DET = DET -- A REMARK ON THE DISTRIBUTIONAL DETERMINANT

by

Stefan Müller
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

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Stefan Müller
Department of Mathematics, Carnegie Mellon University
Pittsburgh, PA 15213-3890, USA

Abstract — A proof of a conjecture of J.M.Ball concerning the distributional determinant is given and an application to nonlinear elasticity is indicated.

INTRODUCTION. — The aim of this note is to prove a conjecture of J.M. Ball concerning the distributional determinant. Consider for illustration a mapping u of an open bounded set $\Omega \subset \mathbf{R}^2$ into $\mathbf{R}^2$. The Jacobian determinant of u is given by

$$\det Du(x) = u^1_{,1}(x)u^2_{,2}(x) - u^1_{,2}(x)u^2_{,1}(x),$$

where upper indices denote components and where lower indices with comma refer to partial derivatives. For the weak continuity properties of the Jacobian determinant and the existence theory in nonlinear elasticity it is of crucial importance that $\det Du$ can be expressed as a divergence. Let

$$\operatorname{Det} Du = (u^1 u^2_{,2})_{,1} - (u^1 u^2_{,1})_{,2}.$$

Clearly $\operatorname{Det} Du = \det Du$ for $u \in C^2$; by approximation that identity is easily seen to hold in the sense of distributions if u merely lies in the Sobolev space $W^{1,2}(\Omega;\mathbf{R}^2)$. By the Sobolev imbedding theorem, $\operatorname{Det} Du$ is well defined as a distribution if $u \in W^{1,4/3}(\Omega;\mathbf{R}^2)$, but the identity det = Det fails in general if one only assumes $u \in W^{1,p}(\Omega;\mathbf{R}^2), \forall p < 2$. Indeed, letting Ω be the unit ball and $u(x) = \frac{x}{|x|}$ one finds that $\det Du = 0$ a.e., while $\operatorname{Det} Du = \pi\delta_0$. The fact that in this example the desired identity fails only by a Dirac mass led Ball [1] to conjecture that $\det Du = \operatorname{Det} Du$ whenever $\operatorname{Det} Du$ is an L^1-function. Here we give a proof of that conjecture, in fact of a slightly stronger result (see Theorem 1 below).

For the statement in the n-dimensional case recall that for an n by n matrix F its adjugate $\operatorname{adj} F$ is the transpose of the matrix of its cofactors (so that $F \operatorname{adj} F = \det F \, \mathrm{Id}$) and that for any function $u : \Omega \subset \mathbf{R}^n \to \mathbf{R}^n$, $u \in W^{1,n-1}(\Omega; \mathbf{R}^n)$, $\operatorname{adj} Du$ satisfies the identity

$$(\operatorname{adj} Du)^j_{1,j} = 0,$$

in the sense of distributions (see [2]), summation over repeated indices being understood. The distributional determinant is then defined as

$$\operatorname{Det} Du = (u^1 (\operatorname{adj} Du)^j_1)_{,j}.$$

Theorem 1 shows that if $u \in W^{1,n^2/(n+1)}$ (so that $\operatorname{Det} Du$ is defined) and if $\operatorname{Det} Du \in L^1$ then $\operatorname{Det} Du = \det Du$.

MAIN RESULT. — **Theorem 1.** Let $\Omega \subset \mathbf{R}^n$ be open, let $1 \leq p < n$ and let $v \in W^{1,p}(\Omega)$. Assume that $\sigma \in L^q(\Omega; \mathbf{R}^n)$ with $\frac{1}{p} - \frac{1}{n} + \frac{1}{q} \leq 1$ and that the distributional divergence of σ satisfies $\operatorname{div} \sigma \in L^1(\Omega)$. Assume finally that the distribution

$$d = \operatorname{div}(v\sigma),$$

satisfies $d \in L^1(\Omega)$. Then

$$d(x) = Dv(x).\sigma(x) + v(x) \operatorname{div} \sigma(x),$$

for a.e. $x \in \Omega$.

Remarks. - 1. The assumption on q guarantees that $v\sigma \in L^1_{loc}$ so that d is well defined; if $p > n, q = 1$ is sufficient.

2. If d is assumed to be a Radon measure rather than an L^1-function then $Dv.\sigma + v \operatorname{div} \sigma$ equals the *regular* part of d a.e.

3. The arguments used in the proof are entirely local, so $W^{1,p}, L^q$ and L^1 could be replaced by $W^{1,p}_{loc}, L^q_{loc}$ and L^1_{loc}.

4. The distributional determinant is obtained as a special case by letting $v = u^1, \sigma^j = (\operatorname{adj} Du)^j_1, p = n^2/(n+1)$ and $q = p/(n-1) = n^2/(n^2-1)$.

5. Theorem 1 allows one to reformulate the existence theory of Ball [1] using the pointwise determinant $\det Du$ rather than the distributional determinant $\operatorname{Det} Du$. In the two-dimensional case one would minimize in a subset of $\mathcal{A} = \{u \in W^{1,4/3} : \operatorname{Det} Du \in L^1\}$, in the three-dimensional in a subset of $\mathcal{A} = \{u \in W^{1,p} : \operatorname{Adj} Du \in L^q, \operatorname{Det} Du \in L^1\}, p \geq 3/2, 1/p + 1/q \leq 4/3$.

The main ingredient in the proof of Theorem 1 is a version of the Lebesgue point theorem for $W^{1,p}$ functions due to Calderon and Zygmund [3] (see Ziemer [4] for refinements and further references). Denoting by $B_r(x_0)$ the ball of radius r around x_0 and by $⨍$ the average we have

Theorem 2 ([3, Thm. 12]). - Let $\Omega \subset \mathbf{R}^n$ be open, $1 \leq p < n$ and $v \in W^{1,p}(\Omega)$. Then, for a.e. $x_0 \in \Omega$,

$$\lim_{r\to 0} \frac{1}{r} \left\{ ⨍_{B_r(x_0)} |v(x) - v(x_0) - Dv(x_0)(x - x_0)|^{p^*} \, dx \right\}^{1/p^*} = 0,$$

where $p^* = \frac{pn}{n-p}$.

PROOFS. — *Proof of Theorem 1 ($\operatorname{div} \sigma = 0$).* - For the sake of simplicity we first deal with the case $\operatorname{div} \sigma = 0$ which is the one relevant to Ball's conjecture. By the Lebesgue point theorem and Theorem 2 we have, for a.e. $x_0 \in \Omega$,

$$\lim_{r\to 0} ⨍_{B_r(x_0)} |d(x) - d(x_0)| \, dx = 0, \tag{1}$$

$$\lim_{r\to 0} ⨍_{B_r(x_0)} |\sigma(x) - \sigma(x_0)|^q \, dx = 0, \tag{2}$$

$$\lim_{r\to 0} \frac{1}{r} \left\{ ⨍_{B_r(x_0)} |v(x) - v(x_0) - Dv(x_0)(x - x_0)|^{p^*} \, dx \right\}^{1/p^*} = 0. \tag{3}$$

Fix such a point x_0, let $\psi \in C_0^\infty(B_1)$, $\psi \geq 0$, $\int_{\mathbf{R}^n} \psi(y) \, dy = 1$ and let $\psi_r(x) = r^{-n} \psi(\frac{x - x_0}{r})$. By (1)

$$\lim_{r\to 0} d(\psi_r) = \lim_{r\to 0} \int \psi_r(x) d(x) \, dx = d(x_0). \tag{4}$$

On the other hand letting

$$P(x) = v(x_0) + Dv(x_0)(x - x_0),$$

and writing B_r instead of $B_r(x_0)$ we have

$$\begin{aligned} d(\psi_r) &= -\int_\Omega (\psi_r)_{,j}(x) v(x) \sigma^j(x) \, dx \\ &= -\int_{B_r} (\psi_r)_{,j}(x)(v(x) - P(x)) \sigma^j(x) \, dx - \int_{B_r} (\psi_r)_{,j} P(x) \sigma^j(x) \, dx \end{aligned}$$

$$
\begin{aligned}
&= -\int_{B_r} (\psi_r)_{,j}(x)(v(x) - P(x))\sigma^j(x)\,dx + \int_{B_r} \psi_r(x)\, Dv(x_0).\sigma(x)\,dx \\
&= -\int_{B_r} (\psi_r)_{,j}(x)(v(x) - P(x))\sigma^j(x)\,dx + \int_{B_r} \psi_r(x)\, Dv(x_0).(\sigma(x) - \sigma(x_0))\,dx \\
&\quad + \int_{B_r} \psi_r(x)\, Dv(x_0).\sigma(x_0)\,dx. \qquad (5)
\end{aligned}
$$

The last term is of course nothing but $Dv(x_0).\sigma(x_0)$. The first term in the last expression is bounded by

$$r^{-n-1} \sup |D\psi| \int_{B_r} |v(x) - P(x)||\sigma(x)|\,dx,$$

which by Hölder's inequality can be estimated by

$$\frac{C}{r} \left\{ \fint_{B_r} |v(x) - P(x)|^{p^*}\,dx \right\}^{1/p^*} \left\{ \fint_{B_r} |\sigma(x)|^q\,dx \right\}^{1/q}$$

and thus converges to zero as $r \to 0$ by (2) and (3). Finally, the second term in the last identity in (5) is bounded by

$$C|Dv(x_0)| \fint_{B_r} |\sigma(x) - \sigma(x_0)|\,dx$$

and therefore converges to zero as $r \to 0$ by (2). It follows that

$$\lim_{r \to 0} d(\psi_r) = Dv(x_0).\sigma(x_0). \qquad (6)$$

It should be noted that the derivation of (6) only depends on the assumptions on v and σ and not on the hypothesis $d \in L^1$. The theorem (for $\operatorname{div} \sigma = 0$) now follows from (4) and (6).

Proof of Theorem 1 (general case). - Since $\operatorname{div} \sigma \in L^1$ we may assume that in addition to (1) to (3) we also have

$$\lim_{r \to 0} \fint_{B_r(x_0)} |\operatorname{div} \sigma(x) - \operatorname{div} \sigma(x_0)|\,dx = 0. \qquad (7)$$

The only difference in the case $\operatorname{div} \sigma \neq 0$ is that in (5) an additional term

$$\int_{B_r} \psi_r(x) P(x)(\operatorname{div} \sigma)(x)\,dx$$

appears. This term may be rewritten as

$$\int_{B_r} \psi_r(x)(P(x) - P(x_0))(\operatorname{div}\sigma)(x)\,dx \quad + \quad \int_{B_r} \psi_r(x)P(x_0)(\operatorname{div}\sigma(x) - \operatorname{div}\sigma(x_0))\,dx$$
$$+ \quad \int_{B_r} \psi_r(x)P(x_0)(\operatorname{div}\sigma)(x_0)\,dx$$

The last term is just $v(x_0)\operatorname{div}\sigma(x_0)$, the first term is bounded by

$$C|Dv(x_0)|r \fint_{B_r} |\operatorname{div}\sigma|(x)\,dx,$$

and the second by

$$C|v(x_0)| \fint_{B_r} |\operatorname{div}\sigma(x) - \operatorname{div}\sigma(x_0)|\,dx,$$

and in view of (7) both terms converge to zero as $r \to 0$. Thus

$$\lim_{r\to 0} d(\psi_r) = Dv(x_0).\sigma(x_0) + v(x_0)\operatorname{div}\sigma(x_0), \tag{8}$$

and the theorem follows from (4).

Proof of Remark 2. - If d is a Radon measure and d_{reg} its regular part then, by the differentiability theorem for measures (see, e.g. [5, Thm. 8.6]),

$$\lim_{r\to 0} d(\psi_r) = d_{reg}(x_0)$$

for a.e. x_0, and the remark follows from (8).

References

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