QUASICONVEX INTEGRANDS AND LOWER SEMICONTINUITY IN L^1

by

Irene Fonseca

Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

and

Stefan Müller
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

and

Institut für Angewandte Mathematik Universität Bonn
Beringstr. 4, W–5300 Bonn 1
Germany

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QUASICONVEX INTEGRANDS AND LOWER SEMICONTINUITY IN L^1

IRENE FONSECA† AND STEFAN MÜLLER††

Abstract. In this paper it is shown that, under mild continuity and growth hypotheses, if f(x,u,.) is quasiconvex and if u_n, $u \in W^{1,1}$ are such that $u_n \to u$ in L^1 then

$$\int_\Omega f(x,u(x),\nabla u(x))\,dx \le \liminf \int_\Omega f(x,u_n(x),\nabla u_n(x))\,dx\ .$$

The proof relies on a blow up argument in connection with a truncation result which allows one to consider uniformly convergent sequences.

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†† Department of Mathematics, Carnegie Mellon University, and Institut für Angewandte Mathematik Universität Bonn Beringstr. 4, W-5300 BONN 1, Germany. The research was partially supported by the National Science Foundation under Grant No. DMS - 9002679 and by SFB256 at the University of Bonn.

1. Introduction.

In this paper we prove that if $u_n, u \in W^{1,1}(\Omega; \mathbb{R}^p)$ are such that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^p)$, if $f(x,u,.)$ is quasiconvex, if f satisfies technical continuity conditions (see Section 2) and f grows at most linearly in the last argument, with possibly degenerate bounds, then

$$\int_\Omega f(x,u(x),\nabla u(x))\, dx \leq \liminf_{n \to +\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))\, dx.$$

This lower semicontinuity result was obtained by Dal Maso [DM] in the scalar case $p = 1$; in the vector-valued case and for $f = f(A)$ convex, by Ball and Murat [BM] and Reshetnyak [R]; when $p > 1$ and $f = f(x,\nabla u)$ quasiconvex the problem was addressed by Fonseca [Fo] and, independently, by Kinderlehrer [K]. For the case where $f = f(x,u,A)$ and $f(x,u,.)$ is convex, Aviles and Giga [AG] obtained lower semicontinuity results.

The main new tool involved in this paper is a careful truncation technique which, together with a blow up argument, enables us to reduce to the case where the sequence u_n converges uniformly. F. Murat has informed us that related truncation arguments are used in the context of renormalized solutions to partial differential equations (see e. g. [BDGM]).

The study of this problem was motivated by the analysis of variational problems for phase transitions and the related question of understanding the relaxation of functionals of the type

$$u \to \int_\Omega f(x,u(x),\nabla u(x))\, dx \tag{1.1}$$

in spaces admitting discontinuous functions u. As an example of that relation consider the family of singular perturbations

$$E_\varepsilon(u) := \int_\Omega W(u(x))\, dx + \varepsilon^2 \int_\Omega h^2(\nabla u(x))\, dx$$

of the nonconvex energy

$$E(u) := \int_\Omega W(u(x))\, dx,$$

where W has two potential wells at a and b. Depending on the constraints or boundary conditions imposed on the admissible functions, often E(.) admits infinitely many minimizers which are piecewise constant functions of bounded variation, $u \in \{a, b\}$ a. e. in Ω. In search for a reasonable selection criteron one studies the properties of the limits of sequences of minimizers for the perturbed problems (see [FT1], [G1], [G2], [KS], [Mo], [OS]). The natural notion of convergence for the functional in this context is Γ-convergence as introduced by De Giorgi [DG] (see [At], [DM], [DD] for more recent expositions).

In the isotropic scalar case, i. e. if $u : \Omega \to \mathbb{R}$ and $h = \|.\|$, using an idea of Modica and Mortola, Modica [Mo] showed that the $\Gamma(L^1)$ limit of the rescaled energies

$$J_\varepsilon(u) := \frac{1}{\varepsilon} E_\varepsilon(u)$$

is given by

$$J_0(u) = \mathcal{F}(u)$$

where

$$\mathcal{F}(u) := \inf_{\{u_n\}} \left\{ \liminf_{n \to +\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))dx \mid u_n \in W^{1,1}((\Omega ; \mathbb{R}), u_n \to u \text{ in } L^1 \right\}$$

is the relaxation in $BV(\Omega ; \mathbb{R})$ of (1.1) and

$$f(x, u, A) = 2\sqrt{W(u)}\ h(A). \tag{1.2}$$

Precisely, if $u \in \{a, b\}$ a. e. and if $\{u = a\}$ has finite perimeter in Ω then

$$\inf_{\{u_\varepsilon\}} \{\liminf J_\varepsilon(u_\varepsilon) \mid u_\varepsilon \in W^{1,1}((\Omega ; \mathbb{R}), u_\varepsilon \to u \text{ in } L^1 \} = \mathcal{F}(u).$$

This result was generalized by [OS] to "anisotropic" functions h with linear growth for which h^2 is convex. In this case the integral representation for the relaxation $\mathcal{F}(.)$ was obtained by Dal Maso [DM] who proved that

$$\mathcal{F}(u) = \int_\Omega f(x,u(x),\nabla u(x))\, dx + \int_{\Sigma(u)} D(x,u^-(x),u^+(x),\nu(x))\, dH_{N-1}(x) +$$

$$+ \int_\Omega f^\infty(x,u(x),dC(u)(x))\, dx \tag{1.3}$$

where H_{N-1} denotes the N-1-dimensional Hausdorff measure and the distributional derivative Du of the function $u \in BV(\Omega;\mathbb{R})$ admits the decomposition into mutually singular Radon measures

$$Du = \nabla u\, L_N \lfloor \Omega + (u^+ - u^-)\nu H_{N-1} \lfloor \Sigma(u) + C(u).$$

Here L_N is the N-dimensional Lebesgue measure, ∇u denotes the absolutely continuous part of Du, i. e. the Radon-Nikodym derivative of Du with respect to L_N, $\Sigma(u)$ is the jump set of Du with normal ν defined for H_{N-1} a. e. $x \in \Omega$ and C(u) is the Cantor part of the derivative (for details we refer the reader to Evans and Gariepy [EG], Federer [Fe], Ziemer [Zi]). In (1.3) f^∞ represents the *recession function* (see Section 2) and $D(x,a,b,\nu)$ is given by

$$D(x,a,b,\nu) = \int_a^b f^\infty(x,s,\nu)\, ds.$$

In the isotropic vector valued case, i. e. if $u : \Omega \to \mathbb{R}^p$ and $h = \|.\|$, Baldo [B] and Fonseca and Tartar [FT1] obtained once again the same representation for the Γ-limit. All the above results confirm Gurtin's [G1], [G2] conjecture that the "preferred" solution has minimal surface energy.

In the anisotropic, vector-valued case and with u subject to the constraint curl u = 0, recent work by Kohn and Müller [KM] seems to indicate that the Modica and Mortola inequality

$$J_\varepsilon(u) \geq \int_\Omega f(x,u(x),\nabla u(x))\, dx$$

with f given by (1.2) is no longer optimal. However, it is clear that

$$u \to \int_\Omega f(x,u(x),\nabla u(x))\, dx$$

still provides a lower bound for the rescaled energies $J_\varepsilon(.)$. In particular, the Γ–limit must be bigger than or equal to $\mathcal{F}(u)$. The issue thus arises, to find an integral representation for $\mathcal{F}(u)$ in the vector-valued case.

Fonseca and Rybka [FR] proved that, when f(x, u, .) is convex and if u takes only the values a and b across a plane with normal ν then

$$\mathcal{F}(u) = \int_\Omega f(x,u(x),0)\, dx + \int_{\Sigma(u)} K(x,a,b,\nu)\, dH_{N-1}(x),$$

where

$$K(x,a,b,\nu) := \inf\{ \int_{Q_\nu} f^\infty(x,\xi(y),\nabla\xi(y))\, dy \mid \xi \in \mathcal{A} \}$$

and

$\mathcal{A} = \{\xi \in W^{1,1}(Q_\nu;\mathbb{R}^p) \mid \xi(y) = b$ if $y.\nu = 1/2$, $\xi(y) = a$ if $y.\nu = -1/2$, and ξ is periodic in the remaining $\nu_1,\ldots,\nu_{N-1}$ directions with period 1$\}$,

where $\{\nu_1,\ldots,\nu_{N-1},\nu = \nu_N\}$ forms an orthonormal basis of $\mathbb{R}^N$ and Q_ν is the cube $\{y \in \mathbb{R}^N \mid |y.\nu_i|$ 1/2, i = 1,...,N$\}$. The characterization of the surface energy density K was inspired by the work of Fonseca and Tartar [FT2].

Independently, Ambrosio and Pallara [AP] showed that $\mathcal{F}(.)$ admits an integral representation with the same structure as in (1.3), and this result together with the work of Fonseca and Rybka [FR] provides a complete characterization of $\mathcal{F}(u)$, namely

$$\mathcal{F}(u) = \int_\Omega f(x,u(x),\nabla u(x))\, dx + \int_{\Sigma(u)} K(x,u^-(x),u^+(x),\nu(x))\, dH_{N-1}(x) + \\ + \int_\Omega f^\infty(x,u(x),dC(u)(x))\, dx. \tag{1.4}$$

To identify the first and the third term on the right hand side of (1.4) [AP] make use of the lower semicontinuity results of Aviles and Giga [AG] whose proofs rely on sophisticated tools from geometric measure theory. Also, f has to satisfy linear growth condition from below, i. e.

$$c\|A\| - C \leq f(x,u,A) \leq C(1 + \|A\|) \tag{1.5}$$

for some c, C > 0, preventing a situation as in (1.2). In addition, we remark that the convexity hypothesis on f(x,u,.) may be too restrictive. Indeed, as shown by Acerbi and Fusco [AF], Dacorogna [D] and Morrey [Mr] the $W^{1,1}$ - weak lower semicontinuous envelope of the functional (1.1) is the integral of the quasiconvexification of the energy density f(x,u,.), and so we expect quasiconvexity as a natural constitutive assumption rather than convexity. This concern is genuine as there are examples of quasiconvex functions with linear growth that are not convex (see Sverák [S] and Zhang [Z]).

In this work we consider quasiconvex integrands and we relax (1.5) to include degenerate lower bounds. Under these conditions we provide an analytical proof of the lower semicontinuity of (1.1) in L^1 thus obtaining the first term in the relaxation $\mathcal{F}(u)$. Our method seems to be appropriate to proving the lower semicontinuity of the third term in (1.4) corresponding to the Cantor part of the measure Du and one might thus conjecture that the representation of $\mathcal{F}(u)$ given by (1.4) is still valid for quasiconvex integrands with possibly degenerate lower bounds.

2. Lower semicontinuity in L^1 for quasiconvex integrands.

Let p, N ≥ 1 and let M^{pxN} denote the vector space of all pxN real matrices.

Definition 2.1([Mr]).
A function $f : M^{pxN} \to \mathbb{R}$ is said to be *quasiconvex* if

$$f(A) \leq \frac{1}{meas(D)} \int_D f(A+\nabla\varphi(x))\, dx$$

for all $A \in M^{pxN}$, for every domain $D \subset \mathbb{R}^N$ and for all $\varphi \in W_0^{1,\infty}(D; \mathbb{R}^p)$.

Remark 2.2. If $|f(A)| \leq C(1 + \|A\|)$ one shows easily by approximation that the inequality holds for all $\varphi \in W_0^{1,1}(D; \mathbb{R}^p)$.

Let $\Omega \subset \mathbb{R}^N$ be an open, bounded domain and let

$$f : \Omega x \mathbb{R}^p x M^{pxN} \to [0, +\infty).$$

We consider the following hypotheses on f:

(H1) f is continuous ;

(H2) f(x,u,.) is quasiconvex ;

(H3) there exists a nonnegative, bounded, continuous function $g : \Omega x \mathbb{R}^p \to [0, +\infty)$, c, C > 0 such that

$$cg(x, u)\|A\| - C \le f(x, u, A) \le Cg(x, u)\,(1 + \|A\|)$$

for all $(x, u, A) \in \Omega x \mathbb{R}^p x M^{pxN}$;

(H4) for all $(x_0, u_0) \in \Omega x \mathbb{R}^p$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| + |u - u_0| < \delta$ implies that

$$f(x_0, u, A) - f(x_0, u_0, A) \ge -\varepsilon\,(1 + \|A\|)$$

and

$$|f(x_0, u, A) - f(x, u, A)| \le \varepsilon\,(1 + \|A\|).$$

Theorem 2.3.

If the assumptions (H1) -(H4) hold and if $u_n, u \in W^{1,1}(\Omega; \mathbb{R}^p)$ are such that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^p)$ then

$$\int_\Omega f(x,u(x),\nabla u(x))\,dx \le \liminf \int_\Omega f(x,u_n(x),\nabla u_n(x))\,dx. \tag{2.1}$$

Remarks 2.4. (i) If (H2) is replaced by convexity and if the growth condition (H3) holds, then the hypothesis $(H4)_1$ presents no restriction. This fact will be examined in Section 4.

(ii) Lower semicontinuity for functions of the type (1.2) follows from Theorem 2.3. Indeed, if

$$f(x, u, A) = 2\sqrt{W(u)}\; h(A),$$

where h is a nonnegative quasiconvex function and

$$c\|A\| - C \le h(A) \le C\,(1 + \|A\|),$$

then we set

$$W_M(u) := \min\{M, W(u)\} \text{ and } f_M(u, A) := 2\sqrt{W_M(u)}\; h(A).$$

It is clear that f_M satisfies (H1)-(H4) and so, if $u_n, u \in W^{1,1}(\Omega; \mathbb{R}^p)$ are such that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^p)$ then

$$\int_\Omega f_M(u(x),\nabla u(x))\,dx \le \liminf \int_\Omega f_M(u_n(x),\nabla u_n(x))\,dx$$

$$\le \liminf \int_\Omega f(u_n(x),\nabla u_n(x))\,dx.$$

Letting $M \to +\infty$ and using the Monotone Convergence Theorem we conclude (2.1).

(iii) As we showed in (ii) the boundedness of g presents no restriction for the examples that we have in mind. This assumption becomes crucial for proving in Proposition 2.6 that the u_n may be considered to be smooth functions, which in turn allows one to apply in $(2.14)_2$ the change of variables formula (2.3) for Lipschitz functions.

It is possible to remove in (H3) the boundedness constraint imposed on g by using a suitable generalization of the change of variables formula (2.3) for $W^{1,1}$ functions. For for the sake of clarity, however, we focus attention on the case where g is bounded.

The main idea of the proof is to use a blow-up argument to localize (2.1) (see (2.5) and step 2 in the proof of Theorem 2.3) and a careful truncation technique for vector-valued functions which allows one to replace L^1 convergence by uniform convergence (see Lemmas 2.8 and step 3 in the proof of Theorem 2.3). Firstly we recall some auxiliary results.

Proposition 2.5.
If $f : M^{p\times N} \to \mathbb{R}$ is quasiconvex and if $|f(A)| \leq C(1 + \|A\|)$ for some constant $C > 0$ and for all $A \in M^{p\times N}$ then there exists a constant $C' = C'(C, N)$ such that

$$|f(A) - f(B)| \leq C' \|A - B\|$$

for all $A, B \in M^{p\times N}$.

Proof. We refer the reader to Dacorogna [D, Chapter 4, Lemma 2.2] or Evans [E]

Proposition 2.6. (i) If Theorem 2.3 holds true for Ω being a ball it holds true for all open, bounded sets Ω.
(ii) Let Ω be a ball. If (H1) and (H3) hold and if if $u_n, u \in W^{1,1}(\Omega; \mathbb{R}^p)$ are such that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^p)$ then there exist $\tilde{u}_n \in C_0^\infty(\mathbb{R}^N; \mathbb{R}^p)$ such that $\|\tilde{u}_n - u\|_{L^1(\Omega)} \to 0$ and

$$\liminf_{n \to +\infty} \int_\Omega f(x,\tilde{u}_n(x),\nabla\tilde{u}_n(x))\, dx = \liminf_{n \to +\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))\, dx.$$

Proof. The proof follows essentially the argument by Acerbi and Fusco [AF] and for completeness it is included in Section 3.

Proposition 2.7.
Let $f : M^{p\times N} \to \mathbb{R}$ be a function satisfying (H1), (H2) and

$$0 \leq f(A) \leq C(1 + \|A\|)$$

for some $C > 0$. If $A_0 \in M^{p \times N}$ and if $u_n \in W^{1,1}(\Omega; \mathbb{R}^p)$ are such that $u_n \to 0$ in $L^1(\Omega; \mathbb{R}^p)$ and $\{\|\nabla u_n\|_{L^1}\}$ is bounded then

$$\text{meas}(\Omega)\, f(A_0) \leq \liminf \int_\Omega f(A_0+\nabla u_n(x))\, dx.$$

Proof. See Section 3.

We will also use the following results. If $u \in W^{1,1}(\Omega; \mathbb{R}^p)$ then for a. e. $x_0 \in \Omega$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{\frac{1}{\varepsilon^N} \int_{B(x_0,\varepsilon)} |u(x) - u(x_0) - \nabla u(x_0)(x - x_0)|^{N/(N-1)} dx \right\}^{(N-1)/N} = 0, \qquad (2.2)$$

and if $w \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R})$ and $g \in L^1(\mathbb{R}^N; \mathbb{R})$ then the *change of variables formula* (or *coarea formula*) holds, namely

$$\int_{\mathbb{R}^N} g(x)\, |\nabla w(x)|\, dx = \int_{-\infty}^{+\infty} \left(\int_{w^{-1}(t)} g(x)\, dH_{N-1}(x) \right) dt. \qquad (2.3)$$

For details see Calderon and Zygmund [CZ], Evans and Gariepy [EG] and Ziemer [Z]. An easy consequence of (2.3) is the following estimate on level sets of $W^{1,\infty}$ functions.

Lemma 2.8.
Let $v \in W^{1,\infty}_{loc}(\mathbb{R}^N; \mathbb{R}^p)$, let $0 < \alpha < \beta < L$ and let $C_0 > 0$ be such that

$$\int_{\{|v| \leq L\} \cap B(0,1)} \|\nabla v(x)\|\, dx \leq C_0.$$

Then

$$\text{ess inf}_{t \in (\alpha, \beta)}\ t\, H_{N-1}(\{x \in B(0, 1) \mid |v(x)| = t\}) \leq \frac{C_0}{\ln(\beta/\alpha)}.$$

Proof. Let $B := B(0, 1)$ and consider a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^N; \mathbb{R})$ such that $\varphi = 1$ in $B(0, 1)$ and its support is contained in $B(0, 2)$. Applying the co-area formula (2.3) to

$$w(x) := \varphi(x)|v(x)| \ \text{ and } \ g(x) := \chi_{[0, L]}(|v(x)|)\, \chi_B(x)$$

we have

$$\int_0^L H_{N-1}(\{x \in B \mid |v(x)| = t\})\, dt = \int_{\{|v| \leq L\} \cap B(0,1)} \|\, D|v(x)|\, \|\, dx$$

$$\leq \int_{\{|v| \leq L\} \cap B(0,1)} \|\nabla v(x)\|\, dx \leq C_0$$

and so, if

$$\text{ess inf}_{t\in(\alpha,\beta)}\ t\,H_{N-1}(\{x\in B \mid |v_n(x)|=t\}) = a$$

then

$$C_0 \ge \int_\alpha^\beta H_{N-1}(\{x\in B \mid |v_n(x)|=t\})\,dt \ge \int_\alpha^\beta \frac{a}{t}\,dt$$

$$= a\ln\left(\frac{\beta}{\alpha}\right).$$

Thus

$$\text{ess inf}_{t\in(\alpha,\beta)}\ t\,H_{N-1}(\{x\in B \mid |v_n(x)|=t\}) \le \frac{C_0}{\ln(\beta/\alpha)}.$$

■

Proof of Theorem 2.3. In the sequel and using Proposition 2.6 we assume Ω is a ball and that $u_n \in C_0^\infty(\mathbb{R}^N; \mathbb{R}^p)$. In addition, suppose without loss of generality that

$$\liminf_{n\to+\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))\,dx = \lim_{n\to+\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))\,dx < +\infty.$$

Step 1.(localization) We first reduce the problem to verifying the pointwise inequality (2.5) below. As f is nonnegative there exists a subsequence such that

$$f(.,u_n(.),\nabla u_n(.)) \to \mu \text{ weakly * in the sense of measures,}$$

where μ is a nonnegative finite measure. Using Radon-Nikodym Theorem, we can write μ as a sum of two mutually singular nonnegative measures

$$\mu = \mu_a(x)L_N + \mu_s$$

where L_N denotes the Lebesgue measure in $\mathbb{R}^N$ and for a. e. $x_0 \in \Omega$

$$\mu_a(x_0) = \lim_{\varepsilon\to 0} \frac{\mu(B(x_0,\varepsilon))}{L_N(B(x_0,\varepsilon))} < +\infty. \tag{2.4}$$

We claim that

$$\mu_a(x_0) \ge f(x_0, u(x_0), \nabla u(x_0)) \quad \text{for a. e. } x_0 \in \Omega. \tag{2.5}$$

Then, considering an increasing sequence of smooth cut-off functions φ_k, with $0 \le \varphi_k \le 1$ and $\sup_k \varphi_k(x) = 1$ in Ω, we obtain

$$\lim_{n\to+\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))\,dx \geq \liminf_{n\to+\infty} \int_\Omega \varphi_k(x)\, f(x,u_n(x),\nabla u_n(x))\,dx$$
$$= \int_\Omega \varphi_k(x)\,d\mu(x) \geq \int_\Omega \varphi_k(x)\mu_a(x)\,dx$$
$$\geq \int_\Omega \varphi_k(x)\, f(x,u(x),\nabla u(x))\,dx.$$

Letting $k \to +\infty$, the result follows now from the Monotone Convergence Theorem. The rest of this section is dedicated to proving claim (2.5).

<u>Step 2</u>.*(blow-up)* We use a blow-up argument in connection with (2.2) to derive a lower bound for $\mu_a(x_0)$. Let x_0 be a Lebesgue point for u, ∇u and such that (2.2) and (2.4) hold and consider the affine functions

$$u_0(x) := u(x_0) + \nabla u(x_0)\, x \text{ and } w_0(x) := \nabla u(x_0)\, x.$$

We abbreviate $B := B(0, 1)$, and we consider a subdomain $B' \subset\subset B$. We claim that there exist sequences $r_n \to 0^+$ and $w_n \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^p)$ such that $w_n \to w_0$ in $L^1(B; \mathbb{R}^p)$ and

$$\mu_a(x_0) \geq \lim_{n\to+\infty} \frac{1}{\text{meas}(B)} \int_{B'} f(x_0+r_nx,u(x_0)+r_nw_n(x),\nabla w_n(x))\,dx. \tag{2.6}$$

Let $\varphi \in C_0(B)$ be a cut-off function such that $0 \leq \varphi \leq 1$ and $\varphi(x) = 1$ if $x \in B'$. By (2.4) we have

$$\mu_a(x_0) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N \text{meas}(B)} \mu(B(x_0,\varepsilon))$$

$$\geq \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^N \text{meas}(B)} \int_{B(x_0,\varepsilon)} \varphi\left(\frac{x - x_0}{\varepsilon}\right) d\mu(x)$$

$$= \limsup_{\varepsilon\to 0} \lim_{n\to+\infty} \frac{1}{\varepsilon^N \text{meas}(B)} \int_{B(x_0,\varepsilon)} \varphi\left(\frac{x - x_0}{\varepsilon}\right) f(x,u_n(x),\nabla u_n(x))\,dx$$

$$= \limsup_{\varepsilon\to 0} \lim_{n\to+\infty} \frac{1}{\text{meas}(B)} \int_B \varphi(x)\, f(x_0+\varepsilon x,u_n(x_0+\varepsilon x),\nabla u_n(x_0+\varepsilon x))\,dx$$

$$\geq \limsup_{\varepsilon\to 0} \limsup_{n\to+\infty} \frac{1}{\text{meas}(B)} \int_{B'} f(x_0+\varepsilon x,u(x_0)+\varepsilon w_{n,\varepsilon}(x),\nabla w_{n,\varepsilon}(x))\,dx \tag{2.7}$$

where

$$w_{n,\varepsilon}(x) := \frac{u_n(x_0+\varepsilon x) - u(x_0)}{\varepsilon}$$

$$= \frac{1}{\varepsilon} [u_n(x_0+\varepsilon x) - u_0(\varepsilon x)] + w_0(x).$$

By (2.2) and Hölder's inequality

$$\lim_{\varepsilon \to 0} \lim_{n \to +\infty} \|w_{n,\varepsilon} - w_0\|_{L^1(B)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_B |u(x_0+\varepsilon x) - u_0(\varepsilon x)| \, dx$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{N+1}} \int_{B(x_0,\varepsilon)} |u(x) - u(x_0) - \nabla u(x_0)(x - x_0)| dx$$

$$= 0.$$

Now (2.6) is obtained by a standard diagonalization argument. Indeed choose a sequence $r_k \to 0$ and choose n_k such that

$$\|w_{n_k,r_k} - w_0\|_{L^1(B)} < 1/k + \lim_{n \to +\infty} \|w_{n,r_k} - w_0\|_{L^1(B)}$$

and

$$\frac{1}{\text{meas}(B)} \int_{B'} f(x_0+r_k x, u(x_0)+r_k w_{n_k,r_k}(x), \nabla w_{n_k,r_k}(x)) \, dx \leq$$

$$1/k + \limsup_{n \to +\infty} \frac{1}{\text{meas}(B)} \int_{B'} f(x_0+r_k x, u(x_0)+r_k w_{n,r_k}(x), \nabla w_{n,r_k}(x)) \, dx.$$

Letting

$$w_k := w_{n_k,r_k}$$

(2.6) follows from (2.7) (one may choose a further subsequence to ensure that the limit on the right hand side of (2.6) exists).

<u>Step 3</u>.*(truncation)* We show that the sequence w_n constructed in Step 2 can be replaced by a uniformly convergent sequence. More precisely, we claim that if $g(x_0, u(x_0)) > 0$ then there exists a sequence $\tilde{w}_n \in W^{1,\infty}_{loc}(\mathbb{R}^N; \mathbb{R}^p)$ such that $\|\tilde{w}_n\|_{1,1,B'} \leq$ Const., $\tilde{w}_n \to w_0$ in $L^\infty(B; \mathbb{R}^p)$ and

$$\mu_a(x_0) \geq \lim_n \frac{1}{\text{meas}(B)} \int_{B'} f(x_0+r_n x, u(x_0)+r_n \tilde{w}_n(x), \nabla \tilde{w}_n(x)) \, dx. \tag{2.8}$$

Let $0 < s < t < 1$ and let $\varphi_{s,t}$ be a cut-off function such that $0 \leq \varphi_{s,t} \leq 1$, $\varphi_{s,t}(\tau) = 1$ if $\tau \leq s$, $\varphi_{s,t}(\tau) = 0$ if $\tau \geq t$, $\|\varphi'_{s,t}\|_\infty \leq C(t - s)^{-1}$. Set

$$\phi^n_{s,t}(x) := \varphi_{s,t}(|w_n(x) - w_0(x)|)$$

and

$$w^n_{s,t}(x) := w_0(x) + \varphi_{s,t}(|w_n(x) - w_0(x)|) \, (w_n(x) - w_0(x)).$$

Clearly

$$\|w^n_{s,t} - w_0\|_\infty \leq t. \tag{2.9}$$

Define

$$h_n(x, s, A) := f(x_0+r_nx, u(x_0)+r_ns, A)$$

and let $L = \|w_0\|_{L^\infty(B)} + 1$. By (H3) and as $g(x_0, u(x_0)) > 0$, g continuous, there exists n_0 such that for all $n \geq n_0$, $|s| \leq L$

$$C (\|A\| + 1) \geq h_n(x, s, A) \geq c\|A\| - C \tag{2.10}$$

for some c, $C > 0$. Also

$$\int_{B'} h_n(x,w^n_{s,t}(x),\nabla w^n_{s,t}(x))\, dx = \int_{B'\cap\{|w_n(x)-w_0(x)|\leq s\}} h_n(x,w_n(x),\nabla w_n(x))\, dx +$$

$$+ \int_{B'\cap\{s<|w_n(x)-w_0(x)|\leq t\}} h_n(x,w^n_{s,t}(x),\nabla w^n_{s,t}(x))\, dx +$$

$$+ \int_{B'\cap\{|w_n(x)-w_0(x)|>t\}} h_n(x,w_0(x),\nabla w_0(x))\, dx, \tag{2.11}$$

and by (2.10) we have

$$-C \leq h_n(x, w_0(x), \nabla w_0(x)) \leq C$$

which implies that

$$\int_{B'\cap\{|w_n(x)-w_0(x)|>t\}} h_n(x,w_0(x),\nabla w_0(x))\, dx \leq C\, \mathrm{meas}\{x \in B \mid |w_n(x) - w_0(x)| > t\}. \tag{2.12}$$

On the other hand, if $s < |w_n(x) - w_0(x)| < t$ then

$$\nabla w^n_{s,t}(x) = \nabla u(x_0) + \varphi_{s,t}(|w_n(x) - w_0(x)|)\, (\nabla w_n(x) - \nabla u(x_0)) +$$

$$+ (w_n(x) - w_0(x))\otimes\varphi'_{s,t}(|w_n(x) - w_0(x)|)\nabla|w_n(x) - w_0(x)|$$

thus, by (2.10), we have

$$\int_{B'\cap\{s<|w_n(x)-w_0(x)|\leq t\}} h_n(x,w^n_{s,t}(x),\nabla w^n_{s,t}(x))\, dx \leq$$

$$\leq C \int_{\{s<|w_n(x)-w_0(x)|\leq t\}} (1 + \|\nabla w_n(x) - \nabla u(x_0)\|)\, dx$$

$$+ C\frac{1}{t - s} \int_{B'\cap\{s<|w_n(x)-w_0(x)|\le t\}} |w_n(x) - w_0(x)|\, |\nabla|w_n(x) - w_0(x)|\,|\, dx. \tag{2.13}$$

We remark that for almost all t we have

$$\lim_{s\to t-} \int_{\{s<|w_n(x)-w_0(x)|\le t\}} (1 + \|\nabla w_n(x) - \nabla u(x_0)\|)\, dx = 0 \qquad (2.14)_1$$

and by the change of variables formula (2.3)

$$\lim_{s\to t-} \frac{1}{t - s} \int_{B'\cap\{s<|w_n(x)-w_0(x)|\le t\}} |w_n(x) - w_0(x)|\, |\nabla|w_n(x) - w_0(x)|\,|\, dx \le$$

$$\le t\, H_{N-1}\{x \in B' \mid |w_n(x) - w_0(x)| = t\}. \qquad (2.14)_2$$

Due to (2.10),

$$\int_{B'\cap\{|w_n(x)-w_0(x)|\le 1\}} |\nabla|w_n(x) - w_0(x)|\,|\, dx \le \int_{B'\cap\{|w_n(x)-w_0(x)|\le 1\}} (\|\nabla w_n(x)\| + C)\, dx$$

$$\le C \int_{B'} [h_n(x,w_n(x),\nabla w_n(x)) + 1]\, dx \le \text{Const.}$$

since the latter sequence is convergent. Hence, by Lemma 2.8 there exists $t_n \in [\|w_n - w_0\|_{L^1}^{1/2},\ \|w_n - w_0\|_{L^1}^{1/3}]$ such that (2.14) holds (with $t = t_n$) and

$$t_n\ H_{N-1}\{x \in B' \mid |w_n(x) - w_0(x)| = t_n\} \le \frac{\text{Const.}}{\ln\|w_n - w_0\|_{L^1}^{-1/6}}.$$

According to (2.14) choose $0 < s_n < t_n$ such that

$$\int_{\{s_n<|w_n(x)-w_0(x)|\le t_n\}} (1 + |\nabla w_n(x) - \nabla u(x_0)|)\, dx = O(1/n),$$

$$\frac{1}{t_n - s_n} \int_{B'\cap\{s_n<|w_n(x)-w_0(x)|\le t_n\}} |w_n(x) - w_0(x)|\, \|\nabla|w_n(x) - w_0(x)|\,\|\, dx \le$$

$$\le t_n\, H_{N-1}\{x \in \Omega \mid |w_n(x) - w_0(x)| = t_n\} + O(1/n)$$

and set

$$\tilde{w}_n(x) := w^n_{s_n,t_n}(x).$$

By (2.9)

$$\|\tilde{w}_n - w_0\|_\infty \le t_n \to 0$$

and by (2.6), (2.11)-(2.14) we conclude that

$$\mu_a(x_0) \geq \lim{}_n \frac{1}{\text{meas}(B)} \int_{B'} f(x_0+r_n x, u(x_0)+r_n w_n(x), \nabla w_n(x))\, dx$$

$$\geq \liminf_n \frac{1}{\text{meas}(B)} \int_{B' \cap \{|w_n(x)-w_0(x)| \leq s\}} h_n(x, w_n(x), \nabla w_n(x))\, dx$$

$$\geq \liminf_n \frac{1}{\text{meas}(B)} \Big\{ \int_{B'} h_n(x, \tilde{w}_n(x), \nabla \tilde{w}_n(x))\, dx$$

$$- O(1/n) - \frac{C}{\ln \|w_n - w_0\|_{L^1}^{-1/6}} - C\, \text{meas}\{x \in B \mid |w_n(x) - w_0(x)| > t_n\} \Big\}$$

$$= \liminf_n \frac{1}{\text{meas}(B)} \int_{B'} h_n(x, \tilde{w}_n(x), \nabla \tilde{w}_n(x))\, dx,$$

since $t_n \geq \|w_n - w_0\|_{L^1}^{1/2}$ and thus

$$\text{meas}\{x \in B \mid |w_n(x) - w_0(x)| > t_n\} \leq \frac{1}{t_n} \|w_n - w_0\|_{L^1} \leq \|w_n - w_0\|_{L^1}^{1/2} \to 0.$$

Finally the bound on $\|\nabla \tilde{w}_n\|_{L^1(B')}$ follows from (2.10).

<u>Step 4</u>.*(Proof of claim (2.5))*. We want to show that

$$\mu_a(x_0) \geq f(x_0, u(x_0), \nabla u(x_0)) \quad \text{for a. e. } x_0 \in \Omega.$$

Let x_0 be a Lebesgue point for u, ∇u and such that (2.2) and (2.4) hold. If $g(x_0, u(x_0)) = 0$ then (2.5) is satisfied trivially as f is a nonnegative function. If $g(x_0, u(x_0)) > 0$ consider a subdomain $B' \subset\subset B$ and let $\varepsilon > 0$. By (2.8) and (H4) we have

$$\mu_a(x_0) \geq \lim{}_n \frac{1}{\text{meas}(B)} \int_{B'} f(x_0+r_n x, u(x_0)+r_n \tilde{w}_n(x), \nabla \tilde{w}_n(x))\, dx$$

$$\geq \lim{}_n \frac{1}{\text{meas}(B)} \Big\{ \int_{B'} f(x_0, u(x_0), \nabla \tilde{w}_n(x))\, dx - \varepsilon \int_{B'} (1 + \|\nabla \tilde{w}_n(x)\|)\, dx \Big\}.$$

By Proposition 2.7 and taking into account that $\{\nabla \tilde{w}_n\}$ is a sequence bounded in L^1 we deduce that

$$\mu_a(x_0) \geq \frac{1}{\text{meas}(B)} \int_{B'} f(x_0, u(x_0), \nabla u(x_0))\, dx - \varepsilon C.$$

Letting $\varepsilon \to 0$, we conclude (2.5) given the arbitrariness of B'.

■

3. Proofs of auxiliary results.

In this section we prove Propositions 2.6 and 2.7. We first recall

Proposition 2.6. (i) If Theorem 2.3 holds true for Ω being a ball it holds true for all open, bounded sets Ω.

(ii) Let Ω be a ball. If (H1) and (H3) hold and if if $u_n, u \in W^{1,1}(\Omega; \mathbb{R}^p)$ are such that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^p)$ then there exists $\tilde{u}_n \in C_0^\infty(\mathbb{R}^N; \mathbb{R}^p)$ such that $\|\tilde{u}_n - u\|_{L^1(\Omega)} \to 0$ and

$$\liminf_{n \to +\infty} \int_\Omega f(x,\tilde{u}_n(x),\nabla\tilde{u}_n(x))\, dx = \liminf_{n \to +\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))\, dx.$$

Proof. (i) As in Acerbi and Fusco [AF], we show that it suffices to prove Theorem 2.3 in the case where Ω is a ball. Indeed, if the result was true whenever the domain is a ball, for an arbitrary open set Ω and using Vitali's Covering Theorem we can write

$$\Omega = \cup\, (a_i + \varepsilon_i\, B(0, 1)) \cup E$$

where $\text{meas}(E) = 0$ and $\{a_i + \varepsilon_i\, B(0, 1)\}$ is a family of mutually disjoint balls. Fixing a positive integer k we have

$$\begin{aligned}\liminf_n \int_\Omega f(x,u_n(x),\nabla u_n(x))\, dx &\geq \\ &\geq \sum_{i=1}^{k} \liminf_n \int_{a_i+\varepsilon_i B(0,1)} f(x,u_n(x),\nabla u_n(x))\, dx \\ &\geq \sum_{i=1}^{k} \int_{a_i+\varepsilon_i B(0,1)} f(x,u(x),\nabla u(x))\, dx.\end{aligned}$$

Letting $k \to +\infty$ and using the Monotone Convergence Theorem we conclude that

$$\int_\Omega f(x,u(x),\nabla u(x))\, dx \leq \liminf \int_\Omega f(x,u_n(x),\nabla u_n(x))\, dx.$$

(ii) As in Acerbi and Fusco [AF], we remark that we can extend $u_n \in W^{1,1}(\Omega; \mathbb{R}^p)$ to $u_n^* \in W^{1,1}(\mathbb{R}^N; \mathbb{R}^p)$. Moreover, as $C_0^\infty(\mathbb{R}^N; \mathbb{R}^p)$ is dense in $W^{1,1}(\mathbb{R}^N; \mathbb{R}^p)$ there exist sequences $v_{n,k} \in C_0^\infty(\mathbb{R}^N; \mathbb{R}^p)$ such that

$$v_{n,k} \to u_n^* \text{ in } W^{1,1}(\mathbb{R}^N; \mathbb{R}^p). \tag{3.1}$$

as $k \to +\infty$. Moreover, we may assume that $v_{n,k}$ and $\nabla v_{n,k}$ converge to u_n and ∇u_n, respectively, almost everywhere. We claim that

$$\lim_k \int_\Omega f(x,v_{n,k}(x),\nabla v_{n,k}(x))\,dx = \int_\Omega f(x,u_n(x),\nabla u_n(x))\,dx. \tag{3.2}$$

Indeed, by (H3)

$$0 \le f(x, u, A) \le C(1 + \|A\|)$$

and thus by applying Fatou's Lemma to $x \to f(x, v_{n,k}(x),\nabla v_{n,k}(x))$ and $C(1 + \|\nabla v_{n,k}(x)\|) - f(x, v_{n,k}(x),\nabla v_{n,k}(x))$ and by observing that

$$\int_\Omega (1 + \|\nabla v_{n,k}(x)\|)\,dx \to \int_\Omega (1 + \|\nabla u_n(x)\|)\,dx$$

one has (3.2). Finally, using (3.1) and (3.2) for all n choose k_n such that

$$\|v_{n,k_n} - u_n\|_{L^1} \le 1/n$$

and

$$\left| \int_\Omega f(x,v_{n,k_n}(x),\nabla v_{n,k_n}(x))\,dx - \int_\Omega f(x,u_n(x),\nabla u_n(x))\,dx \right| \le 1/n.$$

It is clear that, setting

$$\tilde{u}_n := v_{n,k_n},$$

one has

$$\|\tilde{u}_n - u\|_{L^1(\Omega)} \to 0$$

and

$$\lim_n \int_\Omega f(x,\tilde{u}_n(x),\nabla \tilde{u}_n(x))\,dx = \lim_n \int_\Omega f(x,u_n(x),\nabla u_n(x))\,dx.$$

■

We next prove Theorem 2.3 in the special case where $f = f(A)$ and u is an affine function. The proof presented here was obtained in Fonseca [Fo] (see Theorem 4.6 and Remark 4.16) and we are now aware of the fact that Marcellini's [Ma] proof for the case of weak convergence in $W^{1,m}$, $m > 1$, is essentially the same. Yet another proof has been given by Kinderlehrer [K] who uses a subdivision of Ω in small domains in connection with the Vitali covering argument.

Proposition 2.7.
Let $f : M^{p\times N} \to \mathbb{R}$ be a function satisfying (H1), (H2) and

$$0 \le f(A) \le C(1 + \|A\|)$$

for some $C > 0$. If $A_0 \in M^{p\times N}$ and if $u_n \in W^{1,1}(\Omega; \mathbb{R}^p)$ are such that $u_n \to 0$ in $L^1(\Omega; \mathbb{R}^p)$ and $\{\|\nabla u_n\|_{L^1}\}$ is bounded then

$$\text{meas}(\Omega)\, f(A_0) \leq \liminf \int_\Omega f(A_0+\nabla u_n(x))\, dx.$$

Proof. The proof is taken from [Fo]. Related ideas appear in [DG] and [Ma]. We may assume without loss of generality that

$$\liminf \int_\Omega f(A_0+\nabla u_n(x))\, dx = \lim \int_\Omega f(A_0+\nabla u_n(x))\, dx) < +\infty.$$

Due to the growth condition, $\{\|\nabla u_n\|\}$ is bounded in L^1 and so there exists a subsequence and a finite measure μ in Ω such that

$$\|\nabla u_n\| \to \mu \text{ weakly } *,$$

i. e. for every $\varphi \in C_0(\Omega)$

$$\int_\Omega \varphi(x)\, \|\nabla u_n(x)\| dx \;\to\; \int_\Omega \varphi(x)\, d\mu(x). \tag{3.3}$$

Consider an increasing sequence of subdomains Ω_k such that $\overline{\Omega}_k \subset\subset \Omega$ and $\Omega = \cup\, \Omega_k$. Let φ^k be a smooth cut-off function such that $0 \leq \varphi^k \leq 1$, $\varphi^k = 1$ in Ω_k, $\varphi^k = 0$ in $\Omega \setminus \overline{\Omega}_{k+1}$. Setting

$$u_n^k := \varphi^k u_n \in W_0^{1,1}(\Omega; \mathbb{R}^p),$$

as f is quasiconvex we have

$$\begin{aligned} f(A_0)\, \text{meas}(\Omega) &\leq \int_\Omega f(A_0+\nabla u_n^k(x))\, dx \\ &= \int_{\Omega\setminus\Omega_{k+1}} f(A_0)\, dx + \int_{\Omega_{k+1}\setminus\Omega_k} f(A_0+\nabla u_n^k(x))\, dx + \int_{\Omega_k} f(A_0+\nabla u_n(x))\, dx \end{aligned}$$

which implies that

$$f(A_0)\, \text{meas}(\Omega_{k+1}) \leq \int_{\Omega_{k+1}\setminus\Omega_k} f(A_0+\nabla u_n^k(x))\, dx + \int_{\Omega_k} f(A_0+\nabla u_n(x))\, dx.$$

As f is nonnnegative, we deduce that

$$\int_\Omega f(A_0+\nabla u_n(x))\, dx - f(A_0)\, \text{meas}(\Omega_{k+1}) \geq - \int_{\Omega_{k+1}\setminus\Omega_k} f(A_0+\nabla u_n^k(x))\, dx. \tag{3.4}$$

On the other hand,

$$\int_{\Omega_{k+1}\setminus\Omega_k} f(A_0+\nabla u_n^k(x))\,dx \le C \int_{\Omega_{k+1}\setminus\Omega_k} (1 + \|A_0+\nabla u_n^k(x)\|)\,dx$$

$$\le C\,\mathrm{meas}(\Omega_{k+1}\setminus\Omega_k) + C \int_{\Omega_{k+1}\setminus\Omega_k} \|\nabla u_n(x)\|\,dx +$$

$$+ C \int_{\Omega_{k+1}\setminus\Omega_k} |u_n(x)|\,\|\nabla\varphi^k(x)\|\,dx$$

$$\le C\,\mathrm{meas}(\Omega_{k+1}\setminus\Omega_k) + C \int_\Omega (\varphi_{k+1}(x) - \varphi_{k-1}(x))\,\|\nabla u_n(x)\|\,dx$$

$$+ C \int_{\Omega_{k+1}\setminus\Omega_k} |u_n(x)|\,\|\nabla\varphi^k(x)\|\,dx.$$

As $u_n \to 0$ in $L^1(\Omega)$, by (3.3) and (3.4) we obtain

$$\lim_n \int_\Omega f(A_0+\nabla u_n(x))\,dx - f(A_0)\,\mathrm{meas}(\Omega_{k+1}) \ge -C\,\mathrm{meas}(\Omega_{k+1}\setminus\Omega_k) -$$

$$- C \int_\Omega (\varphi_{k+1}(x) - \varphi_{k-1}(x))\,d\mu(x).$$

Finally, summing the above inequality for $k = 2, \ldots, i$, we have

$$(i-1)\lim_n \int_\Omega f(A_0+\nabla u_n(x))\,dx - f(A_0)\sum_{k=2}^{i} \mathrm{meas}(\Omega_{k+1}) \ge$$

$$\ge -C \sum_{k=2}^{i} \left\{\mathrm{meas}(\Omega_{k+1}\setminus\Omega_k) + \int_\Omega (\varphi_{k+1}(x) - \varphi_{k-1}(x))\,d\mu(x)\right\}.$$

Divinding by $(i - 1)$ we find

$$\lim_n \int_\Omega f(A_0+\nabla u_n(x))\,dx - f(A_0)\frac{1}{i-1}\sum_{k=2}^{i} \mathrm{meas}(\Omega_{k+1}) \ge$$

$$\ge -C\,\frac{1}{i-1}\{\mathrm{meas}(\Omega_{i+1}) - \mathrm{meas}(\Omega_2) +$$

$$+ \int_\Omega (\varphi_{i+1}(x) + \varphi_i(x) - \varphi_2(x) - \varphi_1(x))\,d\mu(x)\}$$

$$\ge -C\,\frac{1}{i-1}\{\mathrm{meas}(\Omega) + 4\,\mu(\Omega)\}.$$

Letting $i \to +\infty$ we conclude that

$$\lim_n \int_\Omega f(A_0+\nabla u_n(x))\,dx - f(A_0)\,\mathrm{meas}(\Omega) \ge 0.$$

■

4. Lower semicontinuity for convex integrands.

Suppose that $f : \Omega x \mathbb{R}^p x M^{pxN} \to [0, +\infty)$ satisfies the hypotheses:

(H1) f is continuous ;

(H2') f(x,u,.) is convex ;

(H3) there exists a nonnegative, bounded, continuous function $g : \Omega x \mathbb{R}^p \to [0, +\infty)$, c, C > 0 such that

$$cg(x, u)\|A\| - C \leq f(x, u, A) \leq Cg(x, u) (1 + \|A\|)$$

for all $(x, u, A) \in \Omega x \mathbb{R}^p x M^{pxN}$;

(H4') for all $x_0 \in \Omega x \mathbb{R}^p$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies that

$$|f(x_0, u, A) - f(x, u, A)| \leq \varepsilon (1 + \|A\|).$$

We obtain the following corollary of Theorem 2.3.

Corollary 4.1.

If the assumptions (H1), (H2'), (H3) and (H4') hold and if $u_n, u \in W^{1,1}(\Omega; \mathbb{R}^p)$ are such that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^p)$ then

$$\int_\Omega f(x,u(x),\nabla u(x))\, dx \leq \liminf \int_\Omega f(x,u_n(x),\nabla u_n(x))\, dx.$$

Clearly, in order to apply Theorem 2.3 it suffices to prove that for convex integrands with linear growth (H4') reduces to (H4).

Proposition 4.2.

If f satisfies (H1), (H2') and (H3) then for all $(x_0, u_0) \in \Omega x \mathbb{R}^p$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|u - u_0| < \delta \text{ implies that } f(x_0, u, A) - f(x_0, u_0, A) \geq -\varepsilon (1 + \|A\|).$$

We introduce the *recession function* f^∞ given by

$$f^\infty(x, u, A) := \sup_{t>0} \frac{f(x, u, tA) - f(x, u, 0)}{t}.$$

Note that, for fixed $(x, u, A) \in \Omega x \mathbb{R}^p x M^{pxN}$ and g given by $g(t) := f(x, u, tA) - f(x, u, 0)$, g is a convex function with $g(0) = 0$ and so

$$t \to g(t)/t \text{ is increasing.} \qquad (4.1)_1$$

Therefore

$$f^\infty(x, u, A) = \sup_{t>0} g(t)/t$$

$$= \lim \frac{f(x, u, tA)}{t} \quad \text{as } t \to +\infty. \qquad (4.1)_2$$

If (H1) and (H3) hold and if f(x, u, .) is convex then $f^\infty(x, u, .)$ is convex (and hence continuous), homogeneous of degree one and (see e. g. Fonseca and Rybka [FR], Lemma 2.3)

$$0 \leq f^\infty(x, u, A) \leq Cg(x, u) \|A\|$$

for all $(x, u, A) \in \Omega x \mathbb{R}^p x M^{pxN}$.

The proof of this result is based on the following auxiliary lemmas, where for notational convenience we omit the dependence of f on the variable x.

Lemma 4.3.
If (H2') and (H3) hold then for all $u \in \mathbb{R}^p$

$$\lim_{r \to +\infty} \sup_{\|A\| = 1} \left| \frac{f(u, rA)}{r} - f^\infty(u, A) \right| = 0.$$

Proof. *Step 1*. Fix $u \in \mathbb{R}^p$, assume that $f(u, 0) = 0$ and that Lemma 4.3 fails. Then there exist $\varepsilon > 0$, $r_n \to +\infty$, $A_n \to A$ with $\|A_n\| = 1 = \|A\|$ such that, by (4.1),

$$f^\infty(u, A_n) - \frac{f(u, r_n A_n)}{r_n} = \left| \frac{f(u, r_n A_n)}{r_n} - f^\infty(u, A_n) \right| > \varepsilon.$$

for all n. Using the convexity of f at $r_n A$ we have

$$f^\infty(u, A_n) > \frac{f(u, r_n A_n)}{r_n} + \varepsilon$$

$$\geq \frac{f(u, r_n A)}{r_n} + L_n.(A_n - A) + \varepsilon$$

where, by (H3) and Proposition 2.5, $\{L_n\}$ is a bounded sequence of matrices. Letting $n \to +\infty$ and due to the continuity of $f^\infty(u, .)$ we obtain a contradiction, namely

$$f^\infty(u, A) \geq f^\infty(u, A) + \varepsilon.$$

Step 2. In the general case, we set $g(u, A) := f(u, A) - f(u, 0)$. It is clear that the argument in Step 1 applies to g and that $f^\infty(u, A) = g^\infty(u, A)$ and so

$$0 = \lim_{r \to +\infty} \sup_{\|A\| = 1} \left| \frac{g(u, rA)}{r} - g^\infty(u, A) \right|$$

$$= \lim_{r \to +\infty} \sup_{\|A\| = 1} \left| \frac{f(u, rA)}{r} - f^{\infty}(u, A) \right|.$$

■

Lemma 4.4.
If (H1), (H2') and (H3) hold, for all $u_0 \in \mathbb{R}^p$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|u - u_0| < \delta \text{ implies that } f^{\infty}(u, A) - f^{\infty}(u_0, A) \geq -\varepsilon$$

for all matrices $A \in M^{p\times N}$ such that $\|A\| = 1$.

Proof. *Step 1*. Assume that $f(u, 0) = 0$ and fix $u_0 \in \mathbb{R}^p$ and $\varepsilon > 0$. By (4.1) and by Lemma 4.3 we may choose $r_0 > 2$ such that

$$0 \leq f^{\infty}(u_0, A) - \frac{f(u_0, r_0A)}{r_0} < \varepsilon/2$$

for every A with $\|A\| = 1$. On the other hand, as f is continuous there exists $\delta > 0$ (depending only on ε and r_0) such that

$$|u - u_0| < \delta \quad \text{implies} \quad \sup_{\|A\| = 1} |f(u, r_0A) - f(u_0, r_0A)| < \varepsilon.$$

By (4.1) we have

$$\begin{aligned} f^{\infty}(u, A) &\geq \frac{f(u, r_0A)}{r_0} \\ &\geq \frac{f(u_0, r_0A)}{r_0} - \varepsilon/r_0 \\ &\geq f^{\infty}(u_0, A) - \varepsilon/2 - \varepsilon/r_0 \\ &\geq f^{\infty}(u_0, A) - \varepsilon. \end{aligned}$$

Step 2. As in the proof of the previous lemma, we set $g(u, A) := f(u, A) - f(u, 0)$ and we apply Step 1. The result follows from the fact that $f^{\infty}(u, A) = g^{\infty}(u, A)$.

■

Proof of Proposition 4.2. *Step 1*. Assume that $f(u, 0) = 0$ and fix $u_0 \in \mathbb{R}^p$, $\varepsilon > 0$. By (4.1), Lemma 4.3 and by continuity choose $r_0 > 2$, $\delta > 0$ such that

$$0 \leq f^\infty(u_0, A) - \frac{f(u_0, r_0A)}{r_0} < \varepsilon/2$$

for every A with $\|A\| = 1$, and

$$|u - u_0| < \delta \text{ implies } \sup_{\|A\| \leq 1} |f(u, r_0A) - f(u_0, r_0A)| < \varepsilon.$$

Thus, if $|u - u_0| < \delta$ and if $\|A\| \leq r_0$ we have

$$\begin{aligned} f(u, A) &\geq f(u_0, A) - \varepsilon \\ &\geq f(u_0, A) - \varepsilon(1 + \|A\|) \end{aligned} \qquad (4.2)$$

and by (4.1) if $A = rB$, $\|B\| = 1$, $r > r_0$ then

$$\begin{aligned} \frac{f(u, A)}{\|A\|} = \frac{f(u, rB)}{r} &\geq \frac{f(u, r_0B)}{r_0} \\ &\geq \frac{f(u_0, r_0B)}{r_0} - \varepsilon/r_0 \\ &\geq f^\infty(u_0, B) - \varepsilon/2 - \varepsilon/r_0 \\ &\geq f^\infty(u_0, B) - \varepsilon. \end{aligned}$$

Finally, as $f^\infty(u, .)$ is homogeneous of degree one, by (4.1) we deduce that

$$\begin{aligned} f(u, A) &\geq f^\infty(u_0, A) - \varepsilon \|A\| \\ &\geq f(u_0, A) - \varepsilon \|A\| \end{aligned}$$

which, together with (4.2) yields the result.

Step 2. In the general case we apply Step 1 to the function $g(u, A) := f(u, A) - f(u, 0)$ in order to find $\delta > 0$ such that

$$|f(u, 0) - f(u_0, 0)| < \varepsilon/2 \text{ and } g(u, A) \geq g(u_0, A) - \frac{\varepsilon}{2}(1 + \|A\|)$$

whenever $|u - u_0| < \delta$. Hence

$$\begin{aligned} f(u, A) &\geq f(u, 0) + f(u_0, A) - f(u_0, 0) - \frac{\varepsilon}{2}(1 + \|A\|) \\ &\geq f(u_0, A) - \varepsilon/2 - \frac{\varepsilon}{2}(1 + \|A\|) \end{aligned}$$

$$\geq f(u_0, A) - \varepsilon(1 + \|A\|).$$

■

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