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Abstract

General equilibrium models in which economic agents have finite marginal utility from consumption at the origin lead to financial assets whose prices are continuous but exhibit singular components. In particular, there is no bona–fide "interest rate" in such models, although asset prices can be determined by equilibrium considerations (and uniquely, up to the formation of mutual funds). The singularly continuous processes in question charge precisely the set of time–points at which some agent "drops out" of the economy, or "comes back" into it, between intervals of zero–consumption. Not surprisingly, these processes are governed by local time.

1. Introduction

A primary objective of consumption–based, capital asset pricing theory has been to model the relationship between rates of return and aggregate consumption. In continuous–time models, a number of researchers (e.g., Merton (1973), Breeden (1979, 1986), Cox, Ingersoll & Ross (1985a,b), Lucas (1978), and Duffie & Zame (1989)) have studied this relationship. Two key results which emerge from these papers are that, in equilibrium,

(1.1)   the rate of return from a riskless asset should be the negative of the growth rate of the marginal utility for consumption of a representative agent, and

(1.2)   the excess (above the risk–free rate) rate of return from a risky asset should be proportional to the covariance between the price of that asset and the aggregate consumption, with the constant of proportionality independent of the asset and equal to the relative index of risk aversion for a representative agent.

Under some regularity conditions, including the strict positivity of optimal consumption processes, equilibrium prices in continuous–time, capital asset pricing models have been shown to enjoy these two properties, but the existence of suitable equilibrium prices in a multi–agent economy has until recently been an open question. Price processes reside in an infinite–dimensional space, and one method of proving existence of equilibrium in such spaces is based on a fixed point result of Mas–Colell (1986) (see, e.g., Duffie (1986)). Mas–Colell's theorem assumes "uniform properness" of utility functions, which in the time–additive case requires finite marginal utility at the zero level of consumption. On the other hand, the derivations of statements (1.1) and (1.2) require positivity of consumption at all times, a situation which is known not to prevail when the marginal utility of zero consumption is finite. Araujo & Monteiro (1989a,b) have obtained equilibrium without assuming "uniform properness", but the ramifications of their results for continuous–time, capital asset pricing theory have yet to be explored.

The present paper concerns the existence of equilibrium and the extent to which (1.1) and (1.2) hold. The main result is that equilibrium does exist, but if some agents have finite

marginal utility at zero while others do not, then the riskless asset can fail to have a rate of return in the traditional sense, i.e., there may be no processes $r(t)$ such that the price $P_0(t)$ of the riskless asset satisfies

$$(1.3) \qquad\qquad P_0(t) = P_0(0) \exp(\int_0^t r(s)ds).$$

However, (1.1) holds in a more general sense made precise in Remark 8.3. Likewise, the price processes of the risky assets may not have rates of return in the traditional sense. However, the difference between any risky asset and the riskless asset will have a traditional rate of return, and if we define the "excess rate of return" to be the rate for this difference, then (1.2) will hold. All these difficulties are caused by the fact that some agents may see their optimal consumption fall to zero. If assumptions are made to prevent this, then a process $r(t)$ satisfying (1.3) can be found, and the characterizations (1.1) and (1.2) hold.

Duffie & Zame (1989) were the first to prove the existence of an equilibrium satisfying (1.1) and (1.2) in a continuous–time, consumption–based, capital asset pricing model. They assumed infinite marginal utility at zero for every agent and avoided Mas–Colell's uniform properness condition by a functional analytic argument. Consequently, the anomaly addressed by the present work did not arise. Duffie & Zame's model also included a *spot price process*, which denominated the consumption good in terms of a "numeraire." Such a process obscures the difficulty we address here because rates of return for assets denominated in terms of a numeraire can exist even when their rates denominated in terms of the consumption good fail to.

Karatzas, Lehoczky & Shreve (1990) established the existence of equilibrium by reducing the problem to a finite–dimensional fixed point problem. (Some of the results of Karatzas, *et al.* have been sharpened by Dana & Pontier (1990).) The variables in the finite–dimensional problem of Karatzas *et al.* are the weights needed to form the appropriate

representative agent, an idea borrowed from Huang (1987). The method does not require any conditions on marginal utilities at zero, but existence of equilibrium is obtained only if the model includes a spot price process. (Such a model is referred to as the *moneyed model* in Karatzas, *et al.*) In the model without a spot price (the *moneyless model*), equilibrium is obtained in Karatzas *et al.* only when all agents have infinite marginal utility at zero.

In this paper we consider a multi–agent model without a spot price and with no condition on marginal utilities at zero. For the sake of simplicity, we set up the model in a pure–exchange economy; it is not difficult to combine this paper with Karatzas, Lehoczky & Shreve (1990) to obtain analogous results for a production economy. Martingale methods are used to solve the optimization problems for the individual agents, and so there is no need to introduce a state vector or otherwise attempt to create Markov processes. This was also the case in Duffie (1986), Duffie & Zame (1989) and Karatzas, Lehoczky & Shreve (1990), but not in previous equilibrium papers.

In order to obtain equilibrium, we hypothesize at the outset a riskless asset (called a *bond*) whose price is continuous and of bounded variation, but which is not necessarily absolutely continuous. Thus, there is no "interest rate" which can be used to recover the price process for this bond. The risky assets (called *stocks*) also have continuous, bounded variation price processes. We assume in the main body of the paper that the singularly continuous parts of the stocks prices match that of the bond price; we show in an appendix (Section 11) that failure of this condition would allow arbitrage. Following Karatzas, Lehoczky & Shreve (1990), we reduce the equilibrium problem to a finite–dimensional fixed point problem, whose solution allows us to define a representative agent utility function. Related to the fact that some agents can see their optimal consumption fall to zero, this representative agent utility function may have a discontinuous first derivative. Itô rule computations for such a function introduce semimartingale local times, and these in turn lead to singularly continuous components in the asset prices. Section 8 provides formulae for the equilibrium asset prices and interprets them in light of (1.1) and (1.2). Section 9 shows by example that the singularly continuous components

in the asset prices can be nontrivial, even in apparently innocuous situations.

Finally, we note that by allowing asset prices to be possibly discontinuous semimartingales, Back (1990) constructs a consumption–based, capital asset pricing model even more general than ours. In this context and under the assumption of existence of equilibrium, he obtains a counterpart to (1.2). Our paper presents a rationale for moving at least some distance from the traditional model (with absolutely continuous asset prices) in the direction of the one proposed by Back.

2. Agents and Endowment Processes.

We consider an economy consisting of N agents. Each of these agents receives an *exogenous endowment* process $\varepsilon_n = \{\varepsilon_n(t); 0 \leq t \leq T\}$ which is positive, and progressively measurable with respect to the filtration $\{\mathscr{F}_t\}$. It will be assumed throughout that $\{\mathscr{F}_t\}$ is the augmentation by null sets of the natural filtration

$$\mathscr{F}_t^W = \sigma(W(s); \ 0 \leq s \leq t), \quad t \in [0,T]$$

generated by a d–dimensional Brownian motion $W(t) = (W_1(t),...,W_d(t))^*$ on the complete probability space $(\Omega, \mathscr{F}, P)$. All economic activity takes place on the finite horizon $[0,T]$.

The aggregate endowment $\varepsilon(t) \triangleq \sum_{n=1}^{N} \varepsilon_n(t)$ will be assumed to be a continuous semimartingale of the form

$$(2.1) \qquad \varepsilon(t) = \varepsilon(0) + \int_0^t \varepsilon(s)d\xi(s) + \int_0^t \varepsilon(s)\nu(s)ds + \int_0^t \varepsilon(s)\rho^*(s)dW(s).$$

Here ζ is an $\{\mathscr{F}_t\}$–adapted process with paths which are continuous but singular with respect to Lebesgue measure and of bounded variation on $[0,T]$, and ρ, ν are bounded, $\{\mathscr{F}_t\}$–progressively measurable processes with values in $\mathbb{R}^d$ and $\mathbb{R}$, respectively. We shall

assume that there are positive, finite constants $k < K$ such that

$$(2.2) \qquad k \leq \varepsilon(t) \leq K \ , \quad \forall \ t \in [0,T]$$

holds almost surely.

In order to establish the uniqueness of equilibrium, we shall also need the condition

$$(2.3) \qquad \varepsilon_n(t) > 0 \quad \text{a.s.} \ , \quad \forall \ t \in [0,T] \ \text{and} \ n = 1,...,N.$$

3. Utility Functions

Each agent is endowed with a *utility function* $U_n : (0,\infty) \to \mathbb{R}$ which is of class C^3, strictly increasing and strictly concave, and satisfies $U_n'(\infty) \overset{\Delta}{=} \lim_{c \to \infty} U_n'(c) = 0$.

For the uniqueness of equilibrium, we shall also need the condition

$$(3.1) \qquad c \mapsto cU_n'(c) \ \text{is noncedreasing}, \quad \forall \ n = 1,...,N.$$

This condition amounts to assuming that $-\dfrac{cU_n''(c)}{U_n'(c)}$, the Arrow–Pratt measure of relative risk–aversion, is less than or equal to one.

We shall denote by I_n the inverse of the function U_n'; this is a strictly decreasing mapping of $(0,U_n'(0))$ onto $(0,\infty)$, and we extend it on all of $(0,\infty)$ by setting $I_n(y) = 0$ for $y \geq U_n'(0)$.

In this model, agents derive utility by consuming parts of the aggregate commodity endowment. Because such endowments will typically be random and time–varying, the agents will find it useful to participate in a market which allows them both to hedge their risk and smooth out their consumption. A model for such a market is introduced in the next section; its coefficients will be determined in section 8 by equilibrium considerations, in terms of the endowment processes and utility functions of the individual agents.

4. A Financial Market with Singular Bond Prices.

This financial market has $d + 1$ assets; one of them is a pure discount *bond*, with price $P_0(t)$ at time t which satisfies

$$(4.1) \qquad dP_0(t) = P_0(t)[r(t)dt + dA(t)], \quad P_0(0) = 1.$$

The remaining assets are risky *stocks*, with prices–per–share $P_i(t)$ given by

$$(4.2) \qquad dP_i(t) = P_i(t)[b_i(t)dt + dA_i(t) + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)], \quad 1 \leq i \leq d.$$

The processes $r(\cdot)$, $A(\cdot)$, $b_i(\cdot)$, $A_i(\cdot)$ and $\sigma_{ij}(\cdot)$ will be referred to collectively as the *coefficients* of the model. They are all $\{\mathscr{F}_t\}$–progressively measurable. The processes $r(\cdot)$, $b_i(\cdot)$ and $\sigma_{ij}(\cdot)$ are bounded uniformly in (t,ω), the matrix $\sigma(t) = \{\sigma_{ij}(t)\}_{1 \leq i,j \leq d}$ satisfies the strong nondegeneracy condition

$$(4.3) \qquad \xi^* \sigma(t)\sigma^*(t)\xi \geq \delta\|\xi\|^2 ; \quad \forall\ t \in [0,T],\ \forall\ \xi \in \mathbb{R}^d$$

almost surely (for some given $\delta > 0$), and the processes A, A_i have P–almost every path continuous, of bounded variation on $[0,T]$ (uniformly in ω) and *singular* with respect to Lebesgue measure, with $A(0) = A_i(0) = 0, i = 1,...,d$.

We shall see in section 10 (Appendix) that we have to assume

$$(4.4) \qquad A_i(t) = A(t); \quad \forall\ t \in [0,T],\ i = 1,...,d$$

almost surely, in order to exclude arbitrage opportunities. This condition will be imposed from now on.

The so–called "relative risk process"

(4.5) $$\theta(t) \triangleq (\sigma(t))^{-1}[b(t) - r(t)\underline{1}], \quad 0 \leq t \leq T$$

will be important in the sequel. Let us notice that it is progressively measurable with respect to $\{\mathscr{F}_t\}$ and, thanks to (4.3), bounded.

5. Portfolio and Consumption Policies

Each agent has at his disposal the choice of an $\mathbb{R}^d$–valued *portfolio process* $\pi_n(t) = (\pi_{n1}(t),...,\pi_{nd}(t))^*$, and of a nonnegative *consumption* rate *process* $c_n(t)$, $0 \leq t \leq T$; these processes are $\{\mathscr{F}_t\}$–progressively measurable, and satisfy $\int_0^T \{\|\pi_n(t)\|^2 + c_n(t)\}dt < \infty$, a.s. For every such pair (π_n, c_n), the corresponding wealth process X_n has initial value $X_n(0) = 0$ and obeys the equation[1]

$$dX_n(t) = \sum_{i=1}^{d} \pi_{ni}(t)[b_i(t)dt + dA_i(t) + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)]$$

$$+ (X_n(t) - \sum_{i=1}^{d} \pi_{ni}(t))[r(t)dt + dA(t)] + (\varepsilon_n(t) - c_n(t))dt$$

(5.1)

$$= X_n(t)[r(t)dt + dA(t)] + (\varepsilon_n(t) - c_n(t))dt + \sum_{i=1}^{d} \pi_{ni}(t)dG_i(t)$$

$$+ \pi_n^*(t)[(b(t) - r(t)\underline{1})dt + \sigma(t)dW(t)],$$

where $b(t) \triangleq (b_1(t),...,b_d(t))^*$, $\underline{1} = (1,...,1)^*$ and $G_i(t) \triangleq A_i(t) - A(t)$ for $i = 1,...,d$.

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[1]The interpretation here is that $\pi_{ni}(t)$ represents the amount invested by the nth agent in the ith stock, at time t, for $i = 1,...,d$; the amount $X_n(t) - \sum_{i=1}^{d} \pi_i(t)$ is invested in the bond.

Let us recall the process θ of (4.5), and introduce the exponential martingale

$$(5.2) \qquad Z(t) \triangleq \exp[-\int_0^t \theta^*(s)dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds]; \quad 0 \le t \le T.$$

According to the Girsanov theorem,

$$(5.3) \qquad \tilde{W}(t) \triangleq W(t) + \int_0^t \theta(s)ds ; \quad 0 \le t \le T$$

is then a Brownian motion under the new probability measure $\tilde{P}(A) = E[Z(T)1_A]$, $A \in \mathscr{F}_T$ (cf. Karatzas & Shreve (1988), §3.5). With this notation, and taking (4.4) into account, the solution of (5.1) is given by

$$(5.4) \qquad \beta(t)X_n(t) = \int_0^t \beta(s)(\epsilon_n(s) - c_n(s))ds + \int_0^t \beta(s)\pi_n^*(s)\sigma(s)d\tilde{W}(s); \quad 0 \le t \le T,$$

where

$$(5.5) \qquad \beta(t) \triangleq \frac{1}{P_0(t)} = \exp\{-\int_0^t r(s)ds - A(t)\}.$$

5.1 Remark: The martingale Z of (5.2) satisfies the equation

$$(5.6) \qquad Z(t) = 1 - \int_0^t Z(s)\theta^*(s)dW(s).$$

An application of the integration—by—parts formula to the product of βX_n and Z yields then, in conjunction with (5.4) and (5.6):

(5.7) $\zeta(t)X_n(t) = \int_0^t \zeta(s)[\varepsilon_n(s) - c_n(s)]ds + \int_0^t \zeta(s)[\sigma^*(s)\pi_n(s) - X_n(s)\theta(s)]^* dW(s);\ 0 \le t \le T.$

Here

(5.8) $$\zeta(t) \triangleq \beta(t)Z(t),$$

and it is easily verified that the equation (5.7) is actually equivalent to (5.4).

□

We assume for the present that the process ζ of (5.8) satisfies the condition

(5.9) $$0 < \delta \le \zeta(t) \le \Delta ; \quad \forall\ 0 \le t \le T$$

almost surely, for some finite constants $\Delta > \delta > 0$. This assumption will be justified at the end of Section 7.

5.2 Definition: The portfolio/consumption process pair (π_n, c_n) for the nth agent is called *admissible*, if the corresponding wealth process of (5.4) satisfies

$$\beta(t)X_n(t) + \tilde{E}(\int_t^T \beta(s)\varepsilon_n(s)ds \,|\, \mathscr{F}_t) \ge 0 , \quad \forall\ t \in [0,T],$$

or equivalently (by virtue of the so—called "Bayes rule", p. 193 in Karatzas & Shreve (1988)):

(5.10) $$\zeta(t)X_n(t) + E(\int_t^T \zeta(s)\varepsilon_n(s)ds \,|\, \mathscr{F}_t) \ge 0 ; \quad \forall\ t \in [0,T],$$

almost surely.

□

In particular, it follows from (5.7)–(5.10) and (2.2) that for an admissible pair (π_n, c_n), the process

$$\zeta(t)X_n(t) + \int_0^t \zeta(s)(c_n(s) - \varepsilon_n(s))ds \ ; \quad 0 \leq t \leq T$$

is a local martingale, bounded from below. It is, therefore, also a supermartingale with initial value equal to zero, and this implies

$$(5.11) \qquad E\int_0^T \zeta(s)c_n(s)ds \leq E[\zeta(T)X_n(T) + \int_0^T \zeta(s)c_n(s)ds] \leq E\int_0^T \zeta(s)\varepsilon_n(s)ds.$$

5.3 Proposition: Let $\hat{c}_n$ be a consumption process which satisfies

$$(5.12) \qquad E\int_0^T \zeta(s)\hat{c}_n(s)ds = E\int_0^T \zeta(s)\varepsilon_n(s)ds.$$

Then there exists a portfolio process $\hat{\pi}_n$ such that the pair $(\hat{\varepsilon}_n, \hat{c}_n)$ is admissible, and the corresponding wealth process $\hat{X}_n$ is given by

$$(5.13) \qquad \beta(t)\hat{X}_n(t) = \tilde{E}[\int_t^T \beta(s)(\hat{c}_n(s) - \varepsilon_n(s))ds \,|\, \mathscr{F}_t] \ ; \quad 0 \leq t \leq T.$$

Proof: According to (5.12), the $\tilde{P}$–martingale

$$(5.14) \qquad M_n(t) \triangleq \tilde{E}[\int_0^T \beta(s)(\hat{c}_n(s) - \varepsilon(s))ds \,|\, \mathscr{F}_t] \ ; \quad 0 \leq t \leq T$$

has zero expectation; from the fundamental martingale representation theorem, it admits the

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stochastic integral representation

$$(5.15) \qquad M_n(t) = \int_0^t \beta(s)\pi_n^*(s)\sigma(s)d\tilde{W}(s)$$

for some portfolio process π_n (cf. Karatzas & Shreve (1988), Problem 3.4.16 and proof of Proposition 5.8.6). It follows then, from (5.4), (5.14) and (5.15), that the wealth process $\hat{X}_n$ corresponding to $(\hat{\pi}_n,\hat{c}_n)$ is given by (5.13), and that it satisfies the admissibility requirement of Definition 5.2.

$\square$

6. The nth Agent's Optimization Problem.

Each agent's goal is to maximize the expected discounted utility from consumption

$$(6.1) \qquad E\int_0^T e^{-\int_0^t \mu(s)ds} U(c_n(t))ds,$$

over all admissible pairs (π_n,c_n) which satisfy

$$(6.2) \qquad E\int_0^T e^{-\int_0^t \mu(s)ds} U^-(c_n(t))dt < \infty.$$

Here $\mu : [0,T] \to \mathbb{R}$ is a given bounded, measurable function. A pair $(\hat{\pi}_n,\hat{c}_n)$ that achieves the supremum of (6.1) over such pairs, is called *optimal.*

We can describe the optimal $(\hat{\pi}_n,\hat{c}_n)$ in the manner of Karatzas, Lehoczky & Shreve (1987) and Cox & Huang (1989), as follows: there is a unique positive number y_n for which

(6.3)
$$E \int_0^T \zeta(t) I_n(y_n \zeta(t) e^{\int_0^t \mu(s)ds}) dt = E \int_0^T \zeta(t) \varepsilon_n(t) dt.$$

Then the consumption process

(6.4)
$$\hat{c}_n(t) \triangleq I_n(y_n \zeta(t) e^{\int_0^t \mu(s)ds}); \quad 0 \leq t \leq T$$

satisfies (5.12), and from Proposition 5.3 there exists a portfolio process $\hat{\pi}_n$ such that $(\hat{\pi}_n, \hat{c}_n)$ is admissible — with associated wealth process given by (5.13).

For any other admissible pair (π_n, c_n), the elementary consequence of concavity:

$$U_n(I_n(y)) \geq U_n(c) + y[I_n(y) - c] ; \quad \forall \ y \in [0,\infty), \ c \in [0,\infty)$$

gives (when applied to $y = y_n \zeta(t) e^{\int_0^t \mu(s)ds}$ and $c = c_n(t)$), after multiplying by $\exp\{-\int_0^t \mu(s)ds\}$ and integrating $dt \times dP$:

$$E \int_0^T e^{-\int_0^t \mu(s)ds} U_n(\hat{c}_n(t)) dt - E \int_0^T e^{-\int_0^t \mu(s)ds} U_n(c_n(t)) dt$$

$$\geq y_n \ [E \int_0^T \zeta(t) \hat{c}_n(t) dt - E \int_0^T \zeta(t) c_n(t) dt].$$

But this last term is nonnegative, thanks to (5.11) and (5.12), and the optimality of $(\hat{\pi}_n, \hat{c}_n)$ follows. (By taking $c_n(t)$ to be a suitable constant in the above argument, we see that $\hat{c}_n(\cdot)$ satisfies the requirement (6.2).)

7. Equilibrium and the "Representative Agent".

We shall say that the financial market of section 4 results in **equilibrium**, if in the notation of section 5, we have the following conditions:

(i) *Clearing of the commodity market*:

$$(7.1) \qquad \sum_{n=1}^{N} \hat{c}_n(t) = \varepsilon(t); \quad 0 \le t \le T,$$

(ii) *clearing of the stock markets*:

$$(7.2) \qquad \sum_{n=1}^{N} \hat{\pi}_{ni}(t) = 0 ; \quad 0 \le t \le T, \quad i = 1,...,d ,$$

(iii) *clearing of the bond market*:

$$(7.3) \qquad \sum_{n=1}^{N} \hat{X}_n(t) = 0 ; \quad 0 \le t \le T.$$

In this context, $\hat{c}_n$, $\hat{\pi}_n$ and $\hat{X}_n$ denote the optimal processes for the nth agent.

7.1 Proposition: The conditions $(7.1) - (7.3)$ lead to the a.s. identity

$$(7.4) \qquad \varepsilon(t) = \sum_{n=1}^{N} I_n(y_n \zeta(t) \, e^{\int_0^t \mu(s)ds}) ; \quad 0 \le t \le T,$$

where y_n is defined by (6.3) for $n = 1,...,N$.

Conversely, suppose that there exists a financial market for which the process ζ of (5.8) satisfies (7.4) and (6.3), for suitable positive numbers $y_1,...,y_N$. Then this financial market results in equilibrium.

Proof: For the first claim, simply observe that (7.4) follows from (7.1) and (6.4). For the converse, note that for the financial market in question the optimal consumption processes $\{\hat{c}_n\}_{n=1}^N$ are again given by (6.4), and the corresponding wealth processes $\{\hat{X}_n\}_{n=1}^N$ by (5.13). The condition (7.1) follows directly from (7.4) and (6.4), and leads, in conjunction with (5.13) and (5.14), to (7.3) and $\sum_{n=1}^N \hat{M}_n(t) \equiv 0$, respectively. Now this last condition, together with (5.15) and the nondegeneracy of σ^*, gives (7.2).

□

In order to facilitate the search for an equilibrium financial market, let us introduce for every vector $\Lambda \in (0,\infty)^N$ the function

$$(7.5) \qquad U(c;\Lambda) \triangleq \max_{\substack{c_1 \geq 0, \ldots, c_N \geq 0 \\ \sum_{n=1}^N c_n = c}} \sum_{n=1}^N \lambda_n U_n(c_n) ; \quad 0 < c < \infty.$$

It can be seen as in Karatzas, Lehoczky & Shreve (1990), section 10, that the maximum is achieved at

$$(7.6) \qquad \bar{c}_n = I_n(\frac{1}{\lambda_n} H(c;\Lambda)) ; \quad n = 1,\ldots,N$$

where $H(\cdot;\Lambda)$ is the inverse of the continuous, decreasing function

$$(7.7) \qquad I(y;\Lambda) \triangleq \sum_{n=1}^N I_n(y/\lambda_n) ; \quad 0 < y < \infty.$$

Thus $U(c;\Lambda) = \sum_{n=1}^N \lambda_n U_n(I_n(H(c;\Lambda)/\lambda_n))$, and it follows from this representation that $U(\cdot;\Lambda)$ is continuous and continuously differentiable on $(0,\infty)$ with $U'(c;\Lambda) = H(c;\Lambda)$, and of class C^3

away from the set

(7.8) $\mathscr{A} = (\alpha_1,...,\alpha_N\}, \quad \text{with} \quad \alpha_n \triangleq I(\lambda_n U_n'(0);\Lambda).$

We interpret the function $U(\cdot;\Lambda)$ of (7.5) as the *utility function of a representative agent*, who assigns weights $\lambda_1,...,\lambda_N$ to the individual agents in the economy.

The problem of equilibrium can then be cast as that of determining the "right" way to assign these weights. Indeed, with the identification $\Lambda = (\lambda_1,...,\lambda_N) = (\frac{1}{y_1},...,\frac{1}{y_N})$ the equations (7.4), (6.3) can be written as:

(7.9) $\zeta(t) = e^{-\int_0^t \mu(s)ds} U'(\varepsilon(t);\Lambda) ; \quad 0 \le t \le T$

$$E\int_0^T e^{-\int_0^t \mu(s)ds} U'(\varepsilon(t);\Lambda) I_n(\frac{1}{\lambda_n} U'(\varepsilon(t);\Lambda)dt$$

(7.10)

$$= E\int_0^T e^{-\int_0^t \mu(s)ds} U'(\varepsilon(t);\Lambda) \varepsilon_n(t)dt ; \quad n = 1,...,N,$$

and *constructing equilibrium is equivalent to finding a vector* $\Lambda \in (0,\infty)^N$ *which satisfies (7.10)*.

Once such a vector has been found, the process ζ of the corresponding financial market is given by (7.9) and satisfies the requirement (5.9), thanks to the assumption (2.2) and the continuity of $U'(\cdot;\Lambda)$. The optimal consumption processes of the individual agents are given by (6.4) as

(7.11) $\hat{c}_n(t;\Lambda) \triangleq I_n(\frac{1}{\lambda_n} U'(\varepsilon(t);\Lambda)) ; \quad 0 \le t \le T, \quad n = 1,...,N$

(see also (7.6)).

8. Characterization of Equilibrium Asset Prices.

We quote from Karatzas, Lehoczky & Shreve (1990), Theorem 11.1, the following fundamental result.

8.1 Theorem: There exists a vector $\Lambda \in (0,\infty)^N$ which satisfies (7.10). Furthermore, if the endowment processes satisfy (2.3) and all utility functions $\{U_n\}_{n=1}^N$ obey condition (3.1), this vector is unique up to a multiplicative constant.

□

Consider now the process $\eta(t) \triangleq \zeta(t) \, e^{\int_0^t \mu(s)ds}$; from (5.8), (5.5) and (5.6) it follows that η satisfies the stochastic integral equation

$$(8.1) \qquad \eta(t) = 1 + \int_0^t \eta(s)\{\mu(s) - r(s)\}ds - \int_0^t \eta(s)dA(s) - \int_0^t \eta(s)\theta^*(s)dW(s).$$

On the other hand, (7.9) gives $\eta(t) = U'(\varepsilon(t);\Lambda)$; apply the generalized Itô rule for convex functions of semimartingales (e.g. Karatzas & Shreve (1988), Chapter 3, Theorems 6.22, 7.1 and Problem 6.24), to obtain

$$\eta(t) = U'(\varepsilon(0);\Lambda) + \int_0^t [U''(\varepsilon(s);\Lambda)\varepsilon(s)\nu(s) + \tfrac{1}{2} U'''(\varepsilon(s);\Lambda)\varepsilon^2(s)\|\rho(s)\|^2]ds$$

$$+ \int_0^t U''(\varepsilon(s);\Lambda)\varepsilon(s)d\xi(s) + \int_0^t U''(\varepsilon(s);\Lambda)\varepsilon(s)\rho^*(s)dW(s)$$

$$(8.2)$$

$$+ \sum_{n=1}^N [U''(\alpha_n+;\Lambda) - U''(\alpha_n-;\Lambda)] L_t(\alpha_n)$$

in conjunction with (2.1), where $L_t(a)$ is the local time at a for the semimartingale ε, accumulated up to time t.

We can identify now various terms in the two semimartingale decompositions (8.1), (8.2) for the same process η, to get

$$(8.3) \qquad\qquad U'(\varepsilon(0); \Lambda) = 1$$

$$(8.4) \qquad r(t) = \mu(t) - \frac{U''(\varepsilon(t);\Lambda)\varepsilon(t)\nu(t) + \frac{1}{2} U'''(\varepsilon(t);\Lambda)\|\rho(t)\|^2\varepsilon^2(t)}{U'(\varepsilon(t);\Lambda)},$$

$$(8.5) \qquad\qquad \theta(t) = - \frac{U''(\varepsilon(t); \Lambda)}{U'(\varepsilon(t);\Lambda)} \varepsilon(t)\rho(t)$$

and

$$(8.6) \qquad A(t) = - \int_0^t \frac{U''(\varepsilon(s); \Lambda)}{U'(\varepsilon(s);\Lambda)} \varepsilon(s)d\xi(s) - \sum_{n=1}^N \frac{U''(\alpha_n+;\Lambda) - U''(\alpha_n-;\Lambda)}{U'(\alpha_n;\Lambda)} L_t(\alpha_n).$$

Condition (8.3) determines uniquely the vector Λ amongst those which satisfy the equations (7.10). With Λ thus determined, (8.4) $-$ (8.6) provide the equilibrium values for the processes r, θ and A appearing in the financial market of section 4. The equilibrium market is thus determined uniquely, *up to the formation of mutual funds* (in the sense that the coefficients b, σ are not individually determined, but only modulo the process $\theta(t) = (\sigma(t))^{-1}[b(t) - r(t)\underline{1}]$ that they give rise to).

8.2 Remarks: If $U_n'(0+) = \infty$ for every $n = 1,...,N$ and $\xi \equiv 0$ in (2.1), then (8.6) gives $A \equiv 0$. Formulae (8.4), (8.5) for the equilibrium financial market model agree then with (11.8), (11.9) of Karatzas, Lehoczky & Shreve (1990). In this case, the process of r of (8.4), (4.1) is a genuine interest rate.

On the other hand, if $U_n'(0+) < \infty$ for some $n = 1,...,N$, or if the process ξ in (2.1) is

nontrivial, then the resulting process A of (8.6) is nontrivial as well. The resulting bond price process P_0, in the financial market model of section 4, *does not have then a bona—fide interest rate*. In the following section we present an example of this situation.

8.3 Remark: In light of (8.2), it is reasonable to define the growth rate of the marginal utility for consumption of the representative agent to be the stochastic differential

$$dG(t) \triangleq \frac{1}{U'(\varepsilon(t);\Lambda)} [U''(\varepsilon(t);\Lambda)\varepsilon(t)\nu(t)dt + \frac{1}{2} U'''(\varepsilon(t);\Lambda)\varepsilon^2(t)\|\rho(t)\|^2$$

$$+ U''(\varepsilon(t);\Lambda)\varepsilon(t)d\xi(t) + \sum_{n=1}^{N} (U''(\alpha_n+;\Lambda) - U''(\alpha_n-;\Lambda))dL_t(\alpha_n)].$$

This quantity is equal to $-(r(t) - \mu(t))dt - dA(t)$. In particular, if there is no discounting ($\mu \equiv 0$),

$$(8.7) \qquad \frac{dP_0(t)}{P_0(t)} = - dG(t) ,$$

which is a precise formulation of (1.1). From (8.5), we have

$$(8.8) \qquad b(t) - r(t)\underline{1} = \sigma(t)\theta(t) = - \frac{\varepsilon(t) U''(\varepsilon(t);\Lambda)}{U'(\varepsilon(t);\Lambda)} \sigma(t)\rho(t),$$

a precise formulation of (1.2) (recall (4.4)).

9. An Example.

Let us consider again the Example of section 11 in Karatzas, Lehoczky & Shreve (1990); with $d = 1$, $N = 2$ and $\mu \equiv 0$, we take $U_1(c) = \log c$ and $U_2(c) = \log(1+c)$. Then for any $\Lambda = (\lambda_1,\lambda_2) \in (0,\infty)^2$, we have

$$(9.1) \quad I(y;\Lambda) = \left\{ \begin{array}{ll} \dfrac{\lambda_1 + \lambda_2}{y} - 1; & 0 < y < \lambda_2 \\[2mm] \dfrac{\lambda_1}{y} & ; \quad y \geq \lambda_2 \end{array} \right\}, \quad H(c;\Lambda) = \left\{ \begin{array}{ll} \dfrac{\lambda_1}{c} & ; \quad 0 < c < \alpha(\Lambda) \\[2mm] \dfrac{\lambda_1 + \lambda_2}{1 + c} & c \geq \alpha(\Lambda) \end{array} \right\}$$

with $\alpha(\Lambda) = \lambda_1/\lambda_2$, and the numbers $\bar{c}_n$ of (7.6) are given by

$$(9.2) \quad \bar{c}_1(c;\Lambda) = \left\{ \begin{array}{ll} c & ; \quad 0 < c < \alpha(\Lambda) \\[2mm] \dfrac{\lambda_1(1+c)}{\lambda_1 + \lambda_2} & ; \quad c \geq \alpha(\Lambda) \end{array} \right\}, \quad \bar{c}_2(c;\Lambda) = \left\{ \begin{array}{ll} 0 & ; \quad 0 < c < \alpha(\Lambda) \\[2mm] \dfrac{\lambda_2(1+c)}{\lambda_1 + \lambda_2} - 1; & c \geq \alpha(\Lambda) \end{array} \right\}.$$

The representative agent utility function $U(c;\Lambda) = \lambda_1 U_1(\bar{c}_1) + \lambda_2 U_2(\bar{c}_2)$ becomes then

$$(9.3) \quad U(c;\Lambda) = \left\{ \begin{array}{ll} \lambda_1 \log c & ; \quad 0 < c < \alpha(\Lambda) \\[2mm] (\lambda_1+\lambda_2)\log(1+c) + \lambda_1 \log \dfrac{\lambda_1}{\lambda_1 + \lambda_2} + \lambda_2 \log \dfrac{\lambda_2}{\lambda_1 + \lambda_2} & ; \quad c \geq \alpha(\Lambda) \end{array} \right\},$$

and we observe

$$(9.4) \qquad U''(\alpha(\Lambda)+;\Lambda) - U''(\alpha(\Lambda)-;\Lambda) = \frac{\lambda_2^3}{\lambda_1(\lambda_1+\lambda_2)}.$$

For the aggregate endowment, we consider the process

$\varepsilon(t) \triangleq 1 + \exp[W(t\wedge\tau) - \frac{1}{2}(t\wedge\tau)^2]$, $0 \leq t \leq T$ with $\tau = \inf\{t \in [0,T]; W_t = 1\} \wedge T$, which is a bounded martingale and satisfies

$$(9.5) \qquad d\varepsilon(t) = \varepsilon(t) 1_{\{t\leq\tau\}} dW(t), \quad \varepsilon(0) = 2$$

(i.e., (2.1) with $\xi \equiv 0$, $\nu \equiv 0$, $\rho(t) = \varepsilon(t) 1_{\{t\leq\tau\}}$), as well as (2.2). For a given number $k \in (0,1)$ which also satisfies

$$(9.6) \qquad k \, E \int_0^T \frac{2\varepsilon(t)}{1+\varepsilon(t)} \, dt > T \,,$$

we take

$$(9.7) \qquad \varepsilon_1(t) \overset{\Delta}{=} k \, \varepsilon(t) \,, \quad \varepsilon_2(t) \overset{\Delta}{=} (1-k)\varepsilon(t).$$

With these choices, the equations (7.9), (7.10) become

$$(9.8) \qquad \zeta(t) = U'(\varepsilon(t);\Lambda)$$

$$(9.9) \qquad \lambda_1 T = k \, E \int_0^T \zeta(t)\varepsilon(t) dt$$

$$(9.10) \qquad E \int_0^T (\lambda_2 - \zeta(t))^+ \, dt = (1-k) \, E \int_0^T \zeta(t)\varepsilon(t) dt.$$

According to Theorem 8.1, there exists a *unique* $\Lambda \in (0,\infty)^2$ which satisfies $(9.8) - (9.10)$ and $U'(1;\Lambda) = 2$. We shall deal henceforth with this Λ, and denote the corresponding $\alpha(\Lambda) = \lambda_1/\lambda_2$ simply by α.

Suppose that $\varepsilon(t) \leq \alpha$, $\forall \, t \in [0,T]$ almost surely. Then from (9.3), (9.8) we have $\zeta(t) = \frac{\lambda_1}{\varepsilon(t)}$ and (9.9) gives $k = 1$, a contradiction. On the other hand, suppose that $\varepsilon(t) \geq \alpha$, $\forall \, t \in [0,T]$ almost surely; since $\varepsilon(\cdot)$ reaches values arbitrarily close to one with positive probability, we must have $\alpha \leq 1$. Moreover, $\zeta(t) = \frac{\lambda_1 + \lambda_2}{1 + \varepsilon(t)}$, and (9.8), (9.9) give

$$\frac{\lambda_1 T}{\lambda_1 + \lambda_2} = k \, E \int_0^T \frac{\varepsilon(t)}{1 + \varepsilon(t)} \, dt$$

which, in conjunction with (9.6), yields the contradiction $\frac{\lambda_1}{\lambda_2} = \alpha > 1$. It develops from this

analysis that *the process $\varepsilon(\cdot)$ crosses the level α during the interval* $[0,T]$, *with positive probability.*

From $(8.4) - (8.6)$ we conclude that the equilibrium coefficients of the financial market are given by

$$(9.11) \qquad r(t) = -[1_{\{\varepsilon(t)<\alpha\}} + (\tfrac{\varepsilon(t)}{1+\varepsilon(t)})^2 \, 1_{\{\varepsilon(t)\geq\alpha\}}] \, 1_{\{t\leq\tau\}}$$

$$(9.12) \qquad \theta(t) = [1_{\{\varepsilon(t)<\alpha\}} + \tfrac{\varepsilon(t)}{1+\varepsilon(t)} \, 1_{\{\varepsilon(t)\geq\alpha\}}]\} \, 1_{\{t\leq\tau\}}$$

$$(9.13) \qquad A(t) = -\frac{L_t(\alpha)}{\alpha(1+\alpha)}$$

in this case. From the preceding analysis, it develops that the process (9.13) is non–trivial.

According to (6.4) and (9.2), the optimal consumption processes are given by

$$(9.14) \qquad \hat{c}_1(t) = \varepsilon(t) \, 1_{\{\varepsilon(t)<\alpha\}} + \frac{\alpha(1+\varepsilon(t))}{1+\alpha} \, 1_{\{\varepsilon(t)\geq\alpha\}}$$

$$(9.15) \qquad \hat{c}_2(t) = \frac{\varepsilon(t) - \alpha}{1+\alpha} \, 1_{\{\varepsilon(t)\geq\alpha\}} \, .$$

9.1 Remark: It is perhaps worthwhile to note that $\{t \geq 0;\, \varepsilon(t) = \alpha\}$, the set of time–points charged by the process A of (9.13), coincides with the set of time–points at which switches from one régime to another occur in the formulae (9.14), (9.15).

This actually holds in some generality; with $\xi \equiv 0$, the process A of (8.6) charges the set $\overset{N}{\underset{n=1}{U}} \{t \geq 0;\, \varepsilon(t) = \alpha_n\}$ and is flat away from it. Now for any fixed $n \in \{1,...,N\}$ with $U'_n(0+) < \infty$, $\{t \geq 0;\, \varepsilon(t) = \alpha_n\}$ is precisely the set of time–points at which

$$\hat{c}_n(t) = I_n(\tfrac{1}{\lambda_n} \, U'(\varepsilon(t);\Lambda)),$$

the optimal consumption process for the nth agent, "switches from positive to zero value, or vice–versa" (or equivalently: the set of time–points at which the nth agent "exits from", or "enters into", the economy). It is precisely at these instances of exit or entry that the singularly continuous process A makes itself felt. (Of course, when $\varepsilon(\cdot)$ has a nonzero diffusion coefficient ρ, these "switches" are not clean; every point of the set $\{t \geq 0;\ \varepsilon(t) = \alpha_n\}$ is a cluster point, and it is not possible in general to say, at any one of these points, whether the agent is "exiting" or "entering" the economy.)

10. Appendix.

In this section we show that the condition (4.4), or equivalently

(10.1)
$$G_i \triangleq A_i - A \quad \text{has a.a. paths absolutely continuous} $$
$$\text{with respect to Lebesgue measure, } \forall\ i = 1,...,d ,$$

is *necessary for excluding arbitrage opportunities in the financial market of section 4*. The sufficiency of (10.1) in this regard follows from the inequality (5.11).

Let us start by writing the solution of equation (5.1):

$$\beta(t)X_n(t) = \int_0^t \beta(\theta)(\varepsilon_n(\theta) - c_n(\theta))d\theta + \sum_{i=1}^d \int_0^t \beta(\theta)\pi_{ni}(\theta)dG_i(\theta)$$

(10.2)

$$+ \int_0^t \beta(\theta)\pi_n^*(\theta)(b(\theta) - r(\theta)\underset{\sim}{1})d\theta + \int_0^t \beta(\theta)\pi_n^*(\theta)\sigma(\theta)dW(\theta); \quad t \geq 0.$$

For any given function $\xi : [0,\infty) \to \mathbb{R}$ of bounded variation, let us denote by $\check{\xi}(t)$ its total variation on the interval $[0,t]$. We define

$$(10.3) \qquad C(t) \triangleq t + \sum_{i=1}^{d} \check{F}_i(t) , \qquad F_i(t) \triangleq G_i(t) + \int_0^t (b_i(\theta) - r(\theta)) d\theta ; \quad t \geq 0$$

$$(10.4) \qquad\qquad T(s) \triangleq \inf\{t \geq 0; \ C(t) > s\} ; \quad s \geq 0.$$

10.1 Lemma: (i) For every fixed $s \geq 0$, $T(s)$ is a stopping time of $\{\mathscr{F}_t\}$; the resulting filtration

$$(10.5) \qquad\qquad \{\mathscr{G}_s\} = \{\mathscr{F}_{T(s)}\}, \quad s \geq 0$$

satisfies the usual conditions.

(ii) Almost every path of the process $\{T(s); s \geq 0\}$ is absolutely continuous with respect to Lebesgue measure and is strictly increasing.

(iii) For every $i = 1,...,d$, almost every path of the process $\{\tilde{F}_i(s) \triangleq F_i(T(s)); s \geq 0\}$ is absolutely continuous with respect to Lebesgue measure.

Proof: For (i), cf. Karatzas & Shreve (1988), Exercise 3.4.4 and Problem 3.4.5. For (ii) and (iii), we have from (10.3) almost surely: $C(t_2) - C(t_1) \geq \max(t_2 - t_1, \check{F}_i(t_2) - \check{F}_i(t_1))$, $\forall \ 0 \leq t_1 \leq t_2$. Therefore, for given $0 \leq s_1 < s_2$:

$$s_2 - s_1 = C(T(s_2)) - C(T(s_1)) \geq \max(T(s_2) - T(s_1), \check{F}_i(T(s_2)) - \check{F}_i(T(s_1))).$$

The conclusions on absolute continuity follow now easily from this.

□

Consequently, we can write

$$(10.6) \qquad\qquad T(s) = \int_0^S T'(\nu) d\nu , \quad \tilde{F}_i(s) = \int_0^S \tilde{F}_i'(\nu) d\nu$$

where T', $\tilde{F}'_i$ are $\{\mathcal{G}_s\}$–progressively measurable, locally integrable processes.

On the other hand, the processes $\tilde{\beta}(s) \triangleq \beta(T(s))$, $\tilde{X}(s) \triangleq X_n(T(s))$, $\tilde{\varepsilon}(s) \triangleq \varepsilon_n(T(s))$, $\tilde{c}(s)$ $\triangleq c_n(T(s))$, $\tilde{\pi}(s) \triangleq \pi_n(T(s))$, $\tilde{b}(s) \triangleq b(T(s))$, $\tilde{r}(s) \triangleq r(T(s))$, $\tilde{\sigma}(s) \triangleq \sigma(T(s))$, $\tilde{M}(s) \triangleq W(T(s))$ are all $\{\mathcal{G}_s\}$–progressively measurable. In terms of them, we have the following time–changed version of equation (10.2):

(10.7)
$$\tilde{\beta}(s)\tilde{X}(s) = \int_0^s \tilde{\beta}(\nu)(\tilde{\varepsilon}(\nu) - \tilde{c}(\nu))T'(\nu)d\nu + \sum_{i=1}^d \int_0^s \tilde{\beta}(\nu)\tilde{\pi}_i(\nu)\tilde{F}'_i(\nu)d\nu$$

$$+ \int_0^s \tilde{\beta}(\nu)\tilde{\pi}^*(\nu)\tilde{\sigma}(\nu)d\tilde{M}(\nu) \ ; \ s \geq 0.$$

The process $\{\tilde{M}(s), \mathcal{G}_s; s \geq 0\}$ is a martingale on $(\Omega, \mathcal{F}, P)$ with quadratic variation $T(s)$; therefore, there exists a Brownian motion $\tilde{B}$ on this space (possibly extended, to accommodate an independent, one–dimensional Brownian motion process), such that

(10.8)
$$\tilde{M}(s) = \int_0^s \sqrt{T'(\nu)} \, d\tilde{B}(\nu) \ ; \ s \geq 0$$

(Karatzas & Shreve (1988), Theorem 3.4.2).

Let us take now $\tilde{c}(s) \triangleq 0$, $\tilde{\pi}_i(s) \triangleq k \, \mathrm{sgn}(\tilde{F}'_i(s)) \cdot 1_{\{T'(s)=0\}}$, $s \geq 0$, for some finite constant $k > 0$. The process $\tilde{\pi}_i$ is bounded and $\{\mathcal{G}_s\}$–progressively measurable, and thus the process $\pi_{ni}(t) \triangleq \tilde{\pi}_i(C(t))$ is bounded and $\{\mathcal{F}_t\}$–progressively measurable. If X_n is the wealth process corresponding to consumption $c_n \equiv 0$ and portfolio $\pi_n = (\pi_{n1}, \ldots, \pi_{nd})^*$ as above, the time–changed version $\tilde{X}(s) = X_n(T(s))$ is given, thanks to (10.7) and (10.8), by

(10.9) $$\tilde{\beta}(s)\tilde{X}(s) = \int_0^s \tilde{\beta}(\nu)\tilde{\varepsilon}(\nu)T'(\nu)d\nu + k \int_0^s \tilde{\beta}(\nu) \sum_{i=1}^d |\tilde{F}'_i(\nu)| \, 1_{\{T'(\nu)=0\}}d\nu.$$

Now suppose that we have, for some $i = 1,...,d$: $\text{meas}\{s \geq 0; \tilde{F}'_i(s,\omega) \neq 0$ and $T'(s,\omega) = 0\} > 0$, for every ω in some event of positive probability (here and below, "meas" stands for "Lebesgue measure"). Then by selecting $k > 0$ sufficiently large, we can make $X(\cdot)$ a.s. nonnegative, and arbitrarily large with positive probability. In order to exclude this "arbitrage possibility", we must have

$$(10.10) \qquad \text{meas}\{s \geq 0; \tilde{F}'_i(s,\omega) \neq 0 \text{ and } T'(s,\omega) = 0\} = 0; \quad \forall \, \omega \in \Omega^*, \; i = 1,...,d$$

for some event Ω^* with $P(\Omega^*) = 1$.

10.2 Lemma: (10.10) implies (10.1).

Proof: Fix $\omega \in \Omega^*$ and $\varepsilon > 0$; then there is a $\delta > 0$ such that
$$\sum_{j=1}^{m} [C^{ac}(t'_j,\omega) - C^{ac}(t_j,\omega)] < \varepsilon,$$ for every finite collection of non–overlapping intervals $\{(t_j,t'_j)\}_{j=1}^{m}$ in $[0,T]$ with $\sum_{j=1}^{m}(t'_j - t_j) < \delta$. (Here and in the sequel, the superscript "ac" denotes the absolutely continuous part.) Then for every $i = 1,...,d$, the quantity

$$\sum_{j=1}^{m} |F_i(t'_j,\omega) - F_i(t_j,\omega)| = \sum_{j=1}^{m} |\tilde{F}_i(C(t'_j,\omega),\omega) - \tilde{F}_i(C(t_j,\omega),\omega)|$$

$$= \sum_{j=1}^{m} \left| \int_{C(t_j,\omega)}^{C(t'_j,\omega)} \tilde{F}'_i(\nu,\omega)d\nu \right|$$

$$= \sum_{j=1}^{m} \left| \int_{C(t_j,\omega)}^{C(t'_j,\omega)} 1_{\{T'(\nu,\omega)>0\}} \tilde{F}'_i(\nu,\omega)d\nu \right|$$

(thanks to (10.10)) can be made arbitrarily small, because it amounts to integrating the

integrable function $\tilde{F}_i'(\cdot,\omega)$ over a set with Lebesgue measure

$$\sum_{j=1}^{m} \int_{C(t_j,\omega)}^{C(t_j',\omega)} 1_{\{T'(\nu,\omega)>0\}} \, d\nu = \sum_{j=1}^{m} \int_{C(t_j,\omega)}^{C(t_j',\omega)} C'(T(\nu,\omega))T'(\nu,\omega)d\nu$$

$$= \sum_{j=1}^{m} \int_{t_j}^{t_j'} C'(\theta,\omega)d\theta = \sum_{j=1}^{m} [C^{ac}(t_j',\omega) - C^{ac}(t_j,\omega)] < \varepsilon.$$

Thus the function $F_i(\cdot,\omega)$ is absolutely continuous with respect to Lebesgue measure, and by (10.3) the same is true for the function $G_i(\cdot,\omega)$, for every $i = 1,\dots,d$.

□

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