A LOCAL TRANSLATION OF UNTYPED λ CALCULUS INTO SIMPLY TYPED λ CALCULUS

by

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INTRODUCTION:

There are several ideas behind what follows. The first is Dana Scott's idea that untyped λ Calculus = unityped λ Calculus = simple typed λ Calculus with a retract pair $\Phi \in 0 \longrightarrow (0 \longrightarrow 0)$, $\Psi \in (0 \longrightarrow 0) \longrightarrow 0$ and $\Phi \circ \Psi = I$. The second idea is that it is perfectly possible to have a type structure where everything is a function without $D = D \longrightarrow D$. In addition, the 1st such structure that comes to mind has a wonderful automorphism group (known to us from a representation of the Freyd–Heller group) which permits translation of local properties from the *simple* type structure it contains. Finally, the automorphisms of the type structure and Scott's retracts, defined at higher types, share a functional equation which permits the transfer of local properties of the type structure to the untyped case; namely,

$$\alpha(\mathscr{X})\ \mathscr{Y} = \alpha(\mathscr{X}\ (\alpha^{-1}\ \mathscr{Y})).$$

These ideas will lead us to a method for translating local properties of the simple typed λ calculus such as FD! and the Church–Rosser property. The ideas and results below provide an alternative to Levy's labeled λ–calculus.

λ^T

The type algebra **T** is generated from 0, 1, 2, ... , n, ... (the generators) by the operations

$$\longrightarrow : \mathbf{T} \times \mathbf{T} \longrightarrow \mathbf{T}$$
$$d_0 : \mathbf{T} \longrightarrow \mathbf{T}$$
$$d_1 : \mathbf{T} \longrightarrow \mathbf{T}$$

satisfying the relations

$$d_0\,(\sigma \longrightarrow \tau) = \sigma$$
$$d_1\,(\sigma \longrightarrow \tau) = \tau$$
$$(d_0\sigma) \longrightarrow (d_1\,\sigma) = \sigma.$$

$\longrightarrow$ represents the function space construction, d_0 the domain operation, and d_1 the codomain operation. **T** is obviously the free "surjective pairing" algebra on a countable number of generators.

Each expression of **T** rewrites to a unique normal from having one of the shapes

$$\sigma \longrightarrow \tau$$
$$d_{i_1} \dots d_{i_k}\ m$$

where σ and τ are themselves normal. The depth of an occurrence of a generator m in one of these forms is defined by

$$\text{depth}\ (d_{i_1} \dots d_{i_k} [m]) = -k$$
$$\text{depth}\ (\sigma\,[m] \longrightarrow \tau) = 1 + \text{depth}\ (\sigma\,[m])$$
$$\text{depth}\ (\sigma \longrightarrow \tau\,[m]) = 1 + \text{depth}\ (\tau\,[m])$$

Among the members of **T** are the simple types. These are generated from the generators by $\longrightarrow$ alone. **S** is the algebra of simple types.

Aut (**T**) is the automorphism group of **T**. Each member of Aut (**T**) is completely determined by its action on the generators.

$\lambda^{\mathbf{T}}$ is the λ calculus with Church types from **T**. This is well defined since $\sigma_0 \longrightarrow \tau_0 = \sigma_1 \longrightarrow \tau_1 \Rightarrow \sigma_0 = \sigma_1 \wedge \tau_0 = \tau_1$. $\lambda^{\mathbf{T}}$ has two variants $\lambda^{\mathbf{T}}_{\beta}$ and $\lambda^{\mathbf{T}}_{\beta\eta}$.

Aut (**T**) acts on $\lambda^{\mathbf{T}}$ as follows. Given $\gamma \in$ Aut (**T**)

$$\begin{aligned} \gamma(x^{\sigma}) &= x^{\gamma(\sigma)} \\ \gamma(\mathscr{X}\ \mathscr{Y}) &= \gamma(\mathscr{X})\ \gamma(\mathscr{Y}) \\ \gamma(\lambda\, x\, \mathscr{X}) &= \lambda \gamma(x)\ \gamma(\mathscr{X}) \end{aligned}$$

Clearly $\mathscr{X} \xrightarrow[\beta(\eta)]{\Delta} \mathscr{Y} \Leftrightarrow \gamma(\mathscr{X}) \xrightarrow[\beta(\eta)]{\gamma(\Delta)} \gamma(\mathscr{Y})$

so Aut (**T**) acts on $\lambda^{\mathbf{T}}_{\beta(\eta)}$. Note that $\gamma \in$ Aut (**T**) satisfies the functional equation

$$\gamma(\mathscr{X})\ \mathscr{Y} = \gamma(\mathscr{X}\ (\gamma^{-1}\ \mathscr{Y}))$$

which expresses that γ is "self conjugate."

We shall now define an extension of $\lambda^{\mathbf{T}}$, $\lambda\Gamma^{\mathbf{T}}$. $\lambda\Gamma^{\mathbf{T}}$ has new constants $\Gamma^{\gamma(\sigma)}_{\sigma} \in \sigma \longrightarrow \gamma(\sigma)$ for each $\sigma \in \mathbf{T}$ and $\gamma \in$ Aut (**T**)

with rewrites

$$\Gamma^{\gamma(\sigma)}_{\sigma}\ x\ y \longrightarrow \Gamma^{d_1\gamma(\sigma)}_{d_1\sigma}\ (x(\ \Gamma^{d_0\sigma}_{d_0\gamma(\sigma)}\ y)) \quad \beta \text{ form}$$

$$\Gamma^{\gamma(\sigma)}_{\sigma} \longrightarrow \lambda xy\ \Gamma^{d_1\gamma(\sigma)}_{d_1\sigma}\ (x\ (\ \Gamma^{d_0\sigma}_{d_0\gamma(\sigma)}\ y)) \quad \beta\eta \text{ form.}$$

The action of Aut (T) extends to $\lambda\Gamma^{\mathbf{T}}$ by conjugation

$$\gamma_1 (\Gamma_{\sigma}^{\gamma_0(\sigma)}) = \Gamma_{\gamma_1(\sigma)}^{\gamma_1 \circ \gamma_0 \circ \gamma_1^{-1}(\gamma_1(\sigma))}$$

and to $\lambda\Gamma^{\mathbf{T}}_{\beta(\eta)}$; as before,

$$\mathscr{X} \xrightarrow[\beta(\eta)\Gamma]{\Delta} \mathscr{Y} \Leftrightarrow \gamma(\mathscr{X}) \xrightarrow[\beta(\eta)\Gamma]{\gamma(\Delta)} \gamma(\mathscr{Y})$$

$\mathscr{X} \in \lambda\Gamma^{\mathbf{T}}$ is said to be well put together (w.p.t.) if each occurrence of Γ in $\mathscr{X}$ is in function position. Clearly w.p.t. is closed under $\xrightarrow[\beta\Gamma]{}$. Define [ ]: w.p.t. $\longrightarrow$ **N** by

$$[x] = 1$$

$$[\mathscr{X}\,\mathscr{Y}] = [\mathscr{X}]^2 + [\mathscr{Y}] \quad \mathscr{X} \not\equiv \Gamma$$

$$= \quad 1 + [\mathscr{Y}] \quad \mathscr{X} \equiv \Gamma$$

$$[\lambda x.\mathscr{X}] = [\mathscr{X}]$$

PROPOSITION 1: Let $\mathscr{X}$ be well put together then every $\xrightarrow[\Gamma]{}\!\!>>$ reduction sequence beginning with $\mathscr{X}$ terminates in a unique $\xrightarrow[\Gamma]{}\!\!>>$ normal form.

PROOF: First we calculate $[\Gamma\ \mathscr{X}\ \mathscr{Y}] = ([\mathscr{X}] + 1)^2 + [\mathscr{Y}] = [\mathscr{X}]^2 + 2[\mathscr{X}] + [\mathscr{Y}] + 1 > [\mathscr{X}]^2 + [\mathscr{Y}] + 2 = [\Gamma\ (\mathscr{H}\ (\Gamma\ \mathscr{Y}))]$. Thus any Γ reduction beginning with $\mathscr{X}$ terminates in at most $[\mathscr{X}]$ steps. To complete the proof observe that $\xrightarrow[\Gamma]{}\!\!>$ satisfies the diamond property.

It is easily seen by induction on $\xrightarrow[\Gamma]{}\!\!>>$ that if

$$\begin{array}{ccc} \mathscr{W} & \xrightarrow[\beta]{\Delta} & \mathscr{X} \\ {\scriptstyle\Gamma}\downarrow\!\!\downarrow & & \\ \mathscr{Y} & & \end{array}$$

then there exists a unique residual of Δ in $\mathscr{Y}$, Δ', and a $\mathscr{Z}$ s.t.

$$\begin{array}{ccc} & & \mathscr{X} \\ & & \downarrow\!\!\downarrow {\scriptstyle\Gamma} \\ \mathscr{Y} & \xrightarrow[\beta]{\Delta'} & \mathscr{Z} \end{array}$$

A reduction sequence

$$\mathscr{X}_0 \xrightarrow[\Gamma]{} \!\!> \mathscr{Y}_0 \xrightarrow[\beta]{} \mathscr{X}_1 \xrightarrow[\Gamma]{} \!\!> \cdots \xrightarrow[\beta]{} \mathscr{X}_n \xrightarrow[\Gamma]{} \!\!> \mathscr{Y}_n$$

is said to be Γ complete if each $\mathscr{Y}_i$ is $\xrightarrow[\Gamma]{} \!\!>$ normal.

PROPOSITION 2: Suppose $\mathscr{X}$ is well put together and $\mathscr{X} \xrightarrow[\beta\Gamma]{} \!\!> \mathscr{Y}$ then there is a Γ complete reduction to the $\xrightarrow[\Gamma]{} \!\!>$ normal of $\mathscr{Y}$ from $\mathscr{X}$.

PROOF: By induction on the length of the reduction $\mathscr{X} \xrightarrow[\beta\Gamma]{} \!\!> \mathscr{Y}$. We distinguish two cases.

CASE 1: $\mathscr{X} \xrightarrow[\beta\Gamma]{} \!\!> \mathscr{Z} \xrightarrow[\Gamma]{} \mathscr{Y}$. Immediate by induction hypothesis and proposition 1.

CASE 2: $\mathscr{X} \xrightarrow[\beta\Gamma]{} \!\!> \mathscr{Z} \xrightarrow[\beta]{\Delta} \mathscr{Y}$. By induction hypothesis there is a Γ complete reduction from $\mathscr{X}$ to the Γ normal form of $\mathscr{Z}$, say $\mathscr{W}$. Now $\mathscr{W}$ has a unique residual Δ' of Δ and

Thus the desired reduction is $\mathscr{X} \underset{\text{complete}}{\longrightarrow\!\!\!\rightarrow} \mathscr{W} \xrightarrow{\Delta'} \mathscr{U} \xrightarrow[\Gamma]{} \!\!> \text{normal form}$

$\lambda^{\mathbf{S}}$ is very familiar in both its β and $\beta\eta$ variants. We shall be particularly interested in two sorts of extensions of $\lambda^{\mathbf{S}}$.

The first sort of extension is definitional extension. These have additional constants F of various types and rewrites of the form

$$F\, x_1 \ldots x_t \longrightarrow \mathscr{X} \qquad \beta \text{ form}$$
$$F \longrightarrow \lambda x_1 \ldots x_t\, \mathscr{X} \qquad \beta\eta \text{ form}$$

where $\mathscr{X}$ can contain constants previously defined in a well founded way.

The second sort of extension is extension of the type structure by products.

We shall assume that the reader is familiar with the properties of these objects.

A subgroup $G \subseteq \text{Aut}\,(\mathbf{T})$ is said to be admissible if for each finite $\mathscr{F} \subseteq \mathbf{T}$ there exists $\gamma \in G$ such that $\sigma \in \mathscr{F} \Longrightarrow \gamma\,(\sigma) \in \mathbf{S}$. Obviously, Aut $(\mathbf{T})$ is itself admissible.

Consider now the definitional extension $\lambda\Gamma^{\mathbf{S}}$ obtained from $\lambda^{\mathbf{S}}$ by adding constants $\Gamma^{\tau}_{\sigma} \in \sigma \longrightarrow \tau$ for all σ and τ and rewrites

$$\Gamma^{\sigma_2 \longrightarrow \tau_2}_{\sigma_1 \longrightarrow \tau_1}\, xy \longrightarrow \Gamma^{\tau_2}_{\tau_1}\, (x(\Gamma^{\sigma_1}_{\sigma_2}\, y)) \qquad \beta \text{ form}$$

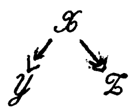
$$\Gamma^{\sigma_2 \longrightarrow \tau_2}_{\sigma_1 \longrightarrow \tau_1} \longrightarrow \lambda xy\, \Gamma^{\tau_2}_{\tau_1}\, (x(\, \Gamma^{\sigma_1}_{\sigma_2}\, y)) \qquad \beta\eta \text{ form}$$

Note that there are no rewrites for Γ^{τ}_{n} and Γ^{n}_{σ}.

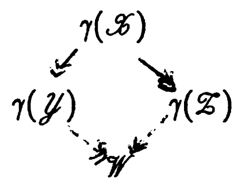
THEOREM 1: Any reduction diagram which can be completed in $\lambda\Gamma^{\mathbf{S}}_{\beta(\eta)}$ can be completed in $\lambda\Gamma^{\mathbf{T}}_{\beta(\eta)}$.

PROOF: Given the $\lambda\Gamma^{\mathbf{T}}_{\beta(\eta)}$ reduction diagram $\mathscr{D}$ let $\mathscr{S}$ be all the members of $\mathbf{T}$ appearing in $\mathscr{D}$. Since Aut $(\mathbf{T})$ is admissible there exists γ s.t. $\sigma \in \mathscr{S} \Longrightarrow \gamma(\sigma)$. Let $\mathscr{D}'$ be the completion of $\gamma(\mathscr{D})$ in $\lambda\Gamma^{\mathbf{S}}_{\beta(\eta)}$ Then the desired completion is $\gamma^{-1}(\mathscr{D}')$. Here we shall calculate 2 examples.

Church–Rosser:

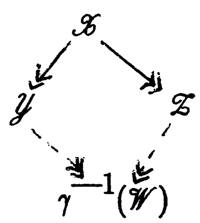
Given 

in $\lambda\Gamma^{\mathbf{T}}_{\beta(\eta)}$

pass to 

in $\lambda\Gamma^{\mathbf{S}}_{\beta(\eta)}$

which can be completed by the Church–Rosser theorem. This "translates" to the completed diagram.



in $\lambda\Gamma^{\mathbf{S}}_{\beta(\eta)}$

STANDARDIZATION:

Given $\mathscr{X} \longrightarrow\!\!>> \mathscr{Y}$ in $\lambda\Gamma^{\mathbf{T}}_{\beta(\eta)}$ we obtain $\gamma(\mathscr{X}) \longrightarrow\!\!>> \gamma(\mathscr{Y})$ in $\lambda\Gamma^{\mathbf{S}}_{\beta(\eta)}$. By the standardization theorem we have $\gamma(\mathscr{X}) \underset{\text{std.}}{\longrightarrow\!\!>} \gamma(\mathscr{Y})$ so translating yields $\mathscr{X} \underset{\text{std.}}{\longrightarrow\!\!>} \mathscr{Y}$, in $\lambda\Gamma^{\mathbf{T}}_{\beta(\eta)}$.

UNTYPED λ CALCULUS:

We shall now interpret the untyped λ calculus in $\lambda\Gamma^{\mathbf{T}}$. Let $\mathscr{X}$ be well put together. Then $|\mathscr{X}|$ is the result of erasing all types and occurrences of Γ in $\mathscr{X}$. For untyped $\mathscr{X}$ define $\mathscr{X}^{\mathbf{T}}$ by

$$x^{\mathbf{T}} = \Gamma^1_0\, x^0$$

$$(\mathscr{X}\mathscr{Y})^{\mathbf{T}} = \Gamma^{\ell \to n}_m\, \mathscr{X}^{\mathbf{T}}\, \mathscr{Y}^{\mathbf{T}} \text{ where } \mathscr{X}^{\mathbf{T}} \in m$$

$$\mathscr{Y}^{\mathbf{T}} \in \ell \text{ and } n \text{ is new}$$

$$(\lambda x \mathscr{X})^{\mathbf{T}} = \Gamma^n_{0 \to m}\, (\lambda x^0\, \mathscr{X}^{\mathbf{T}}) \text{ where}$$

$$\mathscr{X}^{\mathbf{T}} \in m \neq 0 \text{ and } n \text{ is new}$$

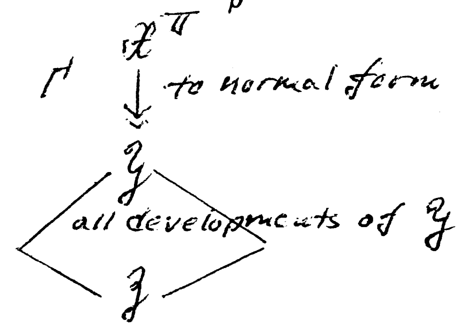
Note that $\mathscr{X}^{\mathbf{T}}$ is well defined and well put together and $|\mathscr{X}|^{\mathbf{T}} = \mathscr{X}$.

THEOREM 2: Any reduction diagram which can be completed in $\lambda\Gamma^{\mathbf{T}}_{\beta}$ can be completed in λ_β.

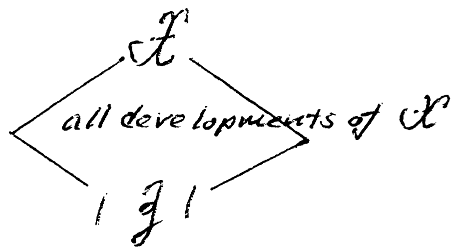
REMARK: The theorem fails for η.

PROOF: The proof is like the proof of theorem 1 except for the use of Γ complete reductions. We calculate two examples. FD!:

Given the untyped $\mathscr{X}$, we have in $\lambda\Gamma^{\mathbf{T}}_{\beta}$ by FD!



Translating this back to λ_β gives



since $|\mathscr{Y}| \equiv \mathscr{X}$

STANDARDIZATION:

We shall take a much longer route than is necessary in order to show the usefulness of Γ complete reductions. First, observe that propositions 1 and 2 hold for $\lambda\Gamma^{\mathbf{S}}_{\beta}$ although the notions of Γ normal are different.

The Γ complete reduction

$$\mathscr{X}_0 \underset{\Gamma}{\twoheadrightarrow} \mathscr{Y}_0 \xrightarrow[\beta]{\Delta_1} \cdots \xrightarrow[\beta]{\Delta_n} \mathscr{X}_n \underset{\Gamma}{\twoheadrightarrow} \mathscr{Y}_n$$

is said to be prestandard if whenever $i < j$ we have Δ_j is not residual of a redex to the left of Δ_i.

PROPOSITION 3: In $\lambda\Gamma^{S}_{\beta}$ if $\mathscr{X} \longrightarrow\!\!> \mathscr{Y}$ then there exists a prestandard Γ complete reduction to the $\xrightarrow[\Gamma]{}\!\!>$ normal form of $\mathscr{Y}$ from $\mathscr{X}$.

PROOF: For $\mathscr{X} \in \lambda\Gamma^{S}$ define $|\mathscr{X}| = \omega$ size of the $\beta\Gamma$ reduction graph of $\mathscr{X}$ + length of $\mathscr{X}$. We prove induction on $|\mathscr{X}|$ that if $\mathscr{X} \xrightarrow[\beta\Gamma]{}\!\!> \mathscr{Y}$ and $\mathscr{Y}$ is $\xrightarrow[\Gamma]{}\!\!>$ normal then there is a Γ complete prestandard reduction from $\mathscr{X}$ to $\mathscr{Y}$.

CASE 1: $\mathscr{X}$ is not $\xrightarrow[\Gamma]{}\!\!>$ normal. Let $\mathscr{Z}$ be the $\xrightarrow[\Gamma]{}\!\!>$ normal form of $\mathscr{X}$. Then $|\mathscr{X}| > |\mathscr{Z}|$ and $\mathscr{Z} \xrightarrow[\beta\Gamma]{}\!\!> \mathscr{Y}$ by proposition 2. Now apply the induction hypothesis.

CASE 2: $\mathscr{X}$ is in $\xrightarrow[\Gamma]{}\!\!>$ normal form. If $\mathscr{X}$ is in head normal form

$$\lambda x_1 \dots x_t \;\; x\, \mathscr{X}_1 \dots \mathscr{X}_s$$
$$\lambda x_1 \dots x_t \;\; \Gamma\, \mathscr{X}_1$$

the proposition follows immediately from the induction hypothesis. If $\mathscr{X}$ has a head redex $\lambda x_1 \dots x_t \;\; \underbrace{(\lambda x.\mathscr{X}_0)\;\mathscr{X}_1} \;\dots \mathscr{X}_s$ we distinguish two cases.

CASE i: $\mathscr{X} \longrightarrow\!\!> \mathscr{Y}$ without contracting a residual of Δ. Then the proposition follows immediately from the induction hypothesis applied to $\lambda x.\mathscr{X}_0$ and the $\mathscr{X}_i$ $i = 1 \dots s$.

CASE ii: Some residual of Δ is contracted. Let

$$\mathscr{X} \xrightarrow[\beta]{\Delta} \mathscr{Z} \equiv \lambda x_1 \dots x_t.\left[\mathscr{X}_1 / x\right] \mathscr{X}_0\, \mathscr{X}_2 \dots \mathscr{X}_s.$$

By permutability of head contractions $\mathscr{Z} \xrightarrow[\beta\Gamma]{}\!\!> \mathscr{Y}$. Since $|\mathscr{X}| > |\mathscr{Z}|$, by induction hypothesis there is a Γ complete prestandard reduction from $\mathscr{Z}$ to $\mathscr{Y}$ say $\mathscr{R}$. The desired reduction is $\mathscr{X} \xrightarrow[\beta]{\Delta} \mathscr{R}$.

COROLLARY: In $\lambda\Gamma^{T}_{\beta}$ if $\mathscr{X}$ is well put together and $\mathscr{X} \longrightarrow\!\!> \mathscr{Y}$ then there exists a prestandard Γ complete reduction from $\mathscr{X}$ to the $\xrightarrow[\Gamma]{}\!\!>$ normal form of $\mathscr{Y}$.

PROOF: Like the proof of theorem 1.

Standardization is now obtained for the untyped case by observing that if $\mathscr{R}$ is Γ complete prestandard then $|\mathscr{R}|$ is standard.

SUBGROUPS OF AUT (T).

A subgroup $G \subseteq \text{Aut}(T)$ is said to be well orbited if whenever $d_{i_1} \ldots d_{i_k}$ n belongs to the G orbit of n we have $k = 0$.

Given the subgroup G, $\lambda\Gamma^{T}_{G}$ is defined like $\lambda\Gamma^{T}$ except that we require $\gamma \in G$.

THEOREM 3: Suppose that G is admissible and well orbited. Then for any untyped term $\mathscr{X}$ the following are equivalent.

$$(1) \quad \exists\, \mathscr{Y} \in \lambda\Gamma^{T}_{G} \quad \mathscr{X} = |\mathscr{Y}|$$
$$(2) \quad \exists\, \mathscr{Y} \in \lambda^{S} \quad \mathscr{X} = |\mathscr{Y}|$$

PROOF: Trivially $(2) \Rightarrow (1)$. Suppose (1) since G is admissible we can assume that each member of T appearing in $\mathscr{Y}$ actually belongs to S. Moreover, we can assume that we are given $\sigma_1 \ldots \sigma_n \in S$ $\gamma_1 \ldots \gamma_n \in G$ such that $\gamma_j \sigma_i \in S$ and whenever $\Gamma^{\gamma(\sigma)}_{\sigma}$ appears in $\mathscr{Y}$ σ is a σ_i and γ is a γ_j. We shall construct an $h \in \text{Hom}(T)$ such that $h\,\sigma_i = h(\gamma_j\,\sigma_i)$. The desired $\mathscr{Y}$ is then obtained from the given one by applying h and then deleting Γ.

To construct h we proceed as follows.

First, we expand S by adding a unary operation g_γ for each $\gamma \in G$. Let $\mathscr{E}$ be a finite set of equations between expressions in the expanded language of S. We define certain rewrite rules applicable to $\mathscr{E}$, its equations, and the expressions in its equations as follows.

$$\begin{array}{lll} g_\gamma(\sigma \longrightarrow \tau) & \longmapsto & (g_\gamma\ \sigma) \longrightarrow (g_\gamma\ \tau) \\ g_{\gamma_1}(g_{\gamma_2}\ \sigma) & \longmapsto & g_{\gamma_1 \circ \gamma_2}\ \sigma \\ g_\gamma\ n \doteq \sigma & \longmapsto & n \doteq g_{\gamma-1}\ \sigma \quad \text{if } \sigma \text{ is not a generator} \\ \sigma_0 \longrightarrow \tau_0 = \sigma_1 \longrightarrow \tau_1 & \longmapsto & \sigma_0 = \sigma_1, \tau_0 = \tau_1 \\ \mathscr{E} & \longmapsto & \left[\sigma | n\right] \mathscr{E} \quad \text{if } n \doteq \sigma \in \mathscr{E} \\ & & \text{and } \sigma \not\longrightarrow\!\!> n, g_\gamma\ n \end{array}$$

Where $\sigma \doteq \tau$ denotes ambiguously $\sigma = \tau$ and $\tau = \sigma$. The last rewrite is called a pivot on $n \doteq \sigma$. Clearly if no pivot is applied the rewriting $\mathscr{E}$ terminates.

Now suppose G is well orbited and we are given $\sigma_1 \ldots \sigma_n \in S$ and $\gamma_1 \ldots \gamma_n \in G$ such that $\gamma_i\ \sigma_j \in S$. Let $\mathscr{E}$ = the set of $g_{\gamma_i}\ \sigma_j = \gamma_i\ (\sigma_j)$. Note that each $E \in \mathscr{E}$ is true of S and G and rewrites preserve truth. Now in the course of rewriting $\mathscr{E}$, for any pivot on $n \doteq \sigma$, we have $n \notin \sigma$, for otherwise rewriting σ and applying d_0 and d_1 we would obtain $d_{i_1} \ldots d_{i_k}$ $n = \gamma(n)$ for some $\gamma \in G$ and $k > 0$. Thus each pivot eliminates a generator from $\mathscr{E}$ and rewriting $\mathscr{E}$ terminates.

Now when rewriting $\mathscr{E}$ terminates the equations in $\mathscr{E}$ have the form $n = n$ or

$n \doteq g_\gamma n$. Let

$n_1 \doteq \sigma_1$

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$n_k \doteq \sigma_k$

be the pivots made on $\mathscr{E}$ in order of occurrence. If $m \in \sigma_i$ then either no pivot is made on m or the pivot made on m is at least the $i+1^{st}$. Now define an $h \in \mathrm{Hom}(T)$ by $h(m) = m$ if no pivot is made on m $h(n_i) = h(\sigma_i)$ where each g_γ in σ is interpreted as the identity. Now apply h to each equation in $\mathscr{E}$ and each pivot, interpreting each g_γ as the identity. The resulting equations are all true. Moreover this interpretation is preserved under rewrite reversal. Thus we have

$$h\,\sigma_i = h(\gamma_j\,\sigma_i)$$

for $1 \leq i, j \leq n$ and h is as desired.

Admissible, well orbited subgroups of Aut (T) are easy to construct. For example, one can easily be constructed from the infinite (rootless) homogeneous binary tree.