THERE IS NO HYPERRECURRENT S,K COMBINATOR

by

Rick Statman

Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

Research Report 91–133 2

June, 1991

THERE IS NO HYPERRECURRENT S,K COMBINATOR

by

RICK STATMAN

JUNE 1991

We work in the S,K combinator calculus with weak β reduction and conversion.

M is said to be *hyperrecurrent* if whenever $M = N$, N is recurrent [5]. We shall show that there is no hyperrecurrent combinator. First we note the following

PROPOSITION 1: N is recurrent $\Leftrightarrow$ Cpl (N) $\twoheadrightarrow$ N

PROOF: We observe first that $M \twoheadrightarrow N \Rightarrow$ Cpl (M) $\twoheadrightarrow$ Cpl (N) by induction on the length of a reduction sequence from M to N.
Namely,

$$\begin{tikzcd}
M \arrow[r, twoheadrightarrow] \arrow[d, twoheadrightarrow] & Cpl(M) \arrow[d, twoheadrightarrow] \\
\underline{P} \arrow[r, twoheadrightarrow] \arrow[d, "\Delta"'] & Cpl(\underline{P}) \arrow[d, dashed, twoheadrightarrow, "\ell"] \\
N \arrow[r, twoheadrightarrow] \arrow[ur, dashed, twoheadrightarrow, "\ell"] & Cpl(N)
\end{tikzcd}$$

by FD! [1]. The proposition follows from the cofinality of Gross–Knuth sequences.

PROPOSITION 2: The following are equivalent

(1) M is hyperrecurrent

(2) $P = M = Q \Rightarrow P \twoheadrightarrow Q$

(3) $P \twoheadrightarrow M \Rightarrow M \twoheadrightarrow P$

PROOF: (1) $\Rightarrow$ (2) by the Church Rosser theorem and (2) $\Rightarrow$ (3) trivially. Suppose (3). Recall the process of making M normal. Namely there is a normal N s.t $NI \underset{\neq \emptyset}{\twoheadrightarrow} M$. We observe that $NI \twoheadrightarrow Q \Rightarrow Q \twoheadrightarrow NI$ by induction on the length of a reduction sequence. Namely,

$$
\begin{tikzcd}
NI \arrow[d, two heads] & \\
R \arrow[r, two heads] \arrow[d, "\Delta"'] & NI \arrow[d, two heads, "l"] \\
Q \arrow[r, dashed, two heads, "\text{strip lemma}"'] & T \arrow[d, two heads, "\text{definition of } N"] \\
 & M \arrow[d, two heads, "\text{by } (3)"] \\
 & NI
\end{tikzcd}
$$

Thus NI is recurrent; so, M is recurrent. Hence $P = M \Rightarrow P \twoheadrightarrow M \Rightarrow M \twoheadrightarrow P$ and M is hyperrecurrent.

Recall the following definition from [3]. M is *cyclically equivalent to* N, $M \sim N$, if $M \twoheadrightarrow N$ and $N \twoheadrightarrow M$. We obtain the following

COROLLARY: M is not hyperrecurrent $\Leftrightarrow$ M/= splits into at least infinitely many singletons and one infinite cyclic quivalence classes.

PROOF: If M is hyperrecurrent then M/= is a cyclic equivalence class. If M is not hyperrecurrent then by proposition 2 for each $N = M$ there exists $P \twoheadrightarrow N$ s.t. P is not recurrent. For each such P there is a normal Q s.t. $QI \underset{\neq \emptyset}{\twoheadrightarrow} P$ so $QI \twoheadrightarrow$ $\mathrm{Cpl}\,(QI) \twoheadrightarrow P$. Since QI is not recurrent $\mathrm{Cpl}\,(QI) \not\twoheadrightarrow QI$. Thus, since QI has a unique redex, QI lies on no reduction cycle. That there is at least one infinite cyclic equivalence class as trivial.

M is said to have the *upward Church Rosser property* if whenever $P \twoheadrightarrow M \twoheadleftarrow Q$ these exists R s.t. $P \twoheadleftarrow R \twoheadrightarrow Q$. Plotkin first gave an example of a λ term without the upward Church Rosser property ([1]).

Clearly if M is hyperrecurrent then it has the upward church Rosser property.

We observe here that it is impossible to construct effectively for each M a term P_M such that $P_M \twoheadrightarrow M$ but $M \not\twoheadrightarrow P_M$ because the Ershov fixed point theorem applies to the relation $\twoheadrightarrow$ (see [4]). We shall construct P_M and Q_M s.t. $P_M \twoheadrightarrow M \twoheadleftarrow Q_M$ and either $M \not\twoheadrightarrow P_M$ or $M \not\twoheadrightarrow Q_M$.

THEOREM: M has the upward Church Rosser property $\Leftrightarrow$ M = an atom.

PROOF: $\Rightarrow$ Set $\Omega \equiv SII\,(SII)$. If N and N$'$ are distinct normal terms then $\Omega\, N \twoheadrightarrow P \Rightarrow P \twoheadrightarrow \Omega\, N$ and P contains no subterm $= \Omega\, N'$. If $M \neq$ an atom and N is normal set $\alpha_N \equiv [x]M\ (\Omega\, N)$ so $\alpha_N \twoheadrightarrow M$

(1) If β has order 0 and $[x]M\,\beta \overset{\text{head}}{\twoheadrightarrow} P$ then P contains a trace of β and every trace of $\beta \equiv \beta$.

Consider a head reduction $\mathcal{R}$ beginning with $[x]M\,\beta$. We partition $\mathcal{R}$ into segments $\mathcal{R}_1\, \mathcal{R}_2 \dots \mathcal{R}_1\, \mathcal{S}$. Each $\mathcal{R}_i$ begins with an applicative combination of terms of the form $[x]Q\ \beta$ and proceeds by simulating, with β for x, a canonical (see [6] pg. 36) head reduction until a term of the form $K\ A\ \beta\, R_1 \dots R_s$ is obtained, where A is an atom. At this stage $s \neq 0$ since $M \neq A$. $\mathcal{R}_i$ then terminates by projecting β and applying the reduction rule for A if a head redex results from the projection. The trailing $\mathcal{S}$ is just a partial such segment.

(2) Suppose

$$\begin{tikzcd}
(G \arrow[d, two heads] & H) \arrow[d, two heads] \arrow[r, two heads, "\text{head}", "n \text{ steps}"'] & J \\
{[x]M} & (\Omega N) &
\end{tikzcd}$$

Then there exist L, P, Q, R such that

$$
\begin{tikzcd}
 & J \equiv (L \arrow[d, two heads] & P) \arrow[d, two heads] \\
\alpha_N \arrow[r, two heads, "\text{head}", "\le n \text{ steps}"'] & (Q & R)
\end{tikzcd}
$$

and J contains a trace of H

The proof is by induction on the length of the reduction G ——>> [x]M. The basis case is just (1) and its proof. Now suppose $G \xrightarrow{\Delta} F$ ——>> [x]M. We simulate the head reduction from GH with FH. This is nothing more than paying attention to the proof of the strip lemma ([1]). The simulation can be obtained vertically by replacing the residuals of Δ by its immediate reduct Δ'. Horizontally, head reductions are skipped when Δ is at the head. All these occurrences are disjoint from traces of H. Applying the induction hypothesis gives

$$
\begin{tikzcd}
(G \arrow[d, "\Delta"'] & H) \arrow[d, equal] \arrow[r, two heads, "\text{head}", "n \text{ steps}"'] & J \arrow[d, two heads, "\text{cpl } \Delta \text{ residuals}"] & \\
(F \arrow[d, two heads] & H) \arrow[d, two heads] \arrow[r, two heads, "\text{head}", "\le n \text{ steps}"'] & (U \arrow[d, two heads] & V) \arrow[d, two heads] \\
{[x]M} & (\Omega N) \arrow[r, two heads, "\text{head}", "\le n \text{ steps}"'] & (Q & R)
\end{tikzcd}
$$

Now it cannot be that some whole term in the head reduction from GH consists of a residual of Δ. Since this contradicts the fact that the corresponding term in the head reduction of FH has a trace of H. Thus $J \equiv LP$ and $L \xrightarrow{\text{cpl } \Delta \text{ residuals}}\!\!\!\!\!\rightarrow U$ and $P \xrightarrow{\text{cpl } \Delta \text{ residuals}}\!\!\!\!\!\rightarrow V$

(3) Suppose α_{N_0} <<—— F ——>> α_{N_1}. Then for some i α_{N_i} ——>> $\alpha_{N_{1-i}}$.

If $\alpha_{N_0} <\!\!<\!\!—\, F \,—\!\!>\!\!> \alpha_{N_1}$ then by the standardization theorem there exist L, P, Q, R s.t.

$$\begin{tikzcd}
F \arrow[d, two heads, "\text{head}"] & & & \\
(\,P \arrow[d, two heads] & Q\,) \arrow[d, two heads] \arrow[r, two heads, "\text{head}"] & (\,L \arrow[d, two heads] & R\,) \arrow[d, two heads] \\
{[x]M} & (\Omega N_i) & {[x]M} & (\Omega N_{1-i})
\end{tikzcd}$$

By (2) there exist U, V s.t. $\alpha_{N_i} \xrightarrow{\text{head}}\!\!> UV$, $U —\!\!>\!\!> [x]M$ and $V —\!\!>\!\!> \Omega\, N_{1-i}$. This proves (3).

Next we consider a head reduction beginning with α_{N_0} and we mark exactly those terms GH s.t. $G —\!\!>\!\!> [x]M$ and $H —\!\!>\!\!> \Omega\, N$ for some normal N

$$\begin{tikzcd}
\alpha_{N_0} \arrow[r, two heads, "\text{head}"] & (\,G_1 \arrow[d, two heads] & H_1\,) \arrow[d, two heads] \arrow[r, two heads, "\text{head}"] & \cdots \arrow[r, two heads, "\text{head}"] & (\,G_n \arrow[d, two heads] & H_n\,) \arrow[d, two heads] \arrow[r, two heads, "\text{head}"] & \cdots \\
 & {[x]M} & (\Omega N_1) & & {[x]M} & (\Omega N_n) &
\end{tikzcd}$$

For $i = 1, 2, \ldots$ we can apply (2) to obtain

$$\begin{tikzcd}
(\,G_i \arrow[d, two heads] & H_i\,) \arrow[d, two heads] \arrow[r, two heads, "\text{head}", "k_i \text{ steps}"'] & (\,G_{i+1} \arrow[d, two heads] & H_{i+1}\,) \arrow[d, two heads] \\
{[x]M} & (\Omega N_i) \arrow[dr, two heads, "\text{head}", "\leq k_i \text{ steps}"'] & {[x]M} & (\Omega N_{i+1}) \\
 & & (\,U_i \arrow[u, two heads] & V_i\,) \arrow[u, two heads]
\end{tikzcd}$$

By (1) all the traces of $\Omega\ N_i$ in $U_i\ V_i$ are $\equiv \Omega\ N_i$. We indicate these by writing $U_i \equiv U_i$ $[\Omega\ N_i]$ and $V_i \equiv V_i\ [\Omega\ N_i]$. In particular, we have either $V_i[x] \longrightarrow\!\!> x$ or $V_i[x] \longrightarrow\!\!> \Omega$ N_{i+1}. Observe that in the first case $N_{i+1} \equiv N_i$ and this applies equally well to $i = 0$ with H_1 for V_1. Suppose now that i is smallest s.t. $V_i[x] \longrightarrow\!\!> \Omega\ N_{i+1}$ and $N_{i+1} \not\equiv N_0$. Then we have

$$\begin{tikzcd}[column sep=small] {[x]\,M\,(\Omega\,N_0)} \arrow[rr, "\text{head}", "\leq k_i\ \text{steps}"'] & & U_i\ [\Omega\ N_0] \arrow[d] & V_i\ [\Omega\ N_0] \arrow[d] \\ & & {[x]M} & (\Omega\ N_{i+1}) \end{tikzcd}$$

by substitution. Hence $i = 1$, and by similar reasoning for all $i \geq 1$ $N_j \equiv N_1$. We conclude

(4) For each normal N there is at most one normal $N' \not\equiv N$ such that $\alpha_N \longrightarrow\!\!> \alpha_{N'}$.

We are now ready to conclude the proof of $\Longrightarrow$. (3) and (4) imply that for any 3 distinct α_N at most 2 pairs are reducts of a common term.

$\Longleftarrow$ Suppose $M =$ the atom A and $P \longrightarrow\!\!> A <\!\!\longleftarrow Q$. Consider the reduction $P \longrightarrow\!\!> A$. A is the trace of a unique occurrence of A in P which we indicate $P[A]$. Then $P <\!\!\longleftarrow P[Q] \longrightarrow\!\!> Q$. This completes the proof.

REFERENCES

[1] Barendregt *The Lambda Calculus*, North Holland 1984.

[2] Klop *Combinatory Reduction Systems*, Mathematisch Centrum 1980.

[3] *Reduction Cycles in Combinatory Logic, in Curry Festschrift* North Holland 1980, Seldin & Hindley eds 193–214

[4] Visser *Numerations, λ–Calculus, and Arithmetic, in Curry Festschrift* 259–284.

[5] Venturini Zilli *Reduction Graphs in the λ Calculus, TCS* ***29*** **(1984)** 251–275.

[6] Curry, Hindley and Seldin *Combinatory Logic II*, North Holland 1972.