation through the Center for Nonlinear Analysis.

References

[1] **Byrd, P. F. and Friedman, M. D.,** ***Handbook of elliptic integrals for engineers*** *and physicists,* Springer Verlag, (1954).

[2] Boyd, R. W., *Nonlinear Optics,* Academic Press, Inc. (1992).

[3] Brandt, H. E., Editor, *Selected papers in Nonlinear Optics,* SPIE Optical Engineering Press, (1991).

[4] Born, M. and Wolf, E., *Principles of Optics,* McMillan, New York (1964).

[5] **Felber, F. S. and Marburger, J. H.,** ***Theory of nonresonant multistable optical*** *devices,* Appl. Phys. Lett. 28, (1976) 731-733.

[6] **Felber, F. S. and Marburger, J. H.,** ***Theory of a lossless nonlinear Fabry-Perot*** *interferometer,* Phys. Rev. A 17, (1978) 335—342.

[7] **Gibbs, H. M.,** ***Optical Bistability: controlling light with light,*** **Academic Press,** Inc. (1985).

[8] Szöke, A., Daneu, V., Goldhar, J. and Kurnit, N. A., *Bistable optical element and its applications,* Appl. Phys. Lett. 15, (1969) 376-379

Captions for figures

Figure 1: The nonlinear Fabry-Perot interferometer.

Figure 2: The geometry. **A** nonlinear medium is placed between two coated glass plates.

Figure 3: Plot of |£/|^2 vs. \$E_T$\$^2$ for n_0 = 3.0, n_2 = 10"^{8}, k_0L = 100, R = 0.7 and 6 = 0. The three curves represent the exact solution (——), the solution under SVEA (——) and the solution under the approximation (25) (——).

Figure 4: Incident, reflected and transmitted waves for the calculation of the reflectivity and phase change of a mirror.

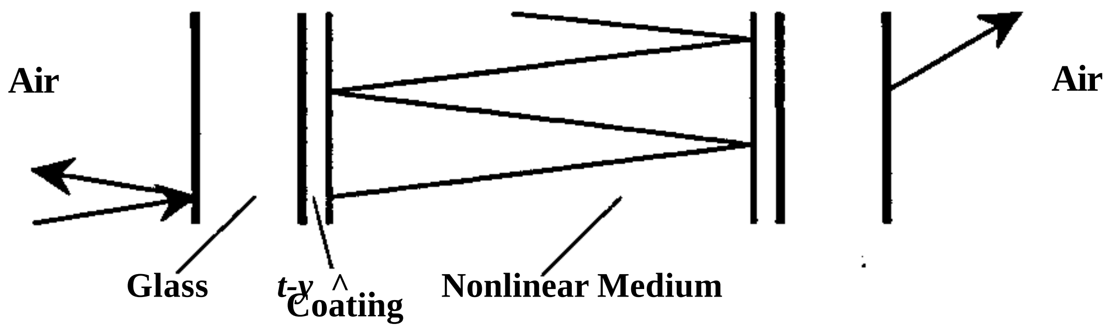

Figure 1

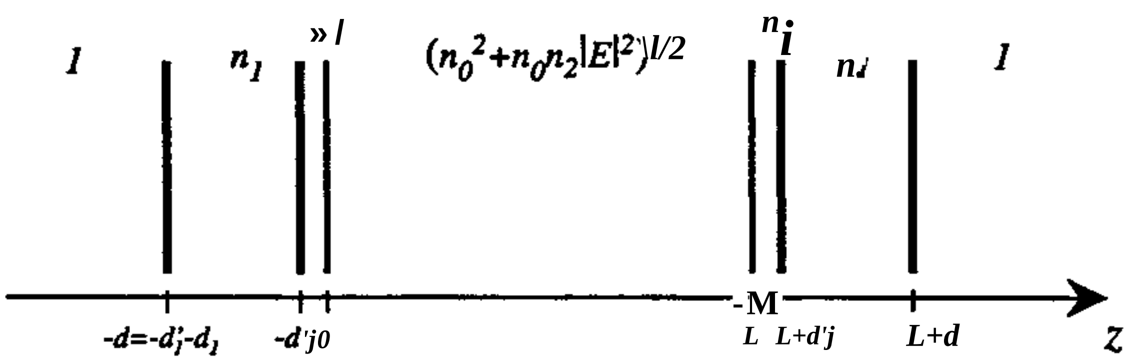

Figure 2

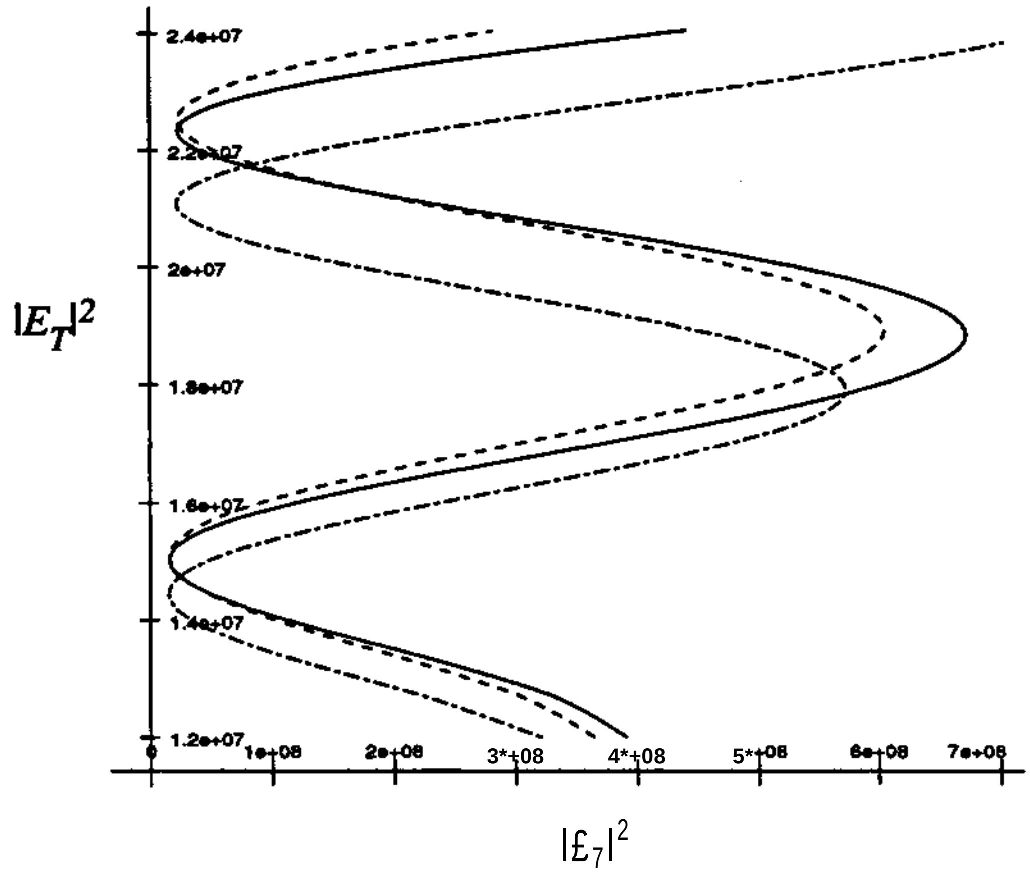

Figure 3

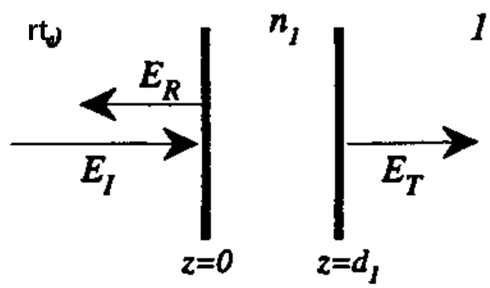

Figure 4