LYAPUNOV EXPONENTS AND STOCHASTIC FLOWS OF LINEAR AND AFFINE HEREDITARY SYSTEMS: A SURVEY

by

Salah-Eldin A. Mohammed
Department of Mathematics
Southern Illinois University at Carbondale
Carbondale, IL 62901-4408
and
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

Research Report No. 91-98 2

January 1991

LYAPUNOV EXPONENTS AND STOCHASTIC FLOWS OF LINEAR AND AFFINE HEREDITARY SYSTEMS: A SURVEY[+‡]

Salah–Eldin A. Mohammed

Department of Mathematics
Southern Illinois University
at Carbondale
Carbondale, IL 62901

and

Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213–3890

[+]Research supported in part by NSF Grant DMS–8907857.

[‡]For "Proceedings of International Conference on Diffusion Processes and Related Problems in Analysis, Vol. II".

Lyapunov Exponents and Stochastic Flows of Linear and Affine Hereditary Systems: A Survey[1]

Salah–Eldin A. Mohammed

§1. The General Problem.

In this article we intend to review known results – and also discuss new ones – concerning the existence of flows and the characterization of Lyapunov exponents for trajectories of stochastic linear and affine hereditary systems. Such systems (also called *stochastic functional differential equations*) are stochastic differential equations in which the differential of the state variable x depends on its current value $x(t)$ at time t as well as its previous values $x(s)$, $t-r \le s < t$. We shall be concerned almost exclusively with the *finite history* case $0 \le r < \infty$.

More specifically, consider the stochastic affine hereditary system

$$\left.\begin{aligned} dx(t) &= \sum_{i=0}^{m} \Big[\int_{-r}^{0} \nu_i(t)(ds)x(t+s)\Big]dZ_i(t) + dQ(t), \quad t > 0 \\ x(0) &= v \;, \quad x(s) = \eta(s) \quad , \quad -r < s < 0 \,. \end{aligned}\right\} \tag{I}$$

The above system lives on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, P)$ satisfying the "usual conditions" (Métivier and Pellaumail [31], Métivier [30], Dellacherie and Meyer [14]). Vectors in $\mathbb{R}^n$ (or $\mathbb{C}^n$) are column vectors, given the Euclidean norm $|\cdot|$. The *noise* in (I) is provided by $(\mathcal{F}_t)_{t\ge0}$–semi–martingales $Z_i : \mathbb{R}^+ \times \Omega \to \mathbb{R}$, $i = 0,1,\ldots,m$, $Q: \mathbb{R}^+ \times \Omega \to \mathbb{R}^n$ with jointly stationary increments. The *memory* is prescribed by stationary $(\mathcal{F}_t)_{t\ge0}$ – adapted measure–valued processes ν_i, $i = 0,\ldots,m$, such that each $\nu_i(t,\omega)$ is an

[1]Research supported in part by NSF Grant DMS–8907857.

n×n–matrix–valued measure on $[-r,0]$. The *solution* $x : [-r,\infty) \times \Omega \to \mathbb{R}^n$ is a measurable $(\mathscr{F}_t)_{t\geq 0}$ – adapted process with $x|(0,\infty) \times \Omega$ having a.a. sample paths cadlag (viz. right–continuous with left limits). The *initial condition* is a (possibly random) pair $(v,\eta) \in \mathbb{R}^n \times \mathscr{X}$, where $\mathscr{X}$ is some Banach space containing all cadlag paths $[-r,0] \to \mathbb{R}^n$ e.g. $\mathscr{X} = C([-r,0], \mathbb{R}^n)$, $D([-r,0], \mathbb{R}^n)$, $\mathbf{L}^2([-r,0], \mathbb{R}^n)$, or a weighted $\mathbf{L}^2$ space $\mathbf{L}^2_\rho((-r,0], \mathbb{R}^n)$ so as to allow for the infinite fading memory case $r = \infty$ (cf. Mizel and Trutzer [32], Coleman and Mizel [9], [10]). In order to observe the dynamics of (I) it is convenient to define the *segment* $x_t \in \mathscr{X}$ by

$$x_t(\cdot,\omega)(s) := x(t+s,\omega), \quad t \geq 0, \quad -r \leq s \leq 0.$$

This idea goes back to Krasovskii [24] (pp. 126–175) in the deterministic case : $Q \equiv 0$, $\nu_i(t,\omega)$ fixed in (t,ω), and $Z_i(t) = t$, $i = 0,1,\ldots,m$, a.s. In this case the existence of solutions and the asymptotic stability of the trajectories $x_t \in \mathscr{X} = C([-r,0], \mathbb{R}^n)$ were studied extensively by J.K. Hale and his school in the sixties (Hale [19], [20]), Krasovskii [24], El'sgol'tz[17], Bellman and Cooke [6] and others. The corresponding issues in the case $\mathscr{X} = \mathbf{L}^2(-r,0], \mathbb{R}^n)$ were studied by Delfour and Mitter [13] in the finite memory case (cf. also Corduneanu and Lakshmikantham [11] and the references therein for systems with infinite memory).

For the stochastic hereditary white–noise case ($Z_0(t) = t$, $Z_i(t)$, $Q(t)$ independent Brownian motions, $\nu_i(t,\omega)$ fixed), the existence of $(\mathscr{F}_t)_{t\geq 0}$ – adapted solutions and their asymptotic stability were treated by several authors, e.g. K. Itô and M. Nisio [21], Kushner [25], Mohammed [33], [34], [36], Mizel and Trutzer [32], Mohammed, Scheutzow and Weizsäcker [40], Scheutzow [45], Kolmanovskii and Nosov [23]. Extensions of the existence results to the case of semimartingale noises Z_i, Q were discussed by Doleans–Dade [15], Métivier and Pellaumail [31], Métivier [30], Protter [42] and others.

Our present discussion will focus on results concerning almost sure asymptotic stability of the *trajectory* $(x(t), x_t) \in \mathscr{E} := \mathbb{R}^n \times \mathscr{X}$ of the stochastic hereditary system (I). In particular the following issues will be discussed:

(i) Existence of *measurable stochastic (semi)–flows* $X : \mathbb{R}^+ \times \Omega \times \mathscr{E} \to \mathscr{E}$ for (I) with the properties:

(a) If x is the solution of (I) with initial data $(v,\eta) \in \mathscr{E}$, then
$X(t,\cdot,(v,\eta)) = (x(t), x_t)$ for all $t \geq 0$ a.s.

(b) Each map $X(t,\omega,\cdot)$, $t \in \mathbb{R}^+$, a.a. $\omega \in \Omega$, is a continuous affine linear operator on $\mathscr{E}$.

(ii) A characterization of almost sure *Lyapunov exponents*

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log \|(x(t), x_t)\|_{\mathscr{E}}$$

for a given natural norm on the state space $\mathscr{E}$, e.g. if $\mathscr{X} = L^2([-r,0], \mathbb{R}^n)$, one usually takes the Hilbert norm

$$\|(v,\eta)\|_{M_2}^2 := |v|^2 + \int_{-r}^{0} |\eta(s)|^2 ds\,, \quad (v,\eta) \in \mathbb{R}^n \times L^2([-r,0], \mathbb{R}^n),$$

on the classical Delfour–Mitter space $\mathscr{E} := M_2 := \mathbb{R}^n \times L^2([-r,0], \mathbb{R}^n)$ (Delfour and Mitter [13]).

(iii) A study of *hyperbolicity* in (I), viz. the case of non–zero Lyapunov exponents. This is of interest for two reasons. In the linear case ($Q \equiv 0$), hyperbolicity leads to an exponential dichotomy with a flow–invariant saddle–point splitting of $\mathscr{E}$ (see §3A,B). When $Z_i(t) = t$ a.s. with $\nu_i(t,\omega)$ fixed, $1 \leq i \leq m$, Q having stationary increments and (I) hyperbolic, it turns out that the affine hereditary equation admits a unique stationary solution (§3,C, Theorem 13).

In the following section we examine the question of the existence of a robust flow for (I).

§2. Classification of Stochastic Hereditary Systems. Existence of Flows.

(A) Linear Equations Driven by White Noise:

Consider first the non–delay case $r = 0$, $\nu_i(t,\omega) = A_i(t,\omega)\,\delta_{\{0\}}$ with $A_i(t,\omega)$, $i = 0,1,\ldots,m$, stationary $n \times n$ matrix–valued processes and $\delta_{\{0\}}$ the Dirac measure at 0. In this case the state space $\mathscr{E}$ may be identified with $\mathbb{R}^n$, and it is well–known that the trajectories $\{x(t) : t \geq 0,\ x(0) = v \in \mathbb{R}^n\}$ admit a measurable flow $X : \mathbb{R}^+ \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ such that $X(t,\cdot,v) = x(t)$ for all $t \geq 0$, a.s. and $X(t,\omega,\cdot)$ is linear (invertible) on $\mathbb{R}^n$ for a.a. $\omega \in \Omega$ and all $t \geq 0$, (Arnold [1], Leandre [26], Jacod [22]).

However, when $r > 0$ in the hereditary system (I), stochastic flows may no longer exist. To be more specific we introduce the following classification of hereditary systems:

Definition:

The hereditary system (I) is said to be *regular* (with respect to the state space $\mathscr{E}$) if the family of its *trajectories*

$$\{(x(t),x_t) : (x(0),\, x_0) = (v,\eta) \in \mathscr{E}\ ,\quad t \geq 0\}$$

admits a (Borel $\mathbb{R}^+ \otimes \mathscr{F} \otimes$ Borel $\mathscr{E}$, Borel $\mathscr{E}$)–measurable version $X : \mathbb{R}^+ \times \Omega \times \mathscr{E} \to \mathscr{E}$ such that, for a.a. $\omega \in \Omega$ and each $t \geq 0$, the map $X(t,\omega,\cdot) : \mathscr{E} \to \mathscr{E}$ is affine continuous linear. The system (I) is called *singular* (w.r.t. $\mathscr{E}$) if it is not regular (w.r.t. $\mathscr{E}$).

Unfortunately *singular hereditary linear systems do exist*, e.g. the one–dimensional stochastic linear delay equation

$$\left.\begin{aligned} dx(t) &= x(t-r)\,dW(t)\ , \qquad t > 0 \\ (x(0),x_0) &= (v,\ \eta) \end{aligned}\right\} \tag{II}$$

driven by a Wiener process W and with a *positive* delay r is singular with respect to either of the state spaces

$$\mathscr{E} := \{(v,\eta) : v \in \mathbb{R}\ ,\ \ \eta \in C([-r,0],\ \mathbb{R})\ ,\ v = \eta(0)\} \cong C([-r,0],\ \mathbb{R})$$

or $\mathscr{E} := \mathbb{R} \times L^2([-r,0],\ \mathbb{R})$. Indeed we have

Theorem 1 (Mohammed [33], [35])

Let $\mathscr{E} \cong C([-r,0],\ \mathbb{R})$ *or* $\mathbb{R} \times L^2([-r,0],\ \mathbb{R})$ *and suppose the delay* r *in (II) is positive. Suppose* $Y : [0,r] \times \Omega \times \mathscr{E} \to \mathbb{R}$ *is any (Borel* $[0,r] \otimes \mathscr{F} \otimes$ *Borel* $\mathscr{E}$*, Borel* $\mathbb{R}$*) – measurable version of the solution field* $\{x(t)\!:\ 0 \le t \le r\ ,\ (x(0),\ x_0) = (v,\eta) \in \mathscr{E}\}$ *to (II). Then, for a.a.* $\omega \in \Omega$ *and each* $t \in (0,r]$*, the map* $Y(t,\omega,\cdot) : \mathscr{E} \to \mathbb{R}$ *is locally unbounded and (hence) non–linear.*

The above pathological phenomenon is peculiar to the delay case $r > 0$. On the other hand when $r = 0$ we have the simple explicit solution

$$x(t,\omega,v) = v e^{\omega(t) - \frac{1}{2} t} \qquad t > 0,\ \ \omega \in \Omega,\ v \in \mathbb{R}.$$

This version of the solution is (a.s.) continuous linear in the initial state $v \in \mathscr{E} = \mathbb{R}$. The pathology in the delay case $r > 0$ is attributed to the *Gaussian nature* of the Wiener process W coupled with the *infinite–dimensionality* of the state space $\mathscr{E}$. A proof of Theorem 1 may be found in (Mohammed [33], pp. 144–147) for the case $\mathscr{E} \cong C([-r,0],\ \mathbb{R})$. Essentially the same proof also covers the case $\mathscr{E} = \mathbb{R} \times L^2(-r,0],\ \mathbb{R})$.

Remarks:

(i) The conclusion of Theorem 1 imposes non–trivial limitations on the applicability of the general existence theorems for stochastic differential equations given in (Métivier and Pellaumail [31], Métivier [30], Protter [42], et al.). The hypotheses in these theorems (e.g. Theorem 6.10, pp. 74–75 in [31]) require that the random coefficients in the s.d.e. admit a.s. Lipschitz versions

$$\Omega \times \mathscr{E} \to \mathbb{R}^n$$

rather than just being random fields

$$L^2(\Omega,\mathscr{E}) \to L^2(\Omega,\mathbb{R}^n)$$

(e.g. as in Mohammed [33], Theorem (2.1), p. 36; cf. also Berger and Mizel [7], Weizsäcker and Winkler [49], Remark (a) p. 274). As an example the one–dimensional stochastic linear hereditary equation

$$\left.\begin{array}{l} dx(t) = \{\int_{-r}^{0} x(t+s)dW(s)\}dW(t)\ , \quad t > 0 \\ (x(0),x_0) \in \mathscr{E} \end{array}\right\} \tag{III}$$

with a Wiener process $\{W(t) : t \geq -r\}$ does not appear to be covered by the existence theorems in [31], [30], [42]. This is because the coefficient in (III) is a random field

$$\mathscr{E} \to L^2(\Omega,\mathbb{R})$$

$$(v,\eta) \mapsto \int_{-r}^{0} \eta(s)\, dW(s)$$

which *does not admit* measurable a.s. locally bounded or linear versions

$$\Omega \times \mathscr{E} \to \mathbb{R}$$

let alone Lipschitz ones! (in case $r > 0$, $\mathscr{E} \cong C([-r,0], \mathbb{R})$, or $\mathbb{R} \times L^2([-r,0], \mathbb{R})$, see Theorem (8.6) p. 28 in Mohammed [33]).

(ii) The erratic behavior in Theorem 1 above suggests similar difficulties in certain types of stochastic linear partial differential equations driven by *multi–dimensional* white noise (cf. Flandoli and Schaumlöffel [18]).

Recent work by V.J. Mizel and the author shows that the conclusion of Theorem 1 also holds for the one–dimensional hereditary equation

$$\left.\begin{array}{l} dx(t) = \int_{-r}^{0} x(t+s)d\nu(s)dW(t), \quad t > 0 \\ (x(0),x_0) \in \mathbb{R} \times L^2([-r,0], \mathbb{R}) \end{array}\right\} \qquad \text{(IV)}$$

where W is a Wiener process and ν is a fixed finite real–valued Borel measure on $[-r,0]$ satisfying the hypotheses

$$\overline{\text{supp } \nu} \subset [-r,0), \quad \liminf_{n\to\infty} \left| \int_{-r}^{0} e^{2\pi i n(\frac{s}{r})} d\nu(s) \right| > 0.$$

On the other hand, (IV) is regular if ν has a C^1 (or even L^2_1) density with respect to Lebesgue measure on $[-r,0]$ (Mohammed and Scheutzow [39], Theorem 4.2). An interesting problem is to classify all finite signed measures ν on $[-r,0]$ for which the hereditary equation (IV) is regular.

Needless to say we do not know a complete characterization of all processes $Z(t) = (Z_0(t), Z_1(t),\ldots,Z_m(t))$, and $\nu(t) = (\nu_0(t), \nu_1(t),\ldots,\nu_m(t))$ for which the hereditary system (I) is regular. On the other hand regularity holds for a large class of linear hereditary systems driven by white noise. Indeed the next result deals with the case: $Q(t) \equiv 0$, $Z_0(t) = t$, $Z_i(t) = W_i(t)$, $i = 1,2,\ldots,m$, are independent one–dimensional Wiener processes;

$$\nu_0(t,\omega) = \sum_{i=1}^{N} H(0,0,\ldots,0,\underset{\substack{\uparrow\\ i\text{th place}}}{\cdot},0,\ldots,0)\,\delta_{\{-d_i\}} + H(0,0,\ldots,0,\underset{\substack{\uparrow\\ (N+1)\text{st place}}}{\cdot},0)\,\delta_{\{0\}} + h(s)ds$$

where $H : (\mathbb{R}^n)^{N+1} \times L^2([-r,0], \mathbb{R}^n) \to \mathbb{R}^n$ is a fixed continuous linear map, $h : [-r,0] \to \mathbb{R}^{n\times n}$ is an $n\times n$–matrix–valued L^2 function, ds is Lebesgue measure on $[-r,0]$;

$\nu_i(t,\omega) = g_i$, $i = 1,2,\ldots,m$, fixed (deterministic) $n\times n$ matrices. This case corresponds to the stochastic linear functional differential system

$$\left.\begin{aligned} dx(t) = {} & H(x(t-d_1), \ldots, x(t-d_N), x(t), x_t)dt \\ & + \sum_{i=1}^{m} g_i(x(t))\, dW_i(t), \quad t > 0 \end{aligned}\right\} \qquad (V)$$

$$(x(0), x_0) = (v, \eta) \in \mathbb{R}^n \times L^2([-r, 0], \mathbb{R}^n) := M_2$$

with several finite delays $0 < d_1 < d_2 < \ldots < d_N \leq r$ in the drift term and *no delays in the diffusion coefficient.* Observe that the above equation (V) is defined on the canonical complete filtered Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ supporting the m–dimensional Brownian motion $W = (W_1, W_2, \ldots, W_m)$. More specifically, Ω denotes the space of all continuous paths $\omega : \mathbb{R} \to \mathbb{R}^m$ with $\omega(0) = 0$ given the compact open topology and the Borel σ–algebra $\mathcal{F}$. For each $t \geq 0$, $\mathcal{F}_t$ is the σ–algebra generated by all evaluations $\{\rho_u : u \leq t\}$, $\rho_u : \Omega \to \mathbb{R}^m$,

$$\rho_u(\omega) := \omega(u) \quad u \in \mathbb{R}, \quad \omega \in \Omega;$$

and P is Wiener measure on Ω.

Theorem 2 (Mohammed [37])

The hereditary system (V) is regular with respect to the state space $\mathcal{E} = M_2 := \mathbb{R}^n \times L^2([-r,0], \mathbb{R}^n)$. *Indeed there is a Borel measurable version* $X : \mathbb{R}^+ \times \Omega \times M_2 \to M_2$ *of the trajectory field* $\{(x(t), x_t) : t \in \mathbb{R}^+, (x(0), x_0) = (v, \eta) \in M_2\}$ *with the following properties:*

(i) For each $(v,\eta) \in M_2$, $X(t,\cdot,(v,\eta)) = (x(t), x_t)$ *for all* $t \in \mathbb{R}^+$, *a.s.*

(ii) For each $t \in \mathbb{R}^+$ *and* $(v,\eta) \in M_2$, $X(t,\cdot,(v,\eta))$ *is* $\mathcal{F}_t$*–measurable and belongs to* $L^2(\Omega, M_2; P)$.

(iii) There is a Borel set $\Omega_0 \subset \Omega$ *of full Wiener measure such that, for all* $\omega \in \Omega_0$, *the map* $X(\cdot,\omega,\cdot) : \mathbb{R}^+ \times M_2 \to M_2$ *is continuous.*

(iv) For each $t \in \mathbb{R}^+$ *and every* $\omega \in \Omega_0$, *the map* $X(t,\omega,\cdot) : M_2 \to M_2$ *is continuous linear; for each* $\omega \in \Omega_0$, *the map* $\mathbb{R}^+ \ni t \mapsto X(t,\omega,\cdot) \in L(M_2)$ *is measurable and*

locally bounded in the uniform operator norm on $L(M_2)$.

(v) For each $t \geq r$ *and all* $\omega \in \Omega_0$*, the map* $X(t,\omega,\cdot) : M_2 \to M_2$ *is compact.*

The proof of the above theorem hinges on a variational technique which reduces the problem to the solution of a random family of classical hereditary differential systems involving *no stochastic integrals.* Note also the compactness of the flow for $t \geq r$. This fact plays an important role in defining hyperbolicity for (V) and the associated exponential dichotomies in §3 (A),(B). Observe also that in (iv) of the above theorem the map $[r,\infty) \to t \mapsto X(t,\omega,\cdot) \in L(M_2)$ is continuous for all $\omega \in \Omega_0$.

A non–linear analogue of Theorem 2 also holds under the following conditions: In (V), take $\mathscr{E} := C([-r,0],\mathbb{R}^n)$, H globally Lipschitz, $g_i : \mathbb{R}^n \to \mathbb{R}^n$ C^2 maps satisfying a Frobenius condition

$$Dg_i(v)g_j(v) = Dg_j(v)g_i(v) \quad , \quad 1 \leq i,j \leq m \, , \quad v \in \mathbb{R}^n \, ;$$

(Mohammed [33], Theorem (2.1), Chapter (V), §2, p. 121). This latter result is proved in [33] using a non–linear variational method originally due to Sussman [47] and Doss [16] in the non–delay case $r = 0$.

(B) Linear Equations Driven by Semimartingales:

The regularity w.r.t. M_2 of a large class of linear hereditary equations of the form

$$\left.\begin{aligned} dx(t) &= \{\int_{[-r,0]} \nu(t)(ds)x(t+s)\}dt + dN(t)\int_{-r}^{0} K(t)(s)x(t+s)ds + L(t)x(t-) \\ &\qquad\qquad t > 0 \\ x(0) &= v \in \mathbb{R}^n, \quad x(s) = \eta(s), \quad -r < s < 0, \quad r \geq 0 \end{aligned}\right\} \qquad \text{(VI)}$$

has recently been established by Mohammed & Scheutzow [39] under the following setting:

In (VI) all processes are defined on a complete filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$ satisfying the usual conditions. Denote by $\mathcal{M}([-r,0],\mathbb{R}^{n\times n})$ the space of all $n\times n$–matrix–valued Borel measures on $[-r,0]$ (or $\mathbb{R}^{n\times n}$–valued functions of bounded variation on $[-r,0]$). Give $\mathcal{M}([-r,0],\mathbb{R}^{n\times n})$ the σ–algebra generated by all evaluations. The space $\mathbb{R}^{n\times n}$ of all $n\times n$ matrices $A=(a_{ij})_{i,j=1}^{n}$ is given the Euclidean norm

$$\|A\|^2 := \sum_{i,j=1}^{n} a_{ij}^2 .$$

The process $\nu : \mathbb{R}\times\Omega \to \mathcal{M}([-r,0],\mathbb{R}^{n\times n})$ is measurable and $(\mathcal{F}_t)_{t\geq 0}$–adapted. Furthermore, for each $\omega\in\Omega$ and $t\geq 0$, let $\bar{\nu}(t,\omega)$ be the positive measure

$$\bar{\nu}(t,\omega)(A) := |\nu|(t,\omega)\{(A-t)\cap[-r,0]\} \tag{5}$$

for all Borel sets $A\subseteq[-r,\infty)$, with $|\nu|$ the total variation measure of ν w.r.t. the Euclidean norm on $\mathbb{R}^{n\times n}$. For each $\omega\in\Omega$ suppose the positive measure

$$\bar{\bar{\nu}}(\omega)(\cdot) := \int_0^\infty \bar{\nu}(t,\omega)(\cdot)dt \tag{6}$$

has a density $\frac{d\bar{\bar{\nu}}(\omega)}{ds}$ with respect to Lebesgue measure on $[-r,\infty)$ which is locally essentially bounded. If

$$\bar{\bar{\nu}}(t,\omega)(\cdot) := \int_0^t \bar{\nu}(u,\omega)(\cdot)\,du\ , \quad t>0\ , \tag{7}$$

suppose further that the map

$$[0,\infty) \to \mathbf{L}^2([-r,0], \mathbb{R})$$

$$t \mapsto \frac{d\overline{\overline{\nu}}(t,\omega)}{ds} \mid [-r,0]$$

is continuous on $[0,\infty)$ for every $\omega \in \Omega$. It is easy to see that this last condition is satisfied in the deterministic case $\nu(t,\omega) = \nu_0$, $t \geq 0$, $\omega \in \Omega$, for a fixed $\nu_0 \in \mathcal{M}([-r,0], \mathbb{R}^{n\times n})$. The process $N : \mathbb{R} \times \Omega \to \mathbb{R}^{n\times n}$ is an $(\mathcal{F}_t)_{t\geq 0}$–semimartingale and $K : \mathbb{R} \times \Omega \to C^1([-r,0],\mathbb{R}^{n\times n})$ is a measurable $(\mathcal{F}_t)_{t\geq 0}$–adapted process w.r.t. the σ–algebra generated by all evaluations on $C^1([-r,0], \mathbb{R}^{n\times n})$. For a.a. $\omega \in \Omega$, the random field $K(t,\omega)(s)$ is jointly C^1 in $(t,s) \in \mathbb{R}^+ \times [-r,0]$. The $(\mathcal{F}_t)_{t\geq 0}$–semimartingale $L : \mathbb{R} \times \Omega \to \mathbb{R}^{n\times n}$ is assumed to admit a representation $L = M + V$ where M is a *continuous* $(\mathcal{F}_t)_{t\geq 0}$–local martingale and V is an $(\mathcal{F}_t)_{t\geq 0}$–adapted process with a.a. paths right continuous and of bounded variation on compact subsets of $\mathbb{R}^+$.

We then have

Theorem 3 (Mohammed & Scheutzow [39])

Under the above hypotheses, the hereditary system (VI) is regular w.r.t. M_2. *In fact its trajectory* $\{(x(t),x_t) : t \geq 0, (x(0),x_0) \in M_2\}$ *has a measurable version* $X : \mathbb{R}^+ \times \Omega \times M_2 \to M_2$ *satisfying assertions (i), (ii), (iv), (v) of Theorem 2 with* $\Omega_0 \in \mathcal{F}$ *a set of full* P*–measure. Also for all* $\omega \in \Omega_0$ *and every* $(v,\eta) \in M_2$, *the path* $X(\cdot,\omega,(v,\eta)) : \mathbb{R}^+ \to M_2$ *is cadlag.*

(C) A Class of Affine Equations:

Consider the affine hereditary system

$$\left.\begin{aligned} dx(t) &= \{\int_{[-r,0]} \mu(ds)x(t+s)\}dt + dQ(t)\ , \quad t > 0 \\ x(s) &= \eta(s) \qquad\qquad -r \leq s \leq 0. \end{aligned}\right\} \tag{VII}$$

Here Q is an $\mathbb{R}^n$–valued semimartingale on a filtered probability space $(\Omega,\mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ with the usual conditions, and $Q(0) = 0$. The memory is driven by a fixed $\mathbb{R}^{n\times n}$–valued Borel measure μ on $[-r,0]$. The initial condition η belongs to the Banach space $D([-r,0], \mathbb{R}^n)$ of all cadlag paths $[-r,0] \to \mathbb{R}^n$ with the supremum norm

$$\|\eta\|_\infty := \sup_{-r\leq s\leq 0} |\eta(s)|.$$

We shall often take η to be an $\mathcal{F}$–measurable random variable with values in $D([-r,0], \mathbb{R}^n)$ which is *allowed to anticipate the driving noise* Q. (See §3,C).

An essential tool in studying the Lyapunov exponents $\overline{\lim_{t\to\infty}} \frac{1}{t} \log \|x_t\|_\infty$ of (VII) is the associated homogeneous deterministic linear hereditary system

$$\left.\begin{aligned} dy(t) &= \{\int_{[-r,0]} \mu(ds)\, y(t+s)\}dt\,, \quad t > 0 \\ y_0 &= \eta\,. \end{aligned}\right\} \qquad \text{(VIII)}$$

Using the integrated form

$$y(t) = \eta(0) + \int_0^t \int_{[-r,0]} \mu(ds)y(u+s)du\,, \quad t > 0, \qquad \text{(VIII)}'$$

we define a strongly continuous semigroup $(\tilde{T}(t))$ on the space $D := D([-r,0],\mathbb{R}^n)$ by setting $\tilde{T}(t)\eta := y_t^{(\eta)}$, $t \geq 0$, where $y^{(\eta)} : [-r,\infty) \to \mathbb{R}^n$ is the unique solution of (VIII) with initial path $\eta \in D([-r,0], \mathbb{R}^n)$. Denote by $F : [-r,\infty) \to \mathbb{R}^{n\times n}$ the fundamental matrix solution of

$$\left.\begin{aligned} \dot{F}(t) &= \int_{[-r,0]} \mu(ds)F(t+s)\,, \quad t > 0\,, \\ F_0 &= \Delta\,, \quad \Delta(s) := \begin{cases} I & s = 0 \\ 0 & -r \leq s < 0 \end{cases} \end{aligned}\right\} \qquad \text{(IX)}$$

where $I,0 \in \mathbb{R}^{n\times n}$ are the identity and zero $n\times n$ matrices respectively. If we extend Q to all of $\mathbb{R}$ by setting $Q(s) = 0$ for all $s < 0$, then the unique trajectory $\{x_t^{(\eta)}: t \geq 0\}$ of the affine hereditary system (VII) is given by

$$x_t^{(\eta)}(u) = \{\tilde{T}(t)\eta\}(u) + Q_t(u) + \int_0^t \dot{F}(t-s+u)Q(s)ds, \text{ a.s.,} \tag{X}$$

for $t \geq 0$, $u \in J := [-r,0]$, $\eta \in D(J,\mathbb{R}^n)$ (Mohammed and Scheutzow [38], Theorem 1). Alternatively, we have

$$x_t^{(\eta)}(u) = \{\tilde{T}(t)\eta\}(u) + \int_0^t F(t-s+u)dQ(s)\ , \ t \geq 0,\ u \in J. \tag{XI}$$

These integral representations immediately imply that (VII) is regular with a stochastic flow $X : \mathbb{R}^+ \times \Omega \times D \to D$ given by

$$X(t,\omega,\eta) = x_t^{(\eta)}(\cdot,\omega)\ ,\quad \eta \in D([-r,0],\mathbb{R}^n),\qquad t \geq 0,\ \omega \in \Omega.$$

This flow has the property that each $X(t,\omega,\cdot) : D \to D$ is a continuous affine linear map.

§3. Lyapunov Exponents. Hyperbolicity.

(A) Linear Equations. White Noise Case.

Let us go back to the setting of §2(A) and reconsider the linear hereditary system (V), viz.

$$\left.\begin{aligned} dx(t) &= H(x(t-d_1),\ldots,x(t-d_N),x(t),x_t)dt \\ &\qquad + \sum_{i=1}^{m} g_i(x(t))dW_i(t)\ ,\quad t > 0 \end{aligned}\right\} \tag{V}$$

$$(x(0),x_0) = (v,\eta) \in M_2 := \mathbb{R}^n \times L^2([-r,0],\mathbb{R}^n)$$

The existence of a.s. Lyapunov exponents

$$\lim_{t\to\infty} \frac{1}{t} \log \|(x(t),x_t)\|_{M_2}$$

for the above linear hereditary system was studied by Mohammed in [37]. The approach adopted in [37] is to show that the version X of the flow constructed in Theorem 2 is a multiplicative linear cocycle over the canonical Brownian shift $\theta : \mathbb{R} \times \Omega \to \Omega$ on Wiener space:

$$\theta(t,\omega)(u) := \omega(t+u) - \omega(t) \quad , \quad u,t \in \mathbb{R}, \ \omega \in \Omega.$$

One then uses the compactness of $X(t,\omega,\cdot) : M_2 \to M_2$, $t \geq r$, together with an infinite-dimensional version of Oseledec's multiplicative ergodic theorem due to Ruelle ([44], [43]). Indeed we have

Theorem 4 (Mohammed [37])

There is an $\mathcal{F}$*-measurable set* $\hat{\Omega}$ *of full* P*-measure such that* $\theta(t,\cdot)(\hat{\Omega}) \subseteq \hat{\Omega}$ *for all* $t \geq 0$ *and*

$$X(t_2,\theta(t_1,\omega),\cdot) \circ X(t_1,\omega,\cdot) = X(t_1 + t_2, \omega,\cdot) \tag{9}$$

for all $\omega \in \hat{\Omega}$ *and* $t_1,t_2 \geq 0$.

The first step in the proof of the above theorem is to approximate the Brownian motion W in (V) by smooth processes $\{W^k\}_{k=1}^{\infty}$:

$$W^k(t) := k \int_t^{t+\frac{1}{k}} W(u)du \ , \ t \geq 0, \ k \geq 1 \ ,$$

and let $X^k : \mathbb{R}^+ \times \Omega \times M_2 \to M_2$ be the stochastic flow of the following retarded functional differential system with random coefficients:

$$\left.\begin{aligned} dx^k(t) = \{ H(x^k(t-d_1),\dots , x^k(t-d_N), \ x^k(t), \ x^k_t) \\ + \sum_{i=1}^m g_i(x^k(t)) \dot{W}^k(t) - \frac{1}{2}\sum_{i=1}^m g_i^2(x^k(t))\}dt, \ t > 0 \end{aligned}\right\} (V^k)$$

$$(x^k(0),x^k_0) = (v,\eta) \in M_2.$$

It can be shown that if $X : \mathbb{R}^+ \times \Omega \times M_2 \to M_2$ is the flow of (V) constructed in Theorem 2, then

$$\lim_{k\to\infty} \sup_{0\le t\le T} \|X^k(t,\omega,\cdot) - X(t,\omega,\cdot)\|_{L(M_2)} = 0 \tag{10}$$

for a.a. $\omega \in \Omega$ and every $0 < T < \infty$ (see Theorem 2 in [37]). The above convergence actually works for all ω in a Borel set $\hat{\Omega}$ of full Wiener measure which is invariant under $\theta(t,\cdot)$ for all $t \ge 0$. The second step in the proof of Theorem 4 is as follows. We fix $\omega \in \hat{\Omega}$ and use uniqueness of solutions to (V^k) in order to obtain the cocycle property for (X^k,θ), viz. equation (9) with X replaced by X^k, $k \ge 1$. We then pass to the limit as $k \to \infty$ using the convergence in (10).

The a.s. Lyapunov exponents

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t,\omega,(v(\omega),\eta(\omega))\|_{M_2}, \quad \text{a.a. } \omega \in \Omega, \ (v,\eta) \in L^2(\Omega,M_2)$$

of the system (V) are characterized by the following Oseledec multiplicative ergodic theorem:

Theorem 5 (Mohammed [37])

Let $X : \mathbb{R}^+ \times \Omega \times M_2 \to M_2$ *be the flow of (V) given in Theorem 2. Then there exist*

(a) a Borel set $\Omega^* \subseteq \Omega$ *such that* $P(\Omega^*) = 1$ *and* $\theta(t,\cdot)(\Omega^*) \subseteq \Omega^*$ *for all* $t \ge 0$,

(b) a fixed (non–random) sequence of real numbers $\{\lambda_i\}_{i=1}^\infty$,

(c) a random family $\{E_i(\omega) : i \ge 1, \ \omega \in \Omega^*\}$ *of (closed) finite–codimensional*

subspaces of M_2*, with the following properties:*

(i) If the **Lyapunov spectrum** $\{\lambda_i\}_{i=1}^{\infty}$ *is infinite, then* $\lambda_{i+1} < \lambda_i$ *for all* $i \geq 1$ *and* $\lim_{i\to\infty} \lambda_i = -\infty$*; otherwise there is a fixed (non-random) integer* $N > 1$ *such that* $\lambda_N = -\infty < \lambda_{N-1} < \ldots < \lambda_2 < \lambda_1$*;*

(ii) each map $\omega \mapsto E_i(\omega), i \geq 1$*, is* $\mathcal{F}$*-measurable into the Grassmannian of* M_2 *(Mañé [ ], Thieullen [ ]);*

(iii) $E_{i+1}(\omega) \subset E_i(\omega) \subset \ldots \subset E_2(\omega) \subset E_1(\omega) = M_2, i \geq 1, \omega \in \Omega^*$*;*

(iv) for each $i \geq 1$*, codim* $E_i(\omega)$ *is fixed independently of* $\omega \in \Omega^*$*;*

(v) for each $\omega \in \Omega^*$ *and* $(v,\eta) \in E_i(\omega)\backslash E_{i+1}(\omega)$,

$$\lim_{t\to\infty} \frac{1}{t} \log\|X(t,\omega,(v,\eta))\|_{M_2} = \lambda_i \; ; \tag{11}$$

(vi) $\lambda_1 = \lim_{t\to\infty} \frac{1}{t} \log\|X(t,\omega,\cdot)\|_{L(M_2)}$ *for all* $\omega \in \Omega^*$ *(Top exponent);* (12)

(v) $X(t,\omega,\cdot)(E_i(\omega)) \subseteq E_i(\theta(t.\omega))$ *for all* $\omega \in \Omega^*, t \geq 0, i \geq 1$ *(Invariance).*

For a proof of the above result see [37] §4, pp. 106–122. The argument in [37] is based on Ruelle's discrete version of Oseledec's multiplicative ergodic theorem in Hilbert space ([44], Theorem (1.1), p. 248 and Corollary (2.2), p. 253). The following strong version of Kingman's subadditive ergodic theorem is also used to construct the shift invariant set Ω^* appearing in Theorem 5 above.

Theorem 6 (Kingman's Subadditive Ergodic Theorem)

Let $f : \mathbb{R}^+ \times \Omega \to \mathbb{R} \cup \{-\infty\}$ *be a measurable process on a complete probability space* $(\Omega,\mathcal{F},P)$ *such that*

(i) $E \sup_{0\leq u\leq 1} f^+(u,\cdot) < \infty$, $E \sup_{0\leq u\leq 1} f^+(1-u,\cdot) < \infty$;

(ii) $f(t_1+t_2,\omega) \leq f(t_1,\omega) + f(t_2,\theta(t_1,\omega))$ *for all* $t_1,t_2 \geq 0$ *and* ***every*** $\omega \in \Omega$.

Then there exists a set $\hat{\hat{\Omega}} \in \mathcal{F}$ *and a measurable* $\tilde{f} : \Omega \to \mathbb{R} \cup \{-\infty\}$ *with the properties:*

(a) $P(\hat{\hat{\Omega}}) = 1$, $\theta(t,\cdot)(\hat{\hat{\Omega}}) \subseteq \hat{\hat{\Omega}}$ *for all* $t \geq 0$;

(b) $\tilde{f}(\omega) = \tilde{f}(\theta(t,\omega))$ *for all* $\omega \in \hat{\hat{\Omega}}$ *and all* $t \geq 0$;

(c) $\tilde{f}^{+} \in L^1(\Omega,\mathbb{R};P)$;

(d) $\lim_{t\to\infty} \frac{1}{t} f(t,\omega) = \tilde{f}(\omega)$ *for every* $\omega \in \hat{\hat{\Omega}}$.

If θ *is ergodic, then there exists* $f^* \in \mathbb{R} \cup \{-\infty\}$ *and* $\tilde{\tilde{\Omega}} \in \mathcal{F}$ *such that*

(a)' $P(\tilde{\tilde{\Omega}}) = 1,\ \theta(t,\cdot)(\tilde{\tilde{\Omega}}) \subseteq \tilde{\tilde{\Omega}}\ ,\ t \geq 0$;

(b)' $\tilde{f}(\omega) = f^* = \lim_{t\to\infty} \frac{1}{t} f(t,\omega)$ *for every* $\omega \in \tilde{\tilde{\Omega}}$.

A proof of Theorem 6 is given in ([37], Lemma 7, pp. 115–117).

The non–random nature of the Lyapunov exponents $\{\lambda_i\}_{i=1}^{\infty}$ of (V) is a consequence of the fact that θ is ergodic. System (V) is said to be *hyperbolic* if $\lambda_i \neq 0$ for all $i \geq 1$. When (V) is hyperbolic the flow satisfies a *stochastic saddle–point property* (or exponential dichotomy) (cf. the deterministic case with $\mathcal{E} = C([-r,0],\mathbb{R}^n)$, $g_i \equiv 0$, $i = 1,\ldots,m$, in Hale [20], Theorem 4.1, p. 181).

Theorem 7 (Mohammed [37])

Suppose the hereditary system (V) is hyperbolic. Then there exist

(a) a set $\tilde{\Omega}^* \in \mathcal{F}$ *such that* $\theta(t,\cdot)(\tilde{\Omega}^*) = \tilde{\Omega}^*$ *for all* $t \in \mathbb{R}$ *and* $P(\tilde{\Omega}^*) = 1$,

(b) a measurable splitting

$$M_2 = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega) \qquad \omega \in \tilde{\Omega}^*$$

with the following properties:

(i) $\mathcal{U}(\omega)$, $\mathcal{S}(\omega)$, $\omega \in \tilde{\Omega}^*$, *are closed linear subspaces of* M_2, dim $\mathcal{U}(\omega)$ *is finite and fixed independently of* $\omega \in \tilde{\Omega}^*$.

(ii) The maps $\omega \mapsto \mathcal{U}(\omega)$, $\omega \to \mathcal{S}(\omega)$ *are* $\mathcal{F}$*–measurable into the Grassmannian of* M_2.

(iii) For each $\omega \in \tilde{\Omega}^*$ *and* $(v,\eta) \in \mathcal{U}(\omega)$, *there exists* $\tau_1 = \tau_1(\omega,v,\eta) > 0$ *and a positive* δ_1, *independent of* (ω,v,η) *such that*

$$\|X(t,\omega,(v,\eta))\|_{M_2} \geq \|(v,\eta)\|_{M_2}\, e^{\delta_1 t}\ ,\ t \geq \tau_1.$$

(iv) For each $\omega \in \tilde{\Omega}^*$ *and* $(v,\eta) \in \mathcal{S}(\omega)$, *there exists* $\tau_2 = \tau_2(\omega,v,\eta) > 0$ *and a positive* δ_2, *independent of* (ω,v,η), *such that*

$$\|X(t,\omega,v,\eta)\|_{M_2} \leq \|(v,\eta)\|_{M_2}\, e^{-\delta_2 t}\ ,\ t \geq \tau_2.$$

(v) For each $t \geq 0$ *and* $\omega \in \tilde{\Omega}^*$,

$$X(t,\omega,\cdot)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t,\omega)) ,$$

$$X(t,\omega,\cdot)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t,\omega)).$$

In particular, the restriction $X(t,\omega,\cdot)\,|\,\mathcal{U}(\omega) : \mathcal{U}(\omega) \to \mathcal{U}(\theta(t,\omega))$ *is a linear homeomorphism onto.*

(B) Linear Equations. Semimartingale Noise.

We use the general setting and hypotheses in §2(B). The object of this section is to extend Theorems 4, 5, 7 to cover the hereditary system (VI) too:

$$\left.\begin{aligned} dx(t) &= \{\int_{[-r,0]} \nu(t)(ds)x(t+s)\}dt + dN(t)\int_{-r}^{0} K(t)(s)x(t+s)ds + L(t)x(t-) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t > 0 \\ x(0) &= v \in \mathbb{R}^n, \quad x(s) = \eta(s), \quad -r < s < 0, \quad r \geq 0 \end{aligned}\right\} \tag{VI}$$

In order to develop a multiplicative ergodic theory for (VI) we need the following set of hypotheses, which are taken from Mohammed and Scheutzow [39]:

Hypotheses (C):

(i) The processes ν, K are stationary ergodic in the sense that there is a measurable ergodic P–preserving flow $\theta : \mathbb{R} \times \Omega \to \Omega$ such that for each $t \geq 0$, $\theta(t,\cdot)$ is $(\mathcal{F}_t,\mathcal{F}_0)$–measurable and

$$\nu(t,\omega) = \nu(0,\theta(t,\omega)) , \quad t \in \mathbb{R}, \quad \omega \in \Omega , \tag{13}$$

$$K(t,\omega) = K(0,\theta(t,\omega)) , \quad t \in \mathbb{R}, \quad \omega \in \Omega. \tag{14}$$

(ii) The processes N, L, M have jointly stationary ergodic increments:

$$\left.\begin{aligned} N(t+h,\omega) - N(t,\omega) &= N(h,\theta(t,\omega)), \quad t,h \in \mathbb{R} , \omega \in \Omega \\ L(t+h,\omega) - L(t,\omega) &= L(h,\theta(t,\omega)), \quad t,h \in \mathbb{R} , \omega \in \Omega \\ M(t+h,\omega) - M(t,\omega) &= M(h,\theta(t,\omega)), \quad t,h \in \mathbb{R} , \omega \in \Omega \end{aligned}\right\}. \tag{15}$$

Semimartingales satisfying Hypothesis (C)(ii) were studied by J. de Sam Lazaro and P.A. Meyer [12], Çinlar, Jacod, Protter and Sharpe [8], Protter [41]. It follows from Hypothesis (C)(ii) that N and L have jointly stationary increments. Conversely, if N and L have jointly stationary increments, one can arrange for (C)(ii) to hold on a suitable probability path space. (See Protter [41], Theorem (2.2), de Sam Lazaro and Meyer [12], Mohammed and Scheutzow [39].)

In view of Theorem 3 we know that equation (VI) is regular w.r.t. M_2 with a measurable flow $X : \mathbb{R}^+ \Omega \times M_2 \to M_2$. It will turn out that this flow satisfies Theorems 4, 5 and 7. This is achieved via a construction in [39] based on the following consequence of Hypothesis (C)(ii):

Theorem 8 (Mohammed & Scheutzow [39]).

Suppose M *satisfies Hypothesis (C)(ii). Then there is an* $(\mathcal{F}_t)_{t \geq 0}$*-adapted version* $\varphi : \mathbb{R}^+ \times \Omega \to \mathbb{R}^{n \times n}$ *of the solution to the matrix equation*

$$\left.\begin{aligned} d\varphi(t) &= dM(t)\varphi(t) \qquad t > 0 \\ \varphi(0) &= I \quad \in \mathbb{R}^{n\times n} \end{aligned}\right\} \tag{XII}$$

and a set $\Omega_1 \in \mathcal{F}$ *such that*

(i) $P(\Omega_1) = 1$ *;*

(ii) $\theta(t,\cdot)(\Omega_1) \subseteq \Omega_1$ *for all* $t \geq 0$ *;*

(iii) $\varphi(t_1+t_2,\omega) = \varphi(t_2,\theta(t_1,\omega))\, \varphi(t_1,\omega)$ *for all* $t_1,t_2 \in \mathbb{R}^+$ *and every* $\omega \in \Omega_1$ *;*

(iv) $\varphi(\cdot,\omega)$ *is continuous for every* $\omega \in \Omega_1$.

A proof of Theorem 8 is given in [39]. The proof is based on a double approximation argument whereby (XII) is replaced by the families of s.d.e.'s

$$\left.\begin{aligned} d\varphi_m^k(t) &= \dot{M}^k(t)\ f_m(\varphi_m^k(t))dt - \tfrac{1}{2}\, d\langle M\rangle(t) f_m(\varphi_m^k(t)), \qquad t \in \mathbb{R}^+ \\ \varphi_m^k(0) &= I \ \in \mathbb{R}^{n\times n}\ , \end{aligned}\right\} \quad (XII)_m^k$$

$$\left.\begin{aligned} d\varphi_m(t) &= o dM(t) f_m(\varphi_m(t)) - \tfrac{1}{2}\, d\langle M\rangle(t) f_m(\varphi_m(t)), \qquad t \in \mathbb{R}^+ \\ \varphi_m(0) &= I \ \in \mathbb{R}^{n\times n}\ , \end{aligned}\right\} \cdot \quad (XII)_m$$

where the $f_m : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ are C^∞–bounded approximations of the identity map $id_{\mathbb{R}^{n\times n}} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ such that $f_m(A) = A$ whenever $\|A\| \le m$; the M^k are smooth $(\mathscr{F}_t)_{t\ge 0}$–adapted mollifiers of M given by

$$M^k(t) := k\int_{t-\frac{1}{k}}^{t} M(u)du, \qquad t \ge 0,$$

and $\circ dM(t)$ denotes Stratonovich differential.

Using results of Mackevičius [27] on S^p–stability of s.d.e.'s, it is shown in ([39], Theorem (3.1)) that the solutions $\varphi^k : \mathbb{R}^+ \times \Omega \to \mathbb{R}^{n\times n}$ of

$$\left.\begin{aligned} d\varphi^k(t) &= \dot{M}^k(t)\varphi^k(t)dt - \tfrac{1}{2}\, d\langle M\rangle(t)\varphi^k(t), \quad t \in \mathbb{R}^+ \\ \varphi^k(0) &= I \in \mathbb{R}^{n\times n} \end{aligned}\right\} \qquad (XII)^k$$

have a subsequence $\{\varphi^{k'}\}_{k'=1}^{\infty}$ which converges a.s. uniformly on compacta to φ. The multiplicative cocycle property for (φ,θ) follows immediately from the corresponding one for $(\varphi^{k'},\theta)$, $k' \ge 1$.

Under Hypotheses (C) one gets the cocycle property (9) for (X,θ), i.e. Theorem 4 holds true for the linear hereditary system (VI), (Mohammed & Scheutzow [39], Theorem (4.2)(vii)). A key point in proving this fact is to observe that the linear system (VI) is equivalent to the following random family of hereditary linear integral equations

$$x(t) = \varphi(t)[v - \int_0^t Z(u)\{K(u)(0)x(u) - K(u)(-r)x(u-r)$$

$$+ \int_{u-r}^{u} \frac{\partial}{\partial u}(K(u)(s-u))x(s)ds\}du]$$

$$+ \varphi(t)Z(t)\int_{-r}^{0} K(t)(s)x(t+s)\,ds + \int_0^t \varphi(t-u,\theta(u,\cdot))\int_{[-r,0]} \nu(u)(ds)x(u+s)du$$

$$+ \int_0^t \varphi(t-u,\ \theta(u,\cdot))dV(u)x(u-) \qquad \text{(XIII)}$$

$$-\int_0^t \varphi(t-u,\theta(u,\cdot))d[M,N](u)\int_{-r}^{0} K(u)(s)x(u+s)ds\ , \qquad t \geq 0,$$

$$x(t) = \eta(t) \quad \text{a.e. } t \in [-r,0)\ ,$$

where Z is a suitably chosen version of $\int_0^t \varphi^{-1}(u)\,dN(u)$ (Theorem (3.2) in [39]) and [M,N] denotes a version of the $\mathbb{R}^{n\times n}$–valued mutual variation of M and N, viz.

$$[M,N]_{ij} := \sum_{m=1}^{n} [M_{im},N_{mj}]\ ,\ M = (M_{ij})_{i,j=1}^{n}\ ,\ N = (N_{ij})_{i,j=1}^{n}\ .$$

(See Lemma (3.1) in [39]). Observe that the above integral equation *has no stochastic integrals.* This fact contributes to the regularity of the hereditary equation (VI). The cocycle property (9) now follows from the uniqueness of the solution to (XIII) (see proof of Theorem (4.2) in [39]).

The existence of a discrete non–random Lyapunov spectrum $\{\lambda_i\}_{i=1}^{\infty}$ for the hereditary equation (VI) (cf. Theorem 5) is proved via Ruelle–Oseledec multiplicative ergodic theorem which requires the following integrability property

$$E \sup_{0\leq t_1,t_2\leq r} \log^+ \|X(t_1,\theta(t_2,\cdot),\cdot)\|_{L(M_2)} < \infty\ . \qquad (16)$$

In [39] the above integrability property is established under the following set of hypotheses on ν, K, N, L:

Hypotheses (I):

(i) The random variables

$$\sup_{-r\leq s\leq 2r}\left|\frac{d\bar{\bar{\nu}}(\cdot)(s)}{ds}\right|^3, \quad \sup_{\substack{0\leq t\leq 2r\\ -r\leq s\leq 0}}\|K(t,\cdot)(s)\|^4,$$

$$\sup_{\substack{0\leq t\leq 2r\\ -r\leq s\leq 0}}\|\frac{\partial}{\partial t}K(t,\cdot)(s)\|^4, \quad \sup_{\substack{0\leq t\leq 2r\\ -r\leq s\leq 0}}\|\frac{\partial}{\partial s}K(t,\cdot)(s)\|^4, \quad \{|V|(2r,\cdot)\}^4,$$

$$\{\langle M_{ij}\rangle(2r,.)\}^4, \quad 1\leq i,j\leq n,$$

are all integrable. Here $|V|$ is the total variation of V w.r.t. the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^{n\times n}$.

(ii) Let N be of the form $N = N^0 + V^0$ where the local $(\mathcal{F}_t)_{t\geq 0}$–martingale $N^0 = (N^0_{ij})^n_{i,j=1}$ and the bounded variation process $V^0 = (V^0_{ij})^n_{i,j=1}$ are such that the random variables

$$\{[N^0_{ij}](2r,\cdot)\}^4, \quad \{[V^0_{ij}](2r,\cdot)\}^4 = (\sum_{0\leq s\leq 2r}|\Delta V^0_{ij}(s)|^2)^4,$$

$$\{|V^0_{ij}|(2r,\cdot)\}^8, \quad i,j = 1,2,\ldots,n$$

are integrable. Note that $\Delta V^0_{ij}(s)$ is the jump of V^0_{ij} at s and $|V^0_{ij}|(2r,\cdot)$ is the total variation of V^0_{ij} over $[0,2r]$.

(iii) There is a *non–random* time $t_0 > 0$ such that

$$|\langle M_{ij},M_{kl}\rangle|(t_0,\cdot) \in L^\infty(\Omega,\mathbb{R}), \quad i,j,k,l = 1,2,\ldots,n.$$

The integrability property (16) is a consequence of

$$E \log^{+} \sup_{\substack{0 \le t_1, t_2 \le r \\ \|(v,\eta)\| \le 1}} |x(t_1, \theta(t_2,\cdot), (v,\eta))| < \infty. \tag{17}$$

The proof of the latter property involves a lengthy argument based on establishing the existence of suitable higher order moments for the coefficients on the right hand side of the random integral equation (XIII). (See Lemmas (5.1), (5.2), (5.3), (5.4), (5.5), (5.6) in [39]).

Since θ is ergodic, the multiplicative ergodic theorem (Ruelle [44]) now gives a fixed discrete set of Lyapunov exponents for the linear system (VI). In fact we have

Theorem 9 (Mohammed and Scheutzow [39])

Under Hypotheses (C) & (I), the statements of Theorems 5 and 7 hold true for the linear hereditary system (VI).

Note that the Lyapunov spectrum of (VI) does not change if one uses the state space $\mathscr{E} = D([-r,0],\mathbb{R}^n)$ with the supremum norm $\|\cdot\|_\infty$ and drops the hypothesis of the L^2–continuity of $t \mapsto \frac{d\bar{\bar{\nu}}(t,\omega)}{ds} | [-r,0]$ $(\omega \in \Omega)$ referred to in §2(B). (See the remark following Theorem (5.3) in [39]).

(C) Affine Systems. Hyperbolicity and Stationary Solutions.

Here we consider the affine hereditary system (VII) under the setting and hypotheses of §2(C):

$$\left.\begin{aligned} dx(t) &= \{\int_{[-r,0]} \mu(ds)x(t+s)\}dt + dQ(t), \quad t > 0 \\ x(s) &= \eta(s) \qquad\qquad -r \le s \le 0. \end{aligned}\right\} \tag{VII}$$

In order to study the Lyapunov spectrum of the affine system (VII) we recall the following classical results of J.K. Hale for the homogeneous ($Q \equiv 0$) deterministic system (VIII):

$$\left.\begin{aligned} dy(t) &= \int_{[-r,0]} \mu(ds)\, y(t+s)dt\,, \quad t > 0\,, \\ y_0 &= \eta \in D([-r,0],\mathbb{R}^n). \end{aligned}\right\} \tag{VIII}$$

Recall that $\tilde{T}(t) : D([-r,0], \mathbb{R}^n) \to D([-r,0], \mathbb{R}^n)$, $t \geq 0$, is the strongly continuous semi–group given by the trajectories of (VIII). Consider the complexification

$$\tilde{T}_{\mathbb{C}}(t)(\eta) := \tilde{T}(t)(\mathrm{Re}\ \eta) + i\ \tilde{T}(t)(\mathrm{Im}\ \eta)\ \ ,\ \ \eta \in D([-r,0], \mathbb{C}^n)\ ,$$

of $\tilde{T}(t)$ and its restriction $T_{\mathbb{C}}(t)$ to the space of continuous maps $C([-r,0],\mathbb{C}^n)$. Note that $T_{\mathbb{C}}(t)$ is simply the complexification of $T(t) := \tilde{T}(t) | C([-r,0],\mathbb{R}^n)$. Denote by A the infinitesimal generator of the strongly continuous semigroup $(T_{\mathbb{C}}(t))_{t\geq 0}$. Then the spectrum, $\sigma(A)$, of A is discrete and consists entirely of eigenvalues with real parts bounded above. (Hale [20], pp. 168–170). Indeed $\sigma(A)$ coincides with the complex roots λ of the characteristic equation

$$\det[\lambda I - \int_{[-r,0]} e^{\lambda s}\mu(ds)] = 0 \tag{18}$$

(Hale [20], pp. 168–170). It follows from the above equation that $\lambda \in \sigma(A)$ iff $\bar{\lambda} \in \sigma(A)$; and for every $\beta \in \mathbb{R}$ the sum of the generalized eigenspaces corresponding to all λ's such that $\mathrm{Re}\ \lambda \geq \beta$ is finite–dimensional ([20], p. 168). If $\mathbb{R}^{n^*}$ stands for the space of all n–row vectors, we shall let A^* be the formal adjoint of A in $C^* := C([0,r],\mathbb{R}^{n^*})$ with respect to the continuous bilinear form $(\cdot,\cdot)$: $C^* \times D \to \mathbb{R}$,

$$(\psi,\varphi) := \psi(0)\varphi(0) + \int_{[-r,0]} \int_s^0 \psi(\xi - s)\mu(ds)\varphi(\xi)d\xi\ , \tag{19}$$

$\psi \in C^*$, $\varphi \in D$. Then $\sigma(A^*) = \sigma(A)$ ([20], p. 169). For a given finite set Λ of conjugate pairs of eigenvalues of A, denote by E_Λ (E^*_Λ) the sum of the corresponding real generalized eigenspaces of A $(A^*$, resp.) corresponding to the eigenvalues in Λ ([38]). Then $E_\Lambda(E^*_\Lambda)$ is a finite–dimensional real subspace of $C(C^*$, resp.) Pick bases $\{\varphi_i\}_{i=1}^d$, $\{\psi_i\}_{i=1}^d$ of E_Λ, E^*_Λ such that $(\psi_i,\varphi_j) = \delta_{ij}$, $1 \leq i,j \leq d = \dim E_\Lambda = \dim E^*_\Lambda$. Let $B = (B_{ij})_{i,j=1}^d$ be the $d\times d$ matrix representation of $A|E_\Lambda$ with respect to $\{\varphi_i\}_{i=1}^d$. The space D admits a $(\tilde{T}(t))_{t\geq 0}$–invariant topological splitting

$$D = E_\Lambda \oplus E'_\Lambda$$

where $E'_\Lambda := \{\varphi : \varphi \in D, (\psi,\varphi) = 0 \text{ for all } \psi \in E^*_\Lambda\}$ ([38]). If $\eta \in D$, we let η^{E_Λ}, $\eta^{E'_\Lambda}$ denote its projections on E_Λ, E'_Λ, respectively. Applying these projections to both sides of (X) and writing $E = E_\Lambda$, $E' = E'_\Lambda$ we get

$$\left.\begin{aligned} x_t^E &= \tilde{T}(t)(\eta^E) + Q_t^E + \int_0^t \dot{F}(t-s+\cdot)^E Q(s)ds \\ x_t^{E'} &= \tilde{T}(t)(\eta^{E'}) + Q_t^{E'} + \int_0^t \dot{F}(t-s+\cdot)^{E'} Q(s)ds , \end{aligned}\right\} \qquad \text{(XIV)}$$

$t \geq 0$, $\eta \in D$ ([38], Theorem 3).

Define the d–dimensional stochastic process

$$y(t) = (\Psi,x_t) := \begin{bmatrix} (\psi_1,x_t) \\ (\psi_2,x_t) \\ \vdots \\ (\psi_d,x_t) \end{bmatrix} \in \mathbb{R}^d.$$

with $\Psi := \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_d \end{bmatrix}$.

Then one gets

$$x_t^E = \sum_{j=1}^{d} \varphi_j Y_j(t) =: \Phi Y(t) \quad , \ \Phi := (\varphi_1,\dots,\varphi_d) \tag{20}$$

$$dY(t) = BY(t)dt + \Psi(0)\, dQ(t)\ , \quad t > 0 \tag{XV}$$

$$Y(t) = e^{tB}(\Psi,\eta) + \int_0^t Be^{(t-s)B}\, \Psi(0)Q(s)ds + \Psi(0)Q(t) \tag{21}$$

(Mohammed and Scheutzow [38], Theorem 6).

By extending the estimate on the complementary subspace in ([20], Theorem 4.1, p. 181) to cover all cadlag initial paths (Mohammed and Scheutzow [38], Theorem 4), one gets:

Theorem 10

For each $\eta \in D$, $\lambda(\eta) := \lim_{t\to\infty} \frac{1}{t} \log \|\tilde{T}(t)\eta\|_\infty$ *exists and the set of Lyapunov exponents* $\{\lambda(\eta) : \eta \in D\}$ *of the homogeneous system (VIII) coincides with the set* $\{\text{Re}\,\lambda : \lambda \in \sigma(A)\}$ *together with possibly* $-\infty$.

We now give an Oseledec theorem which characterizes the a.s. Lyapunov exponents of the affine hereditary system (VII). The proof of the following theorem may be found in (Mohammed and Scheutzow [38] §4, Scheutzow [46], C13, Theorem 1, p. 160–161).

Theorem 11

Let $\beta_1 > \beta_2 > \beta_3 > \dots$ *be an ordering of the real parts of all eigenvalues in* $\sigma(A)$. *Fix* $m \geq 1$ *and let* $E = E_\Lambda$ *where* $\Lambda = \{\beta_i\}_{i=1}^m$. *Define* Φ, B, Ψ, E' *as before. Let* $\beta < \beta_m$ *and assume that* $|Q(t)| = o(e^{(\beta+\varepsilon)t})$ *for all* $\varepsilon > 0$ *as* $t \to \infty$ *a.s. Let* Y^* *stand for the* d*-dimensional process* ($d = \dim E$)

$$Y^*(t) = -\int_t^\infty Be^{(t-s)B}\, \Psi(0)Q(s)ds + \Psi(0)Q(t). \tag{22}$$

For each $1 \leq j \leq m$ *suppose* E_j *is the sum of generalized subspaces corresponding to the eigenvalues with real parts* $\{\beta_i\}_{i=1}^j$. *Assume that* E_j' *is the complementary subspace to* E_j *for* $1 \leq j \leq m$. *Take* $E_0 = \{0\}$. *Then, for a.a.*

$\omega \in \Omega$, one has

$$\lim_{t\to\infty} \frac{1}{t} \log \|x_t(\omega)\|_\infty = \beta_j \quad \textit{if} \quad x_0(\omega) \in \Phi Y^*(0,\omega)+E'_{j-1}\backslash E'_j, \qquad 1 \le j \le m, \tag{23}$$

and

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log \|x_t(\omega)\|_\infty \le \beta \quad \textit{if} \quad x_0(\omega) \in \Phi Y^*(0,\omega)+E'_m. \tag{24}$$

The key to the proof of the above theorem is to identify the Lyapunov exponents of the projection $\{x_t^E : t \ge 0\}$ with those of $\{Y(t) : t \ge 0\}$ and then observe that

$$Y(t) = e^{tB}(Y(0) - Y^*(0)) + Y^*(t)\,, \qquad t \ge 0.$$

Sufficient conditions for equality in (24) are given in the following theorem. Note here that one does not require $Q(t)$ to be zero for $t \le 0$.

Theorem 12 (Mohammed and Scheutzow [38])

Assume all the conditions and notations of Theorem 11. Suppose also that $|Q(t)| = o(e^{-(\beta+\varepsilon)|t|})$ for some $\varepsilon > 0$ as $t \to -\infty$ a.s. Let E^β be the sum of generalized eigenspaces of A corresponding to all eigenvalues with real parts greater than or equal to β. Define the process $Z^*(t) \in (E^\beta)'$ by

$$Z^*(t) = \int_{-\infty}^{t} \dot{F}(t-s+\cdot)^{(E^\beta)'} Q(s)ds + Q_t^{(E^\beta)'}\,, \quad t \ge 0. \tag{25}$$

Let $\omega \in \Omega$ be such that $\overline{\lim_{t\to\infty}} \frac{1}{t} \log \inf_{v\in E^\beta} \|Z^*(t,\omega,v)\| \ge \beta$ and suppose that $x_0(\omega) \in \Phi\, Y^*(0,\omega) + E'_m$. Then

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log \|x_t(\omega)\|_\infty = \beta.$$

Remarks

(i) Under the conditions of Theorem 12 the statements (23) and (24) can be modified by replacing $\Phi Y^*(0,\omega)$ with $\eta(\omega) := Z^*(0,\omega) + \Phi Y^*(0,\omega)$, which is independent of the choice of m.

(ii) Y^* and Z^* may also be represented as

$$Y^*(t) = -\int_t^{\infty} Be^{(t-s)B}\,\Psi(0)(Q(s)-Q(t))ds,\ \ t \geq 0\,, \tag{26}$$

$$Z^*(t) = \int_{-\infty}^{t} \tilde{T}(t-s)\Delta^{(E^{\beta})'}\,dQ(s). \tag{27}$$

When Q has stationary increments, these representations imply that Y^* and Z^* are stationary processes. In fact Y^* is the only stationary solution of the s.o.d.e. (XV) (Mohammed and Scheutzow [38]).

We now consider the *hyperbolic case* when $\mathrm{Re}\,\lambda \neq 0$ for all $\lambda \in \sigma(A)$. In this case, the following result (Mohammed and Scheutzow [38], Theorem 20) establishes the existence of a unique stationary solution for the affine hereditary system (VII).

Theorem 13

Suppose that Q is cadlag and has stationary increments. Assume that the characteristic equation

$$\det(\lambda I - \int_{[-r,0]} e^{\lambda s}\,\mu(ds)) = 0 \tag{18}$$

has no roots on the imaginary axis; i.e. the homogeneous equation (VIII) has no zero Lyapunov exponents. Suppose also that

$$\overline{\lim_{t\to\pm\infty}}\ \frac{1}{|t|}\log|Q(t)| < |\mathrm{Re}\,\lambda| \qquad a.s.$$

for all characteristic roots λ *of (18). Then there is a unique D–valued random variable* η *such that the trajectory* $\{x_t^{(\eta)} : t \geq 0\}$ *of (VII) is a D–valued stationary process. The random variable* η *is measurable with respect to the* σ*–algebra generated by* $\{Q(t) : t \in \mathbb{R}\}$.

If E is the sum of all generalized eigenspaces of A corresponding to all $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda > 0$, then in Theorem 13, the projection $\eta^E(\eta^{E'})$ is measurable with respect to the σ–algebra generated by $Q(t), t \geq 0$ ($Q(t)$, $t \leq 0$, respectively). (See Mohammed and Scheutzow [38], Theorem 20). Furthermore if Q has *independent* increments (e.g. Q is Brownian motion or a Poisson process), then the projections $x_t^{(\eta)^E}, x_t^{(\eta)^{E'}}$, $t \geq 0$, are stationary and independent processes.

We conclude this section by discussing *p–th moment Lyapunov* exponents

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log E \, \|x_t^{(\eta)}\|_\infty^p \quad , \quad p \geq 1 , \tag{28}$$

of (VII). The following result is proved in ([38], Remark (iii) following Theorem 21) by looking at the moment exponents of the projections $x_t^{(\eta)^E}, x_t^{(\eta)^{E'}}$, where E is the sum of the generalized eigenspaces corresponding to all eigenvalues $\lambda \in \sigma(A)$ with the largest real part β_1.

Theorem 14

Let β_1 *be the top a.s. Lyapunov exponent of (VII) and fix* $p \geq 1$. *Assume that* $Q(t) \in L^p(\Omega, \mathbb{R}^n)$ *for all* $t \geq 0$, $|Q(t)| = o(e^{(\beta_1-\epsilon)t})$ *a.s. as* $t \to \infty$ *for some* $\epsilon > 0$ *and* $\overline{\lim_{t\to\infty}} \frac{1}{t} \log E|Q(t)|^p < p\beta_1$. *If* $Y^*(0)$ *is not a.s. constant, then*

$$\lim_{t\to\infty} \frac{1}{t} \log E \, \|x_t^{(\eta)}\|_\infty^p = \lim_{t\to\infty} \frac{1}{t} \log E \, \|x_t^{(\eta)} - E\, x_t^{(\eta)}\|_\infty^p$$

$$= p\beta_1 ,$$

for all $\eta \in D$.

Under the mild non–degeneracy condition that $Y^*(0)$ is not a.s. constant, the above theorem asserts the existence of *only* one p–th moment exponent which is independent of all *random* (possibly *anticipating*) initial conditions in D. This result is in agreement with the affine linear finite–dimensional non–delay case (r = 0) (Arnold, Oeljeklaus and Pardoux [3], Baxendale [5], Arnold, Kliemann and Oeljeklaus [2]).

Note also the following interesting fact in connection with Theorem 14. The affine hereditary system (VII) may be viewed as a *finite–dimensional* stochastic perturbation of the *infinitely degenerate* deterministic homogeneous system (VIII) with *countably many* Lyapunov exponents. However, these finite–dimensional perturbations provide noise that is generically rich enough to account for a *single* moment Lyapunov exponent in the affine system (VII).

Remark:

More work needs to be done in order to characterize p–th moment exponents for general linear hereditary systems (I) with $Q \equiv 0$. In the white noise case with an asymptotically stable linear drift and a small diffusion, estimates on the mean square moment exponent

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log E\|x_t\|_\infty^2$$

may be found in (Mohammed, Scheutzow and Weizsäcker [40], Mohammed [34], [33], Theorems (4.2) & (4.3), pp. 208–222). Similar estimates in a rather special case with a small discrete delay appear in (Mao [29]).

§4. Examples. Upper Bounds on the Top Exponent.

The examples in this section are all one–dimensional and linear. Regularity of the equations is established and estimates on the top a.s. Lyapunov exponent λ_1 are given. Details of the computations are incorporated in ongoing joint work of the author with M. Scheutzow and will appear elsewhere.

Example 1: (A Linear Delay Equation with Poisson Noise)

Consider the one–dimensional linear delay equation

$$\left.\begin{aligned} dx(t) &= x((t-1)-)\,dN(t) \quad , \quad t > 0 \\ x_0 &\in D = D([-r,0],\mathbb{R}) \end{aligned}\right\} \qquad \text{(XVI)}$$

The process $N(t) \in \mathbb{R}$ is a Poisson process with i.i.d. inter–arrival times $\{T_i\}_{i=1}^{\infty}$ which are exponentially distributed with the same parameter μ. The jumps $\{Y_i\}_{i=1}^{\infty}$ of N are i.i.d. and independent of all the T_i's. Writing

$$j(t) := \sup\{j \geq 0 : \sum_{i=1}^{j} T_i \leq t\},$$

and

$$N(t) = \sum_{i=1}^{j(t)} Y_i ,$$

it is easy to see that (XVI) can be solved a.s. in steps giving

$$x^{(\eta)}(t) = \eta(0) + \sum_{i=1}^{j(t)} Y_i\, x((\sum_{j=1}^{i} T_j - 1)-) \quad \text{a.s.} \tag{29}$$

Observe that $\{x_t : t \geq 0\}$ is a Markov process in the state space D (with the supremum norm $\|\cdot\|_{\infty}$). Furthermore the above relation implies that (XVI) is regular in D; i.e. it admits a measurable flow $X : \mathbb{R}^+ \times \Omega \times D \to D$ with $X(t,\omega,\cdot)$ continuous linear for all $t \geq 0$ and a.a. $\omega \in \Omega$ (cf. the singular equation (II) in §2(A)).

The a.s. Lyapunov spectrum of (XVI) may be characterized directly (without appealing to the Oseledec theorem) by interpolating between the sequence of random times:

$$\tau_0(\omega) := 0 \;,$$

$$\tau_1(\omega) := \inf\{n \geq 1 : \sum_{j=1}^{k} T_j \notin [n-1,n] \;\text{ for all }\; k \geq 1\}$$

$$\tau_{i+1}(\omega) := \inf\{n > \tau_i(\omega) : \sum_{j=1}^{k} T_j \notin [n-1,n] \;\text{ for all }\; k \geq 1\}, \quad i \geq 1.$$

(For details see Scheutzow [46], pp. 162–166).

Theorem 15 (Scheutzow [46])

Let $\xi \in D$ *stand for the constant path* $\xi(s) = 1$ *for all* $s \in [-1,0]$. *Suppose* $E \log \|X(\tau_1(\cdot),\cdot,\xi)\| \leq \infty$ *(possibly* $= -\infty$*). Then the a.s. Lyapunov spectrum*

$$\lambda(\eta) := \lim_{t \to \infty} \frac{1}{t} \log \|X(t,\omega,\eta)\|_\infty \quad , \qquad \eta \in D \;, \quad \omega \in \Omega$$

of (XVI) is $\{-\infty, \lambda_1\}$ *where*

$$\lambda_1 = \frac{1}{E\tau_1} \cdot E \log \|X(\tau_1(\cdot),\cdot,\xi)\|_\infty .$$

In fact,

$$\lim_{t \to \infty} \frac{1}{t} \log \|X(t,\omega,\eta)\|_\infty = \begin{cases} \lambda_1 & \eta \notin \operatorname{Ker} X(\tau_1(\omega),\omega,\cdot) \\ -\infty & \eta \in \operatorname{Ker} X(\tau_1(\omega),\omega,\cdot) \end{cases} .$$

If N *has* J *jumps in* $[0,\tau_1]$ *and* $P(\alpha \leq |Y_1 + 1| \leq \beta) = 1$ *for some positive* α,β, *then*

$$\frac{EJ \;\; \log\alpha}{E\tau_1} \leq \lambda_1 \leq \frac{EJ \;\; \log\beta}{E\tau_1}$$

If $P(Y_1 = -1) > 0$, *then* $\lambda_1 = -\infty$.

The computations underlying the proof of the above theorem also work for the one–dimensional hereditary equation

$$dx(t) = \{\int_{[-r,0]} x(t+s)\, d\mu(s)\} dN(t) \qquad \text{(XVII)}$$

where N is as before and μ is a deterministic finite signed measure on $[-r,0]$ with support bounded away from zero (Scheutzow [46], pp. 166–167).

Example 2:

The one–dimensional hereditary equation

$$dx(t) = \{\nu x(t) + \mu x(t-r)\}dt + \{\int_{-r}^{0} x(t+s)\sigma(s)ds\}dW(t), \quad t > 0 \qquad \text{(XVIII)}$$

with real constants ν, μ is a special case of (VI) in §2(B). If $\sigma : [-r,0] \to \mathbb{R}$ is a C^1 deterministic function, then it follows from Theorem 3 (§2(B)) that (XVIII) is regular w.r.t. M_2. Observe that the process $\int_{-r}^{0} x(t+s)\sigma(s)ds$ has C^1 paths in t and so the stochastic differential dW w.r.t. the one–dimensional Brownian motion W in (XVIII) may be interpreted in the Itô or Stratonovich sense *without changing the solution* x. Taking (Stratonovich) differentials of the process $\log \rho(t)$,

$$\rho(t)^2 := x(t)^2 + \int_{t-r}^{t} x(u)^2 du\,, \quad t > 0\,, \qquad (30)$$

and analyzing the resulting expression one gets the following theorem:

Theorem 16 (Mohammed)

In (XVIII) let δ_0 be the unique solution of the equation

$$2(\nu + \delta) + \mu^2 e^{2\delta r} + 1 = 0. \qquad (31)$$

If λ_1 is the top a.s. Lyapunov exponent of (XVIII) (as given by Theorems 9 &

5(vi)), then $\lambda_1 \leq -\delta_0$.

Details of the proof of the above theorem will appear elsewhere.

Example 3:

Let M be a one–dimensional, sample–continuous square integrable martingale with stationary ergodic increments. From the ergodic theorem we have the fixed (non–random) a.s. limit

$$\beta := \lim_{t\to\infty} \frac{\langle M\rangle(t)}{t}.$$

E.g. if M is standard Brownian motion, then $\beta = 1$. Consider the one–dimensional hereditary equation

$$dx(t) = \{\nu x(t) + \mu x(t-r)\}dt + x(t)\, dM(t), \quad t > 0. \qquad \text{(XIX)}$$

This equation satisfies Hypotheses (C), (I) in §3(B). So (XIX) is regular w.r.t. M_2 (Theorem 3). Furthermore an analysis of the process in (30) gives the following estimate for λ_1:

Theorem 17 (Mohammed)

In (XIX) define δ_0 *as in Theorem 16. Then the top a.s. Lyapunov exponent* λ_1 *of (XIX) satisfies*

$$\lambda_1 \leq -\delta_0 + \frac{\beta}{16}.$$

The estimate for λ_1 in the above theorem is clearly not sharp even when $M = W$, one–dimensional standard Brownian motion (cf. the non–delay case $\mu = 0$).

In the special case $M = \sigma W$ for a fixed real σ, the above bound may be sharpened to

$$\lambda_1 \langle \inf\{\kappa(\alpha,\delta) : \ \alpha \rangle 0, \ \ \delta \in \mathbb{R}\}$$

where

$$\kappa(\alpha,\delta) := -\delta + \frac{1}{16\sigma^2}(\mu^2 e^{2\delta r}\alpha + \frac{1}{\alpha} + 2\nu + 2\delta + \sigma^2)^2.$$

The proof of this fact was the result of joint discussion involving S.T. Ariaratnam, L. Arnold, P. Baxendale, H. Crauel, W. Kliemann, N. Sri Namachchivaya, M. Pinsky and V. Wihstutz. Observe that the above estimate agrees with $\lambda_1 = \nu - \frac{1}{2}\sigma^2$ in the non–delay case $\mu = 0$.

It is not clear under what conditions on the parameters ν, μ, β the hereditary equation (XIX) becomes hyperbolic.

Acknowledgement

The author is very grateful to Victor Mizel for his helpful comments on a preliminary version of the mansucript.

REFERENCES

[1] Arnold, L., *Stochastic Differential Equations: Theory and Applications*, John Wiley and Sons, Inc. New York (1974).

[2] Arnold, L., Kliemann, W. and Oeljeklaus, E., Lyapunov exponents of linear stochastic systems, *Lyapunov Exponents*, Springer Lecture Notes in Mathematics **1186** (1989), 85–125.

[3] Arnold, L., Oeljeklaus, E., and Pardoux, E., Almost sure and moment stability for linear Itô equations, in *Lyapunov Exponents*, Springer Lecture Notes in Mathematics **1186**, (ed. L. Arnold and V. Wihstutz) (1986), 129–159.

[4] Arnold, L. and Wihstutz, V. (eds.) *Lyapunov Exponents: Proceedings of a Workshop*, Bremen, November 1984, Springer Lecture Notes in Mathematics **1186** (1986).

[5] Baxendale, P.H., Moment stability and large deviations for linear stochastic differential equations, in: Ikeda, N. (ed.) *Proceedings of the Taniguchi Symposium on Probabilistic Methods in Mathematical Physics*, Katata and Kyoto 1985, pp. 31–54, Tokyo: Kinokuniya, 1987.

[6] Bellman, R. and Cooke, K., *Differential Difference Equations*, Academic Press (1963).

[7] Berger, M.A. and Mizel, V.J., Volterra equations with Itô integrals I and II, *J. of Integral Equations* **2** (1980), 187–245, **4** (1980), 319–337.

[8] Çinlar, E., Jacod, J., Protter, P. and Sharpe, M., Semimartingales and Markov processes, *Z. Wahrsch. Verw. Gebiete* **54**, (1980), pp. 161–219.

[9] Coleman, B. and Mizel, V.J., Norms and semigroups in the theory of fading memory, *Arch. Rat. Mech. Ana.* **2** (1966), 87–123.

[10] Coleman, B. and Mizel, V.J., On the stability of solutions of functional–differential equations, *Arch. Rat. Mech. Ana.*, Volume 30, Number 3 (1968), 173–196.

[11] Corduneanu, C. and Lakshmikantham, V., Equations with unbounded delay: A survey, *Nonlinear Analysis* **4**, no. 5 (1980), 831–877.

[12] de Sam Lazaro, J. and Meyer, P.A., Questions de théorie des flots, *Seminaire de Probab.* IX, Springer Lecture Notes in Mathematics **465** (1975), pp. 1–96.

[13] Delfour, M.C. and Mitter, S.K., Hereditary differential systems with constant delays I. General case., *J. Differential Eqns.* **12** (1972), 213–255. II. A class of affine systems and the adjoint problem, *J. Differential Eqns.* **18** (1975), 18–28.

[14] Dellacherie, C. and Meyer, P.A., *Probabilités et Potential* 1: Chapters 1–4, 2: Chapters 5–8, 2e'me ed. Hermann, Paris (1980).

[15] Doleans–Dade, C., On the existence and unicity of solutions of stochastic integral equations, *Z. Wahrsch. Verw. Gebiete* **36** (1976), 93–101.

[16] Doss, H., Liens entre équations différentielles stochastiques et ordinairés, *Ann. Inst. Henri Poincaré*, Vol. XIII, no°2 (1977), 99–125.

[17] El'sgol'tz, L.E., *Introduction to the Theory of Differential Equations with Deviating Arguments*, Holden–Day, Inc. (1966).

[18] Flandoli, F. and Schaumlöffel, K.–U., Stochastic parabolic equations in bounded domains: Random evolution operator and Lyapunov exponents, *Stochastics and Stochastic Reports*, Vol. 29, Number 4 (1990), 461–485.

[19] Hale, J.K., Sufficient conditions for stability and instability of autonomous functional differential equations, *J. Differential Equations* **1** (1965), 452–482.

[20] Hale, J.K., *Theory of Functional Differential Equations*, Springer–Verlag, New York–Heidelberg–Berlin (1977).

[21] Itô, K. and Nisio, M., On stationary solutions of a stochastic differential equation, *J. Math. Kyoto University*, **4–1** (1964), 1–75.

[22] Jacod, J., Equations differentielles stochastiques linearies: La methode de variation des constantes, *Seminaire de Probabilités* XVI, Lecture Notes in Mathematics **920**, Springer–Verlag (1982), 442–446.

[23] Kolmanovskii, V.B. and Nosov, V.R., *Stability of Functional Differential Equations* (Translated from Russian), Academic Press, London (1986).

[24] Krasovskii, N., *Stability of Motion*, Moscow (1959), translated by J.L. Brenner, Stanford University Press (1963).

[25] Kushner, H., On the stability of processes defined by stochastic differential–difference equations, *J. Differential Equations* **4** (1968), 424–443.

[26] Leandre, R., Flot d'une equation differentielle stochastique avec semi–martingale directrice discontinue, *Seminaire de Probabilités* XIX, Lecture Notes in Mathematics **1123,** Springer–Verlag (1984), 271–275.

[27] Mackevičius, V., S^p–stability of solutions of symmetric stochastic differential equations, *Lietuvos Matematikos Rinkinys*, T. 25, No. 4, (1985), 72–84 (in Russian); English translation: *Lithuanian Math. J.*, (1989), 343–352.

[28] Mañé, R., Lyapunov exponents and stable manifolds for compact transformations, *Springer Lecture Notes in Mathematics* **1007**, (1983), 522–577.

[29] Mao, X., Exponential stability for delay Itô equations, *Proceedings of IEEE International Conference on Control and Applications*, April 1989.

[30] Métivier, M., *Semimartingales, a Course on Stochastic Processes*, Walter de Gruyter, Berlin, New York (1982).

[31] Métivier, M. and Pellaumail, J., *Stochastic Integration*, Academic Press, New York, London, Toronto, Sydney, San Francisco (1980).

[32] Mizel, V.J. and Trutzer, V., Stochastic hereditary equations: existence and asymptotic stability, *Journal of Integral Equations* **7** (1984), 1–72.

[33] Mohammed, S.–E.A., *Stochastic Functional Differential Equations,* Research Notes in Mathematics No. 99, Pitman Advanced Publishing Program, Boston–London–Melbourne (1984).

[34] Mohammed, S.–E.A., Stability of linear delay equations under small noise, *Proceedings of the Edinburgh Mathematical Society* **29** (1986), 233–254.

[35] Mohammed, S.–E.A., Non–linear flows for linear stochastic delay equations, *Stochastics*, Vol. 17, Number 3 (1986), 207–212.

[36] Mohammed, S.–E.A., Unstable invariant distributions for a class of stochastic delay equations, *Proceedings of the Edinburgh Mathematical Society* **31** (1988), 1–23.

[37] Mohammed, S.–E.A., The Lyapunov spectrum and stable manifolds for stochastic linear delay equations, *Stochastics and Stochastic Reports,* **29** (1990), 89–131.

[38] Mohammed, S.–E.A. and Scheutzow, M.K.R., Lyapunov exponents and stationary solutions for affine stochastic delay equations, *Stochastics and Stochastic Reports*, Vol. 29, Number 2 (1990), 259–283.

[39] Mohammed, S.–E.A. and Scheutzow, M.K.R., Lyapunov exponents of linear stochastic functional differential equations driven by semimartingales. Part I: The multiplicative ergodic theory (preprint; 1990).

[40] Mohammed, S.–E.A., Scheutzow, M.K.R. and Weizsäcker, H.v., Hyperbolic state space decomposition for a linear stochastic delay equation, *SIAM Journal on Control and Optimization* **24,** 3 (1986), 543–551.

[41] Protter, Ph.E., Semimartingales and measure–preserving flows, Preprint, Statistics Department, Purdue University (1985).

[42] Protter, Ph.E., Right–continuous solutions of stochastic integral equations, *J. Multivariate Anal.* **7** (1977), 204–214.

[43] Ruelle, D., Ergodic theory of differentiable dynamical systems, *I.H.E.S. Publications* **50** (1979), 275–305.

[44] Ruelle, D., Characteristic exponents and invariant manifolds in Hilbert space, *Annals of Mathematics* **115** (1982), 243–290.

[45] Scheutzow, M.K.R., Qualitative behaviour of stochastic delay equations with a bounded memory, *Stochastics* **12** (1984), 41–80.

[46] Scheutzow, M.K.R., *Stationary and Periodic Stochastic Differential Systems: A study of Qualitative Changes with Respect to the Noise Level and Asymptotics*, Habilitationsschrift, University of Kaiserslautern, W. Germany (1988).

[47] Sussman, H.J., On the gap between deterministic and stochastic ordinary differential equations, *Ann. Prob.*, **6**, 1 (1978), 19–41.

[48] Thieullen, P., Fibres dynamiques asymptotiquement compacts exposants de Lyapunov. Entropie. Dimension. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, **4(1)** (1987), 49–97.

[49] Weizsäcker, H.v. and Winkler, G., *Stochastic Integrals, An Introduction*, Vieweg Advanced Lectures in Mathematics, Friedr. Vieweg and Sohn, Braunschweig/Wiesbaden (1990).

Salah–Eldin A. Mohammed
Department of Mathematics
Southern Illinois University at Carbondale
Carbondale, IL 62901

and

Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213–3890