

EQUIVALENT MARTINGALE MEASURES AND OPTIMAL MARKET COMPLETIONS

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Abstract

Optimal fictitious completions of an incomplete financial market are shown to be associated with exponential *martingales* (not just local martingales) and, therefore, to "an optimal equivalent martingale measure". Results of independent interest, in the theory of continuous-time martingales, are derived as well.

1. INTRODUCTION

This note is a sequel to our recent article Karatzas, Lehoczky, Shreve & Xu [3] - hereafter abbreviated KLSX. It answers affirmatively an interesting, and potentially also important, question raised in that article. In the course of settling this question of mathematical finance, results of independent interest in the theory of continuous-time martingales are obtained as well.

The issue is the following: it is shown in KLSX that the problem of finding a portfolio which maximizes expected utility from terminal wealth in an incomplete financial market \mathfrak{M} , is equivalent to finding a fictitious completion \mathfrak{M}_λ of this market with a certain *minimality property*, in a suitably parametrized family $\{\mathfrak{M}_\nu; \nu \in K(\sigma)\}$ of fictitious completions. Now each such \mathfrak{M}_ν corresponds to an exponential local martingale Z_ν . *For the optimal fictitious completion \mathfrak{M}_λ , is the associated Z_λ a martingale?* It is this question that is being answered affirmatively in the present note (Theorem 3.2), actually in a slightly more general setting.

We recall in section 2 the basic model of KLSX, and introduce the necessary notation. Section 3 contains the abovementioned main result, and section 4 presents its ramifications. Proofs of auxiliary probabilistic results, some of them of independent interest, are collected in section 5.

2. THE MODEL

Let us recall the setting of KLSX, which is that of an *incomplete financial market* \mathfrak{M} with $m+1$ assets. One of them (the "bond") has price $P_0(t)$ governed by

$$(2.1) \quad dP_0(t) = P_0(t)r(t)dt, \quad P_0(0)=1,$$

whereas the remaining m "stocks" have prices-per-share $P_i(t)$ given by

$$(2.2) \quad dP_i(t) = P_i(t) \left[b_i(t)dt + \sum_{j=1}^d \sigma_{ij}(t) dW_j(t) \right], \quad i=1, \dots, m.$$

Here $W=(W_1, \dots, W_d)^*$ is an \mathbb{R}^d -valued Brownian motion, on a complete probability space $(\Omega, \mathfrak{F}, P)$, where \mathfrak{F} is the P -completion of the σ -field generated by W . We shall denote by $\{\mathfrak{F}_t^W\}$ the filtration generated by W and augmented by P -null events. The *interest rate* $r(t)$, the vector of *stock appreciation rates* $b(t)=(b_1(t), \dots, b_m(t))^*$ and the *volatility matrix* $\sigma(t) = \{\sigma_{ij}(t)\}_{\substack{1 \leq i \leq m \\ 1 \leq j \leq d}}$ are progressively measurable with respect to $\{\mathfrak{F}_t^W\}$. It is

assumed that $d \geq m$ (the number of sources of uncertainty is at least as large as the number of stocks available for investment), and that the matrix $\sigma(t)$ has full row rank for every (t, ω) . For simplicity, we shall also suppose that the processes $r(t)$, $b(t)$ and

$$(2.3) \quad \theta(t) = \sigma^*(t) (\sigma(t)\sigma^*(t))^{-1} [b(t)-r(t)1]$$

are *bounded* (uniformly in (t, ω)), where 1 denotes the m -dimensional vector with all components equal to unity. All economic activity is supposed to take place on a finite horizon $[0, T]$.

For a small investor (whose actions cannot influence the market prices),

a portfolio rule $\pi(\cdot)$ is an \mathbb{R}^m -valued, (\mathfrak{F}_t^W) -adapted process with

$$\int_0^T \|\sigma^*(t)\pi(s)\|^2 ds < \infty \quad \text{almost surely,}$$

whose components $\pi_i(t)$ represent the proportions of the investor's wealth invested in the corresponding stock $i=1, \dots, m$ at time t . The wealth process $X^{X, \pi}(\cdot)$, corresponding to a portfolio rule $\pi(\cdot)$ and a given initial capital $x > 0$, is given then by

$$\begin{aligned} dX^{X, \pi}(t) &= X^{X, \pi}(t) \left[\left\{ 1 - \sum_{i=1}^m \pi_i(t) \right\} r(t) dt + \sum_{i=1}^m \pi_i(t) \{ b_i(t) dt + \sum_{j=1}^d \sigma_{ij}(t) dW_j(t) \} \right] \\ (2.4) \qquad &= X^{X, \pi}(t) [r(t) dt + \pi^*(t) \sigma(t) dW_0(t)] \quad ; \quad X^{X, \pi}(0) = x > 0 , \end{aligned}$$

where

$$(2.5) \qquad W_0(t) := W(t) + \int_0^t \theta(s) ds , \qquad \beta(t) := \exp \left\{ - \int_0^t r(s) ds \right\} .$$

The solution to (2.4) is given by

$$(2.6) \qquad \beta(t) X^{X, \pi}(t) = x \cdot \exp \left\{ \int_0^t \pi^*(s) \sigma(s) dW_0(s) - (1/2) \int_0^t \|\sigma^*(s)\pi(s)\|^2 ds \right\} .$$

The optimization problem considered in KLSX was the following: for a given utility function $U: (0, \infty) \rightarrow \mathbb{R}$ (of class C^1 , strictly increasing and strictly concave, with $\lim_{x \rightarrow \infty} U'(x) = 0$ and $\lim_{x \downarrow 0} U'(x) = \infty$), to find a portfolio rule $\hat{\pi}(\cdot)$ which achieves the supremum in

$$(2.7) \qquad V(x) = \sup_{\pi \in \mathcal{A}(x)} EU(X^{X, \pi}(T)) ,$$

i.e., which maximizes expected utility from terminal wealth, amongst all portfolio rules $\pi(\cdot)$ that satisfy $EU^-(X^{X, \pi}(T)) < \infty$ (these rules constitute the class $\mathcal{A}(x)$).

The case of a *complete market* (with $m=d$) has the distinctive feature that "every contingent claim is attainable" ; in other words, for every positive, \mathcal{F}_T^W - measurable random variable B , there exists a level of initial capital $x = x_B > 0$ and an associated portfolio rule $\pi(\cdot)$ such that $X^{x,\pi}(T)=B$, a.s. In fact, with the help of the exponential martingale

$$(2.8) \quad Z_0(t) = \exp \left[- \int_0^t \theta^*(s) dW(s) - (1/2) \int_0^t ||\theta(s)||^2 ds \right],$$

the "price of the contingent claim B at time $t=0$ " is given by the Black & Scholes - type formula $x_B = E[\beta(T)Z_0(T)B]$.

Thanks to this feature, the optimization problem of (2.7) is relatively straightforward to solve in the complete case: the optimal level of terminal wealth is given by

$$(2.9) \quad \xi_0^x = I(y_0(x)\beta(T)Z_0(T)),$$

and there exists a portfolio $\hat{\pi}(\cdot) \in \mathcal{A}(x)$, which achieves the supremum in (2.7) and for which

$$(2.10) \quad X^{x,\hat{\pi}}(T) = \xi_0^x, \quad \text{a.s.}$$

In (2.9) and in the sequel, $I \equiv (U')^{-1}$ is the inverse of the continuous, strictly decreasing function U' , and $y_0 \equiv X_0^{-1}$ is the inverse of the continuous, strictly decreasing function

$$(2.11) \quad X_0(y) \doteq E \left[\beta(T)Z_0(T) I(y\beta(T)Z_0(T)) \right], \quad 0 < y < \infty$$

(under the assumption $X_0(y) < \infty$, for all $y \in (0, \infty)$).

□

In an *incomplete market* (with $m < d$) the analysis becomes much more involved: not every contingent claim is attainable, and the problem of (2.7) is highly non-trivial. In KLSX, our approach to incomplete markets (inspired in part by He & Pearson [1] and Xu [4]) was to consider a family of *fictitiously completed markets* $\{\mathfrak{M}_\nu; \nu \in K(\sigma)\}$, created from \mathfrak{M} by adding to (2.1), (2.2) the fictitious stocks with prices

$$(2.12) \quad dP_k^{(\nu)}(t) = P_k^{(\nu)}(t) \left[b_k^{(\nu)}(t) dt + \sum_{j=1}^d \rho_{kj}(t) dW_j(t) \right]; \quad k=1, \dots, d-m.$$

Here $\rho(t)$ is a fixed $(d-m) \times d$ matrix-valued process, bounded and $\{\mathcal{F}_t^W\}$ -progressively measurable, with orthonormal rows and $\sigma(t)\rho^*(t) \equiv 0$. The components of the \mathbb{R}^{d-m} -valued process $b^{(\nu)}(t) := \rho(t)\nu(t) + r(t)1$ give the appreciation rates of these additional stocks in terms of an \mathbb{R}^d -valued, $\{\mathcal{F}_t^W\}$ -progressively measurable process $\nu(\cdot)$ which satisfies

$$\int_0^T \|\nu(s)\|^2 ds < \infty \quad \text{and} \quad \sigma(t)\nu(t) = 0, \quad \text{for all } 0 \leq t \leq T$$

almost surely. We denote by $K(\sigma)$ the space of such processes. For a given market completion \mathfrak{M}_ν , the analogues of (2.3), (2.8) become $\theta_\nu(t) := \theta(t) + \nu(t)$ and

$$(2.13) \quad Z_\nu(t) := \exp \left[-\int_0^t (\theta(s) + \nu(s))^* dW(s) - (1/2) \int_0^t (\|\theta(s)\|^2 + \|\nu(s)\|^2) ds \right],$$

respectively. Notice that $Z_\nu(\cdot)$ is, in general, only a (nonnegative) local martingale, hence a supermartingale.

3. THE RESULT

The following result was proved in KLSX (Theorem 8.5); it provides a sufficient condition for attainability in the incomplete market \mathfrak{M} .

3.1 THEOREM (KLSX) : Suppose that, for a given positive, \mathfrak{F}_T^W - measurable random variable B , there exists a process $\lambda \in K(\sigma)$ for which

$$(3.1) \quad E[\beta(T)Z_\nu(T)B] \leq E[\beta(T)Z_\lambda(T)B], \quad \text{for all } \nu \in K(\sigma).$$

Then there exists a portfolio rule $\pi(\cdot)$, such that $X^{x,\pi}(T) = B$ holds almost surely, with $x := E[\beta(T)Z_\lambda(T)B]$.

□

The purpose of this note is to complement Theorem 3.1 with the following result.

3.2 THEOREM : *In Theorem 3.1, the process $Z_\lambda(\cdot)$ is a martingale.* □

From Theorem 3.2 we can conclude that the "optimal" fictitious completion \mathfrak{M}_λ of the original incomplete market \mathfrak{M} , corresponding to the process $\lambda \in K(\sigma)$ that satisfies the equivalent conditions discussed in section 4, is associated with an exponential local martingale $Z_\lambda(\cdot)$ as in (2.13) that is actually a *martingale*. Thus, one has the so-called "equivalent martingale measure"

$$\tilde{P}_\lambda(A) := E[Z_\lambda(T) \cdot 1_A], \quad A \in \mathfrak{F}_T^W.$$

We present in this section our approach to Theorem 3.2; this is based on two "key" probabilistic results, Proposition 3.4 and Theorem 3.5, which are of independent interest. The proofs of all auxiliary results are collected in

section 5.

Let us denote by $S[0,T]$ the space of $\{\tilde{y}_t\}$ - progressively measurable processes with values in \mathbb{R}^d that are square-integrable on $[0,T]$ almost surely, and introduce the subspace

$$(3.2) \quad K^\perp(\sigma) \doteq \{\varphi \in S[0,T]; \varphi(t) \in \text{Range}(\sigma^*(t)) \text{ for all } 0 \leq t \leq T, \text{ a.s.}\},$$

which is the orthogonal complement of

$$(3.3) \quad K(\sigma) \doteq \{\nu \in S[0,T]; \sigma(t)\nu(t) = 0 \text{ for all } 0 \leq t \leq T, \text{ a.s.}\}.$$

For every $\psi \in S[0,T]$, the process

$$(3.4) \quad \zeta_\psi(t) \doteq \exp\left[-\int_0^t \psi^*(s) dW(s) - (1/2) \int_0^t \|\psi(s)\|^2 ds\right], \quad 0 \leq t \leq T$$

is a local martingale of $\{\tilde{y}_t^W\}$. With this notation, and from the fact that the process θ of (2.3) belongs to $K^\perp(\sigma)$, we can write the local martingale Z_ν of (2.13) in the form

$$(3.5) \quad Z_\nu = \zeta_\theta \cdot \zeta_\nu, \quad \text{for every } \nu \in K(\sigma).$$

3.3 PROPOSITION : Under the assumptions of Theorem 3.1, there exists a process $\varphi \in K^\perp(\sigma)$ such that the product $\zeta_\varphi \cdot \zeta_\lambda$ is a martingale. \square

3.4 PROPOSITION : Consider a continuous, strictly increasing and $\{\tilde{y}_t^W\}$ - adapted process $\{\tau(t); 0 \leq t \leq T\}$, with $\lim_{t \uparrow T} \tau(t) = \infty$ and $\tau(0) = 0$ almost surely, and denote by

$$(3.6) \quad A(s) = \inf\{t \geq 0; \tau(t) > s\}, \quad 0 \leq s < \infty$$

its inverse. Then the process

$$(3.7) \quad M_s \doteq W_{A(s)}; \quad 0 \leq s < \infty$$

is an \mathbb{R}^d -valued martingale with respect to the filtration

$$(3.8) \quad \{\mathcal{G}_s\} \doteq \tilde{y}_{A(s)}^W,$$

with quadratic variations

$$\langle M^i, M^k \rangle_s = A(s) \cdot \delta_{ik} .$$

Furthermore, $\{\mathfrak{F}_{A(s)}^W\}$ is the augmentation by null sets of the filtration generated by M , i.e.,

$$(3.9) \quad \mathfrak{F}_s^M = \mathfrak{F}_{A(s)}^W, \quad 0 \leq s < \infty. \quad \square$$

3.5 THEOREM : Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space and X a continuous, \mathbb{R}^m -valued process defined on $[0, \infty]$, with continuous sample paths. Assume that \mathfrak{F} is the P -completion of the σ -field generated by X , and denote by $\{\mathfrak{F}_t^X\}$ the augmentation of the filtration generated by X .

Consider two $\{\mathfrak{F}_t^X\}$ -adapted, strictly positive processes M_1, M_2 with continuous sample paths, defined on $[0, \infty]$. Suppose that $M_i = \{M_i(t), \mathfrak{F}_t^X; 0 \leq t < \infty\}$, $i=1,2$ are martingales, and that the product $M_1 \cdot M_2$ is an $\{\mathfrak{F}_t^X\}$ -martingale with last element $M_1(\infty) \cdot M_2(\infty)$; i.e.,

$$(3.10) \quad M_1(t)M_2(t) = E[M_1(\infty)M_2(\infty) | \mathfrak{F}_t^X], \quad \text{a.s.}$$

for every $t \in [0, \infty]$. Then $M_1(\infty)$, $M_2(\infty)$ are last elements for the martingales M_1 and M_2 , respectively:

$$(3.11) \quad M_i(t) = E[M_i(\infty) | \mathfrak{F}_t^X], \quad \text{a.s.}$$

for every $0 \leq t < \infty$, $i=1,2$. □

In other words, if M_1 and M_2 are martingales on $[0, \infty)$, positive on $[0, \infty]$, and their product $M_1 \cdot M_2$ is a martingale all the way up to $t=\infty$, then M_1 and M_2 have this property as well; neither of them can be "defective" at infinity.

The above three results can be used to prove Theorem 3.2, as follows.

Proof of Theorem 3.2: Let φ be as in Proposition 3.3, and consider the time-change

$$(3.12) \quad \tau(t) := \frac{t}{T-t} + \int_0^t (|\varphi(s)|^2 + |\lambda(s)|^2) ds, \quad 0 \leq t < T$$

which satisfies the requirements of Proposition 3.4, and let $A(\cdot)$ be its inverse as in (3.6). Define the processes

$$(3.13) \quad \tilde{W}(s) := W(A(s)), \quad \tilde{\zeta}_\varphi(s) := \zeta_\varphi(A(s)), \quad \tilde{\zeta}_\lambda(s) := \zeta_\lambda(A(s)); \quad 0 \leq s < \infty.$$

From Propositions 3.4 and 3.3:

$$(3.14) \quad \{\mathcal{G}_s\} := \{\mathfrak{F}_{A(s)}^W\} = \{\mathfrak{F}_s^{\tilde{W}}\},$$

$$(3.15) \quad \left\{ \begin{array}{l} \tilde{\zeta}_\varphi, \tilde{\zeta}_\lambda \text{ are defined and positive on } [0, \infty], \\ \text{and are } \{\mathcal{G}_s\}\text{-martingales on } [0, \infty) \end{array} \right\},$$

$$(3.16) \quad \tilde{\zeta}_\varphi \cdot \tilde{\zeta}_\lambda \text{ is a } \{\mathcal{G}_s\}\text{-martingale on } [0, \infty].$$

From (3.13)-(3.16) and Theorem 3.5, we obtain that $\tilde{\zeta}_\lambda$ is a $\{\mathcal{G}_s\}$ -martingale on $[0, \infty]$, and thus $E[\zeta_\lambda(T)] = E[\tilde{\zeta}_\lambda(\infty)] = E[\tilde{\zeta}_\lambda(0)] = 1$. By Girsanov's theorem, the process

$$W_\lambda(t) := W(t) + \int_0^t \lambda(s) ds, \quad 0 \leq t \leq T$$

is then an $\{\mathfrak{F}_t^W\}$ -Brownian motion under the probability measure $P_\lambda(A) := E[\zeta_\lambda(T) 1_A]$, $A \in \mathfrak{F}_T^W$, and from (3.4) and the orthogonality of θ and λ :

$$\zeta_\theta(t) := \exp \left[- \int_0^t \theta^*(s) dW_\lambda(s) - (1/2) \int_0^t |\theta(s)|^2 ds \right], \quad 0 \leq t \leq T.$$

But θ is bounded, and thus ζ_θ is a P_λ -martingale. We obtain then from (3.5):

$$E[Z_\lambda(T)] = E[\zeta_\theta(T) \cdot \zeta_\lambda(T)] = E^{P_\lambda}[\zeta_\theta(T)] = 1,$$

which proves that Z_λ is a martingale. □

4. IMPLICATIONS

For the analogue of the optimization problem of (2.7) in the complete market \mathfrak{M}_ν , $\nu \in K(\sigma)$, the optimal level of terminal wealth becomes

$$(4.1) \quad \xi_\nu^X = I(y_\nu(x)\beta(T)Z_\nu(T)),$$

where $y_\nu := (X_\nu)^{-1}$ is the inverse of the continuous, strictly decreasing function

$$(4.2) \quad X_\nu(y) := E\left[\beta(T)Z_\nu(T) I(y\beta(T)Z_\nu(T))\right], \quad 0 < y < \infty,$$

provided this latter is finite (by analogy with (2.9),(2.11)).

Given any process λ in $K_1(\sigma) := \{\nu \in K(\sigma); X_\nu(y) < \infty \text{ for all } 0 < y < \infty\}$, the random variable ξ_λ^X will be the optimal level of terminal wealth for the problem (2.7) in the original, incomplete market \mathfrak{M} , if the corresponding portfolio (which achieves ξ_λ^X as its terminal wealth) invests in the original $m+1$ assets only:

$$(4.3) \quad \exists \hat{\pi} \in \mathcal{A}(x) \quad \text{s.t.} \quad X^{X, \hat{\pi}}(T) = \xi_\lambda^X, \quad \text{a.s.}$$

But this portfolio will be admissible in any other fictitious completion \mathfrak{M}_ν , whence the *minimality property*

$$(4.4) \quad EU(\xi_\lambda^X) \leq EU(\xi_\nu^X), \quad \text{for any } \nu \in K_1(\sigma),$$

referred to in the introduction.

It was shown in KLSX that the properties (4.3),(4.4) are equivalent to each other, as well as to the additional properties

$$(4.5) \quad E \tilde{U}(y_\lambda(x)\beta(T)Z_\lambda(T)) \leq E \tilde{U}(y_\nu(x)\beta(T)Z_\nu(T)), \quad \text{for all } \nu \in K_1(\sigma),$$

$$(4.6) \quad E[\beta(T)Z_\nu(T)\xi_\lambda^X] \leq E[\beta(T)Z_\lambda(T)\xi_\lambda^X], \quad \text{for all } \nu \in K_1(\sigma).$$

(In (4.5), $\tilde{U}(y) := \max_{x>0} \{U(x)-xy\}$, $0 < y < \infty$ is the convex dual of the concave function U .)

Furthermore, if there exists a process $\lambda \in K_1(\sigma)$ satisfying any one of (4.3)-(4.6), then the portfolio $\hat{\pi}$ of (4.3) is optimal for the incomplete market optimization problem of (2.7).

The existence of such a λ (and $\hat{\pi}$) was established in KLSX under "reasonable" conditions on the utility function U , using convex duality methods. From (4.6) and Theorem 3.2, it follows that the exponential local martingale Z_λ , corresponding to this $\lambda \in K(\sigma)$, is actually a *martingale*.

5. PROOFS

We collect in this section the proofs of Propositions 3.3, 3.4 and Theorem 3.5.

Proof of Proposition 3.3 : From the proof of Theorem 8.5 in KLSX, we know that $\beta X^{x,\pi} Z_\lambda$ is a *martingale*; here x , π and λ are as in the statement of the proof of Theorem 3.1. Now from (2.6),(2.5) and

$$(3.5) \quad Z_\lambda(t) = \zeta_\lambda(t) \cdot \exp \left[- \int_0^t \theta^*(s) dW(s) - (1/2) \int_0^t \|\theta(s)\|^2 ds \right], \quad 0 \leq t \leq T,$$

we deduce that, with $\varphi := \theta - \sigma^* \pi$, the process $x \cdot \zeta_\varphi(t) \zeta_\lambda(t)$ is equal to $\beta(t) X^{x,\pi}(t) Z_\lambda(t)$, a martingale. □

Proof of Proposition 3.4 : The first two claims follow by standard arguments from the Optional Sampling Theorem (cf. [2]; section 3.4, §§ B,C); only (3.9) needs discussion. Let us start by observing that, because M is a (continuous) martingale with respect to $\{\mathcal{G}_s\}$, it is a martingale with respect to its own augmented filtration $\{\mathfrak{F}_s^M\}$. By Theorem 3.4.6 of [2], $W(t) = M_{\tau(t)}$ is a Brownian motion with respect to $\{\mathfrak{F}_{\tau(t)}^M\}$ (as well as relative to the possibly smaller filtration $\{\mathfrak{F}_t^W\}$). But it is well-known that W has the martingale representation property for its own filtration $\{\mathfrak{F}_t^W\}$ (cf. [2], pp. 182-184), and thus by Corollary 4.1 in [5]: $\{\mathfrak{F}_t^W\} = \{\mathfrak{F}_{\tau(t)}^M\}$.

It follows then that M has the martingale representation property for its own filtration $\{\mathfrak{F}_s^M\}$ (cf. [5], Theorem 2). On the other hand, M is a local martingale with respect to the possibly larger filtration $\{\mathcal{G}_s\}$. By applying Corollary 4.1 of [5] once more, we deduce $\{\mathfrak{F}_s^M\} = \{\mathcal{G}_s\}$. □

For the proof of Theorem 3.5, we shall need the following observations and notation.

To begin with, let us assume for concreteness that $M_1(0)=M_2(0)=1$. From Fatou's lemma it follows easily that M_1, M_2 are supermartingales on $[0, \infty]$. Let us denote by

$$(5.1) \quad \mathfrak{F}_t^0 \doteq \sigma(X(s); 0 \leq s \leq t), \quad 0 \leq t \leq \infty$$

the natural (non-augmented) filtration generated by the process X . Then with

$$(5.2) \quad \mathfrak{F}_\infty^0 \doteq \sigma \left(\bigcup_{0 \leq t < \infty} \mathfrak{F}_t^0 \right),$$

\mathfrak{F} is the P -completion of \mathfrak{F}_∞^0 , and the filtration $\{\mathfrak{F}_t^X\}$ of Theorem 3.5 is just the P -augmentation of the filtration $\{\mathfrak{F}_t^0\}$.

Using the martingale property of the process M_1 on $[0, \infty)$ and the Kolmogorov consistency theorem, one constructs (as in [2], p.192) a probability measure \hat{P} on \mathfrak{F}_∞^0 satisfying

$$(5.3) \quad \hat{P}(B) = E[M_1(t) \cdot 1_B] \quad \text{for any } B \in \mathfrak{F}_t^0, \quad 0 \leq t < \infty.$$

It can be shown (see proofs below) that

$$(5.4) \quad \hat{P} \ll P, \quad \text{on } \mathfrak{F}_\infty^0$$

$$(5.5) \quad \hat{P}(A) \geq \tilde{P}(A) := E[M(\infty) \cdot 1_A], \quad \text{for any } A \in \mathfrak{F}_\infty^0.$$

In particular, (5.4) shows that \hat{P} has a unique extension, also denoted by \hat{P} , to \mathfrak{F} . From (5.5) we obtain that

$$(5.5)' \quad \hat{E}(\xi) \geq \tilde{E}(\xi)$$

holds for any positive, \mathfrak{F} -measurable random variable ξ (where \hat{E} and \tilde{E} denote expectations with respect to \hat{P} and \tilde{P} , respectively).

Proof of Theorem 3.5 : We would like to show that the two measures \hat{P}, \tilde{P} of (5.3), (5.5) actually agree on \mathfrak{F} ; from this would follow $EM_1(\infty) = \tilde{P}(\Omega) = \hat{P}(\Omega) = 1$, whence M_1 would be a martingale on $[0, \infty]$. The same method would prove that M_2 is also a martingale on $[0, \infty]$.

To this end, and in view of (5.5), it suffices to exhibit a positive, \mathfrak{F} -measurable random variable h , such that $\hat{E}(h) = \tilde{E}(h)$. It turns out that $h = M_2(\infty)$ is the right choice. Indeed, using successively the assumption (3.10), the definition in (5.5), property (5.5)', Fatou's lemma, (5.3), and (3.10) again, we get

$$1 = E[M_1(\infty)M_2(\infty)] = \tilde{E}M_2(\infty) \leq \hat{E}M_2(\infty) \leq \liminf_{t \rightarrow \infty} \hat{E}M_2(t) = \lim_{t \rightarrow \infty} E[M_1(t)M_2(t)] = 1,$$

and thus $\hat{E}M_2(\infty) = \tilde{E}M_2(\infty) = 1$, as promised. \square

Proof of (5.4) : Take a P -null set $A \in \mathfrak{F}_\infty^0$ and a double sequence $\left\{ \{A_{nk}\}_{k=1}^\infty \right\}_{n=1}^\infty$ of events in the field $\bigcup_{0 \leq t < \infty} \mathfrak{F}_t^0$ which generates \mathfrak{F}_∞^0 , such that

$$(5.6) \quad A \subseteq B_n := \bigcup_{k=1}^\infty A_{nk}, \quad \sum_{k=1}^\infty P(A_{nk}) \leq 2^{-n}, \quad \text{for all } n \geq 1.$$

For every $K \geq 1$ we obtain, for a suitable $S = S_K$ in $(0, \infty)$ and $B_{nK} := \bigcup_{k=1}^K A_{nk}$:

$$(5.7) \quad 2^{-n} \geq P(B_n) \geq P\left(\bigcup_{k=1}^K A_{nk}\right) = \hat{E}\left[(M_1(S))^{-1} 1_{B_{nK}}\right] \geq \hat{E}\left[(M_1(\infty))^{-1} 1_{B_{nK}}\right].$$

We have used the fact that, under the probability measure \hat{P} , the process $1/M_1$ is a martingale on $[0, \infty)$ and a supermartingale on $[0, \infty]$. Now let K tend to infinity in (5.7) to obtain, by monotone convergence and in conjunction with (5.6):

$$\hat{E}\left[\left(M_1(\infty)\right)^{-1} 1_A\right] \leq \hat{E}\left[\left(M_1(\infty)\right)^{-1} 1_{B_n}\right] \leq 2^{-n}, \quad \text{for all } n \geq 1.$$

Because $M_1(\infty)$ is a.s. positive and finite, this last relation shows $\hat{P}(A) = 0$. \square

Proof of (5.5) : If A belongs to \mathfrak{F}_t^0 for some $t \in (0, \infty)$, then (5.5) is just the supermartingale property of M_1 on $[0, \infty]$. Now take an arbitrary $A \in \mathfrak{F}_\infty^0$; for

every $\varepsilon > 0$ there exists a sequence $\{A_n\}_{n=1}^{\infty}$ in $\bigcup_{0 \leq t < \infty} \mathfrak{F}_t^0$ such that $A \subseteq \bigcup_{k=1}^{\infty} A_k$

and $\sum_{k=1}^{\infty} \hat{P}(A_k) \leq \hat{P}(A) + \varepsilon$. For any integer $N \geq 1$, we have from the above obser-

vation, with $C_m := \bigcup_{k=1}^m A_k$ ($1 \leq m \leq \infty$): $E \left[M_1(\infty) \cdot 1_{C_N} \right] \leq \hat{P}(C_N)$.

Letting N increase to infinity, we obtain:

$$E[M_1(\infty) \cdot 1_A] \leq E[M_1(\infty) \cdot 1_{C_{\infty}}] \leq \hat{P}(C_{\infty}) \leq \sum_{k=1}^{\infty} \hat{P}(A_k) \leq \hat{P}(A) + \varepsilon.$$

But this holds for every $\varepsilon > 0$, and (5.5) follows. □

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