

THERE IS NO HYPERRECURRENT S, K COMBINATOR

by

Rick Statman

Department of Mathematics  
Carnegie Mellon University  
Pittsburgh, PA 15213

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We work in the S,K combinator calculus with weak  $\beta$  reduction and conversion.

M is said to be *hyperrecurrent* if whenever  $M = N$ , N is recurrent [5]. We shall show that there is no hyperrecurrent combinator. First we note the following

**PROPOSITION 1:**  $N$  is recurrent  $\Leftrightarrow \text{Cpl}(N) \longrightarrow\!\!\!\!\!\gg N$

**PROOF:** We observe first that  $M \longrightarrow\!\!\!\!\!\gg N \Rightarrow \text{Cpl}(M) \longrightarrow\!\!\!\!\!\gg \text{Cpl}(N)$  by induction on the length of a reduction sequence from M to N.

Namely,

$$\begin{array}{ccc}
 M & \longrightarrow\!\!\!\!\!\gg & \text{Cpl}(M) \\
 \downarrow & & \downarrow \\
 P & \longrightarrow\!\!\!\!\!\gg & \text{Cpl}(P) \\
 \downarrow & \dashrightarrow & \downarrow \\
 N & \longrightarrow\!\!\!\!\!\gg & \text{Cpl}(N)
 \end{array}$$

by FD! [1]. The proposition follows from the cofinality of Gross-Knuth sequences.

**PROPOSITION 2:** The following are equivalent

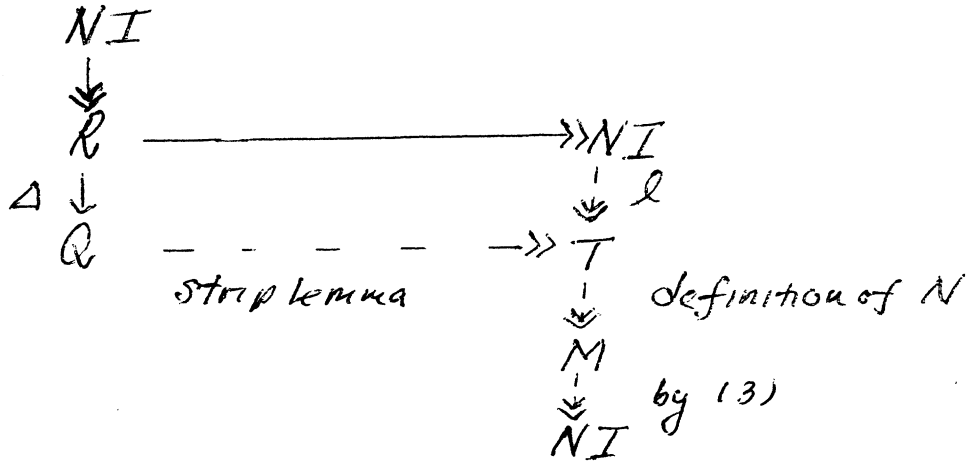
- (1) M is hyperrecurrent
- (2)  $P = M = Q \Rightarrow P \longrightarrow\!\!\!\!\!\gg Q$
- (3)  $P \longrightarrow\!\!\!\!\!\gg M \Rightarrow M \longrightarrow\!\!\!\!\!\gg P$

**PROOF:** (1)  $\Rightarrow$  (2) by the Church Rosser theorem and (2)  $\Rightarrow$  (3) trivially. Suppose (3).

Recall the process of making M normal. Namely there is a normal N s.t  $NI \longrightarrow\!\!\!\!\!\gg M$ . We

observe that  $NI \longrightarrow\!\!\!\!\!\gg Q \Rightarrow Q \longrightarrow\!\!\!\!\!\gg NI$  by induction on the length of a reduction

sequence. Namely,



Thus  $NI$  is recurrent; so,  $M$  is recurrent. Hence  $P = M \Rightarrow P \longrightarrow\!\!\!\longrightarrow M \Rightarrow M \longrightarrow\!\!\!\longrightarrow P$  and  $M$  is hyperrecurrent.

Recall the following definition from [3].  $M$  is *cyclically equivalent* to  $N$ ,  $M \sim N$ , if  $M \longrightarrow\!\!\!\longrightarrow N$  and  $N \longrightarrow\!\!\!\longrightarrow M$ . We obtain the following

**COROLLARY:**  $M$  is not hyperrecurrent  $\Leftrightarrow M/=$  splits into at least infinitely many singletons and one infinite cyclic equivalence classes.

**PROOF:** If  $M$  is hyperrecurrent then  $M/=$  is a cyclic equivalence class. If  $M$  is not hyperrecurrent then by proposition 2 for each  $N = M$  there exists  $P \longrightarrow\!\!\!\longrightarrow N$  s.t.  $P$  is not recurrent. For each such  $P$  there is a normal  $Q$  s.t.  $QI \xrightarrow{\neq \emptyset} P$  so  $QI \longrightarrow\!\!\!\longrightarrow$   $Cpl(QI) \longrightarrow\!\!\!\longrightarrow P$ . Since  $QI$  is not recurrent  $Cpl(QI) \not\longrightarrow\!\!\!\longrightarrow QI$ . Thus, since  $QI$  has a unique redex,  $QI$  lies on no reduction cycle. That there is at least one infinite cyclic equivalence class as trivial.

$M$  is said to have the *upward Church Rosser property* if whenever  $P \longrightarrow\!\!\!\longrightarrow M \leftarrow\!\!\!\longleftarrow Q$  there exists  $R$  s.t.  $P \leftarrow\!\!\!\longleftarrow R \longrightarrow\!\!\!\longrightarrow Q$ . Plotkin first gave an example of a  $\lambda$  term without the upward Church Rosser property ([1]).

Clearly if  $M$  is hyperrecurrent then it has the upward church Rosser property.

We observe here that it is impossible to construct effectively for each  $M$  a term  $P_M$  such that  $P_M \longrightarrow\!\!\!> M$  but  $M \not\longrightarrow\!\!\!> P_M$  because the Ershov fixed point theorem applies to the relation  $\longrightarrow\!\!\!>$  (see [4]). We shall construct  $P_M$  and  $Q_M$  s.t.  $P_M \longrightarrow\!\!\!> M \ll\!-\! Q_M$  and either  $M \not\longrightarrow\!\!\!> P_M$  or  $M \not\longrightarrow\!\!\!> Q_M$ .

**THEOREM:**  $M$  has the upward Church Rosser property  $\Leftrightarrow M =$  an atom.

**PROOF:**  $\Rightarrow$  Set  $\Omega \equiv \text{SII}(\text{SII})$ . If  $N$  and  $N'$  are distinct normal terms then

$\Omega N \longrightarrow\!\!\!> P \Rightarrow P \longrightarrow\!\!\!> \Omega N$  and  $P$  contains no subterm  $= \Omega N'$ . If  $M \neq$  an atom and  $N$  is normal set  $\alpha_N \equiv [x]M (\Omega N)$  so  $\alpha_N \longrightarrow\!\!\!> M$

(1) If  $\beta$  has order 0 and  $[x]M \overset{\text{head}}{\longrightarrow\!\!\!>} P$  then  $P$  contains a trace of  $\beta$  and every trace of  $\beta \equiv \beta$ .

Consider a head reduction  $\mathcal{R}$  beginning with  $[x]M \beta$ . We partition  $\mathcal{R}$  into segments  $\mathcal{R}_1 \mathcal{R}_2 \dots \mathcal{R}_1 \mathcal{S}$ . Each  $\mathcal{R}_i$  begins with an applicative combination of terms of the form  $[x]Q \beta$  and proceeds by simulating, with  $\beta$  for  $x$ , a canonical (see [6] pg. 36) head reduction until a term of the form  $K A \beta R_1 \dots R_s$  is obtained, where  $A$  is an atom. At this stage  $s \neq 0$  since  $M \neq A$ .  $\mathcal{R}_i$  then terminates by projecting  $\beta$  and applying the reduction rule for  $A$  if a head redex results from the projection. The trailing  $\mathcal{S}$  is just a partial such segment.

(2) Suppose

$$\begin{array}{ccc} ( G & H & ) \\ \downarrow & & \downarrow \\ [x]J & M & (\Omega N) \end{array} \xrightarrow[n \text{ steps}]{\text{head}} J$$

Then there exist L, P, Q, R such that

$$\alpha_N \xrightarrow[\leq n \text{ steps}]{\text{head}} J \equiv \begin{pmatrix} L & P \\ \downarrow & \downarrow \\ Q & R \end{pmatrix}$$

and J contains a trace of H

The proof is by induction on the length of the reduction  $G \longrightarrow [x]M$ . The basis case is just (1) and its proof. Now suppose  $G \xrightarrow{\Delta} F \longrightarrow [x]M$ . We simulate the head reduction from GH with FH. This is nothing more than paying attention to the proof of the strip lemma ([1]). The simulation can be obtained vertically by replacing the residuals of  $\Delta$  by its immediate reduct  $\Delta'$ . Horizontally, head reductions are skipped when  $\Delta$  is at the head.

All these occurrences are disjoint from traces of H. Applying the induction hypothesis gives

$$\begin{array}{ccc} \begin{pmatrix} G & H \\ \Delta \downarrow & \parallel \end{pmatrix} & \xrightarrow[\leq n \text{ steps}]{\text{head}} & J \\ & & \downarrow \text{cpl } \Delta \text{ residuals} \\ \begin{pmatrix} F & H \\ \downarrow & \downarrow \end{pmatrix} & \xrightarrow[\leq n \text{ steps}]{\text{head}} & \begin{pmatrix} U & V \\ \downarrow & \downarrow \end{pmatrix} \\ & & \downarrow \downarrow \\ [x]M & \begin{pmatrix} \Omega N \\ \downarrow \end{pmatrix} & \xrightarrow[\leq n \text{ steps}]{\text{head}} & \begin{pmatrix} Q & R \\ \downarrow & \downarrow \end{pmatrix} \end{array}$$

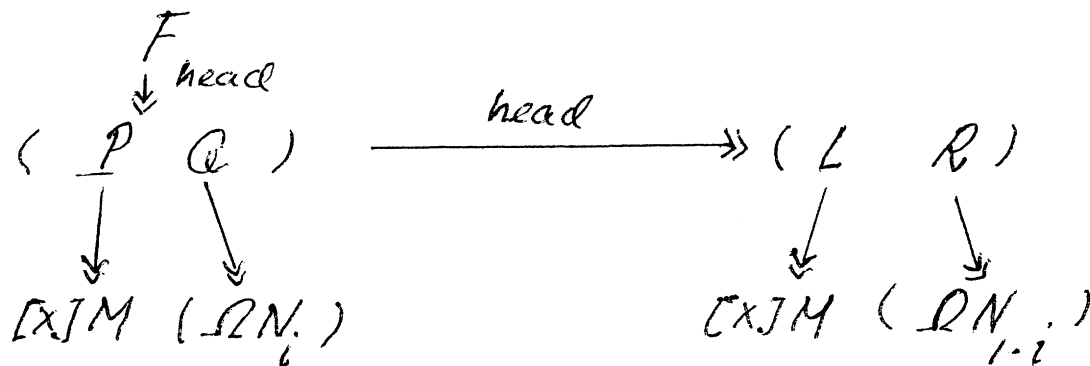
Now it cannot be that some whole term in the head reduction from GH consists of a residual of  $\Delta$ . Since this contradicts the fact that the corresponding term in the head reduction of FH has a trace of H.

Thus  $J \equiv LP$  and  $L \xrightarrow{\text{cpl } \Delta \text{ residuals}} U$  and  $P \xrightarrow{\text{cpl } \Delta \text{ residuals}} V$

(3) Suppose  $\alpha_{N_0} \ll \text{---} F \text{---} \gg \alpha_{N_1}$ . Then for some  $i$   $\alpha_{N_i} \longrightarrow \alpha_{N_{1-i}}$ .

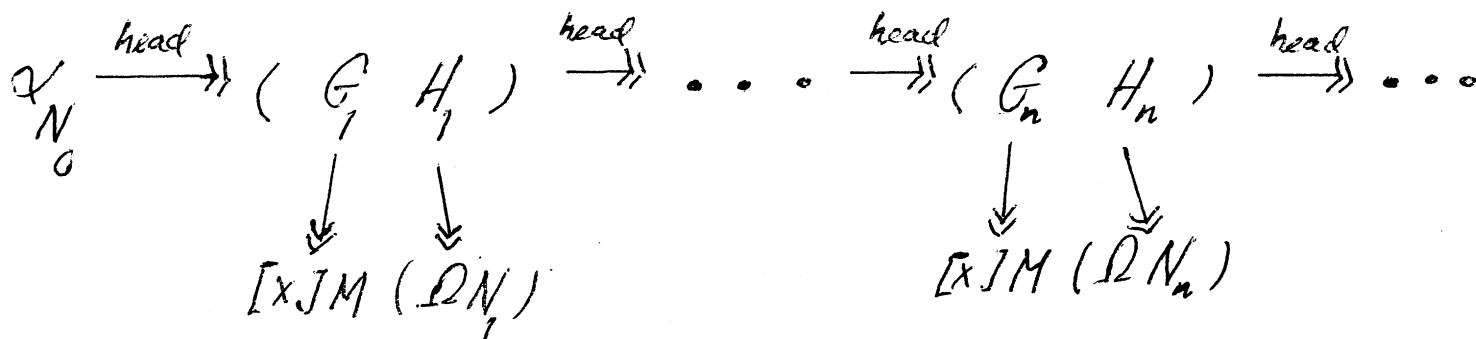
If  $\alpha_{N_0} \ll \text{---} F \text{---} \gg \alpha_{N_1}$  then by the standardization theorem there exist

$L, P, Q, R$  s.t.

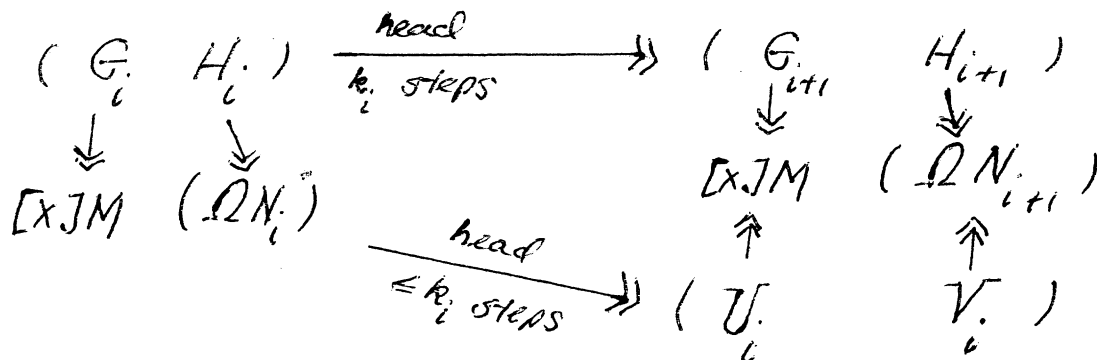


By (2) there exist  $U, V$  s.t.  $\alpha_{N_i} \xrightarrow{\text{head}} UV$ ,  $U \longrightarrow [x]M$  and  $V \longrightarrow \Omega N_{1-i}$ . This proves (3).

Next we consider a head reduction beginning with  $\alpha_{N_0}$  and we mark exactly those terms  $GH$  s.t.  $G \longrightarrow [x]M$  and  $H \longrightarrow \Omega N$  for some normal  $N$



For  $i = 1, 2, \dots$  we can apply (2) to obtain





By (1) all the traces of  $\Omega N_i$  in  $U_i V_i$  are  $\equiv \Omega N_i$ . We indicate these by writing  $U_i \equiv U_i [\Omega N_i]$  and  $V_i \equiv V_i [\Omega N_i]$ . In particular, we have either  $V_i[x] \longrightarrow x$  or  $V_i[x] \longrightarrow \Omega N_{i+1}$ . Observe that in the first case  $N_{i+1} \equiv N_i$  and this applies equally well to  $i = 0$  with  $H_1$  for  $V_1$ . Suppose now that  $i$  is smallest s.t.  $V_i[x] \longrightarrow \Omega N_{i+1}$  and  $N_{i+1} \neq N_0$ . Then we have

$$\begin{array}{ccc}
 [x] M (\Omega N_0) & \xrightarrow[\leq k_i]{\text{head}} & U_i [\Omega N_0] V_i [\Omega N_0] \\
 & \text{s steps} & \swarrow \quad \searrow \\
 & & [x] M \quad (\Omega N_{i+1})
 \end{array}$$

by substitution. Hence  $i = 1$ , and by similar reasoning for all  $i \geq 1$   $N_j \equiv N_1$ . We conclude

(4) For each normal  $N$  there is at most one normal  $N' \neq N$  such that  $\alpha_N \longrightarrow \alpha_{N'}$ .

We are now ready to conclude the proof of  $\Rightarrow$ . (3) and (4) imply that for any 3 distinct  $\alpha_N$  at most 2 pairs are reducts of a common term.

$\Leftarrow$  Suppose  $M = \text{the atom } A$  and  $P \longrightarrow A \leftarrow Q$ . Consider the reduction  $P \longrightarrow A$ .  $A$  is the trace of a unique occurrence of  $A$  in  $P$  which we indicate  $P[A]$ . Then  $P \leftarrow P[Q] \longrightarrow Q$ . This completes the proof.



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- [6] Curry, Hindley and Seldin *Combinatory Logic II*, North Holland 1972.