A SUFFICIENT CONDITION FOR FINITE DECIDABILITY

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Research Report No. 91–128 2

August 1991

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July 3, 1991

Abstract

We prove that a locally finite, finitely generated, congruence modular variety $\mathcal{V}$ whose subdirectly irreducible algebras are all either abelian or linear type 3 above the monolith is finitely decidable if and only if the theory of the finite abelian algebras in $\mathcal{V}$ is decidable.

In this paper we present a fairly restrictive set of conditions on a variety $\mathcal{V}$ which are sufficient to guarantee finite decidability: the decidability of the theory of the finite models in $\mathcal{V}$. Specifically:

Theorem 1 *Let $\mathcal{V}$ be a variety which satisfies the following conditions:*

- *$\mathcal{V}$ is generated by finitely many finite algebras,*
- *$\mathcal{V}$ is congruence permutable,*
- *The non-abelian subdirectly irreducible algebras in $\mathcal{V}$ have linear congruence lattices, in which the centralizer of the monolith μ is 0 or μ, and all covers other than $(0, \mu)$ have type 3.*
- *The theory of the finite abelian algebras in $\mathcal{V}$ is decidable.*

Then $\mathcal{V}$ is finitely decidable.

Since the abelian algebras in such a variety $\mathcal{V}$ are finitely axiomatized relative to the variety (by sentences saying that a Mal'cev term commutes with each basic operation), the stronger claim made in the abstract is also true.

In comparison with the necessary conditions for finite decidability of a finitely generated congruence modular variety (due to J. Jeong [8]), the only restrictive assumption here is the one on the centralizer of the monolith. Under these necessary conditions, this assumption guarantees the linearity of the congruence

*Subject Classifications (1991), *Primary* 03B25, *Secondary* 03C13, 08B26.

lattices of the subdirectly irreducible algebras, and this is the fact which we use most heavily. We will discuss some situations in which it may be possible to weaken this hypothesis at the end of the paper.

Although the conditions which we impose are quite restrictive, they are satisfied by all locally finite finitely decidable varieties of rings, and by all finitely generated finitely decidable distributive varieties (cf. [10], [5], [6] and [7]). Indeed the ideas leading to the proof of the general result stated above were motivated by a desire to understand the results for varieties of rings. Theorem 1 does generalize all known (to the author) sufficient conditions for finite decidability in the congruence modular setting.

Our result is not the end of the story, since a construction due to Idziak can be used to produce non-abelian finitely generated locally finite varieties which are finitely decidable, but which do not satisfy the assumptions of our theorem. We will discuss such examples at the end of the paper.

The proof proceeds in two stages. First we will establish a structure theorem (a "tree decomposition theorem") for the finite non-abelian directly indecomposable algebras in such a variety $\mathcal{V}$. Then we give a semantic embedding of the theory of these into the monadic second order theory of finite trees (which is decidable by a small modification of the arguments in [9] which was previously used by K. Idziak and P.M. Idziak ([4]) for the case of Heyting algebras. The decidability of the theory of all finite algebras in $\mathcal{V}$ then follows from the Feferman-Vaught theorem. The first part of the proof uses commutator theory [2] and tame congruence theory [3] (the latter mainly implicitly through the necessary conditions due to Jeong) to provide a couple of "black boxes" which are needed for the proof. The second part is very direct.

1 The structure theorem

Henceforth $\mathcal{V}$ is to be taken to be a variety which satisfies the conditions above, and $p(x,y,z)$ shall be a Mal'cev term for $\mathcal{V}$. We begin with some notation, definitions, and preliminary lemmas. Several of the lemmas do not require all the restrictions which we have placed on $\mathcal{V}$. It should be clear from the proofs which assumptions are used.

Notation:

1. If X is a set, A_x are algebras, and

$$B \leq \prod_{x \in X} A_x,$$

then for $Y \subseteq X$, B_Y will denote the image of B under the projection onto Y, and for $b \in B$, b_Y will denote the image of b under this same projection.

If $x \in X$ then we use b_x for $b_{\{x\}}$. If $\theta_x \in \operatorname{Con} A_x$ for each $x \in X$ then

$$\theta = \prod_{x \in X} \theta_x$$

may be viewed either as a congruence on $\prod A_x$, or on B, and θ_Y denotes the obvious congruence on B_Y. We say that B is θ-closed if for any f and g from $\prod A_x$ which are θ-related, $f \in B$ if and only if $g \in B$ (i.e. B is a union of θ blocks.)

2. Suppose that A is an algebra whose congruence lattice is a finite chain. Then μ or μ^1 denotes the monolith, and in general μ^{k+1} denotes the congruence which covers μ^k unless we already have that $\mu^k = 1$, in which case $\mu^{k+1} = 1$.

3. A finite subdirect product:

$$B \leq \prod_{x \in X} A_x,$$

is called *irredundant* if for each proper subset $Y \subseteq X$, the projection:

$$\pi_Y : B \rightarrow B_Y$$

is not an isomorphism. Equivalently, for each $x \in X$ there exist b and c in B such that:

$$b_{X-\{x\}} = c_{X-\{x\}} \text{ and } b_x \neq c_x.$$

Lemma 2 *Let X be a finite set, let $A_x \in \mathcal{V}$ be subdirectly irreducible algebras, and suppose that:*

$$A \leq \prod_{x \in X} A_x$$

is an irredundant subdirect product of the algebras A_x. Then A is μ-closed.

Proof: It is sufficient to show that for each $x \in X$, A is

$$0 \times 0 \times \cdots \times \mu_x \times 0 \times \cdots \times 0$$

closed. Since A is an irredundant subdirect product, there exist $f, g \in A$ such that:

$$\begin{aligned} f_{X-\{x\}} &= g_{X-\{x\}} \\ f_x &\neq g_x. \end{aligned}$$

Suppose that $h \in A$, and $k \in \prod_{x \in X} A_x$ is such that:

$$\begin{aligned} h_{X-\{x\}} &= k_{X-\{x\}} \\ (k_x, k_x)) &\in \mu_x \end{aligned}$$

Since:

$$(h_x, k_x) \in \mathrm{Con}_{A_x}(f_x, g_x)$$

and $\mathcal{V}$ is congruence-permutable, there exists a polynomial on A_x in one variable, say $t(y, \mathbf{c})$ such that:

$$h_x = t(f_x, \mathbf{c})$$
$$k_x = t(g_x, \mathbf{c}).$$

Chose $\mathbf{d} \in A$ such that $\mathbf{d}_x = \mathbf{c}$ and define:

$$\begin{aligned} h' &= t(f, \mathbf{d}) \\ k' &= t(g, \mathbf{d}). \end{aligned}$$

Then:

$$\begin{aligned} h'_{X-\{x\}} &= k'_{X-\{x\}} \\ h'_x &= h_x \\ k'_x &= k_x. \end{aligned}$$

Thus:

$$\begin{aligned} p(h, h', k')_{X-\{x\}} &= h_{X-\{x\}} = k_{X-\{x\}} \\ p(h, h', k')_x &= k_x. \end{aligned}$$

Therefore $p(h, h_1, k_1) = k$ so $k \in A$ as required. ∎

Lemma 3 *If D is a finite directly indecomposable member of $\mathcal{V}$, then D is either abelian, or a subdirect product of non-abelian subdirectly irreducible algebras.*

Proof: Let D be a directly indecomposable algebra in $\mathcal{V}$, and suppose that D has an irredundant subdirect representation:

$$D \leq \prod_{x \in X} A_x \times \prod_{y \in Y} A_y,$$

where each A_x is a non-abelian subdirectly irreducible algebra in $\mathcal{V}$, and each A_y is an abelian subdirectly irreducible algebra in $\mathcal{V}$. Let θ^X and θ^Y be the congruences on A which are the kernels of the projections onto X and Y respectively. Then certainly:

$$\theta^X \wedge \theta^Y = 0.$$

Moreover,

$$\theta^X \vee \theta^Y \geq \mu,$$

since if $(f, g) \in \mu$ then

$$h = f_X \cup g_Y \in D$$

by Lemma 2, and

$$f \stackrel{\theta^X}{\equiv} h \stackrel{\theta^Y}{\equiv} g.$$

Let

$$\bar{D} = D/\mu \leq \prod_{x \in X} A_x/\mu_x \times \prod_{y \in Y} A_y/\mu_y$$

and let $\bar{\theta}^X$ and $\bar{\theta}^Y$ be the congruences of D which are the kernels of the composite of the natural map $D \to \bar{D}$ and the projections of $\bar{D}$ onto X and Y respectively. Then we claim:

$$\begin{aligned} \bar{\theta}^X &= \theta^X \vee \mu \\ \bar{\theta}^Y &= \theta^Y \vee \mu \end{aligned}$$

To prove this, suppose that $f, g \in D$ and

$$f \stackrel{\bar{\theta}^X}{\equiv} g.$$

Equivalently, $\bar{f}_X = \bar{g}_X$, or

$$f_x \stackrel{\mu_x}{\equiv} g_x$$

for all $x \in X$. But in this case, by Lemma 2

$$h = g_X \cup f_Y \in D,$$

and

$$f \stackrel{\mu}{\equiv} h \stackrel{\theta^X}{\equiv} g.$$

On the other hand, if for some h,

$$f \stackrel{\mu}{\equiv} h \stackrel{\theta^X}{\equiv} g.$$

then $f_x \stackrel{\mu_x}{\equiv} g_x$ for all $x \in X$. So $f \stackrel{\bar{\theta}^X}{\equiv} g$. This establishes the claim for $\bar{\theta}^X$ and the same reasoning applies to $\bar{\theta}^Y$.

The interval above $\bar{\theta}^X$ is the congruence lattice of a subdirect product of the algebras A_x/μ_x. As we have assumed that these algebras have no abelian covers their type set is $\{3\}$, and this property is preserved by finite subdirect products in modular varieties. On the other hand, the interval above $\bar{\theta}^Y$ is abelian. Therefore $\bar{\theta}^X \vee \bar{\theta}^Y = 1$, and hence $\theta^X \vee \theta^Y = 1$, since we already know that $\theta_X \vee \theta_Y \geq \mu$. Since D is directly indecomposable, one of these two congruences must be 0, and the other 1 which establishes that D is either abelian, or a subdirect product of non-abelian subdirectly irreducible algebras. ■

We depart briefly from our fixed assumptions on $\mathcal{V}$ to prove the following. Recall that a *neutral algebra* is one in which the commutator of two congruences is their intersection. As noted above, neutrality is preserved by finite subdirect products, and implies the distributive law in the lattice of congruences.

Lemma 4 *Let A_x for $x \in X$ be a finite collection of neutral algebras, let B be a subdirect product of these, let A be a subdirectly irreducible algebra, and suppose that $\eta : B \to A$ is a surjective homomorphism. Then η factors through one of the projections $\pi_x : B \to A_x$.*

Proof: The congruence $\ker \eta$ is meet irreducible, hence if $\ker \eta \not\geq \ker \pi_x$ for any x:

$$\begin{aligned} \bigwedge_{x \in X} (\ker \eta \vee \ker \pi_x) &> \ker \eta \\ \ker \eta \vee \bigwedge_{x \in X} \ker \pi_x &> \ker \eta \\ \ker \eta &> \ker \eta. \end{aligned}$$

This contradiction establishes that $\ker \eta > \ker \pi_x$ for some $x \in X$ as claimed. ∎

Given a subdirect product B of algebras A_x, and a proper non-empty subset Y of X, we say that the projections π_Y and π_{X-Y} are complementary, if their kernels are complementary congruences.

Lemma 5 *Let A_x be algebras, $\theta_x \in ConA_x$ be meet irreducible congruences such that for each x, A_x/θ_x is neutral, and let B be a θ-closed subdirect product of the A_x. Suppose that $Y \subseteq X$, is such that π_Y is an isomorphism from B/θ to B_Y/θ_Y and*

$$B' := B_Y/\theta_Y \leq \prod_{y \in Y} A_y/\theta_y$$

is irredundant. If B has no complementary projections, then neither does B'.

Proof: For each $x \in X - Y$ there is a surjective homomorphism

$$\eta_x : B' \to A_x/\theta_x$$

such that for

$$b \in \prod_{X \in X} A_x,$$

with $b_Y/\theta_Y \in B'$:

$$b \in B \iff \eta_x(b_Y/\theta_Y) = b_x/\theta_x \text{ for all } x \in X.$$

(each η_x is just the map which factors π_x through π_Y.) Since each of the algebras A_x/θ_x is neutral and subdirectly irreducible, each η_x factors through some $\pi_{y(x)}$, with a factoring map α_x. Then, for b as above, we have:

$$b \in B \quad \Longleftrightarrow \quad \alpha_x(b_{y(x)}/\theta_{y(x)}) = b_x/\theta_x \text{ for all } x \in X.$$

Suppose that B' has complementary projections given by $\pi_{Z'}$ and $\pi_{Y-Z'}$. Let

$$Z = Z' \cup \{x \in X - Y : y(x) \in Z\}.$$

We claim that π_Z and π_{X-Z} are complementary projections on B. let b and c be elements of B. Then we can find an element d in B such that:

$$d_Y/\theta_Y = b_{Z'}/\theta_{Z'} \cup c_{Y-Z'}/\theta_{Y-Z'}$$

So d agrees with b modulo $\theta_{Z'}$ on Z'. But for each x such that $y(x) \in Z'$,

$$b_x/\theta_x = \alpha_x(b_{y(x)}/\theta_{y(x)}) = \alpha_x(d_{y(x)}/\theta_{y(x)}) = d_x/\theta_x$$

and for other x, $c_x/\theta_x = d_x/\theta_x$. But then since B is θ-closed, this implies that:

$$b_Z \cup c_{X-Z} \in B,$$

and hence π_Z and π_{X-Z} are complementary projections for B. ∎

We will now describe a particular sub-product construction, which we will call the "tree construction". Again, it could be given in somewhat more general terms, but these are not necessary for the situation at hand. We will be able to show that the finite non-abelian directly indecomposable algebras in varieties $\mathcal{V}$ which satisfy our hypotheses can be constructed from the subdirectly irreducible algebras using this construction (the "tree decomposition theorem"). Then it will be a relatively simple matter to describe a decision procedure for this class.

Let X be a (non-empty) set, T a rooted tree with vertices labelled by X, and let X' be X minus the label of the root, and let f the function which assigns to each $x \in X'$ its "parent". Let A_x for $x \in X$ be algebras, $\theta_x \in \operatorname{Con} A_x$ be congruences; and

$$\alpha_x : A_{f(x)} \to A_x/\theta_x$$

be surjective homomorphisms for each $x \in X'$. Then the subalgebra:

$$B_{T,\theta,\alpha} \leq \prod_{x \in X} A_x$$

(or more briefly B) consists of those tuples a such that

$$a_x \in \alpha_x(a_{f(x)})$$

for all $x \in X'$.

It may not be immediately clear why this gives a subdirect product, but this is guaranteed by the condition that each α be surjective, and that a_x is determined only modulo θ_x.

Proposition 6 *Every finite non-abelian directly indecomposable algebra in $\mathcal{V}$ can be obtained by a tree construction from subdirectly irreducible algebras.*

Proof: We remark at the outset that the construction is not unique. The simplest example of this is seen by taking two identical algebras A_1 and A_2 and the subdirect product of these which is just the set of pairs which are μ-related. Then either 1 or 2 can be the root of the tree. Extending this example to three algebras one can see that not even the structure of the tree is determined.

Let B be a non-abelian directly indecomposable algebra in $\mathcal{V}$, and take an arbitrary irredundant subdirect representation:

$$B \leq \prod_{x \in X} A_x.$$

We will show that this representation carries an appropriate tree structure. The idea is to gradually discover the nodes, branches, and labels. This process is by a sort of "genealogical search" – which uncovers the tree, beginning with the nodes which are farthest from the root, and then working down step by step to the root.

We know already that each A_x is non-abelian by Lemma 3, and that B is μ-closed by Lemma 2. Choose $n_1 \geq 1$ such that B is μ^{n_1} but not μ^{n_1+1} closed. Then it must be the case that the representation:

$$B^{(1)} := B/\mu^{n_1} \leq \prod_{x \in X} A_x/\mu^{n_1}.$$

is redundant, or else it would be μ^1-closed (by linearity, all the algebras A_x/μ^{n_1} are subdirectly irreducible), and this would imply that B was μ^{n_1+1} closed.

Now choose $X_1 \subseteq X$ such that:

$$B^{(1)} \cong B^{(1)}_{X_1} \leq \prod_{x \in X_1} A_x/\mu_x^{n_1}.$$

is an irredundant representation. Let $Y_1 = X - X_1$

Thus for each $y \in Y_1$ there is a surjective homomorphism,

$$\eta_y : B^{(1)}_{X_1} \rightarrow A_y/\mu_y^{n_1}$$

Such that:

$$b \in B$$

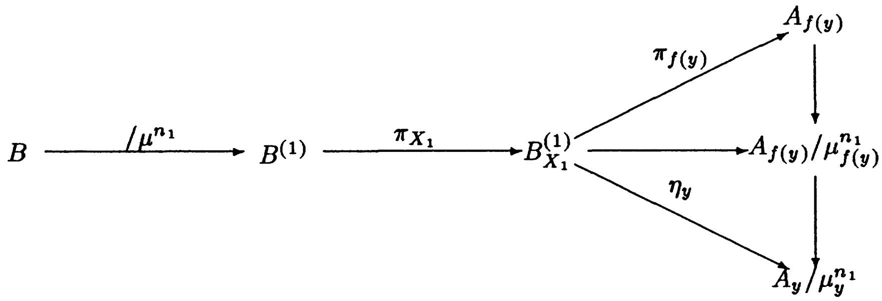
if and only if

$$b_{X_1}/\mu_{X_1}^{n_1} \in B^{(1)} \text{ and } \eta_y(b_{X_1}/\mu_{X_1}^{n_1}) = b_y/\mu_y^{n_1} \text{ for each } y \in Y_1.$$

By Lemma 4 for each $y \in Y_1$ there is an $f(y) \in X_1$ such that η_y factors through the projection onto

$$A_{f(y)}/\mu^{n_1}_{f(y)}.$$

Finally, define $\theta_y = \mu^{n_1}_y$, and α_y to be the composite of the natural map from $A_{f(y)}$ to $A_{f(y)}/\mu^{n_1}_{f(y)}$, and the map which factors η_y. The picture below may make this a little more clear (the right hand column is α_y):



Note that $B^{(1)}_{X_1}$ is also isomorphic to $B_{X_1}/\mu^{n_1}_{X_1}$. Furthermore, from all of the above we have:

$$\begin{aligned} b \in B \quad \Longleftrightarrow \quad & b_{X_1}/\mu^{n_1}_{X_1} \in B^{(1)}_{X_1} \text{ and} \\ & b_y \in \alpha_y(b_{f(y)}) \text{ for each } y \in Y. \quad (*)_1 \end{aligned}$$

This verifies that for those y for which $f(y)$ has been defined, the conditions required of a tree construction hold.

The proof will now proceed inductively, with a successful discovery of a little more information about f, α, and θ at each stage. There is one slight hitch – it is not apparent that the algebra $B^{(1)}$ is indecomposable. Happily, it is indecomposable enough for our purposes, since Lemma 5 guarantees that it does not have complementary projections.

We now attempt to state an inductive hypothesis. After k iterations of the process above, we will have obtained a subset $X_k \subseteq X$, and μ^{n_k}, and defined $f(y)$, α_y and θ_y for all $y \in X - X_k$ such that the following hold:

1. The directed graph of edges $(x, f(x))$ is a forest with roots in X_k,

2. B_{X_k} is μ^{n_k} closed, and the quotient

$$B^{(k)}_{X_k} = B_{X_k}/\mu^{n_k}_{X_k} \leq \prod_{x \in X_k} A_x/\mu^{n_k}_x$$

is irredundant and has no complementary projections.

3.

$$b \in B \iff b_{X_k}/\mu^{n_k}_{X_k} \in B^{(k)}_{X_k} \text{ and}$$
$$b_y \in \alpha_y(b_{f(y)}) \text{ for each } y \in X - X_k.$$

In summary, all of the properties of a tree construction hold at this point, and moreover $B^{(k)}_{X_k}$ is a subdirect product which has no complementary projections. If X_k is a singleton then we have completed a tree construction. Otherwise ...

Since we have an irredundant representation of $B^{(k)}_{X_k}$, we know that it will be μ^m but not μ^{m+1} closed for some $m > 0$ (note that this time μ refers to the product of the minimal congruences of $A_x/\mu^{n_k}_x$ for $x \in X_k$.) Moreover, for no $x \in X_k$ can it be the case that

$$\mu^m_x = 1$$

or else $B^{(k)}_{X_k}$ would decompose through a projection onto $\{x\}$. However, by our choice of m the product:

$$B^{(k)}_{X_k}/\mu^m \leq \prod_{x \in X_k} A_x/\mu^{n_k+m}$$

is redundant. Let $n_{k+1} = n_k + m$, and let

$$B^{(k+1)} = B^{(k)}_{X_k}/\mu^m$$

Choose X_{k+1} a proper subset of X_k such that:

$$B^{(k+1)} \cong B^{(k+1)}_{X_{k+1}} \leq \prod_{X_{k+1}} A_x/\mu^{n_{k+1}}_x,$$

is an irredundant representation, and let $Y_{k+1} = X_{k+1} - X_k$

For $y \in Y_{k+1}$ choose a surjective homomorphism:

$$\eta_y : B^{(k+1)}_{X_{k+1}} \to A_y/\mu^{n_{k+1}}$$

such that:

$$d_{X_k} \in B^{(k+1)} \iff d_{X_{k+1}} \in B^{(k+1)}_{X_{k+1}} \text{ and } \eta(d_{X_{k+1}}) = d_y \text{ for each } y \in Y_{k+1}.$$

By Lemma 4, each η_y factors through the projection onto a coordinate $f(y) \in X_{k+1}$, we may set $\theta_y = \mu^{n_{k+1}}$, and α_y the composition of the natural map from $A_{f(y)}$ onto $A_{f(y)}/\mu^{n_{k+1}}$ followed by the factoring map above. Now,

$$b \in B \iff b_{X_k}/\mu^{n_k}_{X_k} \in B^{(k)}_{X_k} \text{ and}$$
$$b_y \in \alpha_y(b_{f(y)}) \text{ for each } y \in X - X_k.$$

However to determine whether

$$b_{X_k}/\mu_{X_k}^{n_k} \in B_{X_k}^{(k)}$$

holds is equivalent to determining whether

$$d_{X_k} := b_{X_K}/\mu^{n_{k+1}}$$

is in $B^{(k+1)}$. By the definition of α_y, and θ_y for $y \in Y_{k+1}$ this holds if and only if

$$\alpha_y(b_y) = b_{f(y)}/\theta_y$$

and

$$b_{X_{k+1}}/\mu_{X_{k+1}}^{n_{k+1}} \in B_{X_{k+1}}^{(k+1)}$$

which verifies that the part of the inductive hypothesis beginning "$b \in B \iff$" is still satisfied. Finally, $B_{X_{k+1}}^{(k+1)}$ can have no complementary projections or else $B_{X_k}^{(k)}$ would also have them.

Thus we can carry out a complete tree construction inductively, which establishes the stated result. ∎

In passing, we note that the depth of the tree required is at most the length of the longest chain in the congruence lattices of the finite subdirectly irreducible algebras of $\mathcal{V}$. Although this is not required for the subsequent interpretation, it does limit the complexity of the directly indecomposables (and hence of the theory) to a certain extent.

2 Semantic Embeddings

Now we will provide the semantic embedding of the theory of the non-abelian algebras in $\mathcal{V}$ which have a tree decomposition into the monadic second-order theory of finite trees. Since this class contains all the non-abelian finite directly indecomposable algebras, and as the finite abelian algebras in $\mathcal{V}$ have a decidable theory by assumption, this is sufficient to complete the proof of Theorem 1 (by an application of the Feferman-Vaught theorem).

First of all, let $\mathcal{S}\rangle$ be a set of disjoint non-abelian subdirectly irreducible algebras in $\mathcal{V}$ which contains exactly one algebra of each isomorphism type. Let Φ denote the set of all 4-tuples:

$$(A, B, \alpha, \theta)$$

where A and B are elements of $\mathcal{S}\rangle$, θ is a congruence on B, and α is a surjective homomorphism from A to B/θ. A typical element of Φ will be denoted:

$$\alpha = (A_\alpha, B_\alpha, \alpha, \theta_\alpha).$$

For each such α (there are only finitely many) introduce a set constant p_α. Further, let S be the union of $\mathcal{S}\rangle$. We can, and will, interpret the elements of any tree construction as being functions from the set of nodes of the tree into S. That is, we demand that the stalks of a tree construction come from our representative set $\mathcal{S}\rangle$ of non-abelian subdirectly irreducible algebras.

The intended interpretation of a tree construction in the second-order monadic theory of finite trees is that each node will be labelled with the "type" p_α for some α from Φ, which it realizes in the tree construction (for consistency, the root may be labelled by any p_α for which B_α is equal to the stalk at the root) . Elements will be interpreted by $|S|$-tuples of sets which form a partition of the nodes, where the kth element of the tuple specifies the nodes at which the element takes on that value. Now given a finite tree T, specified by its predecessor function f, and constants p_α it is a relatively easy matter to realize this interpretation.

Let T' denote T with its root deleted. First of all, this particular tree with constants represents an algebra obtained from a tree construction if and only if

- the constants p_α form a partition of T, and
- for each $x \in T'$, if $x \in p_\alpha$ and $f(x) \in p_\beta$ then $A_\alpha = B_\beta$.

Secondly, we can define a predicate: *Elem* (for elements) on S-tuples of sets by:

$$Elem(\{X_a\}_{a \in S}) \iff \begin{array}{l} \{X_a\} \text{ partitions } T, \text{ and} \\ \text{if } y \in X_a \cap p_\alpha \text{ then } f(y) \in X_{\alpha^{-1}(a/\theta_\alpha)} \end{array}$$

We have implicitly added the requirement that the partitions X and p are consistent in that if $y \in X_a$ then $y \in p_\alpha$ for some α such that a belongs to B_α.

It is easy to see that we can also interpret the basic operations of $\mathcal{V}$ on tuples satisfying *Elem*, by considering predicates on sequences of $|S|$-tuples which impose consistency requirements (so that the elements which they code come from the same algebras at each node), and which say that the final element takes at each node the value of the operation when evaluated at the rest of the elements. Hence we obtain a semantic embedding of algebras in $\mathcal{V}$ obtained from tree constructions into the monadic second order theory of finite trees (with constants), which is sufficient to establish the decidability of such algebras. For further details of a very similar construction the reader may wish to consult [1]. Then by the Feferman-Vaught theorem, the theory of all finite algebras in $\mathcal{V}$ is decidable, and we have concluded the proof of Theorem 1.

3 Discussion

The most obvious question which arises from the work above is to ask what happens is the assumption of linearity of the congruence lattices of the subdirectly irreducibles is dropped. In this case we can still say a little. By the necessary conditions of Jeong, it is still the case that each such congruence lattice is linear above the centralizer of the monolith (this congruence is denoted ν). An immediate consequence is:

Theorem 7 *If $\mathcal{V}$ is a finitely generated congruence modular variety which satisfies the necessary conditions for finite decidability given in [8], if every finite non-abelian directly indecomposable algebra of $\mathcal{V}$ has a ν-closed representation as a subdirect product of irreducibles, and if the theory of the finite abelian algebras in $\mathcal{V}$ is decidable, then $\mathcal{V}$ is finitely decidable.*

So one approach to further results (which is being pursued by the author and R. Willard) would be to classify those varieties $\mathcal{V}$ which satisfy the conditions above. However, there exists a ring R which has 8 elements, is local and does not have a linear congruence lattice. The variety of left R-modules is finitely decidable. A construction outlined by Idziak then allows one to construct a finitely decidable variety $\mathcal{V}$ whose directly indecomposable algebras are formed from R-modules by the addition of a single element. The directly indecomposable obtained from R itself considered as a left R-module in this way has 9 elements, whereas a ν-closed subdirect product would have at least 17 elements.

Because of examples of this type, it seems unlikely that a necessary and sufficient criterion for the finite decidability of locally finite congruence modular varieties of the form "condition on the congruence lattice of the subdirectly irreducibles" plus "decidability of the abelian part" exists.

Another question, raised by S. Burris, arises from the method of the proof: does there exist a locally finite variety $\mathcal{V}$ which is finitely decidable but for which the theory of the finite directly indecomposable algebras in $\mathcal{V}$ is undecidable?

4 Acknowledgements

I would like to thank Professor S. Burris for organizing the International Workshop on Decidable Varieties, and the financial support of the University of Waterloo for this workshop. Thanks also to Mr. William and Mrs. May Moeser for their kind hospitality, and the use of their swimming-pool, beside which most of the details of the tree decomposition were hammered out. Finally, thanks are due to Ross Willard, who should probably have been a coauthor of this paper.

References

[1] S. Burris and H. Werner, Sheaf constructions and their elementary properties, *Trans. Amer. Math. Soc.*, **248**, 269–309, 1979.

[2] R. Freese and R. McKenzie, *Commutator theory for congruence modular varieties*, London Math. Soc. Lecture Notes, 125, 1987.

[3] D. Hobby and R. McKenzie, *The structure of finite algebra*, Amer. Math. Soc. Contemporary Mathematics, vol. 76, 1988.

[4] K. Idziak and P. M. Idziak. Decidability problem for finite Heyting algebras. *J. Symbolic Logic* **53**, 729–735, 1988.

[5] P. M. Idziak. Reduced sub-powers and the decision problem for finite algebras in arithmetical varieties. *Algebra Universalis* **25**, 365–383, 198 8.

[6] P. M. Idziak. Varieties with decidable finite algebras I: Linearity. *Algebra Universalis* **26**, 234–246, 1989.

[7] P. M. Idziak. Varieties with decidable finite algebras II: Permutability. *Algebra Universalis* **26**, 247–256, 1989.

[8] J. Jeong. On finitely decidable varieties. Ph.D. thesis, University of California (Berkeley), 1991.

[9] M. O. Rabin. Decidability of second order theories and automata on infinite trees. *Trans. Amer. Math. Soc.* **141**, 1–35, 1969.

[10] A.P. Zamyatin, Varieties of associative rings whose elementary theory is decidable, *Soviet Math. Doklady*, **17**, 996–999, 1976.