WEAKLY LIPSCHITZIAN MAPPINGS AND RESTRICTED UNIQUENESS OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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I. Introduction

Our goal in this paper is to provide conditions on a function $f:D\to\mathbb{R}^n$, with D a subset of $\mathbb{R}^{n+1}$, that are weaker than one–sided Lipschitz continuity or monotonicity and that guarantee a restricted property of uniqueness of solutions of the associated initial value problem

$$\dot{x}(t) = f(t,x(t)) \tag{1}$$

$$x(0) = x^o. \tag{2}$$

We call the functions studied here "weakly Lipschitzian"; each such function f has associated with it a number of auxiliary functions $G_j:I_j\to\mathbb{R}$, $j\in\{m+1,\ldots,n\}$ with $I_j\subset\mathbb{R}$ and $m\in\{1,\ldots,n\}$. We show here that when f is weakly Lipschitzian, both classical and Filippov solutions of the initial value problem (1), (2) are unique, provided such solutions are compatible with the domains of the auxilliary functions in a sense to be made precise in section 2.

We arrived at the notion of a weakly Lipschitzian mapping through an earlier study of uniqueness of motions of certain elastic–plastic oscillators [1] in which a physically natural measure of energy separation of solutions was shown to decay in a weaker sense than would be the case were the right–hand side f monotone. For such oscillators, one of the components of the solution necessarily is non–decreasing and there is a concave, increasing response function for the oscillator that controls the evolution of the energy separation. These features of the oscillator led to the observation that the energy separation never exceeds its initial value, even though that separation may increase on some intervals of time, and this yielded uniqueness of solutions. In this article, we show that this observation can be employed through the notion of a weakly Lipschitzian mapping to obtain uniqueness theorems for a broader class of ordinary differential equations than the ones studied in the paper [1].

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In Section 2, we define the notion of a weakly Lipschitzian mapping, and we establish in Theorem 2.1 the uniqueness of classical solutions of (1), (2), i.e., absolutely continuous functions that satisfy the equation (1) almost everywhere and satisfy the initial condition (2), provided that such solutions are compatible with the domains of the auxiliary functions. Our proof of restricted uniqueness involves first proving uniqueness only of a certain number of components of the solution (roughly corresponding to those components that determine the energy separation used in [1]). The proof of uniqueness of the remaining components is then based on the uniqueness of the former components. If the weakly Lipschitzian mapping f happens to be Lipschitzian, one–sided Lipschitzian, or monotone, then the second step is not needed, because the first step treats all of the components of the solution, and no auxiliary functions are used.

In Section 3, we give some examples from mechanics of differential equations in which Theorem 2.1 can be applied to obtain unrestricted uniqueness of solutions. In each of the examples, the form of the right–hand side f of (1) permits us to partition the set of initial data and the domain D of f into finitely many subsets. Each corresponding restriction of f satisfies the hypotheses in Theorem 2.1, and one obtains in this manner local uniqueness of solutions for each restricted problem. Unrestricted uniqueness of solutions of the original problem then follows readily in each example. We note that the example we give of a single damped non–linear oscillator also can be treated using transversality arguments [2], and the example of the elastic–plastic oscillator also can be treated by the methods employed by Gröger, Nečas, and Trávniček [3] in their study of partial differential equations from the theory of plasticity. However, we do not know of a method other than ours that covers both of these examples. Moreover, we know of no other method that yields uniqueness of solutions for the coupled, damped non–linear oscillators that we describe in Section 3.

In Section 4, we describe Filippov's notion of solution [4] of an ordinary differential equation, and we show in Theorem 4.1 that, when f is weakly Lipschitzian and satisfies

Filippov's condition B, the initial–value problem (1), (2) has at most one Filippov solution that is compatible with the domains of the auxiliary functions.

If one wishes also to establish local existence of classical solutions of (1), (2), then one must supplement the assumption that f is weakly Lipschitzian by an additional property. For example, one can assume that f satisfies Carathéodory's conditions [5, Chapter 2, Theorem 1.1]. However, in the case of Filippov solutions the Condition B, which we assume in proving restricted uniqueness of solutions in Theorem 4.1, implies local existence of solutions [4, §3, Theorem 4]. In each of the examples presented in Section 3, local existence of both types of solutions is assured. For the elastic–plastic oscillator in Example 3, a natural extension of the right–hand side is required in order to obtain local existence of Filippov solutions for all choices of initial data; once this extension is made, one can show that the two notions of solutions coincide. (Actually, in all of the examples in Section 3, the two notions of solutions coincide.)

2. Restricted Uniqueness of Classical Solutions

Let $n\in\mathbb{N}$, $D\subset\mathbb{R}^{n+1}$, $f:D\to\mathbb{R}^n$, $x^\circ\in\mathbb{R}^n$, and $T>0$ be given such that $(0,x^\circ)\in D$. An absolutely continuous function $x:[0,T]\to\mathbb{R}^n$ is called a <u>classical solution</u> of the initial–value problem (1), (2) if $(t,x(t))\in D$ for all t in $[0,T]$, $x(0)=x^\circ$, and (1) holds for almost every t in $[0,T]$.

We say that f is <u>weakly Lipschitzian on</u> D if there exist $m\in\{1,\dots,n\}$ and, for each $j\in\{m+1,\dots,n\}$, an increasing mapping $G_j:I_j\to\mathbb{R}$ with I_j an interval in $\mathbb{R}$ satisfying

(WL1) for all $j\in\{m+1,\dots,n\}$, $f_j\geq 0$ and G_j is concave, or $f_j\leq 0$ and G_j is convex;

(WL2) there exists a locally integrable function $L:[0,\infty)\to[0,\infty)$ such that for every (t,x), $(t,\bar{x})$ in D, with $x_j,\bar{x}_j\in I_j$ for all $j\in\{m+1,\dots,n\}$,

$$(Px-P\bar{x})\cdot(Pf(t,x)-Pf(t,\bar{x})) + \sum_{j=m+1}^{n}(G_j(x_j)-G_j(\bar{x}_j))(f_j(t,x)-f_j(t,\bar{x}))$$

$$\leq L(t)\,\|Px-P\bar{x}\|^2; \tag{3}$$

here, for all $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$, we put

$$Px = P(x_1,\ldots,x_n) := (x_1,x_2,\ldots,x_m,0,\ldots,0). \tag{4}$$

We note that if (WL1) and (WL2) hold with $m=n$, then the set $\{m+1,\ldots,n\}$ is interpreted to be the empty set, the sum $\sum_{j=m+1}^{n}$ in (WL2) is zero, and (WL2) is the assertion that f satisfies a one–sided Lipschitz condition on D. If, in addition, L is the zero function, then f is monotone. Thus, monotone functions and (one–sided) Lipschitz continuous functions are weakly Lipschitzian.

THEOREM 2.1: If f is weakly Lipschitzian on D and T is a positive number, there is at most one classical solution $x:[0,T]\to\mathbb{R}^n$ of the initial–value problem (1), (2) that satisfies

$$x_j(t)\in I_j \text{ for all } t\in[0,T],\ j\in\{m+1,\ldots,n\}. \tag{5}$$

The condition (5) gives a precise meaning to the phrase "solutions compatible with the domains of the auxiliary functions" used in the Introduction as well as to the phrase "restricted uniqueness of solutions" that appears in the title of this article. Thus, instead of uniqueness for arbitrary solutions of (1), (2), that would follow from Lipschitz continuity of f, only uniqueness for solutions of (1), (2) compatible with the domains of the auxiliary functions $G_{m+1},\ldots,G_n$ follows from the assumption that f is weakly Lipschitzian.

In order to prove Theorem 2.1, we need three lemmas.

<u>LEMMA</u> <u>2.1</u>: Let an interval $I \subset \mathbb{R}$, numbers $\tau, \tau' \in I$ with $\tau < \tau'$, and $G: I \to \mathbb{R}$ a non–decreasing, concave function be given. There then holds

$$\int_{\tau}^{\tau'} (G(\bar{x}(t)+u(t)) - G(\bar{x}(t))\dot{u}(t)dt \geq 0 \tag{6}$$

for every absolutely continuous function $u:[\tau,\tau'] \to \mathbb{R}$, with $u(\tau)=0$ and $u(t)>0$ for all $t \in (\tau,\tau')$, and for every non–decreasing absolutely continuous function $\bar{x}:[\tau,\tau'] \to I$.

This lemma can be proved using the arguments given on pages 109–113 in the article [1]. There, the counterpart of G was assumed to be positive, to have positive derivative, and non–positive second derivative, but only the implied monotonicity and concavity were used. Wang has proved (Ph.D. Thesis, Carnegie Mellon University, 1991) that when G is C^2, the monotonicity and concavity of G are necessary in order that (6) holds for all functions u and $\bar{x}$ as above.

<u>LEMMA</u> <u>2.2</u>: Let $T>0$ and an interval $I \subset \mathbb{R}$ be given.

(i) If $G: I \to \mathbb{R}$ is a non–decreasing concave function, then there holds

$$\int_{0}^{t} (G(x(\tau))) - G(\bar{x}(\tau))(\dot{x}(\tau) - \bar{x}^{\cdot}(\tau))d\tau \geq 0 \tag{7}$$

for all $t \in [0,T]$ and for all $x, \bar{x}$ absolutely continuous functions from $[0,T]$ into I with $x(0)=\bar{x}(0)$ and $\dot{x}(t) \geq 0$, $\bar{x}^{\cdot}(t) \geq 0$ for almost every $t \in [0,T]$.

(ii) If $G:I\to\mathbb{R}$ is a non–decreasing convex function, then (7) holds for all $t\in[0,T]$ and for all $x, \bar{x}$ absolutely continuous functions from $[0,T]$ into I with $x(0)=\bar{x}(0)$ and $\dot{x}(t)\leq 0$, $\dot{\bar{x}}(t)\leq 0$ for almost every $t\in[0,T]$.

Proof of (i): Let $x, \bar{x}, G$ be given as in (i), and for each $t\in[0,T]$, put $u(\tau):= x(\tau)-\bar{x}(\tau)$, and note that u is absolutely continuous, $u(0)=0$, and for all $t\in[0,T]$,

$$\int_0^t (G(x(\tau)) - G(\bar{x}(\tau)))(\dot{x}(\tau)-\dot{\bar{x}}(\tau))d\tau =$$

$$\int_0^t (G(\bar{x}(\tau)+u(\tau)) - G(\bar{x}(\tau)))\dot{u}(\tau)d\tau. \tag{8}$$

Let $t\in[0,T]$ be given. Because u is continuous, there is a countable set J and a family $((\tau_j,\tau_j')\,|\,j\in J)$ of pairwise disjoint subintervals of $[0,t]$ such that

$$[0,t] \cap u^{-1}(\mathbb{R}\backslash\{0\}) = \bigcup_{j\in J}(\tau_j,\tau_j') \tag{9}$$

and, for all $j\in J$, either

$$u(\tau_j) = 0 \text{ and } u(\tau) > 0 \text{ for all } \tau\in(\tau_j,\tau_j') \tag{10}$$

or

$$u(\tau_j) = 0 \text{ and } u(\tau) < 0 \text{ for all } \tau\in(\tau_j,\tau_j'). \tag{11}$$

Let $j \in J$ be given. If (10) holds, then $\bar{x}$ and u satisfy the hypotheses of Lemma 2.1 and by (6) there holds

$$\int_{\tau_j}^{\tau'_j} (G(\bar{x}(\tau)+u(\tau)) - G(\bar{x}(\tau)))\dot{u}(\tau)d\tau \geq 0. \tag{12}$$

If (11) holds, then put $\tilde{u} := -u$, and note that $\bar{x}=x-u=x+\tilde{u}$, so that $\dot{x}(\tau)\geq 0$ for almost every $\tau \in (\tau_j, \tau'_j)$ and $\tilde{u}(0)=0$, $\tilde{u}(\tau)>0$ for all $\tau \in (\tau_j, \tau'_j)$. Therefore, we have

$$\int_{\tau_j}^{\tau'_j} (G(\bar{x}(\tau)+u(\tau)) - G(\bar{x}(\tau)))\dot{u}(\tau)d\tau =$$

$$\int_{\tau_j}^{\tau'_j} (G(x(\tau)) - G(x(\tau)+\tilde{u}(\tau)))(-\tilde{u}^{\cdot}(\tau))d\tau =$$

$$\int_{\tau_j}^{\tau'_j} (G(x(\tau)+\tilde{u}(\tau)) - G(x(\tau)))\tilde{u}^{\cdot}(\tau)d\tau,$$

and Lemma 2.1 again yields (12). If we put $\mathcal{U} := \bigcup_{j\in J} (\tau_j, \tau'_j)$, then by (9), $u(\tau)=0$ for all $\tau \in [0,t] \setminus \mathcal{U}$, so that $\dot{u}(\tau)=0$ for almost every $\tau \in [0,t]\setminus\mathcal{U}$. We then have by the countably additivity of the integral and relation (12),

$$\int_0^t (G(\bar{x}(\tau)+u(\tau)) - G(\bar{x}(\tau)))\dot{u}(\tau)d\tau =$$

$$\int_{\mathcal{U}} (G(\bar{x}(\tau)+u(\tau)) - G(\bar{x}(\tau)))\dot{u}(\tau)d\tau + \int_{[0,t]\setminus\mathcal{U}} (G(\bar{x}(\tau)+u(\tau)) - G(\bar{x}(\tau)))\dot{u}(\tau)d\tau =$$

$$\sum_{j\in J} \int_{\tau_j}^{\tau'_j} (G(\bar{x}(\tau)+u(\tau)) - G(\bar{x}(\tau)))\dot{u}(\tau)d\tau \geq 0; \tag{13}$$

relations (13) and (8) then tell us that (7) holds.

Proof of (ii): Let $x, \bar{x} : [0,T] \to I$ be given with $\dot{x}(\tau) \le 0$ and $\bar{x}^{\cdot}(\tau) \le 0$ for almost every $\tau \in [0,T]$. Let $G: I \to \mathbb{R}$ be a non–decreasing, convex function, and observe that the relation $\tilde{G}(y) := -G(-y)$ defines a non–decreasing concave function $\tilde{G}:(-I) \to \mathbb{R}$. Hence, if we put $x_* := -x$ and $\bar{x}_* := -\bar{x}$, then $\tilde{G}$, x_*, and $\bar{x}_*$ satisfy the hypotheses in part (i), so that by (7)

$$
\begin{aligned}
0 &\le \int_0^t (\tilde{G}(x_*(\tau)) - \tilde{G}(\bar{x}_*(\tau)))(\dot{x}_*(\tau) - \dot{\bar{x}}_*(\tau)) d\tau \\
&= \int_0^t (-G(-x_*(\tau)) + G(-\bar{x}_*(\tau)))(\dot{x}_*(\tau) - \dot{\bar{x}}_*(\tau)) d\tau \\
&= \int_0^t (-G(x(\tau)) + (G(\bar{x}(\tau)))(-\dot{x}(\tau) + \bar{x}^{\cdot}(\tau)) d\tau
\end{aligned}
$$

for all $t \in [0,T]$, and this yields the desired conclusion in (ii). ■

Lemma 2.3: Let an interval $I \subset \mathbb{R}$ and $T > 0$ be given. If $x, \bar{x} : [0,T] \to I$ are absolutely continuous, $G: I \to \mathbb{R}$ is increasing, and for almost every $t \in [0,T]$ there holds

$$
(G(x(t)) - G(\bar{x}(t)))((\dot{x}(t) - \bar{x}^{\cdot}(t)) = 0, \tag{14}
$$

then for all $t \in [0,T]$,

$$
x(t) - \bar{x}(t) = x(0) - \bar{x}(0). \tag{15}
$$

<u>PROOF</u>: Put

$$E := \{t\in[0,T] \mid \dot{x}(t) \neq \bar{x}^{\cdot}(t)\} \tag{16}$$

and note by (14) that $G(x(t)) = G(\bar{x}(t))$ for almost every $t\in E$. Because G is increasing, it follows that $x(t) = \bar{x}(t)$ for all $t\in E$ and, therefore, $\dot{x}(t) = \bar{x}^{\cdot}(t)$ for almost every $t\in E$. By (16), we conclude that E has measure zero, and, therefore, $\dot{x}(t) = \bar{x}^{\cdot}(t)$ for almost every $t\in[0,T]$. The conclusion (15) is now immediate. ■

<u>Proof of Theorem 2.1</u>: Let $x, \bar{x}$ be classical solutions of (1), (2) that satisfy (5). Relations (1) and (3) then yield for almost every $\tau\in[0,T]$:

$$\begin{aligned}
&(Px(\tau)-P\bar{x}(\tau))(P\dot{x}(\tau)-P\bar{x}^{\cdot}(\tau) + \sum_{j=m+1}^{n} (G_j(x_j(\tau))-G_j(\bar{x}_j(\tau)))(\dot{x}_j(\tau)-\bar{x}_j^{\cdot}(t)) = \\
&= (Px(\tau)-P\bar{x}(\tau))(Pf(\tau,x(\tau))-Pf(\tau,\bar{x}(\tau)) \\
&+ \sum_{j=m+1}^{n} (G_j(x_j(\tau))-G_j(\bar{x}_j(\tau)))(f_j(\tau,x(\tau)-f_j(\tau,\bar{x}((\tau)) \\
&\leq L(\tau)\, \| Px(\tau) - P\bar{x}(\tau)\|^2.
\end{aligned} \tag{17}$$

Integrating the first and last members of (17) from 0 to t and using Lemma 2.2, we obtain for each $t\in[0,T]$:

$$\|Px(t)-P\bar{x}(t)\|^2 \leq \|Px(0)-P\bar{x}(0)\|^2 + \int_0^t 2L(\tau)\|Px(\tau)-P\bar{x}(\tau)\|^2 d\tau,$$

and Gronwall's inequality together with the initial condition (2) yield

$$Px(t) = P\bar{x}(t) \text{ for all } t\in[0,T]. \tag{18}$$

From (17) and (18) we may conclude that

$$\sum_{j=m+1}^{n} (G_j(x_j(\tau))-G_j(\bar{x}_j(\tau)))(\dot{x}_j(\tau)-\dot{\bar{x}}_j(\tau)) \leq 0$$

for almost every $\tau\in[0,T]$, so that for every $t\in[0,T]$ there holds

$$\sum_{j=m+1}^{n} \int_0^t (G_j(x_j(\tau))-G_j(\bar{x}_j(\tau)))(\dot{x}_j(\tau)-\dot{\bar{x}}_j(\tau))d\tau \leq 0. \tag{19}$$

By Lemma 2.2, each of the $n-m$ integrals in (19) is non–negative, and we conclude that for every $j\in\{m+1,...,n\}$ and every $t\in[0,T]$,

$$\int_0^t (G_j(x_j(\tau))-G_j(\bar{x}_j(\tau)))(x_j(\tau)-\bar{x}_j(\tau)d\tau = 0. \tag{20}$$

Relation (20), Lemma 2.3, and (2) then tell us that for every $j\in\{m+1,...,n\}$ and $t\in[0,T]$,

$$x_j(t) - \bar{x}_j(t) = x_j(0) - \bar{x}_j(0) = 0, \tag{21}$$

and (18) together with (21) yield $x=\bar{x}$. ■

3. Examples in which Unrestricted Uniqueness Arises

In each of the examples we present in this section, Theorem 2.1 can be used to obtain unrestricted uniqueness of solutions of (1), (2), because the sign conditions on components of f in (WL1) naturally induce a finite partition of the set of initial data such that initial data in one piece of the partition produce only solutions that remain in a particular region for a short time. Theorem 2.1 then can be used case–by–case to obtain unrestricted uniqueness of solutions, because in each case condition (5) holds for all initial data for that case.

Example 1: For the following damped, non–linear oscillator

$$\ddot{y} = -\dot{y} - y^{1/3} \tag{22}$$

we put $x_1 := \dot{y}$, $x_2 := y$ and obtain the initial–value problem

$$\dot{x}_1 = -x_1 - x_2^{1/3} \tag{23}$$

$$\dot{x}_2 = x_1 \tag{24}$$

$$(x_1(0), x_2(0)) = x^{\circ} = (x_1^{\circ}, x_2^{\circ}). \tag{25}$$

Case 1: $x_2^{\circ} \neq 0$. In this case, the right–hand side f of (23), (24) is locally Lipschitzian at x° and local uniqueness of solutions of (23)–(25) follows from Theorem 2.1 or from classical uniqueness results.

Case 2: $x_2^{\circ} = 0$ and $x_1^{\circ} > 0$. In this case, each solution of (23)–(25) satisfies $x_1(t) \geq 0$ and $x_2(t) \geq 0$ on $[0,T]$ for some $T>0$ (that could, in principle, depend on the solution). We put $n := 2$, $m := 1$, $G_2(x_2) := x_2^{1/3}$, $I_2 := [0,\infty)$ and note that G_2 is increasing and concave on I_2. Moreover, we have

$$f_1(t, x_1, x_2) = -x_1 - x_2^{1/3}$$

$$f_2(t, x_1, x_2) = x_1$$

$$P(x_1,x_2) = (x_1,0)$$

$$Pf(t,x_1,x_2) = (-x_1-x_2{}^{1/3},0)$$

so that, for all $x=(x_1,x_2)$, $\bar{x}=(\bar{x}_1,\bar{x}_2)\in\mathbb{R}^2$ with $x_1\geq 0$, $\bar{x}_1\geq 0$, $x_2\geq 0$, $\bar{x}_2\geq 0$

$$\begin{aligned} & (Px-P\bar{x}) \cdot (Pf(t,x)-Pf(t,\bar{x})) + (G_2(x_2)-G_2(\bar{x}_2))(f_2(t,x)-f_2(t,\bar{x})) \\ = \quad & (x_1-\bar{x}_1)(-x_1-x_2{}^{1/3}+\bar{x}_1+\bar{x}_2{}^{1/3}) + (x_2{}^{1/3}-\bar{x}_2{}^{1/3})(x_1-\bar{x}_1) \\ = \quad & -(x_1-\bar{x}_1)^2 \leq 0. \end{aligned}$$

Therefore, (WL1) and (WL2) are satisfied on $D:= \mathbb{R} \times \{(x_1,x_2)\in\mathbb{R}^2 | x_1\geq 0 \text{ and } x_2\geq 0\}$ with $L=0$. We noted above that in Case 2 each solution x of (23)–(25) satisfies $x_2(t)\geq 0$, i.e., $x_2(t)\in I_2$, for $t\in[0,T]$ for some $T>0$, so that (5) is satisfied locally in time for <u>every</u> solution with initial data x^o satisfying $x_2^o=0$ and $x_1^o>0$. Therefore, Theorem 2.1 applies and (23)–(25) has at most one classical solution in Case 2.

Case 3: $x_2^o=0$ and $x_1^o<0$. In this case, as in Case 2, we put $m=1$ and $G_2(x_2) = x_2{}^{1/3}$, but we must here put $I_2:= (-\infty,0]$, so that G_2 is increasing and convex on I_2. Moreover, every solution x satisfies $x_1(t)\leq 0$ and $x_2(t)\leq 0$ for all $t\in[0,T]$ for some $T>0$, so we may put

$$D:= \mathbb{R} \times \{(x_1,x_2)\in\mathbb{R}^2 \mid x_1\leq 0 \text{ and } x_2\leq 0\}$$

and again verify that (WL1) and (WL2) are satisfied, so that (23)–(25) again has at most one classical solution.

Case 4: $x^{\circ}=(0,0)$. Multiplying (23) by x_1, (24) by $x_2^{1/3}$, adding the resulting equations and integrating, we find that

$$\left(\tfrac{1}{2}x_1^{\,2}+\tfrac{3}{4}x_2^{\,4/3}\right)(t)\le \tfrac{1}{2}x_1^{\circ 2}+\tfrac{3}{4}x_2^{\circ\,\frac{4}{3}}=0$$

for all $t\in[0,T]$. Thus, the only solution of (23)–(25) in Case 4 is $x(t)=(0,0)$, for all $t\in[0,T]$.

Example 2: For the following coupled, damped non–linear oscillators

$$\ddot{y}=-(1+\dot{z}^2)\dot{y}-y^{1/3} \tag{26}$$

$$\ddot{z}=-(1+\dot{y}^2)\dot{z}-z^{1/3} \tag{27}$$

we put $x_1=\dot{y}$, $x_2=\dot{z}$, $x_3=y$, $x_4=z$ to obtain the initial–value problem

$$\dot{x}_1=-(1+x_2)x_1-x_3^{\,1/3} \tag{28}$$

$$\dot{x}_2=-(1+x_1^{\,2})x_2-x_4^{\,1/3} \tag{29}$$

$$\dot{x}_3=x_1 \tag{30}$$

$$\dot{x}_4=x_2 \tag{31}$$

$$x(0)=x^{\circ}=(x_1^{\circ},x_2^{\circ},x_3^{\circ},x_4^{\circ}). \tag{32}$$

The case–by–case analysis is too long to present here in full, so we discuss only one case: $x_1^o>0$, $x_2^o<0$, $x_3^o=x_4^o=0$. From (30), (31), and (32) we conclude that every solution of (28)–(32) in this case remains locally in time in the set

$$U:= \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid x_1\geq 0,\ x_2\leq 0,\ x_3\geq 0,\ x_4\leq 0\}.$$

We put $n:=4$, $m:=2$, $I_3:=[0,\infty)$, $I_4:=(-\infty,0]$, $G_3(x_3):= {x_3}^{1/3}$, $G_4(x_4) = {x_4}^{1/3}$, and note that for all (t,x), $(t,\bar{x})$ with $t\geq 0$ and $x,\bar{x}\in U$

$$(Px-P\bar{x}) \cdot (Pf(t,x)-Pf(t,\bar{x})) + \sum_{j=3}^{4}(G_j(x_j)-G_j(\bar{x}_j))(f_j(t,x)-f_j(t,\bar{x}))$$

$$= (x_1-\bar{x}_1)(-(1+{x_2}^2)x_1-{x_3}^{1/3} + (1+{\bar{x}_2}^2)\bar{x}_1+{\bar{x}_3}^{1/3})$$

$$+(x_2-\bar{x}_2)(-(1+{x_1}^2)x_2-{x_4}^{1/3}+(1+{\bar{x}_1}^2)\bar{x}_2+{\bar{x}_4}^{1/3})$$

$$+({x_3}^{1/3}-{\bar{x}_3}^{1/3})(x_1-\bar{x}_1) + ({x_4}^{1/3}-{\bar{x}_4}^{1/3})(x_2-\bar{x}_2)$$

$$= -(x_1-\bar{x}_1)^2 - (x_1-\bar{x}_1)(x_1{x_2}^2-\bar{x}_1{\bar{x}_2}^2)$$
$$-(x_2-\bar{x}_2)^2 - (x_2-\bar{x}_2)(x_2{x_1}^2-\bar{x}_2{\bar{x}_1}^2)$$

$$\leq -(x_1-\bar{x}_1)^2 - (x_1-\bar{x}_1)^2{x_2}^2 - (x_1-\bar{x}_1)(x_2-\bar{x}_2)\bar{x}_1(x_2+\bar{x}_2)$$
$$-(x_2-\bar{x}_2)^2 - (x_2-\bar{x}_2)^2x_1^2 - (x_2-\bar{x}_2)(x_1-\bar{x}_1)\bar{x}_2(x_1+\bar{x}_1)$$

$$\leq -(x_1-\bar{x}_1)(x_2-\bar{x}_2)[\bar{x}_2(x_1+\bar{x}_1)+\bar{x}_1(x_2+\bar{x}_2)]$$
$$\leq -\tfrac{1}{2}[\bar{x}_2(x_1+\bar{x}_1)+\bar{x}_1(x_2+\bar{x}_2)] \, \| Px-P\bar{x}\|^2, \qquad (33)$$

where we have used the fact that $x_1, \bar{x}_1 \geq 0$ and $x_2, \bar{x}_2 \leq 0$. The relation (33) and the choice of U, G_3, and G_4 tell us that we can choose $D \subset \mathbb{R} \times U$ so that (WL1) and (WL2) are satisfied with L a suitably chosen constant function, and uniqueness in this case follows from Theorem 2.1.

<u>Example</u> <u>3</u>: In the article [1], a uniqueness theorem was proved for the initial–value problem governing forced motions of an elastic–plastic oscillator with work–hardening, and the proof of that theorem motivated both our motion of a weakly–Lipschitzian function as well as our proof of Theorem 2.1. Nevertheless, it is instructive to re–examine this elastic–plastic oscillator in light of Theorem 2.1, because we can understand more immediately than in the article [1] the features of the oscillator that are used in establishing uniqueness. For a given positive–valued, concave strictly increasing function $H:[0,\infty) \to \mathbb{R}$, locally integrable function $g:[0,\infty) \to \mathbb{R}$, and $(v^\circ, \sigma^\circ, w^\circ)$ with $|\sigma^\circ| \leq (H(w^\circ))^{1/2}$, we wish to establish uniqueness of classical solutions of the initial value problem

$$\dot{v}(t) = g(t) - \sigma(t) \tag{34}$$

$$\dot{\sigma}(t) = \begin{cases} \dfrac{H'(w(t))}{1+H'(w(t))} v(t) & \text{if } |\sigma(t)| = (2H(w(t))^{1/2} \\ & \text{and } \sigma(t)v(t) \geq 0, \\ v(t) & \text{otherwise,} \end{cases} \tag{35}$$

$$\dot{w}(t) = \begin{cases} \dfrac{\sigma(t)}{1+H'(w(t))} v(t) & \text{if } |\sigma(t)| = (2H(w(t)))^{1/2} \\ & \text{and } \sigma(t)v(t) \geq 0, \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

$$(v(0), \sigma(0), w(0)) = (v^{o}, \sigma^{o}, w^{o}) \tag{37}$$

subject to the constraint

$$|\sigma(t)| \leq (2H(w(t)))^{1/2}. \tag{38}$$

The case–by–case analysis of initial data is best carried out using the cases: $|\sigma^{o}|<(2H(w^{o}))^{1/2}$, $\sigma^{o}>0$ and $\sigma^{o}<0$. In the first case, the system reduces locally in time to that governing a forced harmonic oscillator

$$\dot{v}(t) = g(t)-\sigma(t)$$
$$\dot{\sigma}(t) = v(t)$$
$$\dot{w}(t) = 0$$

with the constraint (38) in the form of a strict inequality, and uniqueness is immediate from the relation

$$(\tfrac{1}{2}(v-\bar{v})^2 + \tfrac{1}{2}(\sigma-\bar{\sigma})^2)^{\cdot} = 0$$

governing the energy separation of two solutions (v,σ,w) and $(\bar{v},\bar{\sigma},\bar{w})$.

Of the remaining two cases, we treat here only the case $\sigma^{o}>0$; the case $\sigma^{o}<0$ is similar. From the discussion in Section 3 of [1], uniqueness of solutions of (34)–(38) when σ^{o} is positive follows from uniqueness of solutions of the following initial value problem

$$\dot{v}(t) = g(t) - \sigma(t) \tag{39}$$

$$\dot{\sigma}(t) = \begin{cases} \frac{S'(\lambda(t))}{1+S'(\lambda(t))} v(t) & \text{if } \sigma(t)=S(\lambda(t)) \\ & \text{and } v(t) \geq 0 \\ v(t) & \text{otherwise}, \end{cases} \tag{40}$$

$$\lambda^{\cdot}(t) = \begin{cases} \frac{1}{1+S'(\lambda(t))} v(t) & \text{if } \sigma(t)=S(\lambda(t)) \\ & \text{and } v(t) \geq 0 \\ 0 & \text{otherwise}, \end{cases} \tag{41}$$

$$(v(0),\sigma(0),\lambda(0)) = (v^{o},\sigma^{o},0) \tag{42}$$

subject to the constraint

$$0 < \sigma(t) \leq S(\lambda(t)). \tag{43}$$

Here, S is also a positive–valued, increasing, concave function from $[0,\infty)$ into $\mathbb{R}$. We may now put

$$D:= \{(t,v,\sigma,\lambda)\in\mathbb{R}^4 \mid t\geq 0,\ \lambda\geq 0,\ 0<\sigma\leq S(\lambda)\} \tag{44}$$

$I_3:= [0,\infty)$, $G_3:= S$ and note that, from (41),

$$f_3(t,v,\sigma,\lambda) = \begin{cases} \frac{v}{1+S'(\lambda)} & \text{if } \sigma = S(\lambda) \text{ and } v\geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is non–negative and G_3 is increasing and concave. Moreover, denoting by f_1 and f_2 the right–hand sides of (39) and (40), respectively, one can easily verify that for every (t,v,σ,λ), $(t,\bar{v},\bar{\sigma},\bar{\lambda})\in D$ there holds

$$(v-\bar{v})(f_1(t,v,\sigma,\lambda)-f_1(t,\bar{v},\bar{\sigma},\bar{\lambda}))+(\sigma-\bar{\sigma})(f_2(t,v,\sigma,\lambda)-f_2(t,\bar{v},\bar{\sigma},\bar{\lambda})) \\ +(S(\lambda)-S(\bar{\lambda}))(f_3(t,v,\sigma,\lambda)-f_3(t,\bar{v},\bar{\sigma},\bar{\lambda}) \leq 0.$$

Therefore, (WL1) and (WL2) are satisfied and, because (5) is satisfied locally in time for all solutions of (39)–(43), Theorem 2.1 yields uniqueness of solutions of that initial–value problem.

4. Restricted Uniqueness of Filippov Solutions

In this section we indicate how the concepts and arguments in Section 2 can be adapted to yield restricted uniqueness of Filippov solutions of the initial–value problem (1), (2).

Let $n\in\mathbb{N}$, $D\subset\mathbb{R}^{n+1}$, $(t,x)\longmapsto f(t,x)\in\mathbb{R}^n$, $x^o\in\mathbb{R}^n$ and $T>0$ be given such that $(0,x^o)\in D$, f is defined almost everywhere in D and is measurable, and f satisfies Condition B ([4, Section 2]): for every compact subset $C\subset D$, there is an integrable function $t\longmapsto B_C(t)$ such that $|f(t,x)|\leq B_C(t)$ for almost every (t,x) in C. An absolutely continuous function $x\colon [0,T]\to\mathbb{R}^n$ is called a <u>Filippov solution of</u> (1), (2) [4, Section 1] if $(t,x(t))\in D$ for all $t\in[0,T]$, $x(0)=x^o$, for almost every $t\in[0,T]$ and $\delta>0$,

$$|\{x\,|\,(t,x)\in D\}\cap B_\delta(x(t))| > 0, \tag{45}$$

where $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}^n$ and $B_\delta(x(t)):=\{y\in\mathbb{R}^n\,|\,|x(t)-y|<\delta\}$, and for almost every $t\in[0,T]$, there holds

$$\dot{x}(t) \in \bigcap_{\delta>0} \bigcap_{|N|=0} \text{konv } f(t,B_\delta(x(t))\backslash N) \tag{46}$$

where for each subset A of $\mathbb{R}^n$, konv A is the intersection of all the closed half–spaces containing A. The right–hand member of (46) can be thought of as the convex hull of the essential range of $x \longmapsto f(t,x)$ restricted to arbitrarily small $\mathbb{R}^n$–neighborhoods of $x(t)$.

For a mapping f satisfying Condition B, we interpret the assertion f <u>is weakly Lipschitzian on</u> D to mean that the conditions (WL1), (WL2) on f given in Section 2 are to hold for almost every point in D. Our interpretation is consistent with the intent of Filippov that modification of f on a null set in $\mathbb{R}^{n+1}$ should not alter the class of Filippov solutions of (1), (2). We may now reformulate the content of Theorem 2.1 in the context of Filippov solutions.

<u>THEOREM 4.1</u>: If f is weakly Lipschitzian and satisfies Condition B, then for every $T>0$ there is at most one Filippov solution $x:[0,T]\to\mathbb{R}^n$ of the initial–value problem (1), (2) that satisfies

$$x_j(t)\in I_j \text{ for all } t\in[0,T],\ j\in\{m+1,\dots,n\}. \tag{5}$$

Our proof of Theorem 2.1 only has to be modified in the very first step in order to yield a proof of Theorem 4.1. In fact, for Filippov solutions of (1), the equality sign in (17) is not necessarily valid, but one can show nevertheless that the first member of (17) is bounded above by the last. The detailed arguments required to verify this modified form of (17) can be obtained directly from Filippov's article [4] (see the proof of Theorem 9, Section 5). The main fact used in the modification is that a Filippov solution $x:[0,T]\to\mathbb{R}^n$ of (1) satisfies: there exists a null set $N\subset[0,T]$ such that for every $t\in[0,T]\backslash N$ and every $v\in\mathbb{R}^n$,

$$\dot{x}(t) \cdot v \leq \lim_{\delta \to 0} \operatorname{ess\,sup}_{y \in B_\delta(x(t))} (f(t,y) \cdot v) \tag{47}$$

([4, Lemma 2]). This inequality and Condition B permit one to bound to any desired accuracy the first member of (17) by an expression that is of the form given in the left–hand side of (3). Using (3) and taking a limit, one then obtains the modified form of (17).

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