A HYPERBOLIC THEORY FOR THE EVOLUTION OF PLANE CURVES

by

Morton E. Gurtin
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

and

Paolo Podio-Guidugli
Universita di Roma 2
Italy

Research Report No. 89-65 2

November 1989

A hyperbolic theory for the evolution of plane curves

Morton E. Gurtin, Carnegie Mellon University
Paolo Podio-Guidugli, Universita di Roma 2

1. Introduction.

It is the purpose of this paper to develop a theory for the evolution of plane curves which is based on balance laws for mass and momentum in conjunction with constitutive equations appropriate to a phase interface, and which leads to hyperbolic evolution equations. We have three reasons for presenting such a theory:

(1) The form of the balance laws is not at all obvious, and, in fact, represents an intriguing problem in continuum mechanics whose solution requires a nonstandard conceptual framework.

(2) The *parabolic* theory for the evolution of plane curves, which in its simplest form is based on the *curve-shortening equation*[1]

$$V = \kappa \tag{1.1}$$

(relating the normal velocity V and curvature κ) has been extremely successful, providing geometers with great insight; to our knowledge there is no *hyperbolic* version of (1.1).

(3) Crystals of helium in their melt exhibit interfacial oscillations[2] referred to as *melting-freezing waves.* Motivated by Andreev and Parshin's [AP] classical discussion of such waves, a continuum model was developed in [G4][3] for a rigid crystal in an incompressible,[4] inviscid melt. This model, which we shall refer to

---

[1] Cf., e.g., Brakke [B], Gage and Hamilton [GH], and Grayson [Gr], as well as the references therein and those cited in Section 5 of [AG].

[2] Such waves were predicted by Andreev and Parshin [AP] in 1978 and exhibited experimentally by Keshishev, Parshin, and Babkin [KPB] in 1979.

[3] using, as a basis, a framework developed in [G1,G2,G3,AG,GS].

[4] Andreev and Parshin [AP] note that the phase velocity of melting-freezing waves is

as the **CM model**, leads to a free-boundary problem for the evolution of the interface; coupling between the interface and the melt renders this problem difficult, and it would seem useful to have a simple model in which the motion of the interface is governed by a hyperbolic analog of (1.1).

We here develop a theory in which *only the interface is endowed with mathematical structure*:[5] we model the inertia of the melt through an *"effective"* inertia for the interface, with the melt considered only as a source of atoms for the crystallization process; and we characterize this inertia constitutively through the corresponding interfacial mass density. As in the CM model, we restrict our attention to a purely mechanical[6] theory, and, to avoid the geometric complications that accompany evolving surfaces, to a two-dimensional theory in which the interface evolves as a plane curve.

Because of the presence of an "effective inertia", the balance laws for mass and momentum are not obvious. We derive these laws as a consequence of the requirement that the mechanical energy production — the rate at which the kinetic energy is changing minus the power expended by capillary forces — be invariant under Galilean changes in observer.

Constitutive equations, of the form derived in [G1] as a consequence of thermodynamical arguments, are assumed for the relevant interfacial fields. These equations and the underlying balance laws yield a single equation for the evolution of the interface:

$$\rho(\theta)v^{\circ} + \beta(\theta)v = [\psi(\theta) + \psi''(\theta)]\kappa - F. \tag{1.2}$$

Here $\psi(\theta)$, $\rho(\theta)$, and $\beta(\theta)$ are the energy, effective density, and

---

generally well below the sound velocity.

[5] This is the point of view taken by Brower, Kessler, Koplik, and Levine [BK] and Ben-Jacob, Goldenfeld, Langer, and Schon [BG], who use equations involving only the interface to model interfacial evolution governed by bulk diffusion.

[6] As noted by Maris and Andreev [MA], for superconductors such as solid helium and its melt, solidification is "essentially a mechanical process, rather than a thermal process as it is for ordinary materials".

kinetic coefficient for the interface; F is a constant which represents the driving force for crystallization; v° is the time derivative of v following the interface; θ is the angle to the interface-normal $\mathbf{m}$. We assume that

$$\psi(\theta) + \psi''(\theta) > 0, \qquad \rho(\theta) > 0 \tag{1.3}$$

for all values of θ; this ensures that when (1.2) is combined with standard kinematical conditions for the evolution of a plane curve, the resulting partial differential equations are *hyperbolic*.

The equation (1.2) with $\beta(\theta) = 0$, *linearized about a flat interface at equilibrium*, reduces to the classical wave equation:

$$(\psi + \psi'')_0 h_{xx} = h_{tt}. \tag{1.4}$$

This equation has *oscillatory solutions* of the form

$$h(x,t) = Ce^{i\lambda x}e^{-i\omega t} \tag{1.5}$$

with

$$\omega^2 = \frac{(\psi + \psi'')_0 \lambda^2}{\rho_0}. \tag{1.6}$$

As noted by Andreev and Parshin [AP] (cf. [G4]), the CM model also has solutions of the form (1.5), but there ω^2 is proportional[7] to λ^3, rather than λ^2 as predicted by (1.6). Thus the agreement between the simple model developed here and the more detailed CM model *is at most qualitative;* because of the simplified modelling of inertia, this is not unexpected.

For completeness we discuss the form the basic equations take

---

[7] This proportionality of ω^2 to λ^3 is confirmed by the experiments of Keshishev, Parshin, and Babkin [KPB].

when the interface is an evolving *surface* in $\mathbb{R}^3$. There (1.2) is replaced by

$$\rho(m)v^\circ + \beta(m)v = \psi(m)\kappa + \psi_{mm}(m)\cdot L - F, \tag{1.7}$$

where L is the curvature tensor, $\kappa = \mathrm{tr}L$ is twice the mean curvature, and $\psi_{mm}(m)$ is the second gradient of $\psi(m)$ on the surface of the unit ball.

We solve the problem (in $\mathbb{R}^2$ and $\mathbb{R}^3$) of radially symmetric crystallization of an isotropic crystal in an infinite melt. If the phase interface is initially at rest, then:

(i) for $F \geq 0$ the crystal melts completely in finite time;

(ii) for $F < 0$ there is a *critical radius* $R_{crit} := \psi/|F|$ ($R_{crit} := 2\psi/|F|$ in $\mathbb{R}^3$) such that a crystal of radius $R(0) < R_{crit}$ melts in finite time, a crystal of radius $R(0) > R_{crit}$ grows unboundedly as $t \to \infty$.

An analogous problem is discussed in [G4] for the CM model in $\mathbb{R}^3$. The results are qualitatively the same as those described in (i) and (ii). In fact, if we identify F with the constant $\Psi_c + P - \zeta(\Psi + P)$ of the CM model, where Ψ_c and Ψ are the crystal and melt energies, ζ is the ratio of crystal density to melt density, and P is the far-field pressure in the melt, then the critical radii of the two theories coincide. As would be expected, the two theories exhibit quantitative differences: for example, during unbounded growth the radius grows asymptotically as t^2 within the present theory and as t within the CM model.

Although the CM model does exhibit oscillatory behavior, it is not clear whether or not shocks and other propagating discontinuities are possible.[8] To the contrary, such phenomena are generated within the present theory. We study the propagation of fronts across which the curvature is discontinuous. We show that, in the presence of

---

[8] Rogers [1989] shows that such phenomena are not possible within the *linearized* CM theory.

anisotropy, fronts whose amplitude is sufficiently large and of the right sign grow to infinity in finite time, strongly suggesting that the interface develops a corner. Guided by other theories[9] of hyperbolic behavior, this result seems to indicate that there is global existence of classical solutions of (1.2) for initial data that are both sufficiently small and sufficiently smooth, but that smooth solutions corresponding to large data develop singularities in finite time.

---

[9] Cf., e.g., Renardy, Hrusa, and Nohel [RHN].

2. Crystals.

We consider an *infinite* **crystal lattice** modelled as a two-dimensional continuum, in fact as $\mathbb{R}^2$. A **crystal** C is then a compact subset of the lattice with boundary, ∂C, a smooth, simple closed curve. ∂C represents the **interface** between the crystal and its **melt**; we write $m(x)$ for the outward unit normal to ∂C and define a unit tangent $\ell(x)$ so that $\{\ell(x),m(x)\}$ is a positively oriented basis of $\mathbb{R}^2$ (cf. Figure 1). We let ds denote the element of arc length on ∂C and write f_s for the derivative, sometimes partial, of f with respect to arc length on ∂C. (Our convention is that arc length increase in the direction of ℓ.) We then have the Frenet formulas

$$m_s = -\kappa\ell, \qquad \ell_s = \kappa m \tag{2.1}$$

with $\kappa(x)$ the **curvature** of ∂C. We define the **angle** $\theta(x)$, as a smooth function of x, through

$$m = (\cos\theta,\sin\theta), \qquad \ell = (\sin\theta,-\cos\theta). \tag{2.2}$$

Our goal is to model situations in which crystals grow or shrink by processes such as solidification and melting. We therefore consider crystals $C(t)$ that evolve with time t, under the assumption that $\partial C(t)$ is a smooth evolving curve (in the sense of [AG]). We write $V(x,t)$ for the **normal velocity** of $\partial C(t)$ in the direction $m(x,t)$, so that

$$\mathbf{v}(x,t) = V(x,t)m(x,t) \tag{2.3}$$

represents the **velocity** of $\partial C(t)$. Fix t and $x\in\partial C(t)$ and (for τ sufficiently close to t) let $y(\tau)$ denote the curve that passes through x at time t and has

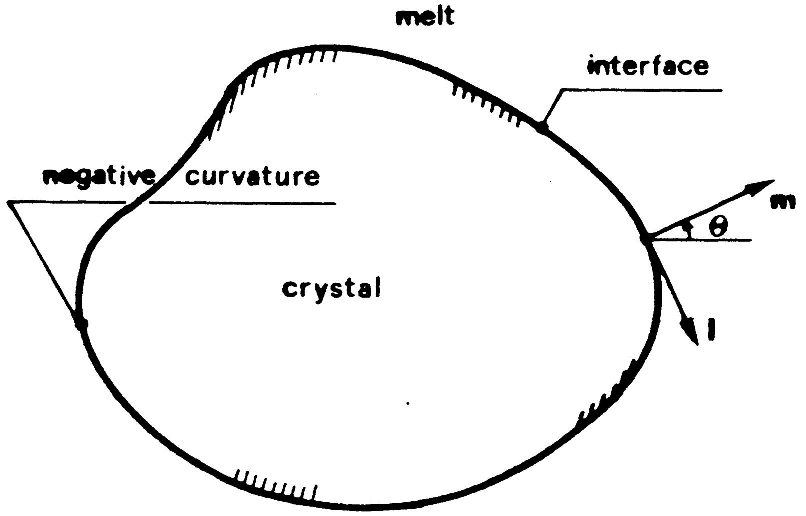

Figure 1

Sign conventions for interfacial motions

$$y^{\cdot}(\tau) = v(y(\tau),\tau)$$

($y^{\cdot}(\tau) = dy(\tau)/d\tau$; we use this notation for functions of time alone). Then the **normal time-derivative** $\Phi^{\circ}(x,t)$ (following $\partial C(t)$) of a scalar or vector function $\Phi(x,t)$ is defined by

$$\Phi^{\circ}(x,t) = (d/d\tau)\Phi(y(\tau),\tau)\big|_{\tau=t}. \tag{2.4}$$

The identities

$$\begin{aligned} \theta^{\circ} &= \boldsymbol{\ell}^{\circ}\cdot\mathbf{m} = -\mathbf{m}^{\circ}\cdot\boldsymbol{\ell} = v_s, \\ \boldsymbol{\ell}^{\circ} &= \mathbf{m}\theta^{\circ} = v_s + \kappa v\boldsymbol{\ell}. \end{aligned} \tag{2.5}$$

are standard.[10]

By an **interfacial chunk** we mean a smoothly evolving curve $\mathcal{A}(t)$ with $\mathcal{A}(t)\subset\partial C(t)$ at each time t; we say that $\mathcal{A}(t)$ **evolves normally** if its endpoints $x_1(t)$ and $x_2(t)$ evolve with the normal velocity of the interface:

$$x_1^{\cdot}(t) = v(x_1(t),t), \qquad x_2^{\cdot}(t) = v(x_2(t),t). \tag{2.6}$$

For any function $\Phi(x,t)$ we write

$$\int_{\partial\mathcal{A}(t)}\Phi = \Phi(x_2(t),t) - \Phi(x_1(t),t). \tag{2.7}$$

We then have the identities[11]

---

[10] Cf., e.g., [AG], eqs. (2.4), (2.6), (2.18).

[11] $(2.8)_2$ is generally not true if the chunk does *not* evolve normally: an additional term is needed to account for the flux of Φ across $\partial\mathcal{A}(t)$ (cf., e.g., [AG], eq. (2.34)).

$$\begin{aligned} \int_{\partial\mathcal{S}(t)} \Phi &= \int_{\mathcal{S}(t)} \Phi_s \, ds, \\ (d/dt)\int_{\mathcal{S}(t)} \Phi \, ds &= \int_{\mathcal{S}(t)} (\Phi^\circ - \Phi\kappa v)\, ds. \end{aligned} \tag{2.8}$$

For convenience, we will generally omit the argument t when writing such integrals.

3. Capillary force. Inertia. Balance laws.

3.1. Basic concepts.

We describe the **micromechanics** of an evolving crystal $C(t)$ by four functions of $x \in \partial C(t)$ and t:

$C(x,t)$	**interfacial stress,**
$b(x,t)$	**interaction force,**
$\rho(x,t)$	(effective) **inertial density,**
$r(x,t)$	**mass supply.**

$C(x,t)$ represents the force *within the interface* exerted across x at time t; if we let $\mathcal{S}^+$ and $\mathcal{S}^-$, respectively, denote right and left neighborhoods of x in $\partial C(t)$, then $C(x,t)$ is the force exerted *on* $\mathcal{S}^-$ *by* $\mathcal{S}^+$. Concerning the remaining functions, $b(x,t)$ represents the net force exerted on the interface by the bulk material of the crystal and the melt; $\rho(x,t)$ gives the inertial density of the interface; $r(x,t)$ represents the *rate* at which mass is supplied to the interface.

We characterize forces by the manner in which they expend power, inertia by the manner in which it affects kinetic energy. In particular, we assume that $C(x,t)$ and $b(x,t)$ expend power through the velocity[12] $v(x,t)$, and that it is this velocity that is appropriate to the kinetic energy of the interface. Precisely, given any normally-evolving interfacial chunk $\mathcal{S}(t)$,

$$\tfrac{1}{2}\int_{\mathcal{S}} \rho|v|^2\, ds. \tag{3.1}$$

---

[12] Cf. [G1], eq. (1.3) and §3.

is the **kinetic energy** of $\ell(t)$,

$$\tfrac{1}{2}\int_{\ell} r|v|^2\,ds. \tag{3.2}$$

is the **rate at which kinetic energy is supplied** to $\ell(t)$, and

$$\int_{\partial\ell} C\cdot v \;+\; \int_{\ell} b\cdot v\,ds \tag{3.3}$$

is the **power expended** on $\ell(t)$. We will refer to

$$\mathcal{E}(\ell)(t) = d/dt\Big\{\tfrac{1}{2}\int_{\ell}\rho|v|^2\,ds\Big\} - \tfrac{1}{2}\int_{\ell} r|v|^2\,ds - \int_{\partial\ell} C\cdot v - \int_{\ell} b\cdot v\,ds \tag{3.4}$$

as the **mechanical energy-production** at time t. The first law of thermodynamics requires that this quantity be balanced by the addition of heat and by changes in the internal energy.

3.2. Invariance under observer changes. The capillary balance law. Balance of interfacial mass.

A basic assumption of our theory is that

> *the mechanical energy-production be invariant under Galilean changes in observer;*

precisely, we assume that, given any normally-evolving interfacial chunk $\ell(t)$ and any time t,

$$\begin{aligned}(d/dt)\Big\{\tfrac{1}{2}\int_{\ell}\rho|v+c|^2\,ds\Big\} - \tfrac{1}{2}\int_{\ell} r|v+c|^2\,ds - \int_{\partial\ell} C\cdot(v+c) - \int_{\ell} b\cdot(v+c)\,ds \\ = (d/dt)\Big\{\tfrac{1}{2}\int_{\ell}\rho|v|^2\,ds\Big\} - \tfrac{1}{2}\int_{\ell} r|v|^2\,ds - \int_{\partial\ell} C\cdot v - \int_{\ell} b\cdot v\,ds\end{aligned} \tag{3.5}$$

for any constant vector c. Here c is the velocity defining an arbitrary observer change, and underlying (3.5) is the tacit assumption that ρ, C, r, and b be invariant, while v transform to $v + c$.

If we expand the left side of (3.5) in terms of $\mathbf{c}$, we find that

$$\mathbf{c}\cdot\Big\{(d/dt)\int_{\mathcal{A}}\rho\mathbf{v}\,ds - \int_{\mathcal{A}} r\mathbf{v}\,ds - \int_{\partial\mathcal{A}}\mathbf{C} - \int_{\mathcal{A}}\mathbf{b}\,ds\Big\} + \tfrac{1}{2}|\mathbf{c}|^2\Big\{(d/dt)\int_{\mathcal{A}}\rho\,ds - \int_{\mathcal{A}} r\,ds\Big\} = 0; \tag{3.6}$$

since $\mathbf{c}$ is *arbitrary*, this yields **balance of interfacial mass**

$$(d/dt)\int_{\mathcal{A}}\rho\,ds = \int_{\mathcal{A}} r\,ds \tag{3.7}$$

and the **capillary balance law**

$$(d/dt)\Big\{\int_{\mathcal{A}}\rho\mathbf{v}\,ds\Big\} - \int_{\mathcal{A}} r\mathbf{v}\,ds = \int_{\partial\mathcal{A}}\mathbf{C} + \int_{\mathcal{A}}\mathbf{b}\,ds. \tag{3.8}$$

The relations (3.7) and (3.8) are required to hold for every normally-evolving interfacial chunk $\mathcal{A}(t)$; using (2.8), we have the **local balance laws**:

$$\begin{aligned} &\rho^{\circ} - \rho\kappa V = r, \\ &\rho\mathbf{v}^{\circ} = \mathbf{C}_s + \mathbf{b}. \end{aligned} \tag{3.9}$$

We have taken the normal velocity as the kinematic variable that characterizes the manner in which power is expended: tangential motion does not expend power. As is consistent with a "constraint" of this type, *we leave as indeterminate the tangential component of the interaction* $\mathbf{b}$, *and therefore concern ourselves only with the normal component of the capillary balance law* $(3.9)_2$.

Using the local balance laws (3.9) in conjunction with (2.8), we can write the mechanical energy production (3.4) in the simple form

$$\mathcal{E}(\mathcal{A})(t) = -\int_{\mathcal{A}}\mathbf{C}\cdot\mathbf{v}_s\,ds. \tag{3.10}$$

For convenience, we decompose the interfacial stress into normal and tangential components:

$$C = \sigma\boldsymbol{\ell} + \xi m \tag{3.11}$$

with $\sigma(x,t)$ the **surface tension**, $\xi(x,t)$ the **surface shear**. Then, writing

$$b = \mathbf{b}\cdot m \tag{3.12}$$

for the **normal interaction**, the normal component of $(3.9)_1$ becomes

$$\rho v^\circ = \xi_s + \sigma K + b. \tag{3.13}$$

4. Constitutive equations.

To state the constitutive equations that form the basis of our theory, we associate, with each evolving crystal, an **interfacial energy** ψ. As **constitutive equations** we allow the interfacial energy, the interfacial stress, the inertial density, and the normal interaction to depend on the orientation of the interface through a dependence on the angle θ, and we allow the kinetics of the interface to affect the normal interaction through a dependence on the normal velocity[13] v:

$$\psi = \psi(\theta), \quad C = C(\theta), \quad \rho = \rho(\theta), \quad b = b(\theta,v). \tag{4.1}$$

We assume that:[14]

(i) $\psi(\theta)$ generates the interfacial stress through the thermodynamic relation[15]

$$C(\theta) = \psi(\theta)\boldsymbol{\ell}(\theta) + \psi'(\theta)m(\theta); \tag{4.2}$$

---

[13] Here v represents the normal velocity of the interface relative to the crystal, so that v is trivially invariant under observer changes.

[14] Cf. [G1,AG], where (4.2) and (4.3) (with $\beta = \beta(\theta,v) \geq 0$) are derived using a thermodynamic argument.

[15] $\psi'(\theta) = d\psi(\theta)/d\theta$.

(ii) the normal interaction is given by a relation of the form

$$b(\theta,v) = -F - \beta(\theta)v, \tag{4.3}$$

with F a constant;

(iii) the following inequalities hold:

$$\rho(\theta), \psi(\theta), \beta(\theta) > 0. \tag{4.4}$$

Trivially, (4.2) implies that $\sigma = \sigma(\theta)$ and $\xi = \xi(\theta)$ with

$$\sigma(\theta) = \psi(\theta), \qquad \xi(\theta) = \psi'(\theta). \tag{4.5}$$

Concerning (4.3), the constant F represents a **driving force** for the crystallization process, while $-\beta(\theta)v$ represents a "drag force" which, by $(4.4)_3$ opposes interfacial motion.[16]

A consequence of (3.9) and the constitutive relations (4.2) and (4.3) is the *balance law for energy*:[17]

$$(d/dt)\Big\{\int_{\partial C}(\tfrac{1}{2}\rho|v|^2 + \psi)ds + F\text{area}(C)\Big\} = \tfrac{1}{2}\int_{\partial C} r|v|^2 ds - \int_{\partial C}\beta v^2 ds, \tag{4.6}$$

with $\partial C = \partial C(t)$. This result allows us to identify the last term as energy dissipated during crystal growth. The derivation of (4.6) is not difficult. First of all we have the standard relation

$$(d/dt)\text{area}(C) = \int_{\partial C} v\, ds, \tag{4.7}$$

as well as the following consequences of (2.5), (4.2), and (4.3):

---

[16] Cf. [G1], Remark 4.1.

[17] Generalizing (7.6) of [AG]. The integral involving ψ over ∂C plus Farea(C) represents a basic Gibbsian functional for the statical theory of crystals (cf. [G2], Sect. 3.2).

$$\begin{aligned} \mathbf{C}\cdot\mathbf{v}_s &= \psi'(\theta)\theta^\circ - \psi(\theta)v\kappa = \psi(\theta)^\circ - \psi(\theta)v\kappa, \\ \mathbf{b}v &= -Fv + \beta(\theta)v^2. \end{aligned} \tag{4.8}$$

The relation (4.6) follows from (3.4) and (3.10) with $\mathscr{c} = \partial C$ (so that $\partial\mathscr{c} = \emptyset$) in conjunction with (2.3), (3.12), (4.7), and (4.8).

5. Partial-differential equations.

The equations of our theory are balance of mass $(3.9)_1$ and the normal capillary-balance (3.13), in conjunction with the constitutive relations (4.2) and (4.3). Balance of mass yields

$$\rho'(\theta)v_s - \rho(\theta)\kappa v = r. \tag{5.1}$$

This equation determines the rate at which mass must be supplied by the melt; since we place no restrictions on r, (5.1) does not restrict the evolution of the crystal in any way.[18] On the other hand, (3.13), by virtue of $(2.5)_3$, (4.3), and (4.4), yields the **evolution equation**

$$\rho(\theta)v^\circ + \beta(\theta)v = [\psi(\theta) + \psi''(\theta)]\kappa - F, \tag{5.2}$$

which forms the basis of our theory. Note that, for an *isotropic interface,* ψ, β, and ρ are constants and (5.2) reduces to

$$\rho v^\circ + \beta v = \psi\kappa - F. \tag{5.3}$$

When the crystallization process takes place in $\mathbb{R}^3$, the interface evolves as a *surface,* rather than as a curve, but apart from notation the theory is identical. Following the notation and terminology of [G1], we write ∇_{surf} for the *surface gradient,* $\mathbf{L} = -\nabla_{\text{surf}}\mathbf{m}$ for the *curvature tensor,* and $\kappa = \operatorname{tr}\mathbf{L}$ for twice the *mean curvature.* Then the only essential changes regarding the theory

---

[18] The melt serves only as a source of atoms for the evolving crystal; [G4] considers a more detailed structure for the melt.

presented thus far are the replacement of $\psi(\theta)$ by $\psi(m)$, $\beta(\theta)$ by $\beta(m)$, $\rho(\theta)$ by $\rho(m)$, and (5.2) by

$$\rho(m)v^{\circ} + \beta(m)v = \psi(m)\kappa + \psi_{mm}(m)\cdot L - F, \tag{5.4}$$

where $\psi_{mm}(m)$ is the second gradient of $\psi(m)$ on the surface of the unit ball.

Consider now the general theory in $\mathbb{R}^2$ as defined by (5.2). *Locally,* the interface may be represented as the graph of a function $y=h(x,t)$, provided the x and y axes are chosen appropriately. Consider the choice indicated in Figure 2 (with orientation such that arc length increases with increasing x) and let

$$p = h_x, \tag{5.5}$$

where a subscript denotes partial differentiation with respect to the corresponding variable. Then

$$p\tan\theta = -1 \tag{5.6}$$

and, considering v as a function $v(x,t)$,

$$\begin{gathered} v = h_t\sin\theta, \qquad \kappa = h_{xx}\sin^3\theta, \\ v^{\circ} = v_t + v_x v\cos\theta = \sin\theta[h_{tt} + 2\sin\theta\cos\theta h_t h_{tx} + (h_t\sin\theta\cos\theta)^2 h_{xx}]. \end{gathered} \tag{5.7}$$

Thus, defining

$$\begin{gathered} B(\theta) = \beta(\theta)/\rho(\theta), \qquad G(\theta) = [\psi(\theta) + \psi''(\theta)]/\rho(\theta), \\ D(\theta) = F/[\rho(\theta)\sin\theta], \end{gathered} \tag{5.8}$$

the evolution equation (5.2) takes the form

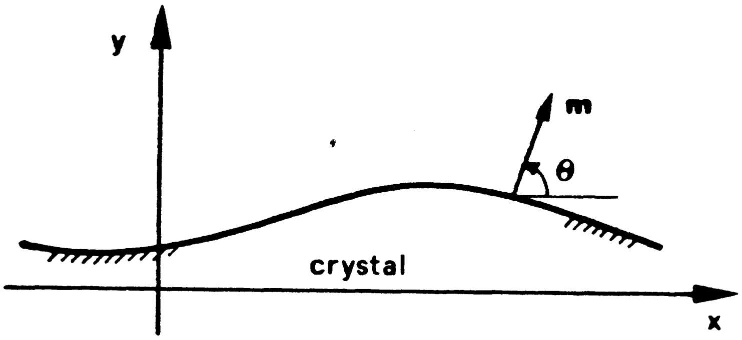

Figure 2

Sign conventions when the interface is a graph $y = h(x,t)$

$$h_{tt} + B(\theta)h_t + 2\sin\theta\cos\theta h_t h_{tx} = \sin^2\theta[G(\theta) - (h_t\cos\theta)^2]h_{xx} - D(\theta), \tag{5.9}$$

with θ a function of h_x through (5.5) and (5.6).

The equation (5.9) is hyperbolic if and only if the discriminant Δ of the coefficients of its principal part is strictly positive. Since $\Delta = 4\sin^2\theta G(\theta)$, it is clear from (4.4) that (5.9) *is hyperbolic if and only if*

$$\psi(\theta) + \psi''(\theta) > 0. \tag{5.10}$$

We will assume that (5.10) *is satisfied for all angles of interest.*

Remark. The inequality (5.10) is essentially a condition of static stability for the interface: it follows from the requirement that straight line-segments locally minimize interfacial energy.[19] When the inequality (5.10) is reversed the partial differential equation (5.9) is elliptic and yields unstable behavior for standard initial-value problems. There is no compelling physical reason to suppose that (5.10) is satisfied; in fact, material scientists often consider energies which violate (5.10) for particular ranges of θ (cf. Gjostein [G], Cahn and Hoffman [CH]). Since $\psi(\theta) > 0$ and periodic, at worst we can have an equation which is elliptic for some but not all values of θ. Such elliptic intervals can be treated by introducing corners in the evolving crystal (cf. [AG]).

6. Some simple solutions.

6.1. Radial solutions for an isotropic crystal.

In view of (5.3) and (5.4), isotropic, radially symmetric crystals evolve according to

$$\rho R^{\cdot\cdot}(t) + \beta R^{\cdot}(t) + \psi R(t)^{-1} = -F, \tag{6.1}$$

---

[19] Cf. Herring [H], Frank [F], Gjostein [G], Gruber (as referred to in [G]), Taylor [T], Fonseca [F], Angenent and Gurtin [AG].

with $R(t)$ the radius of the interface. Assuming that $R^{\cdot}(0)=0$ and appealing to the phase portrait for (6.1), it is not difficult to verify that:

(i) for $F \geq 0$ crystals melt completely in finite time,

(ii) for $F<0$, crystals of radius $R(0)<R_{crit} := \psi/|F|$ melt in finite time, while crystals of radius $R(0)>R_{crit}$ grow unboundedly as $t \to \infty$.

These conclusions are true whether or not $\rho=0$ (provided we drop the initial condition $R^{\cdot}(0)=0$ for $\rho=0$). Further, for $F<0$ and $R(0)>R_{crit}$, $R(t)$ grows (for large t) as t when $\rho=0$ and as t^2 when $\rho \neq 0$.

6.2. Small oscillations about a flat interface.

Assume that

$$F = 0. \tag{6.2}$$

Then flat interfaces ($\kappa=0$) describe equilibrium solutions of the general anisotropic equation (5.2). We now consider *interfacial motions which are close to equilibria of this form.* Precisely, we assume, without loss in generality, that the interface has angle $\theta=\pi/2$ at equilibrium and consider interfacial motions represented as a graph $y=h(x,t)$ with h and its derivatives "small". In view of (5.6), the angle $\theta(x,t)$, to first order in h, is given by

$$\theta = (\pi/2) + h_x. \tag{6.3}$$

Therefore (5.9) *linearized about this equilibrium* has the form

$$h_{tt} + B_0 h_t = G_0 h_{xx}, \tag{6.4}$$

where the subscript zero signifies that the corresponding quantity is to be evaluated at $\theta = \pi/2$. The equation (6.4) has the solution

$$\begin{aligned} h(x,t) &= Ce^{i\lambda x}e^{-(i\omega+\zeta)t}, \\ \omega^2 &= G_0\lambda^2 - \zeta^2, \qquad \zeta = B_0/2, \end{aligned} \tag{6.5}$$

which represents *damped oscillations* of the interface.

6.3. Curvature waves advancing on a flat interface.

We continue to assume that $F = 0$. We consider a front of discontinuous curvature advancing into a flat interface (with $\theta = \pi/2$). Precisely, we consider an interface described by a graph $y = h(x,t)$ and assume that there is a curve $\mathcal{K}$ in the (x,t)-plane which has the form $x = \xi(t)$ and is such that:[20]

(W1) $h(x,t) = 0$ for $x \geq \xi(t)$;

(W2) h and its first derivatives are continuous across $\mathcal{K}$, but second and higher derivatives of h suffer possible jump discontinuities across $\mathcal{K}$;

(W3) $[\kappa] \neq 0$.

Here and in what follows, $[g](t)$ denotes the **jump** in a function $g(x,t)$ across $\mathcal{K}$:

$$[g](t) = g(\xi(t)+0,t) - g(\xi(t)-0,t). \tag{6.6}$$

Because of (W3), we will refer to $\mathcal{K}$ as a **curvature wave**.

Standard kinematical conditions[21] give

$$[h_{tt}] = -c[h_{xt}] = c^2[h_{xx}], \tag{6.7}$$

---

[20] In continuum mechanics second-order waves of this type are usually termed *acceleration waves*. We refer, e.g., to Sect. 2 of [CG] for a discussion of such waves.

[21] Cf., e.g., [CG], eq. (2.5).

where

$$c = d\xi/dt \tag{6.8}$$

is the **velocity of propagation**. Further, by (W1) and (W2),

$$\theta = \pi/2 \quad \text{and} \quad h_t = h_x = 0 \quad \text{on} \quad \mathcal{K}; \tag{6.9}$$

hence (5.6) and (5.7) yield

$$[\theta_t] = [h_{xt}], \qquad [\kappa] = [h_{xx}]. \tag{6.10}$$

As before, we write ϕ_0 for a function $\phi(\theta)$ evaluated at $\theta = \pi/2$. The jump in (5.9) across $\mathcal{K}$ then yields

$$c^2 = G_0 = (\psi + \psi'')_0/\rho_0, \tag{6.11}$$

and the *velocity of propagation is constant.*

We define the **amplitude** of the wave by

$$a = [\kappa] = [h_{xx}]; \tag{6.12}$$

a standard identity[22] then yields

$$2c^2 da/dt = [h_{ttt}] - c[h_{xxt}]. \tag{6.13}$$

Next, we differentiate (5.9) with respect to t and take the jump in the resulting equation; because of (6.9) and (6.10), this yields

$$[h_{ttt}] + B_0[h_{tt}] = G_0[h_{xxt}] + (G')_0[h_{xx}][h_{xt}]. \tag{6.14}$$

In deriving (6.14), we used the fact that, by (W_1), $[h_{xx}\theta_t] = [h_{xx}][\theta_t]$. The relations (6.7) and (6.11)-(6.14) yield a nonlinear differential

---

[22] Cf., e.g., [CG], eq. (2.10).

equation for the amplitude:

$$2da/dt + B_0 a + [(G')_0/c]a^2 = 0. \tag{6.15}$$

For an isotropic crystal, $G(\theta)$ is independent of θ and $(G')_0 = 0$. In this case

$$a(t) = a(0)e^{-\frac{1}{2}B_0 t} \tag{6.16}$$

and curvature waves *decay.*

The results are far more interesting when the crystal is anisotropic. Assume that $(G')_0 \neq 0$. Then (6.15) has the explicit solution

$$a(t) = \frac{a(0)[1 - A]}{1 - Ae^{\frac{1}{2}B_0 t}}, \tag{6.17}$$

$$A = 1 - \lambda/a(0), \qquad \lambda = - B_0 c/(G')_0.$$

An elementary analysis of the solution (6.17) leads to the following conclusions:

(i) if $A = 0$, then $a(t) \equiv a(0)$;
(ii) if $0 < A < 1$, then $a(t) \to \infty$ in the finite time

$$t_\infty = 2(\ln A)/B_0; \tag{6.18}$$

(iii) if $A < 0$ or if $A > 1$, then $a(t) \to 0$ monotonically as $t \to \infty$.

The result (iii) asserts that if, initially, the jump in curvature is sufficiently large and of the right sign, then *this jump becomes infinite in finite time,* strongly suggesting that *the interface develops a corner:* $[\theta] \neq 0$.

Acknowledgment. The work of Gurtin was supported by the Army Research Office and the National Science Foundation; that of Podio-Guidugli by the Ministero della Pubblica Istruzione. This support is gratefully acknowledged.

References.

[AG] Angenent, S. and M. E. Gurtin, Multiphase thermomechanics with interfacial structure. 2. Evolution of an isothermal interface. Arch. Rational Mech. Anal., forthcoming.

[AP] Andreev, A. F. and A. Y. Parshin, Equilibrium shape and oscillations of the surface of quantum crystals, Sov. Phys. JETP **48**, 763-766 (1978).

[B] Brakke, K. A., *The Motion of a Surface by its Mean Curvature,* Princeton University Press (1978).

[BG] Ben-Jacob, E., N. Goldenfeld, J. S. Langer, and G. Schon, String models of interfacial pattern formation, Physica **12D**, 245-252 (1984).

[BK] Brower, R., D. Kessler, J. Koplik, and H. Levine, Simple models of interface growth, Physica **12D**, 241-244 (1984).

[CG] Coleman, B. D. and M. E. Gurtin, Waves in materials with memory. 2. On the growth and decay of one-dimensional acceleration waves, Arch. Rational Mech. Anal. **19**, 239-265 (1965).

[CH] Cahn, J. W. and D. W. Hoffman, A vector thermodynamics for anisotropic surfaces. 2. Curved and faceted surfaces, Act. Metall. **22**, 1205-1214 (1974).

[F] Fonseca, I., Interfacial energy and the Maxwell rule, Rept. 88-18, Dept. Math., Carnegie Mellon, Pittsburgh (1988).

[Fr] Frank, F. C., The geometrical thermodynamics of surfaces, *Metal Surfaces: Structure, Energetics, and Kinetics,* Am. Soc. Metals, Metals Park, Ohio (1963).

[G] Gjostein, N. A., Adsorption and surface energy. 2. Thermal faceting from minimization of surface energy, Act. Metall. **11**, 969-978 (1963).

[Gr] Grayson, M. A., The heat equation shrinks embedded plane curves to points, J. Diff. Geom. **26**, 285-314 (1987).

[GH] Gage, M. and R. S. Hamilton, The heat equation shrinking convex plane curves, J. Diff. Geom. **23**, 69-96 (1986).

[G1] Gurtin, M. E., Multiphase thermomechanics with interfacial structure. 1. Heat conduction and the capillary balance law. Arch.

Rational Mech. Anal. **104**, 195-221 (1988).

[G2] Gurtin, M. E., On the two-phase Stefan problem with interfacial energy and entropy, Arch. Rational Mech. Anal. **96**, 199-241 (1986).

[G3] Gurtin, M. E., Toward a nonequilibrium thermodynamics of two phase materials, Arch. Rational Mech. Anal. **100**, 1275-312 (1988).

[G4] Gurtin, M. E., A mechanical theory for crystallization of a rigid solid in a liquid melt: melting-freezing waves, Arch. Rational Mech. Anal., Forthcoming.

[GS] Gurtin, M. E. and A. Struthers, Multiphase thermomechanics with interfacial structure. 3, Forthcoming.

[H] Herring, C., Some theorems on the free energies of crystal surfaces, Phys. Rev. **82**, 87-93 (1951).

[KPB] Keshishev, K. O., A. Y. Parshin, and A. V. Babkin, Experimental detection of crystallization waves in He^4, JETP Lett. **30**, 56-59 (1979)

[MA] Maris, H. J. and A. F. Andreev, The surface of crystalline helium 4, Phys. Today, 25-30 (Feb. 1987).

[RHN] Renardy, M., W. J. Hrusa, and J. A. Nohel, *Mathematical Problems in Viscoelasticity,* Wiley, New York (1987).

[R] Rogers, R. C., Notes on a linear model of mechanical phase transitions, Forthcoming.

[T] Taylor, J. E., Crystalline variational problems, Bull. Am. Math. Soc. **84**, 568-588 (1978).