SOME PROPERTIES OF FINITELY DECIDABLE VARIETIES

by

Matthew A Valeriote
Department of Mathematics and Statistics
McMaster University
Hamilton, Ontario, Canada L8S 4K1

Ross Willard
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213
and
Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1

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Matthew A. Valeriote † Ross Willard

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Abstract

Let $\mathcal{V}$ be a variety whose class of finite members has a decidable first-order theory. We prove that each finite member $\mathbf{A}$ of $\mathcal{V}$ satisfies the $(3,1)$ and $(3,2)$ transfer principles, and that the minimal sets of prime quotients of type 2 or 3 in $\mathbf{A}$ must have empty tails. The first result has already been used by J. Jeong [9] in characterizing the finite subdirectly irreducible members of $\mathcal{V}$ with nonabelian monolith. The second result implies that if $\mathcal{V}$ is also locally finite and omits type 1, then $\mathcal{V}$ is congruence modular.

1 Introduction

A class $\mathcal{K}$ of structures (in a finite language L) is **decidable** if the set of all first-order L-sentences true in all members of $\mathcal{K}$ is recursive, and is **finitely decidable** if the class of finite members of $\mathcal{K}$ is decidable. In this paper we study **varieties** (that is, classes of algebras closed under subalgebras, products, and homomorphic images) which are finitely decidable.

Through the work of R. McKenzie and M. Valeriote, the structure of decidable locally finite varieties is now completely understood, modulo two special cases. Less is known about (locally finite) finitely decidable varieties.

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A few years ago, P. Idziak [6, 7, 8] determined the structure of those finitely decidable varieties which are finitely generated and congruence distributive. Very recently, J. Jeong [9] made a profound study of the finitely decidable varieties which are locally finite and congruence modular. In this paper we prove a few necessary conditions on finitely decidable varieties without assuming congruence modularity.

Our tools are the tame congruence theory of D. Hobby and R. McKenzie [5] and Yu. Ershov's notion of interpretation (or semantic embedding) of a class $\mathcal{K}'$ of L'-structures into a class $\mathcal{K}$ of L-structures. The reader may find a description of this method in [4], [3], or [11].

Following [11], we shall say that the class $\mathcal{K}$ is ω-**unstructured** if there is a semantic embedding of the class of finite graphs into the class of finite members of $\mathcal{K}$ (and is ω-**structured** otherwise). Here are the properties of this notion that we will use.

Basic Properties 1.1

1. *If $\mathcal{K}$ is ω-unstructured and the language of $\mathcal{K}$ is finite, then $\mathcal{K}$ is finitely undecidable.*

2. *If $\mathcal{K}$ is ω-unstructured and $\mathcal{K} \subseteq \mathcal{K}'$, then $\mathcal{K}'$ is ω-unstructured.*

3. *Suppose $\mathbf{A}$ is a finite algebra, $U \subseteq A$ is the range of some idempotent unary polynomial of $\mathbf{A}$, and $\mathbf{A}|_U$ is the algebra induced by $\mathbf{A}$ on U with the normal indexing (see [5]). If $\mathsf{V}(\mathbf{A}|_U)$ is ω-unstructured, then $\mathsf{V}(\mathbf{A})$ is ω-unstructured. (This is a straightforward variation of Theorem 0.50 in [11]; it will be used in Section 3.)*

We shall assume that the reader is familiar with tame congruence theory. One fact connecting that theory to ω-unstructured varieties, which we shall use in Section 3, is implicit in the proof of Lemma 11.1 from [5]: If $\mathcal{V}$ is an ω-structured variety, then the type set of every finite algebra in $\mathcal{V}$ is contained in $\{1, 2, 3\}$.

2 Transfer principles

The following definition is taken from [11].

Definition 2.1 Let $\mathbf{A}$ be a finite algebra. If $i, j \in \{1, 2, 3\}$ are distinct types, then $\mathbf{A}$ possesses the **(i,j) transfer principle** if for all $\delta, \alpha, \beta \in \text{Con}\,\mathbf{A}$, if $\delta \prec \alpha \prec \beta$ and $\text{typ}(\delta, \alpha) = i$ while $\text{typ}(\alpha, \beta) = j$, then there exists a congruence γ satisfying $\delta \prec \gamma \leq \beta$ and $\text{typ}(\delta, \gamma) = j$. A variety satisfies the (i, j) transfer principle if every finite member does.

Thus the (i, j) and (j, i) transfer principles govern to some extent the relative placement of type i and type j prime quotients in $\text{Con}\,\mathbf{A}$. Every decidable locally finite variety satisfies all six possible transfer principles (this follows from the main theorem of [11]). However, the $(2, 3)$ transfer principle fails in the ring $\mathbf{Z}_4$ (which Zamjatin [12] proved generates a finitely decidable variety), and P. Idziak and M. Valeriote have examples of finitely decidable varieties which fail to satisfy the $(1, 3)$ transfer principle. On the other hand, J. Jeong [9] has proved the $(3, 2)$ transfer principle for all finitely decidable congruence modular varieties.

In this section we prove the $(3, 1)$ and $(3, 2)$ transfer principles for all finitely decidable varieties. For the remainder of this section let $\mathbf{A}$ be a finite algebra with congruences α and β such that

- $0_A \prec \alpha \prec \beta$,
- $\text{typ}(0_A, \alpha) = 3$ and $\text{typ}(\alpha, \beta) \in \{1, 2\}$,
- If γ is a non-zero congruence strictly below β, then $\gamma = \alpha$.

Choose U an $\langle \alpha, \beta \rangle$-minimal set and V a $\langle 0_A, \alpha \rangle$-minimal set. Let $\{0, 1\}$ be the body of V, let V be the range of the idempotent polynomial $e(x)$, and let U be the range of the idempotent polynomial $f(x)$.

LEMMA 2.2 *Suppose $\langle u, v \rangle \in \beta|_U \setminus 0_U$, and let N be the $\beta|_U$-class containing u and v. Then there is a polynomial $g(x)$ of $\mathbf{A}$ satisfying $g(A) \subseteq V$, $g(u) = 0$, $g(v) = 1$, and $g(N) = \{0, 1\}$.*

PROOF. The hypotheses imply $\langle 0, 1 \rangle \in \text{Cg}^{\mathbf{A}}(u, v)$, so there must be a polynomial $g(x)$ of $\mathbf{A}$ satisfying $g(A) \subseteq V$, $g(u) \neq g(v)$, and $0 \in \{g(u), g(v)\}$.

Using the psuedo-join operation described in Lemma 4.17 from [5], we see that for any $t \in V \setminus \{0, 1\}$, the set $\{0, t\}$ is a 2-snag. $\beta \setminus \alpha$ contains no 2-snags (since β is abelian over α), which proves $0/\beta|_V = \{0, 1\}$. Since $g(N)$ is $\beta|_V$-connected and contains 0, it follows that $\{g(u), g(v)\} = g(N) = \{0, 1\}$.

Finally, if $\langle g(u), g(v)\rangle = \langle 1, 0\rangle$ then we could always replace $g(x)$ with $g(x)'$, where $'$ is a unary polynomial of $\mathbf{A}$ whose restriction to $\{0,1\}$ is Boolean complementation. ∎

Now fix an $\langle\alpha,\beta\rangle$-trace N contained in U, two elements a, b of N which are not α-related, and a polynomial $k(x)$ satisfying $k(A) \subseteq V$, $k(a) = 0$, and $k(b) = 1$. (The existence of such a polynomial follows from the previous lemma.)

THEOREM 2.3 *Assuming the conditions stated above, the class of finite diagonal subpowers of* $\mathbf{A}$ *is* ω*-unstructured.*

PROOF. Let $\mathbf{G} = \langle G, E\rangle$ be a finite graph. That is, G is a finite nonempty set and E is a collection of two element subsets of G. Let $X = G \cup E \cup \{p\}$, where p is some new point, and we are assuming that G and E are disjoint.

For $v \in G$, let $f_v : X \longrightarrow \{a, b\}$ be defined as follows:

$$f_v(x) = \begin{cases} b & \text{if } x = v \\ b & \text{if } v \in x \in E \\ a & \text{otherwise} \end{cases}.$$

Also, let $\overline{G} = \{f_v : v \in G\}$ and let F be the set of all functions $h : X \longrightarrow \{a, b\}$ such that $G \cup \{p\} \subseteq h^{-1}(a)$. For each $c \in A$ let $\hat{c}$ denote the constant function $X \longrightarrow \{c\}$, and let $\hat{A} = \{\hat{c} : c \in A\}$.

Define a finite diagonal subpower $\mathbf{A}[\mathbf{G}]$ of $\mathbf{A}$ as follows. If $\mathrm{typ}(\alpha,\beta) = 2$ then $\mathbf{A}[\mathbf{G}] = \mathrm{Sg}^{\mathbf{A}^X}(\overline{G} \cup \hat{A})$, while if $\mathrm{typ}(\alpha,\beta) = 1$ then $\mathbf{A}[\mathbf{G}] = \mathrm{Sg}^{\mathbf{A}^X}(\overline{G} \cup F \cup \hat{A})$. Note that in either case the range of each function in $\mathbf{A}[\mathbf{G}]$ is β-connected, since each generator is.

Ultimately we will show that the graph $\mathbf{G}$ can be recovered from $\mathbf{A}[\mathbf{G}]$ by first-order formulas with parameters, in such a way that the formulas do not depend on $\mathbf{G}$. Note first that the $\{0,1\}$-valued functions in $\mathbf{A}[\mathbf{G}]$ (that is, those $\mu \in \mathbf{A}[\mathbf{G}]$ satisfying $\mathrm{range}(\mu) \subseteq \{0,1\}$) are definable by the formula

$$e(x) = x \;\;\&\;\; p(x, \hat{0}) = \hat{0}$$

where $p(x, y)$ is a pseudo-meet operation for V. Moreover, the $\{0,1\}$-valued functions in $\mathbf{A}[\mathbf{G}]$ inherit the Boolean operations of $\mathbf{A}_{\{0,1\}}$ and thus form a Boolean subalgebra of $\{0,1\}^X$. In fact, the presence in $\mathbf{A}[\mathbf{G}]$ of the functions $k(f_v)$ $(v \in G)$ already ensure that $\{0,1\}^X \subseteq \mathbf{A}[\mathbf{G}]$.

More generally, we shall be interested in elements of $\mathbf{A}[\mathbf{G}]$ which are **two-valued**, that is, whose range is a two-element set. If μ is two-valued, then the **support** of μ is defined to be the set $\{x \in X : \mu(x) \neq \mu(p)\}$. Let H be the set of all two-valued functions μ in $\mathbf{A}[\mathbf{G}]$ whose support is identical to the support of f_v for some $v \in G$.

Now suppose we had a formula $\mathsf{T}(x)$ which defines in $\mathbf{A}[\mathbf{G}]$ a set T satisfying $\overline{G} \subseteq T \subseteq H$. Then we could recover the graph $\mathbf{G}$ from $\mathbf{A}[\mathbf{G}]$ in the following way. Let χ_p be the $\{0,1\}$-valued function which equals 1 only at $x = p$, and let $\mathsf{GEN}(x)$ and $\mathsf{E}(x,y)$ be the following formulas respectively:

$$\text{range}(x) = \{0,1\} \ \ \& \ \ x \wedge \chi_p = \hat{0} \ \ \& \ \ \exists\mu[\mathsf{T}(\mu) \ \ \& \ \ k(\mu) = x],$$

$$\mathsf{GEN}(x) \ \ \& \ \ \mathsf{GEN}(y) \ \ \& \ \ x \neq y \ \ \& \ \ x \wedge y \neq \hat{0}.$$

Then clearly $\mathbf{A}[\mathbf{G}] \models \mathsf{GEN}(\nu)$ if and only if $\nu = k(f_v)$ for some $v \in G$, and $\mathbf{A}[\mathbf{G}] \models \mathsf{E}(k(f_v), k(f_u))$ if and only if $\{v,u\} \in E$. In other words, the formulas $\mathsf{GEN}(x)$ and $\mathsf{E}(x,y)$ recover the graph $\mathbf{G}$ from $\mathbf{A}[\mathbf{G}]$, and thus define a semantic embedding of the class of finite graphs into the class of finite diagonal subpowers of $\mathbf{A}$.

The remainder of the proof is devoted to proving the existence of a first-order definable subset T of $\mathbf{A}[\mathbf{G}]$ satisfying $\overline{G} \subseteq T \subseteq H$. Define

$$\begin{aligned} T_0 &= \{\mu \in \mathbf{A}[\mathbf{G}] : \mu \text{ is two-valued and range}(\mu) \subseteq U\} \\ T_1 &= \{\mu \in T_0 : \text{the support of } \mu \text{ contains exactly one } v \in G\} \\ T_2 &= \{\mu \in T_1 : \text{if range}(\mu) = \{r,s\} \text{ then } \langle r,s\rangle \notin \alpha\} \\ T &= \{h(f_v) : v \in G \text{ and } h \in \mathrm{Pol}_1\mathbf{A} \text{ such that } h(U) = U\}. \end{aligned}$$

Clearly $\overline{G} \subseteq T \subseteq H$ and $T \subseteq T_2 \subseteq T_1 \subseteq T_0$. We shall show that each T_i is definable in $\mathbf{A}[\mathbf{G}]$ and that $T_2 = T$.

Claim 1 *T_0 and T_1 are definable in* $\mathbf{A}[\mathbf{G}]$.

PROOF. Suppose $\mu \in \mathbf{A}[\mathbf{G}]$. The range of μ is contained in U if and only if $f(\mu) = \mu$. And we claim that if the range of μ is contained in U, then μ is two-valued if and only if

> μ is not constant, and for every pair of polynomials $g_1(x)$ and $g_2(x)$ of $\mathbf{A}$ having range contained in V, if $\text{range}(g_1(\mu)) = \text{range}(g_2(\mu)) = \{0,1\}$ then either $g_1(\mu) = g_2(\mu)$ or $g_1(\mu) = g_2(\mu)'$.

(Here, as before, $'$ is a unary polynomial of $\mathbf{A}$ whose restriction to $\{0,1\}$ is Boolean complementation). Clearly if μ is two-valued then it satisfies the displayed condition. Conversely, suppose the range of μ is contained in U and contains at least three distinct values r, s, t (which necessarily belong to the same $\beta|_U$-class). By Lemma 2.2 there exist polynomials $g_i(x)$ of $\mathbf{A}$ $(i = 1, 2, 3)$ satisfying

$$g_1(r) = 0, \quad g_1(s) = 1, \quad \text{range}(g_1(\mu)) = \{0,1\},$$

$$g_2(r) = 0, \quad g_2(t) = 1, \quad \text{range}(g_2(\mu)) = \{0,1\},$$

$$g_3(s) = 0, \quad g_3(t) = 1, \quad \text{range}(g_3(\mu)) = \{0,1\}.$$

Then either g_1 and g_2 or g_2 and g_3 witness the failure of the displayed condition.

This proves that T_0 is definable. To see that T_1 is definable, let $\chi_G = \hat{1}|_G \cup \hat{0}|_{X \setminus G}$; then an element μ of T_0 belongs to T_1 if and only if there is a polynomial $g(x)$ of $\mathbf{A}$ such that $\text{range}(g(\mu)) = \{0,1\}$, $g(\mu) \wedge \chi_p = \hat{0}$, and $g(\mu) \wedge \chi_G$ is an atom of the Boolean algebra $\{0,1\}^X$. ∎

Claim 2 $T_2 = T$.

PROOF. It must be shown that $T_2 \subseteq T$. Suppose $\mu \in T_2$. Let $\{r, s\}$ be the range of μ, with $\mu(p) = r$, and let v be the unique vertex for which $\mu(v) = s$. By hypothesis, $\langle r, s\rangle \in \beta|_U \setminus \alpha|_U$. By the minimality of U (that is, by Theorem 2.8 from [5]), to prove $\mu \in T$ it will be enough to show that $\mu = h(f_v)$ for some polynomial $h(x)$ of $\mathbf{A}$ whose range is contained in U.

Let N' be the $\langle\alpha, \beta\rangle$-trace in U containing $\{r, s\}$. By Corollary 5.2 of [5] there exist polynomials mapping N bijectively to N' and vice versa, so we may as well assume that $N' = N$.

CASE 1. $\text{typ}(\alpha, \beta) = 1$.

By the definition of $\mathbf{A}[\mathbf{G}]$ in this case,

$$\mu = t(f_{v_1}, \ldots, f_{v_n}; h_1, \ldots, h_k)$$

for some $n + k$-ary polynomial t of $\mathbf{A}$, distinct vertices $v_1, \ldots, v_n \in G$, and nonconstant generators $h_1, \ldots, h_k \in F$. We may assume that $t(A^{n+k}) \subseteq U$

and hence $t(N^{n+k}) \subseteq N$. So $t|_N$ is in the clone of $\mathbf{A}|_N$ and therefore depends on at most one variable modulo α.

$\mu(p) = r$ implies $t(a, a, \ldots, a; a, \ldots, a) = r$. Since $\mu(v) = s$, and because of the nature of the generators, it follows that $v = v_i$ for some $i = 1, \ldots, n$, say $v = v_1$, and that $t(b, a, \ldots, a; a, \ldots, a) = s$. So $t|_N$ depends on its first variable modulo α, and hence on no others. Let $h(x) = t(x, a, \ldots, a; a, \ldots, a)$. It follows that $h(\{a, b\}) = \{r, s\}$ and $\mu(x) \overset{\alpha}{\equiv} h(f_v)(x)$ for all $x \in X$. These facts imply $\mu = h(f_v)$.

CASE 2. $\mathrm{typ}(\alpha, \beta) = 2$.

By the definition of $\mathbf{A}[\mathbf{G}]$ in this case,

$$\mu = t(f_{v_1}, \ldots, f_{v_n})$$

for some n-ary polynomial of $\mathbf{A}$ and distinct vertices $v_1, \ldots, v_n \in G$. As before, we can assume that $t|_N$ is a member of the clone of $\mathbf{A}|_N$ and hence satisfies the term condition modulo α. Also as before, we can assume that $v = v_1$ and hence

$$\begin{aligned} t(a, a, \ldots, a) &= r \\ t(b, a, \ldots, a) &= s. \end{aligned}$$

Furthermore, if $2 \leq i \leq n$ then $\mu(v_i) = r$ which implies

$$t(a, \ldots, \underset{\substack{\uparrow \\ i}}{b}, \ldots, a) = r = t(a, \ldots, \underset{\substack{\uparrow \\ i}}{a}, \ldots, a).$$

So by the term condition modulo α in $\mathbf{A}|_N$, $t|_{\{a,b\}}$ depends on only its first variable modulo α. Hence as before, $\mu = h(f_v)$ where $h(x) = t(x, a, \ldots, a)$. ∎

To prove Theorem 2.3 it remains only to show that T_2 is definable relative to T_1. Again the argument splits into cases depending on the type of $\langle \alpha, \beta \rangle$.

Claim 3 *Suppose* $\mathrm{typ}(\alpha, \beta) = 2$. *Let M be the $\langle \alpha, \beta \rangle$-body of U. Then:*

(i) *$\alpha|_M = 0_M$ and hence $T_2 = \{\mu \in T_1 : range(\mu) \subseteq M\}$.*

(ii) *T_2 is definable.*

PROOF. Suppose $\alpha|_M \neq 0_M$. Then we can find a $\langle 0_A, \alpha\rangle$-minimal set V' whose body $\{0', 1'\}$ is contained in M. Let e' be an idempotent polynomial of $\mathbf{A}$ whose range is V', and let $V'' = \text{range}(fe')$. As $0'$ and $1'$ are fixed points of both e' and f, $fe'(\alpha|_{V'}) \not\subseteq 0_A$ and so V'' is a $\langle 0_A, \alpha\rangle$-minimal set. Note that $V'' \subseteq U$ and that the $\langle 0_A, \alpha\rangle$-body of V'' (namely, $\{0', 1'\}$) is contained in M.

Using Lemma 4.30 from [5], it follows that $V'' = U$. Thus it is possible to find an element t in the $\langle 0_A, \alpha\rangle$-tail of V'' which is β-related to $0'$. But we have already seen in the proof of Lemma 2.2 that that is impossible. This proves the first item.

The second item follows easily from Lemmas 4.20 and 4.25 in [5]: if $\mu \in \mathbf{A}[\mathbf{G}]$, then the range of μ is contained in M if and only if its range is contained in U and $d(\mu, \mu, \hat{a}) = \hat{a}$. ∎

Claim 4 *Suppose* $\text{typ}(\alpha, \beta) = 1$. *Define* $\Upsilon = \{\nu \in \mathbf{A}[\mathbf{G}] : \textit{range}(\nu) \subseteq U\}$. *Let* μ *be an element of* T_1, *let* $\{r, s\}$ *be the range of* μ *with* $\mu(p) = r$, *and let* $v \in G$ *be the unique vertex satisfying* $\mu(v) = s$. *Then:*

- **(i)** $\mu = p(f_v, h_1, h_2, h_3)$ *for some 4-ary polynomial* $p(x, y, z, w)$ *of* $\mathbf{A}$ *and generators* $h_1, h_2, h_3 \in F$.
- **(ii)** $\mu \notin T_2$ *(that is,* $\langle r, s\rangle \in \alpha$*) if and only if* $\mu = p(\nu_0, \nu_1, \nu_2, \nu_3)$ *for some 4-ary polynomial* $p(x, y, z, w)$ *of* $\mathbf{A}$ *with range contained in* U *and elements* $\nu_i \in \Upsilon$ *such that none of the four polynomials* $p(x, \nu_1, \nu_2, \nu_3)$, $p(\nu_0, x, \nu_2, \nu_3)$, *etc. of* $\mathbf{A}[\mathbf{G}]$ *are one-to-one when restricted to* Υ.
- **(iii)** T_2 *is definable.*

PROOF. The third item follows from the second, as there are only finitely many 4-ary polynomials of $\mathbf{A}$.

By an argument similar to the proof of Case 1 in Claim 2,

$$\mu = t(f_v, g_1, \ldots, g_n)$$

for some $n+1$-ary polynomial t of $\mathbf{A}$ and nonconstant generators $g_1, \ldots, g_n \in \overline{G} \cup F$, and

$$\begin{aligned} t(a, a, \ldots, a) &= r \\ t(b, a, \ldots, a) &= s. \end{aligned}$$

Let $\mu_1 = t(f_v, \hat{a}, \ldots, \hat{a})$. If $\mu_1 = \mu$ then we would be done, but there is no guarantee that this will be the case. At least the following is true: $\mu_1(x) = \mu(x)$ for all $x \in G \cup \{p\}$. (This follows from the above equations, our assumption that $\mu \in T_1$, and the form of the generators in $\overline{G} \cup F$.) Suppose $e \in E$ and $\mu_1(e) \neq \mu(e)$. Then either $v \in e$ but $\mu(e) = r$, which implies

$$t(b, c_1, \ldots, c_n) = r \text{ for some } c_i \in \{a, b\}, \tag{1}$$

or else $v \notin e$ but $\mu(e) = s$, which implies

$$t(a, d_1, \ldots, d_n) = s \text{ for some } d_i \in \{a, b\}. \tag{2}$$

Pick $\langle c_1, \ldots, c_n \rangle$ and $\langle d_1, \ldots, d_n \rangle$ witnessing equations (1) and (2), if such exist.

Define

$$\begin{aligned} E_1 &= \{e \in E : \mu(e) = \mu_1(e)\} \\ E_2 &= \{e \in E : v \in E \text{ and } \mu(e) = r\} \\ E_3 &= \{e \in E : v \notin E \text{ and } \mu(e) = s\} \end{aligned}$$

and for $i = 1, \ldots, n$ define $h_i : X \longrightarrow \{a, b\}$ to be the function

$$h_i = \hat{a}|_{G \cup \{p\} \cup E_1} \cup \hat{c}_i|_{E_2} \cup \hat{d}_i|_{E_3}.$$

By construction, $h_i \in F \subseteq \mathbf{A}[\mathbf{G}]$ and $\mu = t(f_v, h_1, \ldots, h_n)$. Furthermore, because there are only four possibilities for each $\langle c_i, d_i \rangle$, one of which is $\langle a, a \rangle$, the set $\{h_1, \ldots, h_n\}$ contains at most three distinct nonconstant functions. This prove the first item of the Claim.

To prove the second item, suppose first that $\langle r, s \rangle \in \alpha$. By what has just been shown, $\mu = p(h_0, h_1, h_2, h_3)$ for some 4-ary polynomial p and generators $h_i \in \overline{G} \cup F$. We can assume that each h_i is nonconstant and that the range of p is contained in U. Since $\operatorname{typ}(\alpha, \beta) = 1$, $p|_N$ depends on at most one variable modulo α. In fact, because the range of each h_i is not α-connected while the range of μ is α-connected, and by the minimality of U, $p|_N$ cannot depend on exactly one variable modulo α; hence $p|_N$ is constant modulo α. It follows (again by minimality of U) that for any $c_0, c_1, c_2, c_3 \in N$, the unary polynomials $p(x, c_1, c_2, c_3)$, $p(c_0, x, c_2, c_3)$, etc. of $\mathbf{A}$ collapse $\beta|_U$ into $\alpha|_U$. This implies that each of the four unary polynomials $p(x, h_1, h_2, h_3)$, $p(h_0, x, h_2, h_3)$, etc. of

$\mathbf{A}[\mathbf{G}]$ maps Υ into its proper subset $\{\nu \in \Upsilon : \text{range}(\nu) \text{ is } \alpha\text{-connected}\}$, and so fails to be one-to-one on Υ. Thus p and $h_0, \ldots, h_3$ witness the condition stated in the Claim.

Conversely, suppose that $\mu = p(\nu_0, \nu_1, \nu_2, \nu_3)$ for some polynomial p of $\mathbf{A}$ with range contained in U and elements $\nu_i \in \Upsilon$ such that none of the four functions $p(x, \nu_1, \nu_2, \nu_3)$, $p(\nu_0, x, \nu_2, \nu_3)$, etc. are one-to-one when restricted to Υ. For each $i \leq 3$ let N_i be the $\beta|_U$-class containing the range of ν_i. As $\text{typ}(\alpha, \beta) = 1$, $p(x, y, z, w)$ restricted to $N_0 \times \cdots \times N_3$ depends on at most one variable modulo α. Since $p(x, \nu_1, \nu_2, \nu_3)$ is not one-to-one on Υ, there must exist $c_i \in N_i$ such that the polynomial $p(x, c_1, c_2, c_3)$ of $\mathbf{A}$ is not one-to-one when restricted to U. So by minimality of U, this polynomial collapses $\beta|_U$ into $\alpha|_U$, which proves that p restricted to $N_0 \times \cdots \times N_3$ cannot depend on exactly the first variable modulo α. The same argument in the other three variables establish that p is constant modulo α on $N_0 \times \cdots \times N_3$. So the range of μ is α-connected, as desired. ▮

This completes the proof that T_2 is definable in $\mathbf{A}[\mathbf{G}]$. By Claim 2 and the remarks preceding Claim 1, the graph $\mathbf{G}$ can be defined in $\mathbf{A}[\mathbf{G}]$ by formulas (with parameters) which do not depend on $\mathbf{G}$. In other words, these formulas define a semantic embedding of the class of finite graphs into the class of finite diagonal subpowers of $\mathbf{A}$, and Theorem 2.3 is proved. ▮

COROLLARY 2.4 *Every finitely decidable variety satisfies the* $(3,1)$ *and* $(3,2)$ *transfer principles.*

PROOF. Suppose $\mathcal{V}$ is a variety which fails to satisfy one of these transfer principles. Then by Lemma 5.3 from [11], there is a finite algebra $\mathbf{A}$ in $\mathcal{V}$ which satisfies the assumptions at the beginning of this section. So by Theorem 2.3 and the Basic Properties listed in the Introduction, $\mathcal{V}$ is finitely undecidable. ▮

3 Empty tails

In [5] it was shown that if $\mathcal{V}$ is a decidable variety and $\mathbf{A}$ is a finite member of $\mathcal{V}$, then all minimal sets of type 2 or 3 prime quotients in Con $\mathbf{A}$ must have empty tails. From that it followed that any decidable locally finite variety which omits type 1 must be congruence modular. In this section we shall prove these same results under the assumption that $\mathcal{V}$ is finitely decidable.

THEOREM 3.1 *Suppose* $\mathbf{A}$ *is a finite algebra belonging to an* ω*-structured variety. For every prime quotient* $\delta \prec \alpha$ *in* Con $\mathbf{A}$, *if* $\operatorname{typ}(\delta, \alpha) \neq 1$ *then the* $\langle\delta, \alpha\rangle$*-minimal sets have empty tails.*

PROOF. If the theorem were false, we could pick an algebra $\mathbf{A}$ and a prime quotient $\langle\delta, \alpha\rangle$ which falsifies the claim, and such that the cardinality of $\mathbf{A}$ is as small as possible. By Lemma 11.1 from [5], $\operatorname{typ}\{\mathbf{A}\} \subseteq \{1,2,3\}$. Let U be a $\langle\delta, \alpha\rangle$-minimal set, and let B and T be its body and tail respectively, with $T \neq \emptyset$. The minimality of $|A|$ implies several facts:

1. $\delta = 0_A$. (For if not, then $\mathbf{A}/\delta$ would provide a smaller counterexample.)

2. If $\langle\beta, \gamma\rangle$ is any prime quotient in Con $\mathbf{A}$ with $\beta > 0_A$, then the $\langle\beta, \gamma\rangle$-minimal sets have empty tails (by the same reasoning).

3. $U = A$. (Otherwise, $\mathbf{A}|_U$ would be a smaller counterexample, by Basic Property 3 in the introduction to this paper.)

4. $\mathbf{A}$ is subdirectly irreducible.

To see why this last item is true, suppose $\mathbf{A}$ were not subdirectly irreducible; pick $\beta \in$ Con $\mathbf{A}$ maximal with respect to not containing α, and let $\gamma = \beta \vee \alpha$. Obviously γ is the unique cover of β and $\langle\beta, \gamma\rangle$ is perspective with $\langle 0, \alpha\rangle$. Thus $\mathbf{A}$ is also $\langle\beta, \gamma\rangle$-minimal and $\operatorname{typ}(\beta, \gamma) = \operatorname{typ}(0, \alpha) \in \{2,3\}$. It follows that the $\langle\beta, \gamma\rangle$-tail of $\mathbf{A}$ is nonempty, since otherwise $\mathbf{A}_A$ would be Mal'cev, which it is not ($\mathbf{A}$ is minimal with nonempty tail relative to one of its prime quotients; see the proof of Theorem 8.5 in [5]). This contradicts item (2).

We now consider cases.

CASE 1. $\operatorname{typ}(0, \alpha) = 3$.

Thus $|B| = 2$, say, $B = \{0,1\}$. As $\mathbf{A}$ is subdirectly irreducible, the $(3,1)$ and $(3,2)$ transfer principles (proved in Section 2) imply that $\operatorname{typ}\{\mathbf{A}\} = \{3\}$. Pick congruences β and γ satisfying $\beta \subseteq B^2 \cup T^2$, $\gamma \not\subseteq B^2 \cup T^2$, and $\beta \prec \gamma$. By Lemma 2.17 from [5], and because $\beta \neq 0_A$, there must exist an element $2 \in T$ such that $\{0,2\}$ or $\{1,2\}$ is a $\langle\beta, \gamma\rangle$-minimal set. We may assume that it is $\{0,2\}$. Choose a $\langle 0,\alpha\rangle$-pseudo-join operation $q(x,y)$ and note that q satisfies $q(x,0)=x$ for all $x\in A$, and $q(x,2)=2$ for each $x\in\{0,2\}$. The next lemma will supply the needed contradiction.

LEMMA 3.2 *Suppose, in general, that* $\mathbf{A}$ *is a finite algebra,* α *is an atom in* Con $\mathbf{A}$, *and* $\mathbf{A}$ *is* $\langle 0,\alpha\rangle$*-minimal. Let* N *be a* $\langle 0,\alpha\rangle$*-trace and* T *the* $\langle 0,\alpha\rangle$*-tail, and assume there exist elements* $0\in N$ *and* $2\in T$ *such that:*

- **(i)** $\{0,2\}$ *is the image of an idempotent unary polynomial* e, *and* $\mathbf{A}|_{\{0,2\}}$ *is polynomially equivalent to a Boolean algebra.*
- **(ii)** $\mathbf{A}$ *has a polynomial* $q(x,y)$ *satisfying* $q(x,0)=x$ *for all* $x\in A$ *and* $q(x,2)=2$ *for each* $x\in\{0,2\}$.

Then the class of finite diagonal subpowers of $\mathbf{A}$ *is* ω*-unstructured.*

PROOF. Fix an element $1\in N$ distinct from 0. Observe that $q(1,2)=2$ (as $q(1,2)\overset{\alpha}{\equiv}q(0,2)=2$), so the polynomial $h(x):=q(1,x)$ satisfies $h(0)=1$ and $h(2)=2$. Also observe that $e(N)=\{0\}$.

Claim 5 *If* $p(x_1,\ldots,x_n;y_1,\ldots,y_m)$ *is a polynomial of* $\mathbf{A}$ *such that, for some* $\bar{c}\in\{0,1\}^n$, $p(\bar{x};\bar{c})$ *restricted to* $\{0,1\}$ *is nonconstant, then there is an* $i\in\{1,\ldots,n\}$ *such that for all* $\bar{a}\in\{0,2\}^n$ *and* $\bar{d}\in\{0,2\}^m$, $a_i=2$ *implies* $p(\bar{a},\bar{d})\in T$.

PROOF. We need only to prove this when $n=1$. If the claim were false for some specific values of $\bar{c}$ and $\bar{d}$, then by using the polynomial h we can assume that $\bar{c}=\bar{0}$, and so by suppressing those variables y_i for which $c_i=d_i$ we can further assume that $\bar{d}=\bar{2}$. Then the polynomial $b(x,y):=p(x,y,\ldots,y)$ satisfies $b(0,0)\neq b(1,0)$ and $b(2,2)\in B$. Now let $f(x)=b(x,e(x))$. Then $f(0)\neq f(1)$ and yet $f(2)\in B$, which contradicts the $\langle 0,\alpha\rangle$-minimality of $\mathbf{A}$ (and proves the claim). ∎

To finish the proof of the Lemma, let $\mathbf{G}=\langle G,E\rangle$ be a finite graph and let X be the disjoint union of G and a two-element set $\{a,b\}$. For each $v\in G$ and $e\in E$ define $f_v: X\longrightarrow A$ and $f_e: X\longrightarrow A$ by

$$f_v(x) = \begin{cases} 2 & \text{if } x=v \\ 0 & \text{otherwise} \end{cases}$$

$$f_e(x) = \begin{cases} 2 & \text{if } x\in e \\ 1 & \text{if } x=b \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathbf{A}[\mathbf{G}]$ be the subalgebra of $\mathbf{A}^X$ generated by the set

$$\{f_v : v \in G\} \ \cup \ \{f_e : e \in E\} \ \cup \ \{\hat{a} : a \in A\}.$$

Clearly the set $\mathcal{B}$ of all functions in $\mathbf{A}[\mathbf{G}]$ whose range is included in $\{0, 2\}$ is definable in $\mathbf{A}[\mathbf{G}]$. Moreover, $\mathcal{B}$ is a Boolean subalgebra of $\{0,2\}^X$ and its Boolean operations are definable in $\mathbf{A}[\mathbf{G}]$. (For definiteness, we assume that $0 < 2$.) Let $\chi_G = \hat{2}|_G \cup \hat{0}|_{\{a,b\}}$; then χ_G is in $\mathcal{B}$ (χ_G is the join of the f_v's) and the set $\{f_v : v \in G\}$, which is precisely the set of atoms in $\mathcal{B}$ lying below χ_G, is definable in $\mathbf{A}[\mathbf{G}]$ by a formula $\mathsf{G}(x)$ using χ_G and the constant functions as parameters.

We claim, for distinct $v, w \in G$, that $\{v, w\} \in E$ iff there exists $g \in \mathbf{A}[\mathbf{G}]$ such that $\text{range}(g|_G) \subseteq \{0, 2\}$, $g^{-1}(2) \cap G = \{v, w\}$, and $g(a) \neq g(b)$. Certainly if $\{v, w\} = e \in E$ then $g := f_e$ has these properties. Conversely, suppose $g \in \mathbf{A}[\mathbf{G}]$ has these properties. Pick a polynomial $p(x_1, \ldots, x_n; y_1, \ldots, y_m)$ of $\mathbf{A}$ and nonconstant generators f_{e_i}, f_{v_j} such that $g = p(f_{e_1}, \ldots, f_{e_n}; f_{v_1}, \ldots, f_{v_m})$. Then $g(a) \neq g(b)$ implies $p(\bar{0}; \bar{0}) \neq p(\bar{1}; \bar{0})$. It follows from Claim 5 that for some i,

$$e_i \subseteq \{u \in G \,:\, g(u) \in T\} = \{u \in g \,:\, g(u) = 2\} = \{v, w\}$$

and hence $\{v, w\} = e_i \in E$ as desired.

What remains is to show that the edge relation is first-order definable in $\mathbf{A}[\mathbf{G}]$. To this end, let $S = \{u \in A : q(u, 2) = 2\}$. Note that $N \cup \{2\} \subseteq S$ and that Σ, the set of functions in $\mathbf{A}[\mathbf{G}]$ whose range is contained in S, is definable in $\mathbf{A}[\mathbf{G}]$. We claim that the relation "$f(a) = f(b)$" restricted to those f in Σ is definable. The defining formula $\mathsf{Equal}_{ab}(x)$ is:

$$\bigvee_{u \in S} q(x, \chi_G) = q(\hat{u}, \chi_G).$$

Furthermore, the formula $\mathsf{Good}(x)$ given by

$$x \in \Sigma \quad \& \quad e(q(x, \chi_G^c)) = q(x, \chi_G^c)$$

defines the functions in $\mathbf{A}[\mathbf{G}]$ which are $\{0,2\}$-valued on G and S-valued on $\{a, b\}$. (χ_G^c is the Boolean complement of χ_G in $\{0,2\}^X$.) These remarks imply that the formula $\mathsf{E}(x, y)$, which is the conjunction of $\mathsf{G}(x)$ & $\mathsf{G}(y)$ & $x \neq y$ and

$$\exists z[\mathsf{Good}(z) \;\; \& \;\; q(z, \chi_G^c) \wedge \chi_G = x \vee y \;\; \& \;\; \neg \mathsf{Equal}_{ab}(z)],$$

recovers the edge relation of the original graph. This proves Lemma 3.2, and completes the proof of Case 1. ∎

CASE 2. $\mathrm{typ}(0, \alpha) = 2$.

Let B be the $\langle 0, \alpha \rangle$-body of $\mathbf{A}$ and T the tail. Let $d(x, y, z)$ be a $\langle 0, \alpha \rangle$-pseudo-Mal'cev operation, let β be the largest congruence of $\mathbf{A}$ satisfying $\beta \subseteq B^2 \cup T^2$, and let γ be a cover of β. $\langle \beta, \gamma \rangle$ is nonabelian by Lemma 4.27 of [5], so $\mathrm{typ}(\beta, \gamma) = 3$. $\beta > 0_A$ and therefore by our choice of $\mathbf{A}$, the $\langle \beta, \gamma \rangle$-minimal sets must have empty tails. As in Case 1 there must exist $0 \in B$ and $2 \in T$ such that $\{0, 2\}$ is a $\langle \beta, \gamma \rangle$-minimal set. Let e be an idempotent unary polynomial whose range is $\{0, 2\}$ and set $q(x, y) = d(x, e(x) \wedge e(y), y)$. Then $q(x, 0) = d(x, 0, 0) = x$ for all $x \in A$, while $q(x, 2) = d(x, x, 2) = 2$ for each $x \in \{0, 2\}$. So Lemma 3.2 again can be applied, which proves that Case 2 is impossible. Theorem 3.1 is proved. ∎

COROLLARY 3.3 *Suppose $\mathcal{V}$ is a locally finite variety in a finite language. If $\mathcal{V}$ is finitely decidable and omits type 1, then $\mathcal{V}$ is congruence modular.*

4 Conclusion

Let $\mathcal{V}$ be a locally finite ω-structured variety. We would like to summarize what is currently known about the structure of $\mathcal{V}$, in order to suggest problems for further study. We shall concentrate on the structure of the finite subdirectly irreducible members of $\mathcal{V}$, where the results are sharpest.

Suppose first that $\mathbf{A}$ is a finite subdirectly irreducible member of $\mathcal{V}$ whose monolith μ is nonabelian. Thus $\mathrm{typ}(0_A, \mu) = 3$ and so by the $(3,1)$ and $(3,2)$ transfer principles proved in Section 2, all prime quotients of $\mathbf{A}$ must be of type 3. From this J. Jeong [9] was able to prove that the congruence lattice of $\mathbf{A}$ is a chain. (P. Idziak [7] had earlier proved this result assuming $\mathcal{V}$ is congruence distributive.)

Suppose next that $\mathbf{A}$ is a finite subdirectly irreducible member of $\mathcal{V}$ whose monolith μ is abelian. In this case, less is known but much is conjectured. Let ν be the largest solvable congruence of $\mathbf{A}$. In the nicest of all possible worlds, we would hope that the following are true:

1. ν is abelian.

2. $\mathrm{typ}\{0_A, \nu\} = \mathrm{typ}(0_A, \mu)$.

3. ν is comparable to every congruence of $\mathbf{A}$.

4. The interval from ν to 1_A is a chain.

If item (1) were known to be true, then it would follow that all locally solvable congruences of members of $\mathcal{V}$ are abelian. In particular, the locally solvable algebras in $\mathcal{V}$ would form an abelian subvariety $\mathcal{A}$ of $\mathcal{V}$. It is shown in [11] that any locally finite abelian ω-structured variety satisfies the $(1,2)$ and $(2,1)$ transfer principles. It can then be shown (using results from [1] and [10]) that $\mathcal{A}$ would decompose as a varietal product of a strongly abelian subvariety and an affine subvariety. (This decomposition of $\mathcal{A}$ was announced in [11], where it is attributed to Valeriote, but his original proof was messy and has not been published.)

If the $(1,2)$ and $(2,1)$ transfer principles hold throughout $\mathcal{V}$ (not just in $\mathcal{A}$), then item (2) follows automatically. Idziak and Valeriote have recently investigated the necessity of these transfer principles, and can show the following: if $\mathbf{B}$ is a finite member of $\mathcal{V}$ and $0_B \prec \alpha \prec \beta$ is a failure of the $(1,2)$-transfer principle, and if M is a $\langle 0_B, \alpha\rangle$-trace and N is an $\langle \alpha, \beta\rangle$-trace, then there must be a unary polynomial f of $\mathbf{B}$ such that $f(N) \subseteq M$ and f is nonconstant on N. (The proof is obtained by modifying the proof of Lemma 8.4 in [11].) We expect that this result will prove to be useful in the further study of the $(1,2)$ transfer principle, but we do not regard it as evidence that the $(1,2)$ transfer principle will necessarily hold. (Idziak and Valeriote can prove the same result for failures of the $(1,3)$ transfer principle, and the result is automatically true for failures of the $(2,3)$ transfer principle [11, Lemma 6.2], and yet it is known that these principles can fail in finitely decidable varieties.)

K. Kearnes was the first to ask whether item (3) should be true. Idziak and Valeriote have announced a proof of this for the case $\mathrm{typ}(0_A, \mu) = 2$. We do not know whether it must be true when $\mathrm{typ}(0_A, \mu) = 1$.

In light of the structure of finite subdirectly irreducibles with nonabelian monolith mentioned above, item (4) is equivalent to the claim that ν has at most one cover. We conjecture that this will always be the case, at least if $1 \notin \mathrm{typ}\{\mathbf{A}\}$.

Our motivation for all four items has been the study of ω-structured locally finite congruence modular varieties made by J. Jeong [9]. In addition to the result concerning finite subdirectly irreducibles with nonabelian monolith mentioned above, he proved that the finite subdirectly irreducibles with abelian monolith in such varieties must satisfy items (1), (2) and (4). As well, he proved that such varieties must be congruence permutable and each finitely generated subvariety must be residually small.

Less is known about conditions which are sufficient for a variety $\mathcal{V}$ to be finitely decidable. Idziak [6] has shown that if $\mathcal{V}$ is arithmetical, finitely generated, and its subdirectly irreducible members have linear congruence lattices, then $\mathcal{V}$ is finitely decidable. Recently M. Albert [2] has generalized this result in the following way. Suppose $\mathcal{V}$ is congruence permutable, finitely generated, and residually small. If:

1. each nonabelian subdirectly irreducible member of $\mathcal{V}$ has a linear congruence lattice;

2. each nonabelian subdirectly irreducible member of $\mathcal{V}$ with abelian monolith μ satisfies $\nu = \mu$ (same notation as above);

then $\mathcal{V}$ is finitely decidable if and only if the variety of R-modules associated with the largest abelian subvariety of $\mathcal{V}$ is finitely decidable.

Ultimately we hope that our research will lead to an answer to the question "Which finitely generated varieties (in a finite language) are finitely decidable?" where the answer will be stated in terms of the structure of $\mathcal{V}$ and the finite decidability of certain classes of modules associated with $\mathcal{V}$. We can pose the same question for locally finite varieties (even arbitrary varieties) but then we no longer have confidence that a solution will be found. Even restricted to locally finite discriminator varieties, the problem is as hard as determining which locally finite universal classes of algebras are finitely decidable.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, MCMASTER UNIVERSITY, HAMILTON, ONTARIO, CANADA, L8S 4K1.

DEPARTMENT OF MATHEMATICS, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PENNSYLVANIA 15213 – and –

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO, CANADA, N2L 3G1.