A FREE BOUNDARY PROBLEM RELATED TO SINGULAR STOCHASTIC CONTROL

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Abstract It is desired to control a multi-dimensional Brownian motion by adding a (possibly singularly) continuous process to its $n^{\underline{th}}$ components so as to minimize an expected infinite-horizon discounted running cost. The Hamilton-Jacobi-Bellman characterization of the value function is a variational inequality which has a unique twice continuously differentiable solution. The optimal process is constructed by solving the Skorokhod problem of reflecting the Brownian motion along a free boundary in the (0,0,..., -1) direction.

1. INTRODUCTION

This paper concerns the regularity of solutions u to the nonlinear partial differential equations

$$(1.1) \qquad \max\left\{u(x) - \Delta u(x) - h(x) \quad , \quad \frac{\partial}{\partial x_n} u(x) - 1\right\} = 0 \quad , \quad x \in R^n$$

with a forcing term h which is convex in the x_n variable. Under appropriate smoothness and growth conditions on h, the solution to (1.1) is shown to be twice continuously differentiable. Moreover, we prove smoothness of the free boundary of the region in which the elliptic equation $u-\Delta u-h=0$ holds.

Equation (1.1) is related to a singular stochastic control problem. Briefly, the problem is to optimally control an n-dimensional Brownian motion by pushing only along the (0,0,..., -1) direction. In this context, the solution to (1.1) is the value function for the discounted infinite-horizon control problem in which h is the running cost and the displacement caused by the push is equal to its cost. This problem is formulated and solved in Section 6.

The term "singular" is loosely used to describe stochastic control problems in which the displacement of the state caused by the optimal control process in singularly continuous. See Shreve [22] for an introduction to this concept. In singular stochastic control literature, the C^2 regularity of the value function has been called the "principle of smooth-fit", by Benes, Shepp and Witsenhausen [1] It has been instrumental in the analysis of several one-dimensional problems [8], [11], [12], [13], [16], [17], [18], [23]. Equation (1.1) is a multi-dimensional extension of the equation studied in [1]. Another generalization of this equation is obtained by allowing the controller to push in any direction. In this case, the related nonlinear partial differential equation has the form

$$\max\left\{u(x) - \Delta u(x) - h(x) \quad , \quad |\nabla u(x)| - 1\right\} = 0 \quad , \quad x \in R^n. \tag{1.2}$$

For (1.2) in two dimensions with a convex forcing term h, the authors recently obtained regularity results similar to those described above [24]. For dimensions higher than two, the C^2-regularity of solutions to (1.2) and the smoothness of the free boundary are still unknown. However, Evans [9] studied the equation (1.2) in a bounded domain with a non-convex h, and by using a penalization method obtained existence and uniqueness of a solution u in the class $C^{1,1}$, (differentiable with Lipschitz continuous derivative). In fact, for a general h this interior regularity result is sharp. The boundary regularity of u was improved by Ishii and Koike [14], again via penalization.

Our approach to (1.1) is to solve the obstacle problem

$$\max\left\{v(x) - \Delta v(x) - \frac{\partial}{\partial x_n} h(x) \quad , \quad v(x) - 1\right\} = 0 \quad , \quad x \in R^n. \tag{1.3}$$

We then construct the unique solution u of (1.1) by integrating v along the x_n direction. Since the solution to equation (1.3) is known to be of class $C^{1,1}$, this procedure together with several estimates on the free boundary yields the declared regularity of u. In the context of one-dimensional stochastic control, the connection between (1.1) and (1.3) goes back to Bather and Chernoff [2] and has been given probabilistic explanations by Karatzas and Shreve [18], and El-Karoui and Karatzas [8] and analytical derivations by Karatzas [15], Chow, Menaldi and Robin [7], and Menaldi and Robin [20].

The paper is organized as follows. Equation (1.3) is studied in the next section and Lipschitz continuity of the free boundary is obtained in Section 3. Section 4 is devoted to the construction and the uniqueness of a smooth solution to (1.1) and the smoothness of the free boundary is then improved in Section 5. We establish the connection between the singular stochastic control and (1.1) in Section 6.

2. OBSTACLE PROBLEM.

In this section, we study the solutions to equation (1.3). The following assumptions are used in our analysis.

(2.1) h *is three times continuously differentiable and* $0 = h(0) \leq h(x)$

(2.2) h, *together with its gradient and second derivatives, grows at most polynomially as* $|x|$ *tends to infinity.*

(2.3) *there is an* $\alpha > 0$ *such that*

$$\frac{\partial^2}{\partial x_n^2} h(x) > \alpha \max\left\{ \left|\frac{\partial^2}{\partial x_n \partial x_i} h(x)\right| \quad , \quad 1\right\}$$

for every $x \in R^n$, $i = 1,\dots, n-1$.

Theorem 2.1. *There is a unique polynomially growing, locally Lipschitz, continuously differentiable solution to* (1.3). *We shall henceforth denote this solution by* v.

The $C^{1,1}$ regularity of solutions to (1.3), in a bounded domain, is proved by Brezis and Kinderlehrer [4], and a modification of their proof yields the above result. The proof proceeds by introducing the penalized version of (1.3):

$$(2.4)^{\epsilon} \qquad v^{\epsilon}(x) - \Delta v^{\epsilon}(x) + \frac{1}{2\epsilon}\left[(v^{\epsilon}(x) - 1)^+\right]^2 = \frac{\partial}{\partial x_n} h(x) \quad , \quad x \in R^n,$$

where $\epsilon > 0$ is a small parameter and $a^+ = \max\{0,a\}$. Using standard methods from the theory of partial differential equations, the following lemma can be proved, and Theorem 2.1 follows immediately. (A similar result is proved in [24]).

Lemma 2.2. *For* $R > 0$, *there are constants* $C, m, C(R) \geq 0$, *independent of* ϵ, *such that*

$$(2.5) \qquad |v^{\epsilon}(x)| + |\nabla v^{\epsilon}(x)| \leq C(1 + |x|^m) \quad , \quad x \in R^n$$

$$(2.6) \qquad |D^2 v^{\epsilon}(x)| \leq C(R) \quad , \quad |x| \leq R.$$

Moreover, as ϵ *tends to zero,* v^{ϵ} *converges to the unique solution of* (1.3) *in the weak** *topology of* $W^{2,\infty}_{\ell oc}(R^n)$.

3. GEOMETRIC PROPERTIES OF THE FREE BOUNDARY.

In this section, we prove the Lipschitz continuity of the boundary of the region

$$(3.1) \qquad \mathcal{C} = \{x \in R^n : v(x) < 1\}$$

Further regularity of the free boundary is proved in Section 5. We start with an estimate on the gradient of v^{ϵ}.

Lemma 3.1. *There is a constant* $\beta > 0$, *independent of* ϵ, *such that*

$$(3.2) \qquad \frac{\partial}{\partial x_n} v^{\epsilon}(x) \geq \beta |\nabla v^{\epsilon}(x)|$$

and

$$(3.3) \qquad \frac{\partial}{\partial x_n} v^{\epsilon}(x) > 0$$

for every $\epsilon > 0$, $x \in R^n$.

Proof. Set $w^i(x) = \frac{\partial}{\partial x_i} v^{\epsilon}(x)$. Differentiate $(2.4)^{\epsilon}$ with respect to x_i to obtain

$$[1 + \frac{1}{\epsilon} (v^{\epsilon}(x) - 1)^{+}]w^i(x) - \Delta w^i(x) = \frac{\partial^2}{\partial x_n \partial x_i} h(x).$$

Assumption (2.3), together with the maximum principle, yields that

$$w^n(x) > \alpha \max\{|w^i(x)|,0\},$$

where α is the constant appearing in (2.3). Now it is easy to obtain (3.2) with $\beta = (1 + \frac{n-1}{\alpha^2})^{-1/2}$.

□

Let

$$(3.4) \qquad \mathscr{C}^{\epsilon} = \{x \in R^n : v^{\epsilon}(x) < 1\}.$$

Since $\frac{\partial}{\partial x_n} v^{\epsilon}$ never vanishes on R^n, by using the implicit function theorem we can parametize the level curves of v^{ϵ}. In particular, there is a real-valued function q^{ϵ} such that

$$(3.5) \qquad \mathscr{C}^{\epsilon} = \{x \in R^n ; x_n < q^{\epsilon}(x_1, \dots, x_{n-1})\}$$

We next estimate the Lipschitz constant of q^{ϵ}.

Lemma 3.2. *There is a constant* $k > 0$, *independent of* ϵ, *such that*

$$(3.6) \qquad |q^{\epsilon}(0)| + |\nabla q^{\epsilon}(y)| \leq k$$

for every $y \in R^{n-1}$, $\epsilon > 0$.

Proof. Using the two representations of $\mathscr{C}^{\epsilon}$, (3.4) and (3.5), we obtain two expressions for the outward unit normal vector $\nu^{\epsilon}(x)$ at any boundary point $x \in \partial\mathscr{C}^{\epsilon}$. In paticular,

$$\nu^{\epsilon}(x) = \frac{\nabla v^{\epsilon}(x)}{|\nabla v^{\epsilon}(x)|} \tag{3.7}$$

$$= \frac{(-\nabla q^{\epsilon}(\bar{x}),1)}{[1 + |\nabla q^{\epsilon}(\bar{x})|^2]^{1/2}}$$

where

$$\bar{x} = (x_1, \ldots, x_{n-1}).$$

Hence, for every $y \in R^{n-1}$,

$$\frac{\partial}{\partial x_n} v^{\epsilon}(y,q^{\epsilon}(y)) = |\nabla v^{\epsilon}(y,q^{\epsilon}(y))| \; [1 + |\nabla q^{\epsilon}(y)|^2]^{-1/2}$$

and

$$\frac{\partial}{\partial y_i} q^{\epsilon}(y) = -\frac{\partial}{\partial x_i} v^{\epsilon}(y,q^{\epsilon}(y))(\frac{\partial}{\partial x_n} v^{\epsilon}(y,q^{\epsilon}(y)))^{-1}.$$

We now use the estimate (3.2) to arrive at

$$|\frac{\partial}{\partial y_i} q^{\epsilon}(y)| \le \frac{1}{\beta}\,. \tag{3.8}$$

To complete the proof of (3.6), it sufficies to show that

$$\sup_{\epsilon>0} |q^{\epsilon}(0)| < \infty. \tag{3.9}$$

Suppose that (3.9) does not hold. Then there exists a sequence, denoted by ϵ again, such that $|q^{\epsilon}(0)|$ converges to infinity. We analyze two cases separately.

<u>a)</u> <u>$q^{\epsilon}(0) \to +\infty$</u>. In view of (3.8), we have

$$\lim_{\epsilon} q^{\epsilon}(y) = +\infty \tag{3.10}$$

for every $y \in R^{n-1}$. Also (3.5) together with $(2.4)^{\epsilon}$ implies that

$$v^{\epsilon}(x) - \Delta v^{\epsilon}(x) = \frac{\partial}{\partial x_n} h(x) \quad , \; x_n < q^{\epsilon}(\bar{x}).$$

Using the above equation, (3.10), and the convergence of v^{ϵ} to v, we obtain

(3.11) $$v(x) - \Delta v(x) = \frac{\partial}{\partial x_n} h(x) \quad , \quad x \in R^n$$

and

(3.12) $$v(x) \leq 1 \quad , \quad x \in R^n .$$

Since $\frac{\partial^2}{\partial x_n^2} h(x) \geq \alpha$ (c.f. (2.3)), (3.11) and the maximum principle imply that $\frac{\partial}{\partial x_n} v(x) \geq \alpha$ for every $x \in R^n$. But this contradicts with (3.12).

<u>b) $q^{\epsilon}(0) \mapsto -\infty$</u>. Arguing as in the previous case, we obtain

(3.13) $$v(x) - \Delta v(x) \leq \frac{\partial}{\partial x_n} h(x) \quad , \quad x \in R^n$$

and

(3.14) $$v(x) = 1 \quad , \quad x \in R^n .$$

Thus, $v(x) - \Delta v(x) = v(x) = 1 \leq \frac{\partial}{\partial x_n} h(x)$ for every $x \in R^n$. But $\nabla h(0) = 0$.

□

In view of (3.6), we can extract a subsequence ϵ converging to zero, such that $q^{\epsilon}(y)$ converges to a Lipschitz continuous function $q(y)$, uniformly for bounded y. Moreover q satisfies the estimate (3.6). Due to the convergence of v^{ϵ} to v, we have the following corollary.

<u>Corollary 3.3</u> *We have*

$$\mathscr{C} = \{x \in R^n : v(x) < 1\} = \{x \in R^n : x_n < q(\bar{x})\}.$$

In particular, $q(\bar{x})$ obtained as a limit of $q^{\epsilon}(\bar{x})$ along a subsequence is independent of the subsequence.

<u>Proof</u>: Let x^* be an element of the right-hand side. Then there is a positive constant $\epsilon_0 > 0$, such that

$$v^{\epsilon}(x) - \Delta v^{\epsilon}(x) = \frac{\partial}{\partial x_n} h(x)$$

and

$$v^{\epsilon}(x) \le 1$$

whenever $\epsilon \le \epsilon_0$ and $|x-x^*| \le \epsilon_0$. By the convergence of v^{ϵ} to v, we obtain that

$$v(x) - \Delta v(x) = \frac{\partial}{\partial x_n} h(x)$$

and

$$v(x) \le 1$$

on $|x-x^*| \le \epsilon_0$. Also the maximum principle together with the convexity of h in the x_n- variable yields that $\frac{\partial}{\partial x_n} v(x) > 0$ on $|x-x^*| \le \epsilon_0$. Hence $v(x^*) < 1$. The reverse inclusion is obvious.

□

4. C^2 REGULARITY OF u.

We construct a solution to (1.1) by integrating v along the x_n direction. We start with defining $\gamma(y)$ by

(4.1) $$\gamma(y) = -k|y|^2 + q(0) - \frac{1}{4} k \quad , \quad y \in R^{n-1}$$

where k is as in (3.6). Since q satisfies (3.6), we have

$$q(y) \ge -k|y| + q(0).$$

Hence, for $y \in R^{n-1}$

(4.2) $$\gamma(y) \le q(y)$$

We define $u(x)$ as follows

(4.3) $$u(x) = w(\bar{x}) + \int_{\gamma(\bar{x})}^{x_n} v(\bar{x},\xi)d\xi$$

where $\bar{x} = (x_1, \dots, x_{n-1})$, w is the unique polynomially growing solution of

(4.4) $$w(y) - \Delta w(y) = H(y) \quad , \quad y \in R^{n-1}$$

and

$$H(y) = h(y,\gamma(y)) + \frac{\partial}{\partial x_n} v(y,\gamma(y))[1 - |\nabla\gamma(y)|^2]$$

$$- 2 \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} v(y,\gamma(y)) \frac{\partial}{\partial y_i} \gamma(y)$$

$$- v(y,\gamma(y))\Delta\gamma(y).$$

Note that H is locally Lipschitz continuous, and due to (2.5) it grows at most polynomially. Hence by elliptic regularity

$$w \in C^{2,\alpha}_{\ell oc}(R^{n-1}) \tag{4.5}$$

for any $\alpha < 1$ (see [3] Section 2.5.5.2, pp. 117 for similar results).

<u>Theorem 4.1</u>. *The function u, defined by (4.3), is twice continuously differentiable. Moreover, u is the unique solution of (1.1) satisfying*

$$0 \le u(x) \le C(1 + |x|^m) \tag{4.6}$$

for suitable constants $C,m \ge 0$.

<u>Proof</u>. For notational simplicity we use the subscript i to denote the partial derivative with respect to x_i. We directly calculate

$$u_n(x) = v(x) \tag{4.7}$$

$$u_i(x) = w_i(\bar{x}) + \int_{\gamma(\bar{x})}^{x_n} v_i(\bar{x},\xi)d\xi - \gamma_i(\bar{x})v(\bar{x},\gamma(\bar{x})) \quad , \ i = 1,\dots, n-1 \tag{4.8}$$

$$u_{nj}(x) = v_j(x) \quad , \ j = 1,\dots, n \tag{4.9}$$

$$\begin{aligned} u_{ij}(x) = {} & w_{ij}(\bar{x}) + \int_{\gamma(\bar{x})}^{x_n} v_{ij}(\bar{x},\xi)d\xi - \gamma_i(\bar{x})v_j(\bar{x},\gamma(\bar{x})) \\ & - \gamma_j(\bar{x})v_i(\bar{x},\gamma(\bar{x})) - \gamma_i(\bar{x})\gamma_j(\bar{x})v_n(\bar{x},\gamma(\bar{x})) \\ & - \gamma_{ij}(\bar{x})v(\bar{x},\gamma(\bar{x})) \quad , \ i,j = 1,\dots, n-1 \end{aligned} \tag{4.10}$$

The above formulae, together with (1.3), imply that u solves (1.1).

Since $v \in C^{1,1}_{\ell oc}(R^n)$, $w \in C^{2,\alpha}_{\ell oc}(R^n)$, to prove the C^2 regularity of u it suffices to show that

(4.11)
$$F(x) = \int_{\gamma(\bar{x})}^{x_n} v_{ij}(\bar{x},\xi)d\xi$$

is continuous for every $i,j = 1,\dots, n-1$. Approximate the above integral by

$$F_\delta(x) = \int_{\gamma(\bar{x})}^{(q(\bar{x})-\delta)\wedge x_n} v_{ij}(\bar{x},\xi)d\xi + \int_{q(\bar{x})\wedge x_n}^{x_n} v_{ij}(\bar{x},\xi)d\xi$$

where $\delta > 0$ is a small parameter. Since $v(x) = 1$ whenever $x_n \geq q(\bar{x})$, the second integral in the above expression is zero. Also

$$v(x) - \Delta v(x) = \frac{\partial}{\partial x_n} h(x)$$

if $x_n < q(\bar{x})$. Hence v_{ij} is continuous on this region, due to the assumption (2.1) and the interior regularity of elliptic equations. Combining this with the continuity of q, we conclude that F_δ is continuous, for every $\delta > 0$. We estimate the difference $|F_\delta(x) - F(x)|$ by

$$|F_\delta(x) - F(x)| \leq \int_{(q(\bar{x})-\delta)\wedge x_n}^{q(\bar{x})} |v_{ij}(\bar{x},\xi)|d\xi$$

$$\leq \delta \|v_{ij}\|_{L^\infty(B_{R(x)})}$$

where $R(x) = (|\bar{x}|^2 + q(\bar{x})^2)^{1/2}$. Recall that v satisfies (2.6). Hence,

$$|F_\delta(x) - F(x)| \leq \delta C(R(x)),$$

and F is continuous.

We claim that

(4.12)
$$|u(x)| \leq C(1 + |x|^m)$$

for suitable constants $C,m \geq 0$. Indeed, the integral term in (4.3) satisfies the above estimate due to (2.5). Also w solves (4.4) and the right-hand side of (4.4), H, grows at most polynomially. Thus, we can show that w satisfies (4.12) by using the integral representation of w in terms of H (or equivalently the Feynman-Kac formula).

To prove the positivity of u and the uniqueness of solutions to (1.1), we need the following comparison result.

Lemma 4.2. *Suppose that* $\underline{u}, \overline{u} \in C^2(R^n)$ *are sub and supersolutions to* (1.1), *respectively. Further assume that there are* $C^*, m \geq 0$ *such that*

$$(\underline{u}(x))^+ + (-\overline{u}(x))^+ \leq C^*(1 + |x|^m) \tag{4.13}$$

and

$$\frac{\partial}{\partial x_n} \overline{u}(x) < 1 \quad \textit{whenever} \quad x_n \leq -C^*(1 + |\overline{x}|) \tag{4.14}$$

Then

$$\underline{u}(x) \leq \overline{u}(x) \tag{4.15}$$

We relegate the proof of the above lemma to the end of the proof of the theorem.

Choose $\underline{u}(x) = 0$, $\overline{u}(x) = u(x)$ in Lemma 4.2. In view of (3.6) and (4.12), $\overline{u}$ satisfies the hypothesis of the lemma. Also the positivity of h yields that $\underline{u}$ is a subsolution to (1.1). Thus, $u(x) = \overline{u}(x) \geq \underline{u}(\overline{x}) = 0$.

In view of Lemma 4.2, to prove the uniqueness of solutions to (1.1) it suffices to verify (4.14) for any $\overline{u} \in C^2(R^n)$ satisfying both (1.1) and (4.6). Let such a $\overline{u}$ be given. For $y \in R^{n-1}$, define $x_n(y)$ and $p(y)$ by

$$x_n(y) = \inf\{x_n : (\overline{u} - \Delta\overline{u} - h)(y, x_n) < 0\} \tag{4.16}$$

$$p(y) = \inf\{x_n : \frac{\partial}{\partial x_n} h(y, x_n) > 1\} \tag{4.17}$$

We claim that $x_n(y) \geq p(y)$ for each $y \in R^{n-1}$. Indeed if this inequality does not hold for some y^*, then there is $\delta > 0$ such that

$$G(y^*, p(y^*) - \delta) < 0$$

where

$$G(x) = \overline{u}(x) - \Delta\overline{u}(x) - h(x)$$

By continuity of G, there is a neighbourhood of $(y^*, p(y^*) - \delta)$ on which G is negative. Thus, (1.1) yields that $\bar{u}_n \equiv 1$ and $\Delta\bar{u}_n \equiv 0$ on this neighbourhood. Using these and (2.3), we obtain that

$$\frac{\partial}{\partial x_n} G(y^*, p(y^*) - \delta) = (\bar{u}_n - \Delta\bar{u}_n - \frac{\partial}{\partial x_n} h)(y^*, p(y^*) - \delta)$$

$$= 1 - \frac{\partial}{\partial x_n} h(y^*, p(y^*) - \delta)$$

$$= \int_{p(y^*)-\delta}^{p(y^*)} \frac{\partial^2}{\partial x_n^2} h(y^*, \xi) d\xi$$

$$\geq \alpha\delta$$

Hence, the above inequality and the argument leading to it imply that

$$G(y^*, x_n) < 0$$

and

$$\bar{u}_n(y^*, x_n) = 1$$

for all $x_n \leq p(y^*) - \delta$. But this contradicts the positivity of $\bar{u}$. Hence $x_n(y) \geq p(y)$.

Replace the estimate (3.2) by the assumption (2.3) in the argument leading (3.8), to conclude that

$$\sup_y |\nabla p(y)| < \infty.$$

Since $p(0) > 0$, the above inequality implies that

$$x_n(y) \geq p(y) \geq -C|y|$$

for some constant $C \geq 0$. Thus, (4.14) is satisfied by $\bar{u}$.

□

<u>Proof of Lemma 4.2</u>.

Consider the auxiliary function

$$(4.18)\quad \phi_{\epsilon,\delta}(x) = (1 - \epsilon)\underline{u}(x) - \bar{u}(x) - \delta\left[\sum_{i=1}^{n-1} \xi(\epsilon x_i) + \xi(\epsilon[x_n + \eta(\bar{x})])\right]$$

where $\epsilon, \delta > 0$ are small parameters.

$$\xi(r) = e^{|r|} - |r| - \frac{1}{2} r^2 \quad , \quad r \in R$$

and η is a smooth function satisfying

$$(4.19) \quad \begin{cases} \text{(i)} \quad \sup_{y \in R^{n-1}} [\,|\nabla\eta(y)| + \|D^2\eta(y)\|\,] < \infty \\ \\ \text{(ii)} \quad \eta(y) \geq C^*(|y| + 1) \text{ for all } \; y \in R^{n-1} \end{cases}$$

with the constant C^* appearing in (4.14).

Since ξ grows exponentially, (4.13) implies that $\phi_{\epsilon,\delta}$ achieves its maximum, say at $x^* = x^*(\epsilon,\delta)$. Then,

$$\bar{u}_n(x^*) = (1 - \epsilon)\underline{u}_n(x^*) - \epsilon\delta\xi'(\epsilon[x_n^* + \eta(\vec{x}^*)]).$$

Observe that $\underline{u}_n(x) \leq 1$ and $\xi'(r) \geq 0$ if $r \geq 0$. Thus, we have

$$\bar{u}_n(x^*) \leq 1 - \epsilon \quad \text{if} \quad x_n^* \geq -\eta(\vec{x}^*).$$

But if $x_n^* \leq -\eta(\vec{x}^*)$ then $\bar{u}_n(x^*) < 1$ due to (4.14) and (4.19) (ii). Hence $\bar{u}_n(x^*) < 1$ and since $\bar{u}$ is a supersolution to (1.1), we conclude that

$$(4.20) \qquad \bar{u}(x^*) - \Delta\bar{u}(x^*) - h(x^*) \geq 0.$$

Also

$$(4.21) \qquad 0 \geq \Delta\phi_{\epsilon,\delta}(x^*) = (1 - \epsilon)\Delta\underline{u}(x^*) - \Delta\bar{u}(x^*) - \delta k_\epsilon(x^*),$$

where

$$k_\epsilon(x) = \epsilon^2 \sum_{i=1}^{n-1} \xi''(\epsilon x_i^*) + \epsilon^2[1 + |\nabla\eta(\vec{x}^*)|^2]\xi''(\epsilon[x_n^* + \eta(\vec{x}^*)])$$

$$+ \epsilon \, \nabla\eta(\vec{x}^*)\xi'(\epsilon[x_n^* + \eta(\vec{x}^*)]).$$

By using the inequalities $|\xi'| \leq \xi'' \leq 5\xi$ and (4.19) (i), we obtain that

$$k_\epsilon(x) \le 5\epsilon^2 \sum_{i=1}^{n-1} \xi(\epsilon x_i^*) + C\epsilon\xi(\epsilon[x_n^* + \eta(\vec{x}^*)]). \tag{4.22}$$

Combine (4.20), (4.21), (4.22) and the fact that $\underline{u}$ is a subsolution to (1.1) to obtain

$$0 \ge (1 - \epsilon)[\underline{u}(x^*) - h(x^*)] - [\bar{u}(x^*) - h(x^*)] -$$

$$- \delta[5\epsilon^2 \sum_{i=1}^{n-1} \xi(\epsilon x_i^*) + C\epsilon\xi(\epsilon[x_n^* + \eta(\vec{x}^*)])]$$

$$= \phi_{\epsilon,\delta}(x^*) + \epsilon h(x^*) +$$

$$+ \delta[(1 - 5\epsilon^2) \sum_{i=1}^{n-1} \xi(\epsilon x_i^*) + (1 - C\epsilon)\xi(\epsilon[x_n^* + \eta(\vec{x}^*)])].$$

Since $h, \xi \ge 0$, for $\epsilon \le \min\{5^{-1/2}, C^{-1}\}$ we have $\phi_{\epsilon,\delta}(x^*) \le 0$. Let ϵ, δ go to zero to conclude that $\underline{u}(x) \le \bar{u}(x)$.

□

Remarks

(i) The proof of Lemma 4.2 is an adaptation of the uniqueness proof of Evans [9]. Evans proved the uniqueness of solutions to (1.2) in a bounded domain with Dirichlet boundary condition. In the case of an unbounded domain, (4.6) replaces the Dirichlet condition, and the uniform lower bound imposed in (4.6) is necessary. We give the following simple one-dimensional example to illustrate this point. Let $h(x) = \frac{1}{2} x^2$, $n = 1$. Then (1.1) is

$$\max\{u(x) - u_{xx}(x) - \frac{1}{2} x^2 \quad , \quad u_x(x) - 1\} = 0 \quad , \quad x \in R$$

and the unique positive solution is given by

$$u(x) = \begin{cases} \frac{1}{2} x^2 + 1 - e^{x-2} & , \quad x \le 2 \\ x & , \quad x \ge 2 \end{cases}$$

Also for any $k \le -1/2$, $u(x) = x + k$ solves (1.1).

(ii) The solution u has the following representation (see [18]).

(4.23) $$u(x) = U(x) + \int_{-\infty}^{x_n} [v(\bar{x},\xi) - V(\bar{x},\xi)]d\xi$$

where U,V are the unique polynomially growing solutions to

(4.24) $$U(x) - \Delta U(x) = h(x) \quad , \quad x \in R^n$$

and

(4.25) $$V(x) - \Delta V(x) = \frac{\partial}{\partial x_n} h(x) \quad , \quad x \in R^n .$$

5. FURTHER REGULARITY OF THE FREE BOUNDARY.

In this section we obtain the smoothness of the free boundary $\partial\mathcal{C}$ by verifying the hypothesis of a theorem of Cafarelli [5] (also see pp. 129 and pp. 162 in [10]). Let p be given by (4.17). Then (2.1) and the implicit function theorem imply that p is twice continuously differentiable. We need the following additional assumption:

(5.1) $$\sup_{y \in R^{n-1}} |\Delta p(y)| < \infty .$$

Lemma 5.1. *For every* $y \in R^{n-1}$

(5.2) $$p(y) < q(y)$$

Proof: First, consider the following equation

(5.3) $$z(x) - \Delta z(x) = \frac{\partial}{\partial x_n} h(x) \quad , \quad x_n < p(\bar{x})$$

with the boundary condition

(5.4) $$z(y,p(y)) = 1 \quad , \quad y \in R^{n-1} .$$

We claim that there is a constant $K > 0$ such that

(5.5) $$\sup_{y \in R^{n-1}} z(y,p(y) - \epsilon) \leq 1 - K\epsilon$$

for sufficiently small $\epsilon > 0$. We prove (5.5) by constructing an appropriate supersolution to (5.3) and (5.4).

Set

$$c_1 = \sup_{y \in R^{n-1}} |\Delta p(y)|$$

$$c_2 = \frac{1}{2}\,[(c_1^2 + 4)^{1/2} - c_1],$$

and for $x \in R^n$, define

$$\varphi(x) = 1 + \alpha(x_n - p(\bar{x})) + c_1\alpha[1 - \exp c_2(x_n - p(\bar{x}))],$$

where α is as in (2.3). We calculate directly

(5.6)

$$\varphi(x) - \Delta\varphi(x) = 1 + \alpha(x_n - p(\bar{x})) + c_1\alpha + \alpha\Delta p(\bar{x})[1 - c_1c_2 \exp c_2(x_n - p(\bar{x}))]$$
$$+ (c_2^2 + c_2^2|\nabla p(\bar{x})|^2 - 1)c_1\alpha \exp c_2(x_n - p(\bar{x})).$$

Since $c_2 > 0$ and $c_1c_2 < 1$, we have

$$1 - c_1c_2 \exp c_2(x_n - p(\bar{x})) \geq 0 \quad \text{if} \quad x_n \leq p(\bar{x}),$$

and consequently for any $x_n \leq p(\bar{x})$ the following inequality holds:

$$\alpha\Delta p(\bar{x})[1 - c_1c_2 \exp c_2(x_n - p(\bar{x}))] \geq -\alpha c_1[1 - c_1c_2 \exp c_2(x_n - p(\bar{x}))].$$

Substitute the above inequality into (5.6) and then use (2.3), to obtain

$$\varphi(x) - \Delta\varphi(x) \geq 1 + \alpha(x_n - p(\bar{x})) + c_1\alpha - c_1\alpha[1 - c_1c_2 \exp c_2(x_n - p(\bar{x}))]$$
$$+ (c_2^2 + c_2^2|\nabla p(\bar{x})|^2 - 1)c_1\alpha \exp c_2(x_n - p(\bar{x}))$$
$$\geq 1 + \alpha(x_n - p(\bar{x})) + c_1\alpha[c_2^2 + c_2c_1 - 1]\exp c_2(x_n - p(\bar{x}))$$
$$= 1 + \alpha(x_n - p(\bar{x}))$$
$$\geq \frac{\partial}{\partial x_n} h(x).$$

Hence φ is a supersolution to (5.3), and φ satisfies the boundary condition (5.4). Since φ is growing linearly, we can use the maximum principle to arrive at

$$z(x) \leq \varphi(x) \quad , \quad x_n \leq p(\bar{x}).$$

In particular,

$$z(y,p(y) - \epsilon) \leq \varphi(y,p(y) - \epsilon)$$

$$= 1 - \alpha\epsilon + c_1\alpha[1 - \exp(-c_2\epsilon)]$$

$$\leq 1 - \alpha\epsilon + c_1c_2\alpha\epsilon$$

$$= 1 - \alpha(1 - c_1c_2)\epsilon.$$

Thus, (5.5) holds with $K = (1 - c_1c_2)\alpha > 0$.

We continue by showing that $p \leq q$. We then obtain the strict inequality by using (5.5). Now suppose $p(y^*) > q(y^*)$ for some $y^* \in R^{n-1}$. Then, in a neighbourhood of $(y^*,p(y^*))$, we have $v \equiv 1$ and $\Delta v \equiv 0$. Hence (1.3) implies that there is $\delta > 0$ such that

$$0 \geq (v - \Delta v - \frac{\partial}{\partial x_n} h)(y^*,p(y^*) - \delta)$$

$$= 1 - \frac{\partial}{\partial x_n} h(y^*,p(y^*) - \delta)$$

$$= \int_{p(y^*)-\delta}^{p(y^*)} \frac{\partial^2}{\partial x_n^2} h(y^*,\xi)d\xi > 0 \quad .$$

Therefore, $p \leq q$.

Since $v_n \geq 0$, we have $v(y,p(y)) \leq v(y,q(y)) \leq 1$. Hence v is a subsolution to (5.3), (5.4), and the maximum principle implies that $v(x) \leq z(x)$ whenever $x_n \leq p(\bar{x})$. In particular, (5.5) yields

$$v(y,p(y) - \epsilon) \leq 1 - K\epsilon.$$

Using the identities $v(y,q(y)) = 1$, $v_n(y,q(y)) = 0$, and the Lipschitz continuity of v_n, rewrite the above inequality as

$$1 - K\epsilon \geq v(y,p(y) - \epsilon)$$

$$= v(y,q(y)) - \int_{p(y)-\epsilon}^{q(y)} v_n(y,\xi)d\xi$$

$$= 1 + \int_{p(y)-\epsilon}^{q(y)} \int_{\xi}^{q(y)} v_{nn}(y,r)dr \, d\xi$$

$$\geq 1 - c(y)[q(y) - p(y) + \epsilon]^2,$$

where $c(y) > 0$ depends on the sup-norm of v_{nn} in a neighbourhood of y. Hence

$$q(y) - p(y) \geq \frac{\sqrt{k\epsilon}}{c(y)} - \epsilon$$

for every sufficiently small positive ϵ.

□

Theorem 5.2. *Suppose that* $\frac{\partial}{\partial x_n} h \in C^{m+\alpha}(R^n)$ *for some* $m \geq 2$, $\alpha \in (0,1)$. *Then the free boundary* $\partial\mathcal{C}$ *is of class* $C^{m+1+\alpha}$.

Proof. Let $x^* \in \partial\mathcal{C}$. The previous lemma together with (2.3) yield

$$\frac{\partial}{\partial x_n} h(x) > 1 \geq v(x)$$

for every x sufficiently close to x^*. Hence there is $\delta = \delta(x^*) > 0$ such that

$$H(x) = \frac{\partial}{\partial x_n} h(x) - v(x) > 0 \quad , \quad x \in B_\delta(x^*)$$

and

$$\max\{-\Delta v(x) - H(x),\ v(x) - 1\} = 0 \quad , \quad x \in B_\delta(x^*).$$

Hence by Theorem 3 [5], q is continuously differentiable and v is twice continuously differentiable in $\mathcal{C}$ up to the boundary (see also Theorem 3.10, pp. 162 [10]).

The result of Kinderlehrer and Nirenberg [19] applies to this situation, yielding the stated result (see also Theorem 1.1, pp. 129 [10]).

□

Remark. For a one-dimensional, time-inhomogenous stopping time problem, van Moerbeke [25] proved the smoothness of the free boundary under structural assumptions quite different from ours. He also obtained results like Lemma 5.1 (Section 2.6 [26]).

6. THE SINGULAR CONTROL PROBLEM.

Consider the stochastic process $x_t = (x_t^1, \ldots, x_t^n) \in R^n$ defined by

$$(6.1)\qquad x_t^i = x^i + \sqrt{2}\,W_t^i \quad , \quad i = 1,\dots, n-1 \quad , \quad t \ge 0$$

$$(6.2)\qquad x_t^n = x^n + \sqrt{2}\,W_t^n - \xi(t) \quad , \quad t \ge 0$$

where $x = (x^1,\dots, x^n)$ is the initial condition, $W_t = (W_t^1,\dots, W_t^n)$ is an n-dimensional standard Brownian motion, and $\xi(t)$, the control process, is non-decreasing, left-continuous, and adapted to the augmentation by null sets of the filtration generated by W, with $\xi(0) = 0$.

For a given initial condition x, the control problem is to pick a control process so as to minimize the pay-off functional

$$(6.3)\qquad J(x,\xi(\cdot)) = E \int_0^\infty e^{-t}[h(x_t)dt + d\xi(t)]$$

where E is the mathematical expectation. Finally we define the value function $u^*(x)$ to be the infimum over all control processes satisfying the conditions listed above, i.e.,

$$(6.4)\qquad u^*(x) = \inf_{\xi(\cdot)} J(x,\xi(\cdot)).$$

Theorem 6.1. *The value function* u^* *is the unique solution of* (1.1) *satisfying* (4.6). *Moreover, the infimum in* (6.4) *is achieved by the left-continuous process* ξ^* *given by*

$$(6.5)\quad \xi^*(t) = \max_{0\le\tau\le t} [x_n + \sqrt{2}\,W_\tau^n - q(x^1 + \sqrt{2}\,W_\tau^1,\dots, x^{n-1} + \sqrt{2}\,W_\tau^{n-1})]^+, \ t>0,$$

where q *is as in Section* 3.

Proof. Let u be the solution of (1.1) satisfying (4.6). An application of Itô's rule for semimartingales [21] yields

$$(6.6)\quad u(x) = E\, e^{-t\wedge\tau_m} u(x_{t\wedge\tau_m}) + E\int_0^{t\wedge\tau_m} e^{-s}[u(x_s) - \Delta u(x_s) - h(x_s)]ds$$

$$+ E\int_0^{t\wedge\tau_m} e^{-s}h(x_s)ds + E\int_0^{t\wedge\tau_m} e^{-s}\frac{\partial}{\partial x_n}u(x_s)d\xi(s)$$

$$+ E\sum_{0\le s<t\wedge\tau_m} e^{-s}[u(x_s) - u(x_{s+}) + \frac{\partial}{\partial x_n}u(x_s)(\xi(s+) - \xi(s))]$$

where $t \wedge \tau_m = \min\{t, \tau_m\}$ and τ_m is given by

$$\tau_m = \inf\{s \geq 0 : |x_s| > m\}.$$

Using the equation (1.1) and the convexity of u in the x_n-variable, we obtain

$$u(x) \leq E\left\{e^{-t\wedge\tau_m} u(x_{t\wedge\tau_m}) + \int_0^{t\wedge\tau_m} e^{-s}[h(x_s)ds + d\xi(s)]\right\} \tag{6.7}$$

for any control process $\xi(\cdot)$. First let $\xi \equiv 0$. Then, $x_t = x + \sqrt{2}W_t$, and due to the growth condition (4.6), we have

$$\lim_{t\uparrow\infty} \lim_{m\to\infty} E\, e^{-t\wedge\tau_m} u(x + \sqrt{2}W_{t\wedge\tau_m}) = 0$$

Hence

$$u(x) \leq U(x) = E\int_0^\infty e^{-s} h(x + \sqrt{2}W_s)ds. \tag{6.8}$$

Therefore for an arbitray $\xi(\cdot)$, we have

$$\begin{aligned} E\, e^{-t\wedge\tau_m} u(x_{t\wedge\tau_m}) &\leq E\, e^{-t\wedge\tau_m} U(x_{t\wedge\tau_m}) \\ &= E\, e^{-t\wedge\tau_m} u^*(x_{t\wedge\tau_m}) + \\ &\quad E\, e^{-t\wedge\tau_m}[U(x_{t\wedge\tau_m}) - u^*(x_{t\wedge\tau_m})]. \end{aligned} \tag{6.9}$$

We estimate each term separately. Since our goal is to establish the inequality

$$u(x) \leq J(x,\xi(\cdot)), \tag{6.10}$$

we may assume that $J(x,\xi(\cdot)) < \infty$. This implies that

$$\lim_{t\to\infty} \lim_{m\to\infty} E\int_{t\wedge\tau_m}^\infty e^{-s}[h(x_s)ds + d\xi(s)] = 0.$$

The integral in the above expression is equal to $E\, e^{-t\wedge\tau_m} J(x_{t\wedge\tau_m}, \xi^{t,m}(\cdot))$, where $\xi^{t,m}(\cdot)$ is just a translation of $\xi(\cdot)$. Hence,

$$\ell im_{t\to\infty}\ \ell im_{m\to\infty}\ E\, e^{-t\wedge\tau_m} u^*(x_{t\wedge\tau_m}) = 0. \tag{6.11}$$

We continue by showing that $F(x) = U(x) - u^*(x)$ is non-decreasing in the x_n-variable. Set $e_n = (0,0,\ldots, 0,1) \in R^n$. For $\epsilon > 0$,

$$[U(x) - u^*(x)] - [U(x - \epsilon e_n) - u^*(x - \epsilon e_n)] \geq$$

$$\geq \inf_{\xi(\cdot)} [U(x) - J(x,\xi(\cdot)) - U(x - \epsilon e_n) + J(x - \epsilon e_n, \xi(\cdot))]$$

$$= \inf_{\xi(\cdot)} E \int_0^\infty e^{-s}[h(x + \sqrt{2}W_s) - h(x + \sqrt{2}W_s - \xi(s)e_n) - h(x + \sqrt{2}W_s - \epsilon e_n)$$

$$+ h(x + \sqrt{2}W_s - \xi(s)e_n - \epsilon e_n)]ds.$$

Convexity of h in the x_n-variable implies that the integrand in the last expression is non-negative. Thus, $\frac{\partial}{\partial x_n} F(x) \geq 0$. Using this, we obtain

$$\ell im \sup_{t\to\infty}\ \ell im \sup_{m\to\infty}\ E\, e^{-t\wedge\tau_m}[U(x_{t\wedge\tau_m}) - u^*(x_{t\wedge\tau_m})] = \tag{6.12}$$

$$= \ell im \sup_{t\to\infty}\ \ell im \sup_{m\to\infty}\ E\, e^{-t\wedge\tau_m} F(x + \sqrt{2}W_{t\wedge\tau_m} - \xi(t \wedge \tau_m)e_n)$$

$$\leq \ell im \sup_{t\to\infty}\ \ell im \sup_{m\to\infty}\ E\, e^{-t\wedge\tau_m} F(x + \sqrt{2}W_{t\wedge\tau_m})$$

$$\leq \ell im \sup_{t\to\infty}\ \ell im \sup_{m\to\infty}\ E\, e^{-t\wedge\tau_m} U(x + \sqrt{2}W_{t\wedge\tau_m})$$

$$= 0.$$

The last equality follows from the fact that U grows at most polynomially. Combine (6.9), (6.11), (6.12) to conclude that

$$(6.13)\qquad \lim_{t\to\infty}\lim_{m\to\infty} E\, e^{-t\wedge\tau_m} u(x_{t\wedge\tau_m}) = 0$$

for any ξ with $J(x,\xi(\cdot)) < \infty$. Using (6.13), (6.7) and the finiteness of $J(x,\xi(\cdot))$, we obtain (6.10). Hence,

$$(6.14)\qquad u(x) \le u^*(x).$$

To complete the proof of the theorem, it suffices to show that

$$(6.15)\qquad u(x) \ge J(x,\xi^*(\cdot)).$$

Let x_t^* be the solution of (6.1) and (6.2) with control process ξ^* given by (6.5). The following follows from (6.5)

$$(6.16)\qquad \begin{cases} \text{(i)} & x_t^* \in \overline{\mathscr{C}} \quad \text{for} \quad t > 0 \\ \text{(ii)} & x^*, \xi^* \text{ are continuous on } t > 0 \\ \text{(iii)} & \xi^*(0+) = [x_n - q(\bar{x})]^+ \\ \text{(iv)} & \int_0^t d\xi^*(s) = \int_0^t 1_{\{s\,:\,x_s^* \in \partial\mathscr{C}\}} d\xi^*(s) \end{cases}$$

where 1_A is the indicator set A. Using (6.6), (6.16), (1.1), and the positivity of u, we obtain

$$u(x) = E\, e^{-t\wedge\tau_m} u(x^*_{t\wedge\tau_m}) + \int_0^{t\wedge\tau_m} e^{-s}[h(x_s^*) + d\xi^*(s)]\cdot$$

Let t,m go to infinity in the above expression to arrive at (6.15). □

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