

DECIDABLE DISCRIMINATOR VARIETIES  
WITH LATTICE STALKS

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# Decidable discriminator varieties with lattice stalks

Ross Willard

## Abstract

We determine those universal classes of lattices which generate a decidable discriminator variety when augmented by a ternary discriminator term. They are the locally finite universal classes whose finite members are almost homogeneous.

## 1 Introduction

We are interested in determining which varieties have a decidable first-order theory. For locally finite varieties, R. McKenzie and M. Valeriote [8] have succeeded in reducing this problem to two special subproblems:

**PROBLEM 1:** *For which finite rings  $R$  with unit is the variety of unitary left  $R$ -modules decidable?*

**PROBLEM 2:** *Which locally finite discriminator varieties (in a finite language) are decidable?*

In this paper we continue our investigation [11, 9] of Problem 2.

The *ternary discriminator* on the set  $A$  is the function  $t_A : A^3 \rightarrow A$  given by  $t_A(x, y, z) = x$  if  $x \neq y$ ,  $t(x, x, z) = z$ . A *discriminator variety* is a variety  $\mathcal{V}$  for which there is a term  $t(x, y, z)$  in the language of  $\mathcal{V}$  which defines the ternary discriminator on the universe of every nontrivial subdirectly irreducible member of  $\mathcal{V}$ .

There is a canonical way to generate discriminator varieties. Let  $\mathcal{K}$  be a universal (i.e., definable by a set of universal first-order sentences) class of algebras in the language  $L$ , let  $t$  be a ternary operation symbol not occurring

in  $L$ , and let  $L(t)$  denote the language  $L \cup \{t\}$ . If  $\mathbf{A} \in \mathcal{K}$  then  $\mathbf{A}^t$  denotes the  $L(t)$ -algebra  $\langle \mathbf{A}, t_A \rangle$ , and  $\mathcal{K}^t$  denotes  $\{\mathbf{A}^t : \mathbf{A} \in \mathcal{K}\}$ . Then the variety  $V(\mathcal{K}^t)$  generated by  $\mathcal{K}^t$  is a discriminator variety, and every discriminator variety is definitionally equivalent to one of that form.

In Section 4 we consider discriminator varieties of the form  $V(\mathcal{K}^t)$  where  $\mathcal{K}$  is a universal class of lattices, and describe which classes produce decidable discriminator varieties. One characterization is the following. Let  $\mathbf{Q}$  be the chain of rational numbers under their usual ordering; let  $\mathbf{N}_\omega$  be the unique countably infinite lattice of height 3 having exactly one 4-element chain; let  $\mathbf{M}_{\omega,2}$  be the unique countably infinite modular lattice of height 3 having exactly two coatoms (see Figure 1); and let  $\mathbf{M}_{2,\omega}$  be the dual of  $\mathbf{M}_{\omega,2}$ . Then for any universal class  $\mathcal{K}$  of lattices,  $V(\mathcal{K}^t)$  is decidable if and only if  $\mathcal{K}$  omits each of  $\mathbf{Q}$ ,  $\mathbf{N}_\omega$ ,  $\mathbf{M}_{\omega,2}$ , and  $\mathbf{M}_{2,\omega}$ .

The result we shall use to establish undecidability is the following special case of Lemma 4.1 from [11].

**LEMMA 1.1** *Let  $\mathbf{A}$  be an algebra in the language  $L$ , and  $\mathbf{S}$  a subalgebra of  $\mathbf{A}$ . Suppose there exist first-order  $L$ -formulas  $\mu(x)$ ,  $\tau(x)$ , and  $\psi(z)$  such that, setting  $M = \mu^{\mathbf{A}}|_{\mathbf{S}}$ ,  $T = M \cap \tau^{\mathbf{A}}|_{\mathbf{S}}$ , and  $\text{Aut}_M \mathbf{S} = \{\sigma \in \text{Aut } \mathbf{S} : \sigma(M) = M\}$ ,*

- (i)  $M$  is infinite while  $T$  is finite;
- (ii)  $M = \bigcup \{\sigma(T) : \sigma \in \text{Aut}_M \mathbf{S}\}$ ;
- (iii)  $\psi^{\mathbf{A}} \neq \emptyset$  but  $\psi^{\mathbf{A}}|_{\mathbf{S}} = \emptyset$ .

*Then the class  $\mathcal{P}_s(\{\mathbf{S}, \mathbf{A}\}^t)$  is hereditarily undecidable.*

The reader may already see how Lemma 1.1 can be applied to each of the four forbidden lattices mentioned above (for  $\mathbf{Q}$  one must choose  $\mathbf{A}$  to be a proper sublattice of  $\mathbf{Q}$ ).

Most of this paper is devoted to establishing that  $V(\mathcal{K}^t)$  is decidable assuming that  $\mathcal{K}$  omits  $\mathbf{Q}$ ,  $\mathbf{N}_\omega$ ,  $\mathbf{M}_{\omega,2}$  and  $\mathbf{M}_{2,\omega}$ . The proof requires an improvement of existing technology for analyzing the first-order theory of discriminator varieties. Specifically, we isolate a version of homogeneity which is weaker than the one considered in [11], and we prove its relevance to the general Problem 2. This is the real contribution of this paper.

We adopt the following conventions.  $\mathbf{A} \leq_{bp} \prod_{x \in X} \mathbf{A}_x$  means that  $\mathbf{A}$  is a Boolean product of the family  $(\mathbf{A}_x)_{x \in X}$  (see [3] or [4]). We refer to the

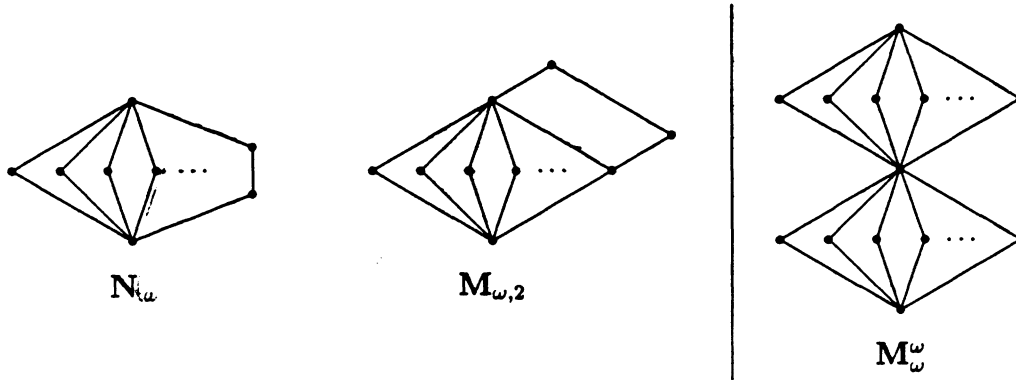


Figure 1

members of  $(\mathbf{A}_x)_{x \in X}$  as the *stalks* of  $\mathbf{A}$ . For a class  $\mathcal{K}$ ,  $\Gamma^{\alpha}(\mathcal{K})$  denotes the class of all Boolean products with stalks from  $\mathcal{K}$ . When referring to a particular Boolean product of  $(\mathbf{A}_x)_{x \in X}$  we shall assume that the Boolean topology on  $X$  is specified.

If  $\mathbf{A} \leq_{bp} \prod_{x \in X} \mathbf{A}_x$  then  $\mathbf{A}$  is a subuniverse of the  $L(t)$ -algebra  $\prod_{x \in X} \mathbf{A}_x^t$ . We denote the corresponding subalgebra of  $\prod_{x \in X} \mathbf{A}_x^t$  by  $\mathbf{A}(t)$ . Clearly  $\mathbf{A}(t) \in V(\mathcal{K}^t)$ . Conversely:

**THEOREM 1.2** (Bulman-Fleming, Werner [1]) *Suppose  $\mathcal{K}$  is a universal class of algebras.*

- (i) *If  $\mathbf{B} \in V(\mathcal{K}^t)$  and  $|B| > 1$ , then  $\mathbf{B} \cong \mathbf{A}(t)$  for some  $\mathbf{A} \in \Gamma^{\alpha}(\mathcal{K})$ .*
- (ii) *If moreover  $\mathbf{B}$  is countable, then the Boolean product  $\mathbf{A} \leq_{bp} \prod_{x \in X} \mathbf{A}_x$  referred to in (i) can be chosen so that the Boolean topology on  $X$  has only countably many clopen sets.*

If  $\mathcal{K}$  is a class of algebras in the language  $L$ , then  $\text{Th}_{\forall}(\mathcal{K})$  denotes the set of all universal (first-order)  $L$ -sentences true in all members of  $\mathcal{K}$ .  $\mathcal{K}_{fin}$  is the class of all finite members of  $\mathcal{K}$ . A lattice has height  $n$  if it contains an  $(n+1)$ -element chain but no  $(n+2)$ -element chain. We say that an algebra is  $n$ -generated if it is generated by a subset of cardinality at most  $n$ .

## 2 Homogeneity

A locally finite algebra  $\mathbf{A}$  shall be called *homogeneous* if every isomorphism between finite subalgebras of  $\mathbf{A}$  can be extended to an automorphism of  $\mathbf{A}$ . If  $\mathcal{K}$  is a locally finite universal class of algebras in a finite language, then we say that  $\mathcal{K}$  is homogeneous provided each of its finite (equivalently, each of its countable) members is homogeneous. In [2] S. Burris, R. McKenzie, and M. Valeriote established the following connection between homogeneous classes and decidable discriminator varieties: if  $\mathcal{K}$  is a locally finite universal class of algebras in a finite language, and if  $\mathcal{K}$  is homogeneous in the above sense and is finitely axiomatizable, then  $V(\mathcal{K}^t)$  is decidable.

In this section we present several “finite perturbations” of the notion of  $\mathcal{K}$  being homogeneous, each involving the existence of a finite set  $\mathcal{K}_0$  of finite members of  $\mathcal{K}$  satisfying

- (1)  $S(\mathcal{K}_0) \subseteq I(\mathcal{K}_0)$ .
- (2) (If the language of  $\mathcal{K}$  contains constant symbols): Every 0-generated member of  $\mathcal{K}$  is in  $I(\mathcal{K}_0)$ .

**Definition 2.1** Suppose  $\mathcal{K}$  is a locally finite universal class of algebras in a finite language,  $\mathcal{K}_0$  is a finite set of finite members of  $\mathcal{K}$  satisfying conditions (1) and (2) above, and  $\mathbf{A} \in \mathcal{K}$ . A *maximal  $\mathcal{K}_0$ -subuniverse of  $\mathbf{A}$*  is a subuniverse  $A_0$  such that (i) either  $A_0 = \emptyset$  or  $\mathbf{A}_0$  is isomorphic to some member of  $\mathcal{K}_0$ , and (ii)  $A_0$  is maximal (among subuniverses of  $\mathbf{A}$  ordered by inclusion) with respect to property (i).

Observe that if  $\mathcal{K}_0$  satisfies conditions (1) and (2), then every algebra in  $\mathcal{K}$  has at least one maximal  $\mathcal{K}_0$ -subuniverse; and if  $A_0$  is a maximal  $\mathcal{K}_0$ -subuniverse of  $\mathbf{A}$  and  $\mathbf{B} \leq \mathbf{A}$ , then  $A_0 \cap B$  can be extended to a maximal  $\mathcal{K}_0$ -subuniverse of  $\mathbf{B}$  (in particular, if  $A_0 \subseteq B$  then  $A_0$  is a maximal  $\mathcal{K}_0$ -subuniverse of  $\mathbf{B}$ ).

The perturbations of homogeneity that we have in mind are most easily stated in terms of “extension conditions,” by which we mean the following. Suppose  $\mathbf{A}$  is an algebra,  $A_0$  is a subuniverse, and  $\mathbf{B}$  and  $\mathbf{B}'$  are subalgebras of  $\mathbf{A}$  satisfying  $B \cap A_0 = B' \cap A_0$ . Then the *extension condition*  $ext(\mathbf{A}, A_0, \mathbf{B}, \mathbf{B}')$  is the claim that every isomorphism  $\sigma : \mathbf{B} \cong \mathbf{B}'$  satisfying  $\sigma|_{B \cap A_0} = id_{B \cap A_0}$  can be extended to an automorphism  $\hat{\sigma}$  of  $\mathbf{A}$  satisfying

$\hat{\sigma}|_{A_0} = \text{id}_{A_0}$ . So for example, a locally finite algebra  $\mathbf{A}$  is homogeneous if and only if  $\text{ext}(\mathbf{A}, \text{Sg}^{\mathbf{A}}(\emptyset), \mathbf{B}, \mathbf{B}')$  holds for all finite subalgebras  $\mathbf{B}$  and  $\mathbf{B}'$  of  $\mathbf{A}$ .

**Definition 2.2** Let  $\mathcal{K}$  be a locally finite universal class of algebras in a finite language, and  $n \in \{0, 1, 2, 3\}$ .  $\mathcal{K}$  satisfies  $H_n$  if there exists a finite set  $\mathcal{K}_0$  (called a *witnessing set*) of finite members of  $\mathcal{K}$  satisfying the conditions (1) and (2) above and

- (if  $n = 0$ ) for every  $\mathbf{A} \in \mathcal{K}_{fin}$  there exists a maximal  $\mathcal{K}_0$ -subuniverse  $A_0$  of  $\mathbf{A}$  such that for all subalgebras  $\mathbf{B}, \mathbf{B}'$  of  $\mathbf{A}$  satisfying  $A_0 \subseteq B$  and  $A_0 \subseteq B'$ ,  $\text{ext}(\mathbf{A}, A_0, \mathbf{B}, \mathbf{B}')$  holds.
- (if  $n = 1$ ) for every  $\mathbf{A} \in \mathcal{K}_{fin}$  and every maximal  $\mathcal{K}_0$ -subuniverse  $A_0$  of  $\mathbf{A}$ , and for all subalgebras  $\mathbf{B}, \mathbf{B}'$  of  $\mathbf{A}$  satisfying  $A_0 \subseteq B$  and  $A_0 \subseteq B'$ ,  $\text{ext}(\mathbf{A}, A_0, \mathbf{B}, \mathbf{B}')$  holds.
- (if  $n = 2$ ) for every  $\mathbf{A} \in \mathcal{K}_{fin}$  and every maximal  $\mathcal{K}_0$ -subuniverse  $A_0$  of  $\mathbf{A}$ , and for all subalgebras  $\mathbf{B}, \mathbf{B}'$  of  $\mathbf{A}$  for which  $B \cap A_0 = B' \cap A_0$ , if  $B \cap A_0$  is itself a maximal  $\mathcal{K}_0$ -subuniverse of both  $\mathbf{B}$  and  $\mathbf{B}'$  then  $\text{ext}(\mathbf{A}, A_0, \mathbf{B}, \mathbf{B}')$  holds.
- (if  $n = 3$ ) for every  $\mathbf{A} \in \mathcal{K}_{fin}$  and every subuniverse  $A_0$  of  $\mathbf{A}$ , if there exists  $\mathbf{C} \in \mathcal{K}$  and a maximal  $\mathcal{K}_0$ -subuniverse  $C_0$  of  $\mathbf{C}$  such that  $\mathbf{A} \leq \mathbf{C}$  and  $A_0 = A \cap C_0$ , then for all subalgebras  $\mathbf{B}, \mathbf{B}'$  of  $\mathbf{A}$  for which  $B \cap A_0 = B' \cap A_0$ ,  $\text{ext}(\mathbf{A}, A_0, \mathbf{B}, \mathbf{B}')$  holds.

For the sake of completeness, we shall also say that  $\mathcal{K}$  satisfies  $H_4$  if  $\mathcal{K}$  is homogeneous. Clearly  $H_3 \Rightarrow H_2 \Rightarrow H_1 \Rightarrow H_0$  since (for  $n < 3$ )  $H_n$  requires fewer extension conditions than does  $H_{n+1}$  with respect to a given set  $\mathcal{K}_0$ . And  $H_4$  implies  $H_3$  because if  $\mathcal{K}$  is homogeneous then we can always let  $\mathcal{K}_0$  be the set (possibly empty) consisting of one member from each isomorphism class of the 0-generated members of  $\mathcal{K}$ . ( $\mathcal{K}_0$  will be finite because the language of  $\mathcal{K}$  is finite.)

On the other hand,  $H_3$  is strictly weaker than  $H_4$  since, for example, if  $\mathcal{K}_0$  is any finite set of finite algebras (in a finite language) then  $\text{IS}(\mathcal{K}_0)$  satisfies  $H_3$  ( $\text{S}(\mathcal{K}_0)$  is its witnessing set). In our earlier paper [11] we called a (locally finite universal) class *almost locally homogeneous* if it satisfied  $H_3$ , and proved that if  $\mathcal{K}$  satisfies  $H_3$  and is finitely axiomatizable, then  $\text{V}(\mathcal{K}^{\dagger})$  is decidable. In Section 3 we will improve that result by replacing  $H_3$  with  $H_2$ .

An example of a class satisfying  $H_2$  but not  $H_3$  is the smallest universal class  $\mathcal{K}$  of lattices containing the lattice  $\mathbf{M}_\omega^\omega$  pictured in Figure 1. A set witnessing  $H_2$  for  $\mathcal{K}$  is the set consisting of a 5-element chain and its subchains.  $\mathcal{K}$  does not satisfy  $H_3$ , for if  $\mathcal{K}_0$  were a witnessing set for  $H_3$  and  $C_0$  were a maximal  $\mathcal{K}_0$ -subuniverse of  $\mathbf{M}_\omega^\omega$ , then because  $C_0$  is finite it would be possible to find a finite sublattice  $\mathbf{A}$  of  $\mathbf{M}_\omega^\omega$  and an atom  $x$  and a coatom  $y$  of  $\mathbf{M}_\omega^\omega$  which belong to  $A$  but not to  $C_0$ . Put  $A_0 = A \cap C_0$ ,  $B = \{x\}$  and  $B' = \{y\}$ ; then the extension condition  $ext(\mathbf{A}, A_0, \mathbf{B}, \mathbf{B}')$  should hold, but doesn't.

$H_0$  is equivalent to the following condition: there exist locally finite universal classes  $\mathcal{K}_1, \dots, \mathcal{K}_r$  in the languages  $L_1, \dots, L_r$  respectively, where each  $L_i$  is an expansion of  $L$  by finitely many constant symbols and each  $\mathcal{K}_i$  is homogeneous, such that  $\mathcal{K} = \bigcup_{i=1}^r \mathcal{K}_i|_L$ . This is a fairly natural condition, and one might suspect that any decidability result for homogeneous classes could be lifted to classes satisfying  $H_0$ . But this is false: for example, if  $\mathcal{K}$  is the smallest universal class of lattices containing the lattice  $\mathbf{N}_\omega$ , then  $\mathcal{K}$  is locally finite, finitely axiomatizable, and satisfies  $H_0$ , yet  $V(\mathcal{K}^t)$  is undecidable. This  $\mathcal{K}$  is also an example of a class which satisfies  $H_0$  but not  $H_1$ .

We do not know whether  $H_1$  is strictly weaker than  $H_2$ . For all particular (locally finite universal) classes  $\mathcal{K}$  we have examined so far, it turns out that the following are equivalent: (i)  $V(\mathcal{K}^t)$  is decidable; (ii)  $\mathcal{K}$  satisfies  $H_1$ ; (iii)  $\mathcal{K}$  satisfies  $H_2$  and is finitely axiomatizable. We wonder if this is true in general.

### 3 Decidability

Until further notice, assume that  $\mathcal{K}$  is a locally finite universal class of algebras in a finite language  $L$  containing no constant symbols, and that  $\mathcal{K}_0$  is a finite set of finite members of  $\mathcal{K}$  which satisfies  $S(\mathcal{K}_0) \subseteq I(\mathcal{K}_0)$ .

Because  $L$  has no constant symbols, it is possible to find a finite  $L$ -algebra  $\mathbf{H}$  and a collection  $\mathcal{C}$  of subuniverses of  $\mathbf{H}$  satisfying

1.  $\emptyset \in \mathcal{C}$ .
2.  $I(\{\mathbf{C} \leq \mathbf{H} : \mathbf{C} \in \mathcal{C} \text{ and } \mathbf{C} \neq \emptyset\}) = I(\mathcal{K}_0)$ ;
3. For every  $\mathbf{C} \in \mathcal{C}$  with  $\mathbf{C} \neq \emptyset$ , every  $\mathbf{A} \in \mathcal{K}_0$  and every embedding  $\alpha : \mathbf{C} \hookrightarrow \mathbf{A}$  there exists a  $\mathbf{D} \in \mathcal{C}$  with  $\mathbf{C} \subseteq \mathbf{D}$  and an isomorphism  $\beta : \mathbf{A} \cong \mathbf{D}$  which satisfies  $\beta\alpha(c) = c$  for all  $c \in \mathbf{C}$ .



Fix such a pair  $(\mathbf{H}, \mathcal{C})$ . The next definition is taken from [11].

**Definition 3.1**

- (1)  $\mathcal{K}/\mathbf{H}$  denotes the class of all  $\mathbf{A} \in \mathcal{K}$  for which the set  $A_0 := A \cap H$  is a common subuniverse of  $\mathbf{A}$  and  $\mathbf{H}$  and, if nonempty,  $\mathbf{A}_0$  inherits the same operations from  $\mathbf{A}$  as it does from  $\mathbf{H}$ .
- (2) Suppose  $\mathbf{A}, \mathbf{B} \in \mathcal{K}/\mathbf{H}$ . An  *$\mathbf{H}$ -embedding from  $\mathbf{A}$  to  $\mathbf{B}$*  is an embedding  $\sigma : \mathbf{A} \hookrightarrow \mathbf{B}$  which satisfies  $\sigma(a) = a$  for all  $a \in A \cap H$  and  $\sigma(a) \notin H$  for all  $a \in A \setminus H$ . In this case write  $\sigma : \mathbf{A} \xrightarrow{\mathbf{H}} \mathbf{B}$ . An  *$\mathbf{H}$ -isomorphism* is an isomorphism which is an  $\mathbf{H}$ -embedding in the above sense.
- (3) If  $\mathbf{A}, \mathbf{B} \in \mathcal{K}/\mathbf{H}$ , write  $\mathbf{A} \xrightarrow{\mathbf{H}} \mathbf{B}$  to mean there exists an  $\mathbf{H}$ -embedding  $\sigma : \mathbf{A} \xrightarrow{\mathbf{H}} \mathbf{B}$ , and  $\mathbf{A} \cong^{\mathbf{H}} \mathbf{B}$  to mean there exists an  $\mathbf{H}$ -isomorphism from  $\mathbf{A}$  to  $\mathbf{B}$ .

Remark: If  $\mathbf{A}$  and  $\mathbf{B}$  are algebras in some expansion of  $\mathbf{L}$  and if  $\mathbf{A}|_{\mathbf{L}}, \mathbf{B}|_{\mathbf{L}} \in \mathcal{K}/\mathbf{H}$ , then by  $\mathbf{A} \xrightarrow{\mathbf{H}} \mathbf{B}$  ( $\mathbf{A} \cong^{\mathbf{H}} \mathbf{B}$ ) we mean the obvious thing, namely, there exists an embedding (isomorphism) from  $\mathbf{A}$  to  $\mathbf{B}$  which is an  $\mathbf{H}$ -embedding ( $\mathbf{H}$ -isomorphism) from  $\mathbf{A}|_{\mathbf{L}}$  to  $\mathbf{B}|_{\mathbf{L}}$ .

**Definition 3.2**

- (1)  $\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$  is the class of those  $\mathbf{A} \in \mathcal{K}/\mathbf{H}$  for which  $A \cap H$  is both an element of  $\mathcal{C}$  and a maximal  $\mathcal{K}_0$ -subuniverse of  $\mathbf{A}$ .
- (2)  $\Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$  is the class of all Boolean products  $\mathbf{A} \leq_{bp} \prod_{x \in X} \mathbf{A}_x$  which satisfy the following properties:
  - (i)  $\mathbf{A}_x \in \mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$  for all  $x \in X$ .
  - (ii) For all  $a \in A$  and  $c \in \bigcup \mathcal{C}$ , the set  $a^{-1}(c)$  is clopen in  $X$ .

The next lemma follows from the proof of Lemma 3.4 in [11].

**LEMMA 3.3** *With the above assumptions, for every countable member  $\mathbf{A}$  of  $\Gamma^a(\mathcal{K})$  there exists  $\mathbf{A}' \in \Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$  having the same underlying Boolean space as  $\mathbf{A}$  and satisfying  $\mathbf{A}(t) \cong \mathbf{A}'(t)$ . ■*

As in [2] and [11], the idea now is to show that if the stalks  $A_x$  of  $A \in \Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$  are “ $\mathbf{H}$ -homogeneous” in some specified sense, then one can achieve a satisfactory Feferman-Vaught-type analysis of  $A(t)$ .

First, we impose one further requirement on the choice of  $\mathbf{H}$  and  $\mathcal{C}$ , namely,

4.  $(\mathcal{C}, \subseteq)$  is a tree with  $\emptyset$  being the root.

Next, we strengthen the notion of *quantifier-free  $\mathbf{H}$ - $n$ -type* found in [11]. Let  $A \in \mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$  with  $A_0 = A \cap H$ , and suppose  $\bar{a} = \langle a_1, \dots, a_n \rangle \in A^n$  ( $n \geq 0$ ). Define  $C_A(\bar{a})$  to be the  $\subseteq$ -least member  $C$  of  $\mathcal{C}$  satisfying these properties:

1.  $\text{Sg}^A(\{a_1, \dots, a_n\} \cup C) \cap A_0 = C$ .
2. (if  $n > 0$ )  $C$  is a maximal  $\mathcal{K}_0$ -subuniverse of  $\text{Sg}^A(\{a_1, \dots, a_n\} \cup C)$ .

Clearly  $C := A_0$  satisfies these properties, so  $C_A(\bar{a})$  exists because  $\mathcal{C}$  is a tree. With  $C_A(\bar{a})$  so defined,  $M_A(\bar{a})$  denotes the subuniverse of  $A$  generated by  $\{a_1, \dots, a_n\} \cup C_A(\bar{a})$ . Thus if  $n > 0$  then  $M_A(\bar{a})$  is the smallest subalgebra of  $A$  which belongs to  $\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$  and whose universe contains  $\{a_1, \dots, a_n\}$ , while if  $n = 0$  then  $C_A(\bar{a}) = M_A(\bar{a}) = \emptyset$ .

**Definition 3.4** Fix an infinite sequence  $b_1, b_2, \dots$  of constant symbols not occurring in  $L$ . With  $A$  and  $\bar{a}$  as above, the *quantifier-free  $(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$ - $n$ -type of  $\bar{a}$  in  $A$*  is (some specification of) the  $\mathbf{H}$ -isomorphism type of the  $L \cup \{b_1, \dots, b_n\}$ -algebra  $\langle M_A(\bar{a}); a_1, \dots, a_n \rangle$  (assuming  $n > 0$ ; if  $n = 0$  then it is  $\emptyset$ ).  $\text{typ}(\bar{a})$  will denote the quantifier-free  $(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$ - $n$ -type of  $\bar{a}$  in  $A$ .

For example,  $\text{typ}(\bar{a})$  can be taken to be what model theorists call the “quantifier-free  $n$ -type of  $\bar{a}$  over the set  $C_A(\bar{a})$ .” Here are some properties of quantifier-free  $(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$ - $n$ -types.

**LEMMA 3.5** Suppose  $A, B \in \mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$ ,  $a_1, \dots, a_n, c \in A$  and  $b_1, \dots, b_n, d \in B$ .

- (i) If  $\langle A; a_1, \dots, a_n \rangle \xrightarrow{\mathbf{H}} \langle B; b_1, \dots, b_n \rangle$  then  $\text{typ}(\bar{a}) = \text{typ}(\bar{b})$ .
- (ii) If  $\text{typ}(\bar{a}, c) = \text{typ}(\bar{b}, d)$  then  $\text{typ}(\bar{a}) = \text{typ}(\bar{b})$ .

PROOF. (i) is left to the reader. (ii) is trivial if  $n = 0$ , so suppose  $\text{typ}(\bar{a}, c) = \text{typ}(\bar{b}, d)$  with  $n > 0$ . By definition this means

$$\langle \mathbf{M}_{\mathbf{A}}(\bar{a}, c); a_1, \dots, a_n, c \rangle \stackrel{\mathbf{H}}{\cong} \langle \mathbf{M}_{\mathbf{B}}(\bar{b}, d); b_1, \dots, b_n, d \rangle. \quad (1)$$

Hence  $\langle \mathbf{M}_{\mathbf{A}}(\bar{a}); a_1, \dots, a_n \rangle \stackrel{\mathbf{H}}{\hookrightarrow} \langle \mathbf{A}; a_1, \dots, a_n \rangle$  on the one hand, and

$$\begin{aligned} \langle \mathbf{M}_{\mathbf{A}}(\bar{a}); a_1, \dots, a_n \rangle &\stackrel{\mathbf{H}}{\hookrightarrow} \langle \mathbf{M}_{\mathbf{A}}(\bar{a}, c); a_1, \dots, a_n \rangle \\ &\stackrel{\mathbf{H}}{\cong} \langle \mathbf{M}_{\mathbf{B}}(\bar{b}, d); b_1, \dots, b_n \rangle && \text{by equation (1)} \\ &\stackrel{\mathbf{H}}{\hookrightarrow} \langle \mathbf{B}; b_1, \dots, b_n \rangle \end{aligned}$$

on the other. So by (i),  $\text{typ}(\bar{a}) = \text{typ}(\bar{b})$ . ■

Suppose  $\mathbf{A} \in \mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$  and  $\bar{a} \in A^n$ . Then  $\mathbf{M}_{\mathbf{A}}(\bar{a})$  is  $N$ -generated, where  $N = n + |H|$ . There are only finitely many  $N$ -generated algebras in  $\mathcal{K}$  up to isomorphism (observation of Weispfennig [10]). Therefore for each  $n \geq 0$  there are only finitely many quantifier-free  $(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$ - $n$ -types realized in  $\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$ . Let them be called  $p_{n,1}, \dots, p_{n,\lambda(n)}$ . The notation  $p_{n+1,i} \vdash p_{n,j}$  shall mean the obvious thing, i.e., for all  $\mathbf{A} \in \mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$  and  $a_1, \dots, a_n, c \in A$ , if  $\text{typ}(\bar{a}, c) = p_{n+1,i}$  then  $\text{typ}(\bar{a}) = p_{n,j}$ . By Lemma 3.5(ii), for each  $i = 1, \dots, \lambda(n+1)$  there is a (unique)  $j$  such that  $p_{n+1,i} \vdash p_{n,j}$ . Similarly, if  $\phi(x_1, \dots, x_n)$  is a quantifier-free  $L(\mathbf{t})$ -formula then  $p_{n,i} \vdash \phi$  means  $\mathbf{A}^\dagger \models \phi(\bar{a})$  whenever  $\mathbf{A} \in \Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$ ,  $\bar{a} \in A^n$ , and  $\text{typ}(\bar{a}) = p_{n,i}$ .

For each  $n > 0$  fix a finite set  $\mathcal{K}_n$  of finite members of  $\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$  with the following properties:

1. For each  $n > 0$  and  $i = 1, \dots, \lambda(n)$  there exists an  $\mathbf{A} \in \mathcal{K}_n$  and  $\bar{a} \in A^n$  such that  $\text{typ}(\bar{a}) = p_{n,i}$  and  $\mathbf{M}_{\mathbf{A}}(\bar{a}) = A$ .
2. Conversely, for every  $\mathbf{A} \in \mathcal{K}_n$  there exist  $\bar{a} \in A^n$  such that  $\mathbf{M}_{\mathbf{A}}(\bar{a}) = A$ .
3. For all  $\mathbf{A}, \mathbf{B} \in \mathcal{K}_n$ , if  $\mathbf{A} \stackrel{\mathbf{H}}{\cong} \mathbf{B}$  then  $\mathbf{A} = \mathbf{B}$ .
4.  $\mathcal{K}_n \subseteq \mathcal{K}_{n+1}$  for all  $n > 0$ .

Also let  $\mathcal{K}_\omega = \bigcup_{n>0} \mathcal{K}_n$ .

Now suppose  $\mathbf{A} \leq_{bp} \prod_{x \in X} \mathbf{A}_x$  is in  $\Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$ . For each  $\mathbf{B} \in \mathcal{K}_\omega$  let  $U_{\mathbf{B}} = \{x \in X : \mathbf{B} \stackrel{\mathbf{H}}{\hookrightarrow} \mathbf{A}_x\}$ . Because of item (ii) in the definition of

$\Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$ ,  $U_{\mathbf{B}}$  is an open subset of  $X$ . As in [11],  $\vec{X}(\mathbf{A})$  denotes  $\langle X, (U_{\mathbf{B}})_{\mathbf{B} \in \mathcal{K}_\omega} \rangle$  and  $\vec{X}(\mathbf{A})^*$  denotes the dual Boolean algebra with distinguished family of ideals indexed by  $\mathcal{K}_\omega$ . Finally, if  $\bar{a} \in A^n$  and  $1 \leq i \leq \lambda(n)$  then by  $\llbracket p_{n,i}(\bar{a}) \rrbracket$  we mean the set of those  $x \in X$  for which the quantifier-free  $(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$ - $n$ -type of  $\langle a_1(x), \dots, a_n(x) \rangle$  in  $\mathbf{A}_x$  is  $p_{n,i}$ .

The next three lemmas provide the ingredients needed for the general argument from [11] to work.

**LEMMA 3.6** *Suppose  $\mathbf{A} \leq_{bp} \prod_{x \in X} \mathbf{A}_x$  is in  $\Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$  and  $\bar{a} \in A^n$  ( $n \geq 0$ ). Then  $(\llbracket p_{n,i}(\bar{a}) \rrbracket)_{i=1}^{\lambda(n)}$  is a clopen partition of  $X$ .*

**PROOF.** It is enough to show that each  $\llbracket p_{n,i}(\bar{a}) \rrbracket$  is open. This follows from Lemma 3.5(i) and item (ii) in the definition of  $\Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$ . ■

**LEMMA 3.7** *Let  $\mathcal{F} = \{\mathbf{B}_1, \dots, \mathbf{B}_k\}$  be a finite subset of  $\mathcal{K}_\omega$ . Suppose each  $\mathbf{B}_i$  is  $n_i$ -generated, and let  $N = n_1 + \dots + n_k + |H| + 1$ . Then for every  $\mathbf{A} \leq_{bp} \prod_{x \in X} \mathbf{A}_x$  in  $\Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$  for which the Boolean algebra  $X^*$  of clopen sets of  $X$  is countable, there is a subalgebra  $\mathbf{A}' \leq \mathbf{A}$  such that*

- (i)  $\mathbf{A}' \in \Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$  (with the same topology on  $X$ ).
- (ii)  $U'_{\mathbf{B}} = U_{\mathbf{B}}$  for each  $\mathbf{B} \in \mathcal{F}$ .
- (iii)  $\mathbf{A}'_x$  is  $N$ -generated for each  $x \in X$ .

**PROOF.** This is a minor variation of Lemma 3.5 in [11]. To ensure that  $\mathbf{A}'$  is in  $\Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$  it is enough to have  $\mathbf{A}' \leq \mathbf{A}$ ,  $\mathbf{A}'$  closed under patchwork, and  $A_x \cap H \subseteq A'_x$  for each  $x \in X$ . The inclusion of  $|H|$  in the formula for  $N$  is what allows us to meet this last demand. ■

**LEMMA 3.8** *Suppose, in addition to everything else assumed of  $\mathcal{K}$  and  $\mathcal{K}_0$ , that  $\mathcal{K}$  satisfies the homogeneity condition  $H_2$  and that  $\mathcal{K}_0$  is a witnessing set. Suppose further that  $\mathbf{A} \in \mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$ , that  $\bar{a} \in A^n$ , and that  $p_{n+1,i}$  is a quantifier-free  $(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$ - $(n+1)$ -type satisfying  $p_{n+1,i} \vdash \text{typ}(\bar{a})$ . Pick  $\mathbf{B} \in \mathcal{K}_{n+1}$  and  $\langle \bar{b}, d \rangle \in B^{n+1}$  satisfying  $M_{\mathbf{B}}(\bar{b}, d) = B$  and  $\text{typ}(\bar{b}, d) = p_{n+1,i}$ . Then*

$$(\exists c \in A)(\text{typ}(\bar{a}, c) = p_{n+1,i}) \quad \text{if and only if} \quad \mathbf{B} \xrightarrow{\mathbf{H}} \mathbf{A}.$$

PROOF. ( $\Rightarrow$ ) is obvious.

( $\Leftarrow$ ) The claim is trivial if  $n = 0$ , so assume  $n > 0$ . Let  $A_0 = A \cap H$  and suppose  $\gamma : \mathbf{B} \xrightarrow{\mathbf{H}} \mathbf{A}$ . Also let  $M = M_{\mathbf{A}}(\bar{a})$ ; thus  $M \leq \mathbf{A}$ ,  $M \in \mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$ , and there exists  $\alpha : M \xrightarrow{\mathbf{H}} \mathbf{B}$  satisfying  $\alpha(\bar{a}) = \bar{b}$ . Finally let  $M' = \gamma\alpha(M)$  and  $D = \text{Sg}^{\mathbf{A}}(M \cup \gamma(B) \cup A_0)$ . We have:

1.  $M$ ,  $M'$  and  $D$  are finite members of  $\mathcal{K}$  with  $M \leq D$  and  $M' \leq D$ .
2.  $A_0$  is a maximal  $\mathcal{K}_0$ -subuniverse of  $D$ .
3.  $M \cap A_0 = M' \cap A_0 (= C_{\mathbf{A}}(\bar{a}))$  and the common intersection is a maximal  $\mathcal{K}_0$ -subuniverse of both  $M$  and  $M'$ .
4.  $\gamma\alpha : M \cong M'$  satisfies  $\gamma\alpha|_{M \cap A_0} = \text{id}_{M \cap A_0}$ .

Because  $\mathcal{K}_0$  witnesses  $H_2$  for  $\mathcal{K}$ ,  $\gamma\alpha$  extends to an automorphism  $\mu$  of  $D$  satisfying  $\mu|_{A_0} = \text{id}_{A_0}$ . Let  $\eta = \mu^{-1}\gamma$  and  $c = \eta(d)$ . Then  $\eta : \mathbf{B} \xrightarrow{\mathbf{H}} \mathbf{A}$  and  $\eta(\bar{b}) = \bar{a}$ , so by Lemma 3.5(i) and our choice of  $\mathbf{B}$ ,  $\text{typ}(\bar{a}, c) = \text{typ}(\eta(\bar{b}), \eta(d)) = \text{typ}(\bar{b}, d) = p_{n+1, i}$ .  $\blacksquare$

Here is the principal result of this section.

**THEOREM 3.9** *Suppose  $\mathcal{K}$  is a locally finite universal class of algebras in a finite language. If  $\text{Th}_{\forall}(\mathcal{K})$  is decidable (for example, if  $\mathcal{K}$  is finitely axiomatizable) and  $\mathcal{K}$  satisfies  $H_2$ , then  $\forall(\mathcal{K}^{\dagger})$  is decidable.*

PROOF. The proof is nearly identical to the proofs of Theorems 2.7 and 3.3 in [11], so we give only a sketch. Both the hypotheses and the conclusion are invariant under the transformation of  $\mathcal{K}$  which replaces each constant by a constant unary operation, so we may assume that the language  $L$  of  $\mathcal{K}$  contains no constant symbols. Let  $\mathcal{K}_0$  witness the property  $H_2$  for  $\mathcal{K}$ , and choose  $\mathbf{H}$  and  $\mathcal{C}$  as above. Because  $\text{Th}_{\forall}(\mathcal{K})$  is decidable, there exist effective representations of (1) the quantifier-free  $(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$ - $n$ -types realized in  $\mathcal{K}$ , and (2) the members of  $\mathcal{K}_{\omega}$ . As well, there are effective procedures which:

1. Given  $n \geq 0$ , list the quantifier-free  $(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$ - $n$ -types  $p_{n,1}, \dots, p_{n,\lambda(n)}$ .
2. Given  $n > 0$ , list the members of  $\mathcal{K}_n$ .

3. Given  $n > 0$  and  $1 \leq i \leq \lambda(n)$ , specify a member  $\mathbf{B}$  of  $\mathcal{K}_n$  for which there exists  $\bar{b} \in B^n$  satisfying  $\text{typ}(\bar{b}) = p_{n,i}$  and  $M_{\mathbf{B}}(\bar{b}) = B$ .
4. Given  $n \geq 0$ ,  $1 \leq i \leq \lambda(n+1)$ , and  $1 \leq j \leq n$ , determine whether  $p_{n+1,i} \vdash p_{n,j}$ .
5. Given  $n \geq 0$ ,  $1 \leq i \leq \lambda(n)$ , and a quantifier-free  $L(t)$ -formula  $\phi(x_1, \dots, x_n)$ , determine whether  $p_{n,i} \vdash \phi$ .
6. Given  $\mathbf{B} \in \mathcal{K}_\omega$ , determine the least integer  $n$  such that  $\mathbf{B}$  is  $n$ -generated.

These facts together with Lemmas 3.6 and 3.8 guarantee (as in [11]) the following Feferman-Vaught-like theorem: There is a recursive procedure which, given an arbitrary  $L(t)$ -formula  $\phi(x_1, \dots, x_n)$ , produces a formula  $\Phi(X_1, \dots, X_{\lambda(n)})$  in the first-order language for Boolean algebras with a family of distinguished ideals indexed by  $\mathcal{K}_\omega$ , such that for all  $\mathbf{A} \in \Gamma^\alpha(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$  and all  $\bar{a} \in A^n$ ,

$$\mathbf{A}(t) \models \phi(\bar{a}) \quad \text{if and only if} \quad \vec{X}(\mathbf{A})^* \models \Phi(\llbracket p_{n,1}(\bar{a}) \rrbracket, \dots, \llbracket p_{n,\lambda(n)}(\bar{a}) \rrbracket).$$

Lemma 3.7 then implies (again as in [11]) that there is an effectively computable function from the set of  $L(t)$ -sentences to the set of positive integers, written  $\phi \mapsto n(\phi)$ , with the property that, for each  $\phi$ ,  $V(\mathcal{K}^t) \models \phi$  if and only if  $V((\mathcal{K}_{n(\phi)})^t) \models \phi$ . Since  $\mathcal{K}_{n(\phi)}$  is finite,  $V((\mathcal{K}_{n(\phi)})^t)$  is decidable (uniformly in the “input parameter”  $\mathcal{K}_{n(\phi)}$ ) by the result of Burris and Werner [4]. Hence  $V(\mathcal{K}^t)$  is decidable.  $\blacksquare$

## 4 Lattices

We shall use the following terminology. An  $M_\lambda$  is any lattice of height 2 having at least two atoms. If  $\mathbf{L}$  is a lattice, then an  $M_\lambda$  in  $\mathbf{L}$  is an  $M_\lambda$  which is a sublattice of  $\mathbf{L}$ . An  $M_\lambda$  is *free in*  $\mathbf{L}$  if:

1. It is an interval sublattice  $\mathbf{I}[a, b]$  of  $\mathbf{L}$ , and
2. Is such that each atom of  $\mathbf{I}[a, b]$  is both meet-irreducible and join-irreducible in  $\mathbf{L}$ .

**Definition 4.1** A *skeleton* of  $\mathbf{L}$  is a subset  $S \subseteq L$  satisfying:

- (1) If  $x \in L \setminus S$  then  $x$  is an atom of some free  $M_\lambda$  in  $\mathbf{L}$ .
- (2) For each free  $M_\lambda$  in  $\mathbf{L}$ , exactly one of its atoms belongs to  $S$ .

Every lattice has a skeleton. Skeletons are sublattices, and any two skeletons of the same lattice are isomorphic. The following facts are easily proved from the definition: (i) if  $\mathbf{L}$  is a lattice,  $S$  is a skeleton of  $\mathbf{L}$ , and  $a$  and  $b$  are elements of  $S$  such that  $a \prec b$  in  $\mathbf{S}$ , then  $a \prec b$  in  $\mathbf{L}$ ; (ii) if  $\mathbf{L}$  has a finite skeleton, then  $\mathbf{L}$  is locally finite.

Here are two more obvious consequences. Let  $\mathbf{L}$  be a lattice with a skeleton  $S$ , and let  $(b_i)_{i < \eta}$  be the distinct elements of  $S$  which are atoms of free  $M_\lambda$ s in  $\mathbf{L}$ . For each  $i < \eta$  let  $X_i$  be the set of all atoms of the free  $M_\lambda$  containing  $b_i$  and let  $\lambda_i = |X_i|$ . Then: (iii) the isomorphism type of  $\mathbf{L}$  is determined by the isomorphism type of the pointed skeleton  $\langle \mathbf{S}; (b_i)_{i < \eta} \rangle$  and the sequence  $(\lambda_i)_{i < \eta}$  of "dimensions"; (iv) for each family of bijections  $(\sigma_i : X_i \rightarrow X_i)_{i < \eta}$  satisfying  $\sigma_i(b_i) = b_i$  there is an automorphism  $\sigma$  of  $\mathbf{L}$  satisfying  $\sigma|_S = \text{id}_S$  and  $\sigma|_{X_i} = \sigma_i$  for all  $i < \eta$ .

If  $\mathcal{K}$  is a class of lattices then we say that  $\mathcal{K}$  has *bounded skeletons* if there is a finite upper bound to the cardinality of the skeletons of members of  $\mathcal{K}$ .

**LEMMA 4.2** *Suppose  $\mathcal{K}$  is a universal class of lattices.  $\mathcal{K}$  has bounded skeletons if and only if  $\mathcal{K}$  omits each of  $\mathbf{Q}$ ,  $\mathbf{N}_\omega$ ,  $\mathbf{M}_{\omega,2}$ , and  $\mathbf{M}_{2,\omega}$ .*

**PROOF.** Each of the above four lattices is its own skeleton, so  $(\Rightarrow)$  follows. Conversely, assume that  $\mathcal{K}$  is a universal class of lattices which omits each of the above four lattices.

**Claim 1** 1. *There is a finite upper bound to the height of members of  $\mathcal{K}$ .*

2. *For sufficiently large  $n$ , the following is true: if  $\mathbf{L} \in \mathcal{K}$ ,  $a, b_1, \dots, b_n, c$  are distinct elements of  $L$  with  $a \prec b_i \prec c$  for all  $i$ , and  $\{a, b_1, \dots, b_n, c\}$  is an  $M_\lambda$  in  $\mathbf{L}$ , then  $\mathbf{I}[a, c]$  is an  $M_\lambda$  in  $\mathbf{L}$ .*

3. *For sufficiently large  $n$ , if  $\mathbf{L} \in \mathcal{K}$  and  $\mathbf{I}[a, c]$  is an  $M_\lambda$  in  $\mathbf{L}$  with at least  $n$  elements, then  $\mathbf{I}[a, c]$  is a free  $M_\lambda$  in  $\mathbf{L}$ .*

**PROOF.** These are easy consequences of the compactness theorem. (1) is equivalent to the fact that  $\mathcal{K}$  omits  $\mathbf{Q}$ . Because  $\mathcal{K}$  omits  $\mathbf{N}_\omega$ , for sufficiently

large  $n$  the hypotheses of (2) imply  $b_i \prec c$  for all  $i$ . (If not, say  $b_1 \not\prec c$ , then because the height of  $\mathbf{L}$  is bounded we could pick  $x \in L$  such that  $b_1 \prec x \prec c$ . Then  $b_i \not\prec x$  would imply  $x \wedge b_i = a$  for  $i \neq 1$ , so  $\{a, b_1, \dots, b_n, x, c\}$  would be isomorphic to an arbitrarily large sublattice of  $\mathbf{N}_\omega$  of height 3.) Now let  $X = \{b \in L : a \prec b \prec c\}$ . If  $I[a, c] \neq X \cup \{a, c\}$ , then because  $\mathbf{L}$  has finite height we could pick a minimal element  $y$  of  $I[a, c] \setminus (X \cup \{a, c\})$ , which must be incomparable to each element of  $X$ . Hence  $a \prec y$  (by minimality) and  $\{a, b_1, \dots, b_n, y, c\}$  is an  $M_\lambda$ . But then the previous argument implies  $y \prec c$ , so  $y \in X$ , a contradiction. Thus  $I[a, c] = X \cup \{a, c\}$ , so  $I[a, c]$  is an  $M_\lambda$ , which establishes (2). To show (3), suppose  $\mathbf{L} \in \mathcal{K}$  has an interval  $I[a, c]$  which is an arbitrarily large (possibly infinite)  $M_\lambda$  but which is not free in  $\mathbf{L}$ . As the height of  $\mathbf{L}$  is bounded, there is an atom  $b$  of  $I[a, c]$  and an element  $x \in L \setminus I[a, c]$  such that either  $x \prec b$  or  $b \prec x$ . If  $b \prec x$  then  $I[a, c] \cup \{x, x \vee c\}$  is the universe of a sublattice of  $\mathbf{L}$  which either contains  $\mathbf{M}_{\omega, 2}$  as a sublattice (if  $I[a, c]$  is infinite) or is isomorphic to an arbitrarily large sublattice of  $\mathbf{M}_{\omega, 2}$  of height 3. Both alternatives contradict the fact that  $\mathcal{K}$  omits  $\mathbf{M}_{\omega, 2}$ . A similar argument using  $\mathbf{M}_{2, \omega}$  works in case  $x \prec b$ . ■

**Claim 2** *There is a positive integer  $N$  with the following property: if  $S$  is a skeleton of some member of  $\mathcal{K}$ , then as a lattice  $\mathbf{S}$  satisfies*

(\*) *If  $a, b_1, \dots, b_n, c$  are distinct elements of  $S$  with  $a \prec b_i \prec c$  for each  $i$ , and if  $\{a, b_1, \dots, b_n, c\}$  is an  $M_\lambda$  in  $\mathbf{S}$ , then  $n \leq N$ .*

PROOF. Suppose instead that  $n$  can be arbitrarily large. Choose  $\mathbf{L} \in \mathcal{K}$  such that  $\mathbf{S} \leq \mathbf{L}$  and  $S$  is a skeleton of  $\mathbf{L}$ . By a remark following Definition 4.1, each  $b_i$  covers  $a$  in  $\mathbf{L}$  as well as in  $\mathbf{S}$ . So by items (2) and (3) from the previous claim, the interval sublattice  $I[a, c]$  of  $\mathbf{L}$  is a free  $M_\lambda$  in  $\mathbf{L}$  (assuming  $n$  is sufficiently large). But then only one of its atoms should be in  $S$ , which is not the case. ■

Until further notice,  $N$  will be a fixed positive integer satisfying the assertion in the previous claim.

**Claim 3** *Let  $\mathcal{C}$  be the class of all lattices  $\mathbf{S} \in \mathcal{K}$  which satisfy the condition (\*) given in Claim 2. There is a finite upper bound to the cardinality of the members of  $\mathcal{C}$ .*



PROOF. Note that  $\mathcal{C}$  is closed under taking interval sublattices. We shall prove the claim by induction on the height of  $\mathbf{S}$ . If  $\mathbf{S}$  has height 2, then  $|S| \leq N + 2$  by condition (\*). Suppose  $h > 2$  and the claim is true for all members of  $\mathcal{C}$  of height less than  $h$ . Let  $\mathbf{S} \in \mathcal{C}$  be of height  $h$ ; let 0 and 1 be its least and greatest elements respectively, and let  $A$  be its set of atoms. For each  $a \in A$  the interval  $I[a, 1]$  is a member of  $\mathcal{C}$  of height less than  $h$ , so the cardinality of  $I[a, 1]$  is bounded. Thus assuming the claim to be false for the members of  $\mathcal{C}$  of height  $h$ , it must be possible to choose  $\mathbf{S}$  as above with  $|A|$  arbitrarily large (perhaps infinite). Do so, and then pick a subset  $A_0$  of  $A$  which is maximal with respect to the property that the join of any two distinct elements of  $A_0$  is 1. By condition (\*),  $|A_0| \leq N$  and so  $|A \setminus A_0|$  can be arranged to be sufficiently large.

By the maximality of  $A_0$ , for each  $x \in A \setminus A_0$  there exists  $a \in A_0$  such that  $x \vee a < 1$ . Because  $|A_0|$  is bounded, a fixed  $a_0 \in A_0$  can be chosen with the property that the set  $X = \{x \in A \setminus A_0 : a_0 \vee x < 1\}$  is sufficiently large. For each  $x \in X$  there is a coatom  $c \in I[a_0, 1]$  satisfying  $x \leq c$ . Again because  $I[a_0, 1]$  is bounded it is possible to choose a fixed coatom  $c_0 \in I[a_0, 1]$  dominating a sufficiently large subset of  $X$ . But that contradicts the fact that  $I[0, c_0]$ , being a member of  $\mathcal{C}$  of height less than  $h$ , must have bounded cardinality. ■

The previous two claims imply that  $\mathcal{K}$  has bounded skeletons, proving ( $\Leftarrow$ ) of Lemma 4.2. ■

Here is the main result of the paper.

**THEOREM 4.3** *For any universal class  $\mathcal{K}$  of lattices, the following are equivalent:*

- (1)  $V(\mathcal{K}^t)$  is decidable.
- (2)  $V(\mathcal{K}^t)$  is not hereditarily undecidable.
- (3)  $\mathcal{K}$  omits  $\mathbf{Q}$ ,  $\mathbf{N}_\omega$ ,  $\mathbf{M}_{\omega,2}$  and  $\mathbf{M}_{2,\omega}$ .
- (4)  $\mathcal{K}$  has bounded skeletons.
- (5)  $\mathcal{K}$  is locally finite and satisfies  $H_1$ .
- (6)  $\mathcal{K}$  is locally finite, finitely axiomatizable, and satisfies  $H_2$ .

PROOF. (1)  $\Rightarrow$  (2) and (6)  $\Rightarrow$  (5) are trivial. (3)  $\Leftrightarrow$  (4) is a restatement of Lemma 4.2, while (6)  $\Rightarrow$  (1) follows from Theorem 3.9. We will show  $\neg(3) \Rightarrow \neg(2)$  &  $\neg(5)$  and then (4)  $\Rightarrow$  (6).

Suppose (3) fails; so  $\mathcal{K}$  contains one of  $\mathbf{Q}$ ,  $\mathbf{N}_\omega$ ,  $\mathbf{M}_{\omega,2}$  or  $\mathbf{M}_{2,\omega}$ .

CASE 1. Assume  $\mathbf{N}_\omega \in \mathcal{K}$ . For future reference, name the elements of the unique 4-element chain in  $\mathbf{N}_\omega$  by  $0 \prec a \prec b \prec 1$ , let the atoms of  $\mathbf{N}_\omega$  other than  $a$  be  $a_i$  ( $i < \omega$ ), and put  $M_\omega = N_\omega \setminus \{b\}$  and  $M_\omega^- = M_\omega \setminus \{a\}$ .

We first show that  $V(\mathcal{K}^t)$  is hereditarily undecidable by verifying the hypotheses of Lemma 1.1. (This was already done in [9]; for the sake of completeness, we repeat the proof here.) Let  $\mathbf{A} = \mathbf{N}_\omega$  and  $\mathbf{S} = \mathbf{M}_\omega$ . Note that the automorphism group of  $\mathbf{M}_\omega$  acts transitively on its set of atoms. Let  $\mu(x)$ ,  $\tau(x)$  and  $\psi(z)$  be formulas asserting respectively that  $x$  is an atom,  $x$  belongs to a 4-element chain, and  $z$  is a coatom belonging to a 4-element chain. Then  $\mu$ ,  $\tau$  and  $\psi$  witness the hypotheses of Lemma 1.1.

Next suppose that  $\mathcal{K}_0$  is a finite set of finite lattices witnessing  $H_1$  for  $\mathcal{K}$ , and let  $L_0$  be a maximal  $\mathcal{K}_0$ -subuniverse of  $\mathbf{N}_\omega$ . If  $L_0$  could be embedded in  $\mathbf{M}_\omega^-$  (i.e., if  $\{0, a, b, 1\} \not\subseteq L_0$ ), say  $L_0 \cong L_1 \leq \mathbf{M}_\omega^-$ , then  $L_1$  must itself be a maximal  $\mathcal{K}_0$ -subuniverse of  $\mathbf{N}_\omega$ . Choose  $a_i \notin L_1$ . Then the isomorphism from  $\mathbf{Sg}(L_1 \cup \{a_i\})$  to  $\mathbf{Sg}(L_1 \cup \{a\})$  sending  $a_i$  to  $a$  and fixing all other elements cannot be extended to an automorphism of  $\mathbf{N}_\omega$ , violating  $H_1$ .

Hence every maximal  $\mathcal{K}_0$ -subuniverse of  $\mathbf{N}_\omega$  must contain  $\{0, a, b, 1\}$ . Let  $L_0$  be a maximal  $\mathcal{K}_0$ -subuniverse of  $\mathbf{N}_\omega$  of greatest cardinality, and put  $L'_0 = L_0 \setminus \{b\}$ . As before we can find a  $L_1 \leq \mathbf{M}_\omega^-$  such that  $L_1 \cong L'_0$ . Since  $L'_0 \in \text{IS}(\mathcal{K}_0) = \text{I}(\mathcal{K}_0)$ , we can extend  $L_1$  to a maximal  $\mathcal{K}_0$ -subuniverse  $L_2$  of  $\mathbf{N}_\omega$ . By the previous argument,  $\{a, b\} \subseteq L_2 \setminus L_1$ , so  $|L_2| \geq |L_1| + 2 = |L_0| + 1$ , contradicting our choice of  $L_0$ . This proves that  $\mathcal{K}$  does not satisfy  $H_1$ .

CASE 2. Assume  $\mathbf{M}_{\omega,2} \in \mathcal{K}$ . (A dual argument will take care of the case when  $\mathbf{M}_{2,\omega} \in \mathcal{K}$ .) Let  $b$  be the join-irreducible coatom of  $\mathbf{M}_{\omega,2}$ , let  $c$  be the other coatom, and let  $a$  be the unique lower cover of  $b$ .

To show that  $V(\mathcal{K}^t)$  is hereditarily undecidable, let  $\mathbf{A} = \mathbf{M}_{\omega,2}$  and  $\mathbf{S} = \mathbf{I}[0, c]$ , and let  $\mu(x)$ ,  $\tau(x)$  and  $\psi(z)$  be formulas asserting respectively that  $x$  is an atom,  $x$  is meet-reducible, and  $z$  is a join-irreducible coatom. Then  $\mu$ ,  $\tau$  and  $\psi$  witness the hypotheses of Lemma 1.1.

An argument like the one in Case 1 can be given to show that  $\mathcal{K}$  does not satisfy  $H_1$ . One uses  $\mathbf{I}[0, c] \setminus \{a\}$  and  $\{0, a, b, c, 1\}$  in place of  $M_\omega^-$  and  $\{0, a, b, 1\}$ . The details are left to the reader.

CASE 3. Assume  $\mathbf{Q} \in \mathcal{K}$ . We repeat the proof from [9] that  $\mathcal{K}$  is hereditarily undecidable. Let  $\mathbf{A} = \{x \in \mathbf{Q} : |x| \geq 1\}$  and  $S = \mathbf{A} \setminus \{1\}$ . Note that  $\mathbf{S} \cong \mathbf{Q}$  so the automorphism group of  $\mathbf{S}$  is transitive. Let  $\mu(x)$ ,  $\tau(x)$  and  $\psi(z)$  be formulas asserting respectively that  $x = x$ ,  $x$  has an upper cover, and  $z$  has a lower cover. Then  $\mu$ ,  $\tau$  and  $\psi$  witness the hypotheses of Lemma 1.1.

To show that  $\mathcal{K}$  does not satisfy  $H_1$ , suppose that  $\mathcal{K}_0$  witnesses  $H_1$  for  $\mathcal{K}$  and let  $L_0$  be a maximal  $\mathcal{K}_0$ -subuniverse of  $\mathbf{A}$ . Thus  $L_0$  is an  $n$ -element chain, and every  $n$ -element subchain of  $\mathbf{A}$  is a maximal  $\mathcal{K}_0$ -subuniverse of  $\mathbf{A}$ . In particular,  $L_1 = \{-n, -(n-1), \dots, -2, -1\}$  is a maximal  $\mathcal{K}_0$ -subuniverse of  $\mathbf{A}$ . But the unique isomorphism from  $L_1 \cup \{1\}$  to  $L_1 \cup \{2\}$  does not extend to an automorphism of  $\mathbf{A}$ , contradicting  $H_1$ . This completes the proof that  $\neg(3) \Rightarrow \neg(2) \ \& \ \neg(5)$ .

(4)  $\Rightarrow$  (6). Let  $N$  be an upper bound to the cardinality of skeletons of members of  $\mathcal{K}$ . Let  $\mathcal{H}$  be the class of *all* lattices whose skeletons have at most  $N$  elements.  $\mathcal{H}$  is a finitely axiomatizable locally finite universal class and  $\mathcal{K} \subseteq \mathcal{H}$ . Recall that if  $\mathbf{L} \in \mathcal{H}$ ,  $S$  is a skeleton of  $\mathbf{L}$ ,  $b_1, \dots, b_r$  are the distinct elements of  $S$  which are atoms of free  $M_\lambda$ s in  $\mathbf{L}$ , and  $\lambda_i$  is the cardinal number of the set of atoms of the free  $M_\lambda$  in  $\mathbf{L}$  containing  $b_i$ , then the pointed skeleton  $\langle \mathbf{S}; b_1, \dots, b_r \rangle$  together with the sequence of “dimensions”  $\langle \lambda_1, \dots, \lambda_r \rangle$  determines  $\mathbf{L}$  up to isomorphism. More significantly, if  $\mathbf{L}'$  is another member of  $\mathcal{H}$  with pointed skeleton  $\langle \mathbf{S}'; b'_1, \dots, b'_r \rangle \cong \langle \mathbf{S}; b_1, \dots, b_r \rangle$  and dimensions  $\langle \lambda'_1, \dots, \lambda'_r \rangle$ , then  $\lambda_i \leq \lambda'_i$  for all  $i$  implies that  $\mathbf{L}$  can be embedded in  $\mathbf{L}'$ .

Since there are up to isomorphism only finitely many pointed skeletons of members of  $\mathcal{H}$ , it follows (as in [11, Claim 5.6]) that the poset of isomorphism classes in  $\mathcal{H}_{fin}$  under embeddability has no infinite antichains, and hence every universal subclass of  $\mathcal{H}$  ( $\mathcal{K}$  in particular) is finitely axiomatizable.

Now we must show that  $\mathcal{K}$  satisfies  $H_2$ . Let  $\mathcal{K}'$  be the class of all  $\mathbf{L} \in \mathcal{K}$  such that no free  $M_\lambda$  in  $\mathbf{L}$  has more than  $N$  atoms. The next claim should be obvious.

**Claim 4**  $\mathcal{K}' = I(\mathcal{K}_0)$  for some finite set  $\mathcal{K}_0$  of finite lattices.

For the remainder of this section, let  $\mathcal{K}_0$  be a set witnessing the previous claim.

**Claim 5**  $S(\mathcal{K}_0) \subseteq I(\mathcal{K}_0)$ .

PROOF. It must be shown that if  $L_1 \leq L \in \mathcal{K}'$  and  $\{a, b_1, \dots, b_n, c\}$  is a free  $M_\lambda$  in  $L_1$  with  $a < b_i < c$  (and  $b_i \neq b_j$  for  $i \neq j$ ) then  $n \leq N$ . If the interval in  $L$  from  $a$  to  $c$  is itself a free  $M_\lambda$  in  $L$ , then  $n \leq N$  by our choice of  $L$ . Otherwise, we can show  $n \leq N$  in the following way.

Choose a skeleton  $S$  of  $L$ . Assume for simplicity that  $S \cap \{b_1, \dots, b_n\} = \{b_1, \dots, b_k\}$  for some  $k \leq n$ . So if  $k < i \leq n$  then  $b_i$  must be an atom of a free  $M_\lambda$  in  $L$ , say  $I[u_i, v_i]$ , and there must be an atom  $d_i \neq b_i$  of  $I[u_i, v_i]$  which belongs to  $S$ . Working now in  $L$ , it follows that  $a \leq u_i < b_i, d_i < v_i \leq c$ , and either the first or the last inequality is strict (since  $I[a, c]$  is not a free  $M_\lambda$  in  $L$ ). It can then be shown that  $b_i \neq d_j$  for  $1 \leq i \leq k < j \leq n$  and  $d_i \neq d_j$  for  $k < i < j \leq n$ . So  $b_1, \dots, b_k, d_{k+1}, \dots, d_n$  are distinct elements of  $S$ , which proves  $n \leq |S| \leq N$ . ■

**Claim 6** *If  $L \in \mathcal{K}$  and  $L_0$  is a maximal  $\mathcal{K}_0$ -subuniverse of  $L$ , then  $L_0$  contains a skeleton of  $L$ .*

PROOF. Assume  $L \in \mathcal{K}$  and let  $L_0$  be a maximal  $\mathcal{K}_0$ -subuniverse of  $L$ . Choose a skeleton  $S$  of  $L$  with the property that if  $I[a, c]$  is a free  $M_\lambda$  of  $L$  at least one of whose atoms belongs to  $L_0$ , then the atom of  $I[a, c]$  contained in  $S$  also belongs to  $L_0$ . Let  $L_1 = L_0 \cup S$ .  $L_1$  is a sublattice of  $L$ , so if we can show that  $L_1 \in \mathcal{K}'$  then it will follow from the maximality of  $L_0$  that  $S \subseteq L_0$ .

Suppose  $\{a, b_1, \dots, b_n, c\}$  is a free  $M_\lambda$  of  $L_1$ , with  $a < b_i < c$ . Then  $a < b_i < c$  in  $L$  as well, so if the interval in  $L$  from  $a$  to  $c$  is *not* a free  $M_\lambda$  in  $L$  then  $\{b_1, \dots, b_n\} \subseteq S$ , which would imply  $n \leq |S| \leq N$ . On the other hand, if the interval in  $L$  from  $a$  to  $b$  is a free  $M_\lambda$  in  $L$  then  $\{b_1, \dots, b_n\} \subseteq L_0$  by our choice of  $S$ , so  $\{a, b_1, \dots, b_n, c\}$  is a free  $M_\lambda$  in  $L_0$  and hence  $n \leq N$  as  $L_0 \in \mathcal{K}'$ . This prove  $L_1 \in \mathcal{K}'$ . ■

We will now complete the proof of Theorem 4.3 by showing that  $\mathcal{K}_0$  witnesses  $H_2$  for  $\mathcal{K}$ . Let  $L$  be a finite member of  $\mathcal{K}$ , and let  $L_0$  be a maximal  $\mathcal{K}_0$ -subuniverse of  $L$ . Suppose that  $B$  and  $B'$  are sublattices of  $L$  satisfying  $B \cap L_0 = B' \cap L_0 (=: B_0)$  and  $B_0$  is a maximal  $\mathcal{K}_0$ -subuniverse of both  $B$  and  $B'$ . Finally, suppose that  $\sigma$  is an isomorphism from  $B$  to  $B'$  satisfying  $\sigma(x) = x$  for all  $x \in B_0$ .

By the previous claim,  $L_0$  contains a skeleton of  $L$ . Suppose  $x \in B$  and  $\sigma(x) \neq x$ . Then  $x \notin L_0$ , so  $x$  is an atom of some free  $M_\lambda$   $I[a, b]$  in  $L$ . Because  $L_0$  contains a skeleton of  $L$ ,  $L_0$  contains at least one atom  $c$  of  $I[a, b]$ . We

claim that  $B_0$  must also contain an atom  $c_x$  of  $\mathbf{I}[a, b]$ . For if this were false then we could put  $B_1 = \text{Sg}(B_0 \cup \{x\})$  and  $L_1 = \text{Sg}(B_0 \cup \{c\})$  and have  $\mathbf{B}_0 < \mathbf{B}_1 \leq \mathbf{B}$  and  $\mathbf{B}_1 \cong \mathbf{L}_1 \leq \mathbf{L}_0 \in \mathbf{I}(\mathcal{K}_0)$ , contradicting the fact that  $B_0$  is a maximal  $\mathcal{K}_0$ -subuniverse of  $\mathbf{B}$ . So  $c_x \in B_0$  exists as claimed. Now since  $\{x, c_x\} \subseteq B$  and since  $a$  and  $b$  belong to every skeleton of  $\mathbf{L}$ , it follows that  $\{a, b\} \subseteq B \cap L_0 (= B_0)$  and hence  $a = \sigma(a) < \sigma(x) < \sigma(b) = b$ .

Thus if  $x \in B$  and  $\sigma(x) \neq x$ , then  $x$  and  $\sigma(x)$  are atoms in the same free  $M_\lambda$  of  $\mathbf{L}$ . It follows that  $\sigma$  can be extended to an automorphism  $\hat{\sigma}$  of  $\mathbf{L}$  satisfying  $\hat{\sigma}|_{L_0} = \text{id}_{L_0}$ . This verifies  $\text{ext}(\mathbf{L}, L_0, \mathbf{B}, \mathbf{B}')$  and proves that  $\mathcal{K}_0$  witnesses  $\text{H}_2$  for  $\mathcal{K}$ .  $\blacksquare$

## 5 Conclusion

Theorem 4.3 and its proof bear striking similarities to our analysis of unary algebras in [11]. (An algebra is *unary* if the arity of each of its fundamental operations is 1.) We can strengthen these similarities.

Let  $\mathcal{K}$  be a locally finite universal class of unary algebras in a finite language and let  $\mathbf{A} \in \mathcal{K}$ . In [11] the following recipe for converting  $\mathbf{A}$  to a vertex-colored poset was given. For  $a \in A$  let  $[a] = \{x \in A : \text{Sg}^{\mathbf{A}}(x) = \text{Sg}^{\mathbf{A}}(a)\}$  and let  $[\mathbf{A}] = \{[a] : a \in A\}$ . Defining  $[a] \leq [b]$  to mean  $\text{Sg}^{\mathbf{A}}(a) \subseteq \text{Sg}^{\mathbf{A}}(b)$  turns  $[\mathbf{A}]$  into a poset (of finite height, since  $\mathcal{K}$  is locally finite).

For  $a \in A$  define  $D_a = \{x \in A : [x] < [a]\}$ . We say that  $[a], [b] \in [\mathbf{A}]$  have the same color if  $D_a = D_b$  and there exists an isomorphism  $\sigma : \text{Sg}^{\mathbf{A}}(a) \cong \text{Sg}^{\mathbf{A}}(b)$  satisfying  $\sigma|_{D_a} = \text{id}_{D_a}$  and  $\sigma([a]) = [b]$ . A color class in  $[\mathbf{A}]$  is the set of all  $[a] \in [\mathbf{A}]$  of one fixed color.

Let us call a color class *free* if each of its elements is maximal (i.e., has no upper cover in the poset  $[\mathbf{A}]$ ). Then by a *base for  $\mathbf{A}$*  we mean a subset  $\mathcal{B} \subseteq [\mathbf{A}]$  satisfying (1) if  $[a] \in [\mathbf{A}] \setminus \mathcal{B}$  then  $[a]$  is an element of a free color class in  $[\mathbf{A}]$ ; (2) for each free color class in  $[\mathbf{A}]$ , exactly one of its elements is a member of  $\mathcal{B}$ . Bases coordinatize members of  $\mathcal{K}$  in the same way that skeletons coordinatize lattices. More precisely, suppose  $\mathbf{A} \in \mathcal{K}$ ,  $\mathcal{B}$  is a base for  $\mathbf{A}$ ,  $B = \bigcup \mathcal{B}$ ,  $([b_i])_{i < \eta}$  are the distinct elements of  $\mathcal{B}$  belonging to free color classes of  $[\mathbf{A}]$ , and  $\lambda_i$  is the cardinality of the free color class containing  $[b_i]$ . Then  $\mathbf{A}$  is determined up to isomorphism by the isomorphism type of  $(\mathbf{B}; (b_i)_{i < \eta})$  and the sequence of “dimensions”  $(\lambda_i)_{i < \eta}$ .

By slightly extending the analysis in [11] we can prove the following theorem.

**THEOREM 5.1** *Suppose  $\mathcal{K}$  is a locally finite universal class of unary algebras in a finite language  $\mathcal{L}$ . Let  $N$  be a positive integer bounding the maximum cardinality of the 1-generated members of  $\mathcal{K}$ . Then the following are equivalent:*

- (1)  $V(\mathcal{K}^t)$  is decidable.
- (2)  $V(\mathcal{K}^t)$  is not hereditarily undecidable.
- (3)  $\mathcal{K}$  omits each member of a specific finite set  $\mathcal{F}_{\mathcal{L},N}$  of countably infinite  $\mathcal{L}$ -algebras. ( $\mathcal{F}_{\mathcal{L},N}$  depends on  $\mathcal{L}$  and  $N$  but not on  $\mathcal{K}$ .)
- (4)  $\mathcal{K}$  has bounded bases.
- (5)  $\mathcal{K}$  satisfies  $H_1$ .
- (6)  $\mathcal{K}$  is finitely axiomatizable and satisfies  $H_2$ .

Surely there is a general theorem, incorporating Theorems 4.3 and 5.1 as special cases, waiting to be discovered. Its proof will undoubtedly come from model theory. In that field G. Cherlin, L. Harrington, A. H. Lachlan and others have already made a profound study of the countable stable homogeneous relational structures in a finite language (see e.g. [7, 5, 6]). They found that for a given language the above-mentioned structures fall into finitely many families in each of which the members are determined up to isomorphism by a finite sequence of dimensions. Motivated by their work, we propose the following two problems for further study.

**PROBLEM 1.** Given a locally finite universal class  $\mathcal{K}$  of algebras in a finite language and a positive integer  $r$ , describe the countable homogeneous members of  $\mathcal{K}$  of complete rank at most  $r$ .

**PROBLEM 2.** Given a locally finite universal class  $\mathcal{K}$  of algebras in a finite language, describe the homogeneous universal subclasses of  $\mathcal{K}$ . Are they finitely axiomatizable?

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