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A Nonlinear Elliptic Equation Arising from Gauge Field Theory and Cosmology

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Abstract

We study radially symmetric solutions of a nonlinear elliptic partial differential equation in $\mathbf{R}^2$ with critical Sobolev growth, i.e., the nonlinearity is of exponential type. This problem arises from a wide variety of important areas in theoretical physics including superconductivity and cosmology. Our results lead to many interesting implications for the physical problems considered. For example, for the self-dual Chern–Simons theory, we are able to conclude that the electric charge, magnetic flux, or energy of a non-topological N-vortex solution may assume any prescribed value above an explicit lower bound. For the Einstein-matter-gauge equations, we find a necessary and sufficient condition for the existence of a self-dual cosmic string solution. Such a condition imposes an obstruction for the winding number of a string in terms of the universal gravitational constant.

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1 Introduction

The purpose of this paper is to present a fairly systematic study of the radially symmetric solutions of the equation

$$\Delta u + p(|x|)q(e^u) = 4\pi N\delta(x), \qquad x \in \mathbf{R}^2, \tag{1.1}$$

where N is a positive integer and $\delta(x)$ is the Dirac distribution concentrated at the origin. The equation (1.1) arises from several important areas in theoretical

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physics and its solutions are recognized as representing N non-interacting vortices or particles superimposed at the origin when a critical condition, called the Bogomol'nyi condition, is fulfilled by the coupling parameters. In the context of the classical Ginzburg–Landau theory for low-temperature superconductivity, the solutions of (1.1) realize topological defects in a superconductor known as the phenomenon of partial destruction of superconductivity by a magnetic field. In the context of the self-dual Chern–Simons model discovered recently by Hong, Kim and Pac (1990) and Jackiw and Weinberg (1990), the solutions of (1.1) give rise to topological and non-topological vortices carrying both electric and magnetic charges which are useful or even crucial to several issues in theoretical physics such as high-temperature superconductivity, the quantum Hall effect, and the proton decay problem in grand unified theories of forces. In the context of cosmology, the solutions of (1.1) are cosmic strings in the coupled Einstein-matter-gauge theory (Comtet & Gibbons 1988), (Linet 1988, 1990), and are believed to be produced in relatively later phase transition stages of the early universe after the Big Bang and are responsible for the galaxy formation (Kibble 1980), (Vilenkin 1985). Furthermore (1.1) also appears when one uses the method of Bogomol'nyi (1976) or Jackiw and Weinberg (1990) to get stationary neutral or charged vortices in the self-dual Abelian Higgs or Chern–Simons system in (2+1) spacetime dimensions with more general potential functions (Lohe 1981), (Lee & Nam 1991), (Yang 1991). These profound physical origins of (1.1) motivate our present analytic work.

Since in the Ginzburg–Landau model, the non-interacting superconducting vortices are completely understood (Jaffe & Taubes 1980), (Wang & Yang 1992a), we shall focus our attention in the study of (1.1) to the other areas mentioned above. For the Chern–Simons system, topological vortices are better understood and can actually be constructed through a numerically efficient method (Spruck & Yang 1991). On the other hand, the existence of non-topological vortices has only been proved in the radial case (Spruck & Yang 1992a) and the determination of the electric charge, magnetic flux, angular momentum and energy of such a solution has remained open. In this paper we shall solve this open problem. In fact we are able to prove the existence of non-topological Chern–Simons vortices that can realize any *prescribed* charge, flux, angular momentum or energy values above an explicit lower bound. This result confirms not only the fractal nature of those physical quantities for non-topological vortices conjectured in the work of (Jackiw & Weinberg 1990), (Jackiw, Pi & Weinberg 1990), (Jackiw & Pi 1991), but says also that we actually have a continuous spectrum to realize them. Such a result seems to be quite unexpected. Another interesting result is that we are able to prove that radially symmetric topological vortices are uniquely determined by the vortex number N. For solutions without radial symmetry, the uniqueness still remains open. For the cosmic string solutions of the Einstein-matter-gauge equations, the first existence theorem was established in (Spruck & Yang 1992b) by the shooting argument used in (Spruck & Yang 1992a) and the total vortex or string number N has to verify a certain restriction. Here we obtain a *necessary* and *sufficient* condition for the existence of a finite-energy solution. Our condition imposes an explicit upper bound for N (the winding number of a string) in terms of Newton's universal gravitational constant, G, which excludes the existence of solutions with large N numbers. This is the second unexpected result. Besides, we shall also prove the existence of a class of solutions that approach the asymmetric vacuum expectation value at infinity. Solutions of this type have been rather elusive to obtain by other methods. All these solutions are regular. Note that the major difference between the solutions of our problem and those constructed in the recent work of Smoller, Wasserman, Yau and McLeod (1992) is that their solutions are spherically symmetric with the absence of matter field while the ones found in the present paper are cylindrically symmetric with matter coupling. Our method here is based on a shooting argument for a reduced boundary value problem defined in the entire $\mathbf{R}$. The shooting data are given at $-\infty$ which yield a clear understanding of the structure of the problem.

A brief outline of the paper is in order. In Sect. 2 we recall the self-dual Chern–Simons system and the Einstein-matter-gauge equations and introduce all necessary physical quantities. In Sect. 3 we set up the mathematical problem to be studied in the paper and state our main results. The conditions imposed on the functions p and q in (1.1) come from the physical problems discussed in Sect. 2. In Sect. 4 we present our detailed proofs. In Sect. 5 we state some of the interesting physical implications of our solutions.

2 The Physical Models

The purpose of this section is to introduce the model equations and physical quantities to be investigated in the paper.

2.1. The Chern–Simons Vortices.

Let ϕ be a complex scalar field, called the Higgs field, and A_j ($j = 1, 2$) a vector gauge field. Both ϕ and A_j are assumed to be defined in $\mathbf{R}^2$. Let $D_j\phi = \partial_j\phi - \mathrm{i}A_j\phi$

be the gauge-covariant derivative and $F_{jk} = \partial_j A_k - \partial_k A_j$ the magnetic field induced from A_j. Then the energy density for the stationary solutions of the self-dual Chern–Simons theory is (Jackiw & Weinberg 1990)

$$\mathcal{E} = \frac{\kappa^2}{4}\frac{F_{12}^2}{|\phi|^2} + |D_j\phi|^2 + \frac{1}{\kappa^2}|\phi|^2(1-|\phi|^2)^2, \tag{2.1}$$

where $\kappa > 0$ is a constant. The Chern–Simons Gauss law implies that the electric charge density ρ obeys the relation $\rho = \kappa F_{12}$. The self-dual solutions of the model are governed by the following Bogomol'nyi type equations (Jackiw & Weinberg 1990):

$$\begin{aligned} D_1\phi + \mathrm{i}D_2\phi &= 0, \\ F_{12} + \tfrac{2}{\kappa^2}|\phi|^2(|\phi|^2-1) &= 0, \end{aligned} \qquad x \in \mathbf{R}^2. \tag{2.2}$$

Using the elliptic L^p-theory in (2.2) and the finite energy condition $\int_{\mathbf{R}^2}\mathcal{E}dx < \infty$ in (2.1), we can derive the boundary condition (Spruck & Yang 1991)

$$|\phi(x)| \to 1 \quad \text{or } |\phi(x)| \to 0 \qquad \text{as } |x| \to \infty.$$

The former is called topological which gives rise to quantized magnetic and electric charges, etc., while the latter, non-topological (Jackiw & Weinberg 1990), (Jackiw, Pi & Weinberg 1990), (Jackiw & Pi 1991). We shall show in this paper that, in contrast to topological solutions, non-topological solutions can carry arbitrary charges and energies above a certain level.

The first equation in (2.2) says that, locally, ϕ is the product of a complex analytic function and a non-vanishing function. Thus the zeros of ϕ are discrete and all have integral multiplicities. We will be interested in a solution of (2.2) so that the origin is the only zero of ϕ and the multiplicity of the zero is an arbitrary integer, N say. Such a solution describes N vortices clustered together. It is straightforward that the substitution $u = \ln|\phi|^2$ reduces (2.2) into the elliptic equation (Jackiw & Weinberg 1990)

$$\Delta u = \frac{4}{\kappa^2}e^u(e^u - 1) + 4\pi N\delta(x), \qquad x \in \mathbf{R}^2. \tag{2.3}$$

2.2. Self-Dual Strings in the Einstein-Matter-Gauge Theory.

Let $g_{\mu\nu}$ be the metric tensor of a four-dimensional Minkowskian spacetime with signature $(-+++)$, $R_{\mu\nu}$ the Ricci tensor, and R the scalar curvature. Then the Einstein tensor takes the form

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R.$$

The standard $U(1)$ matter-gauge Lagrangian in the Bogomol'nyi coupling is defined in the expression

$$\mathcal{L} = \frac{1}{4} g^{\mu\mu'} g^{\nu\nu'} F_{\mu\nu} F_{\mu'\nu'} + \frac{1}{2} g^{\mu\nu} (D_\mu \phi)(D_\nu \phi)^* + \frac{1}{8}(|\phi|^2 - 1)^2,$$

where, again, ϕ is a complex scalar matter field, $D_\mu \phi = \partial_\mu \phi - \mathrm{i} A_\mu \phi$ the gauge-covariant derivative, A_μ a gauge vector field, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the electromagnetic field.

The Einstein-matter-gauge equations are

$$\begin{aligned} G_{\mu\nu} &= -4\pi G T_{\mu\nu}, \\ \frac{1}{\sqrt{|g|}} D_\mu (g^{\mu\nu} \sqrt{|g|} [D_\nu \phi]) &= \frac{1}{2}(|\phi|^2 - 1)\phi, \\ \frac{1}{\sqrt{|g|}} \partial_{\mu'} (g^{\mu\nu} g^{\mu'\nu'} \sqrt{|g|} F_{\nu\nu'}) &= \frac{\mathrm{i}}{2} g^{\mu\nu} (\phi [D_\nu \phi]^* - \phi^* [D_\nu \phi]), \end{aligned} \tag{2.4}$$

where g is the determinant of the metric $\{g_{\mu\nu}\}$, G the universal gravitational constant which is of the order 10^{-40}, and

$$T_{\mu\nu} = g^{\mu'\nu'} F_{\mu\mu'} F_{\nu\nu'} + \frac{1}{2}(D_\mu \phi [D_\nu \phi]^* + [D_\mu \phi]^* D_\nu \phi) - g_{\mu\nu} \mathcal{L},$$

the energy-momentum tensor of the matter-gauge sector.

We assume from now on the string ansatz that

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -dt^2 + dz^2 + g_{jk} dx^j dx^k, \qquad j,k = 1,2, \end{aligned}$$

where $t = x^0, z = x^3$, $\{g_{jk}\}$ is the metric tensor of a two-dimensional Riemannian manifold M, and that A_μ, ϕ depend only on the coordinates on M and

$$A_\mu = (0, 0, A_1, A_2).$$

Then $T_{\mu\nu}$ verifies

$$\begin{aligned} T_{00} &= \mathcal{E}, \quad T_{33} = -\mathcal{E}, \quad T_{03} = T_{0j} = T_{3j} = 0, \\ T_{jk} &= g^{j'k'} F_{jj'} F_{kk'} + \frac{1}{2}(D_j \phi [D_k \phi]^* + [D_j \phi]^* D_k \phi) - g_{jk} \mathcal{E}, \end{aligned}$$

where

$$\mathcal{E} = \frac{1}{4} g^{jj'} g^{kk'} F_{jk} F_{j'k'} + \frac{1}{2} g^{jk} (D_j \phi)(D_k \phi)^* + \frac{1}{8}(|\phi|^2 - 1)^2$$

is the energy density of the matter-gauge sector. Besides, if we use K to denote the Gaussian curvature of the two-manifold (M, *{gjk})*, the Einstein tensor is simplified under local isothermal coordinates into the form

r* /** is
"""LJOO = ^33 = A,

$G_{MI'}$ = 0 for other values of /x, i/.

Denote by *Vj* the covariant derivative with respect to the metric *{gjk}* and J* the current vector

$$J_k = \frac{1}{2}A_k - \frac{i}{4}(\phi^*[D_k\phi] - \phi[D_k\phi]^*).$$

Then, in terms of the skew-symmetric Levi-Civita tensor *tjk* with C12 = $\sqrt{\backslash f \backslash g \backslash}$, we have

$$\mathcal{E} = \frac{1}{4}g^{jj'}g^{kk'}\left(F_{jk} + \frac{1}{2}\epsilon_{jk}(|\phi|^2 - 1)\right) (F?_k> + \backslash eM\backslash 4>\backslash^2 - 1))$$

$$+\frac{1}{4}g^{jk}(D_j\phi + i\epsilon_j^{j'}D_{j'}\phi)(D_k\phi + i\epsilon_k^{k'}D_{k'}\phi)^* \qquad (2.5)$$

$$+\nabla_j(\epsilon^{jk}J_k),$$

which suggests the following curved-space version of the Bogomol'nyi equations

$$Dj4> + \backslash t)D_k<t> = 0,$$

$$F_{jk} + \frac{1}{2}\epsilon_{jk}(|\phi|^2 - 1) = 0, \qquad x \in M. \qquad (2.6)$$

From (2.6) it follows that *Tjk* = 0 (j, fc = 1,2). Thus the Einstein equations become a single scalar equation

$$K - 4\pi G\, Vjie\wedge Jk) = 0, \qquad x \in M. \qquad (2.7)$$

It is straightforward to examine that a solution triplet ({#,*}, ^>, {Aj}) of the coupled system (2.6)-(2.7) also verifies the full Einstein-matter-gauge equations (2.4). Such a solution is called a self-dual cosmic string.

In local isothermal coordinates, it is easy to show as in the Chern-Simons case that the zeros of <*f*> of a solution of (2.6) are isolated and all have integral multiplicities. Thus, if we are interested in solutions such that *M* is conformally flat, i.e., (Af, *{gjk})* = (R^2i {e^j*}), aad ^ a t ^ ^^ exactly one zero, which is the origin of R^2, with a given multiplicity *N*, then the substitution u = In \<*f*>\$^2$ reduces (2.6) into the equation

$$\Delta u = e^{\eta}(e^u - 1) + 4\pi N\delta(x).$$

On the other hand, (2.6) also gives us

$$\begin{aligned}\nabla_j(\epsilon^{jk}J_k) &= -\tfrac{1}{4}(|\phi|^2-1)+\tfrac{1}{4}e^{-\eta}\Delta|\phi|^2\\ &= -\tfrac{1}{4}(e^u-1)+\tfrac{1}{4}e^{-\eta}\Delta e^u.\end{aligned}$$

Thus the standard formula $K = -(e^{-\eta}\Delta\eta)/2$ applied in (2.7) yields a curvature equation for the unknown conformal factor η:

$$\Delta\eta = 2\pi G[e^\eta(e^u-1)-\Delta e^u].$$

Combining the equations for u and η, we see immediately that

$$H \equiv \frac{1}{2\pi G}\eta - u + e^u + 2N\ln|x|$$

is a harmonic function in $\mathbf{R}^2$. Therefore, if we are only interested in radially symmetric solutions, H must be a constant. Without loss of generality, we take $H=0$. We then express η in terms of u. This procedure shows that the coupled equations for u and η are equivalent to the single equation

$$\Delta u = |x|^{-4\pi NG}e^{2\pi G(u-e^u)}(e^u-1)+4\pi N\delta(x), \qquad x\in\mathbf{R}^2. \tag{2.8}$$

The integer N is called the winding number of the string solution.

2.3. Non-Interacting Vortices in Generalized Systems.

A special feature of these generalized models is that the Higgs potential functions can be adjusted to realize in a wide range rather different magnetic excitation patterns (Wang & Yang 1992b). The existence of topological solutions is well known (Lohe & van der Hoek 1983), (Yang 1991). The method adopted in the present paper applies also to the proofs of existence and analysis of properties of non-topological solutions. For example, we mention here the radially symmetric generalized self-dual Chern–Simons vortices governed by the equation

$$\Delta u = -e^u h(e^u)\int_{e^u}^1 h(t)\,dt + 4\pi N\delta(x), \qquad x\in\mathbf{R}^2, \tag{2.9}$$

where $h(t)\geq 0$ $(t\geq 0)$ is a smooth function satisfying the typical conditions:

(h1) There holds $h(1)>0$.

(h2) The set $h^{-1}(0)\cap[0,1]$ has measure zero.

Important examples of h are

$$h(t)=t^m, \qquad t\geq 0, \quad m=0,1,2,..., \tag{2.10}$$

and

$$h(t) = \frac{1}{(a+t)^2}, \qquad t \geq 0, \quad a > 0. \tag{2.11}$$

Clearly, when $h = 1$, equation (2.9) coincides with (2.3).

When the Chern-Simons Higgs system is considered in a symmetric gravitational background, self duality may again be derived and the governing equation reads.

$$\Delta u = p(|x|)e^u(e^u - 1) + 4\pi N\delta(x), \qquad x \in \mathbf{R}^2. \tag{2.12}$$

In the case that $p(|x|)$ takes some special forms, the equation (2.12) is shown to be integrable by Schiff (1991). We can apply our method to (2.12) as well to study topological and non-topological solutions for a general metric function p.

3 Main Results

Now we shall study the radially symmetric solutions of the equation (1.1), which, upon setting $r = |x|$ and $u = u(r)$, is equivalent to

$$u_{rr}(r) + \frac{1}{r}u_r(r) + p(r)q(e^{u(r)}) = 0, \qquad r > 0,$$

$$u(r) = 2N \ln r + O(1) \qquad \text{for small } r > 0.$$

Under the new variables

$$t = \ln r, \qquad U(t) = u(r), \tag{3.1}$$

the problem becomes

$$U''(t) + f(t)g(U(t)) = 0, \qquad -\infty < t < \infty, \tag{3.2}$$

$$U(t) = \alpha t + O(1) \qquad \text{as } t \to -\infty, \tag{3.3}$$

where

$$f(t) = e^{2t}p(e^t), \quad g(u) = q(e^u), \quad \alpha = 2N.$$

Motivated by the physical models discussed in Sect. 2, we shall assume that f and g satisfy the following conditions:

(H1) $f, g \in C^1(\mathbf{R})$ and

$$\int_{-\infty}^{0} |tf(t)|\, dt < \infty,$$

$$\sup_{u \in \mathbf{R}} \left\{ |g(u)| + |g'(u)| \right\} < \infty.$$

(H2) $f(\cdot) > 0$ in $\mathbf{R}$, $\lim_{t\to\infty} f(t) = \infty$, $g(0) = 0$, and $g(u) > 0$ for all $u < 0$.
(H3) $f'(t) \geq 0$ for all $t \in \mathbf{R}$.
(H4) There exists $M > 0$ such that $g'(u) > 0$ when $u < -M$ and

$$\int_{0}^{\infty} f(t) g(-Mt)\, dt < \infty.$$

(H5) If one defines

$$M_0 = \inf \left\{ M > 0 \,\middle|\, \int_0^\infty f(t) g(-Mt)\, dt < \infty \right\},$$

then

$$\int_0^\infty f(t) g(-M_0 t)\, dt = \infty.$$

In addition, for every $c > 0$,

$$\inf_{t>0} \frac{f(t-c)}{f(t)} > 0.$$

(H6) Let $G_0(u) = \int_{-\infty}^{u} g(w)\, dw$. (Note that the assumptions (H2)–(H4) imply that $\int_{-\infty}^{0} g(u)\, du < \infty$, so that $G_0(u)$ is well defined.) Define

$$F_1(t) = \frac{f'(t)}{f(t)} \qquad \text{and} \qquad G_1(u) = \frac{G_0(u)}{g(u)}.$$

Then both $f_1 = \lim_{t\to\infty} F_1(t)$ and $g_1 = \lim_{u\to-\infty} G_1(u)$ exist and are finite.

(H7) The functions F_1 and G_1 defined in the assumption (H6) satisfy $F_1(t) \geq f_1$ for all $t \in \mathbf{R}$ and $G_1(u) \geq g_1$ for all $u \in (-\infty, 0)$.

(H8) There exists $\delta > 0$ such that $g'(u) \leq 0$ in $[-\delta, 0]$.

Our main result on (3.2)–(3.3) is the following.

Theorem 3.1. *Consider the differential equation (3.2) with the boundary condition (3.3) where $\alpha \geq 0$ is a given constant and $f(\cdot)$ and $g(\cdot)$ satisfy (H1)–(H3). Then*

(1) There exists at least one solution of (3.2)–(3.3) such that $u \leq 0$, $u' \geq 0$, $u'' \leq 0$ in $\mathbf{R}$ (the equal signs hold only if $\alpha = 0$), and

$$\lim_{t\to\infty} u(t) = 0. \tag{3.4}$$

If in addition (H8) is fulfilled, then there exists a unique non-positive solution satisfying (3.4).

(2) Assume also (H4)–(H6). Then, for every β in $(\alpha + 2f_1 g_1, \infty)$, there exists at least one solution u of (3.2)–(3.3), such that $u < 0$, $u'' < 0$ in $\mathbf{R}$ and

$$\lim_{t\to\infty} u'(t) = -\beta. \tag{3.5}$$

If in addition (H7) holds, then for any non-positive solution of (3.2) satisfying $\liminf_{t\to\infty} u(t) < 0$, there exists some $\beta \in (\alpha + 2f_1 g_1, \infty)$ to achieve (3.5).

The proof will be given in the next section.

If we apply this theorem to the Chern–Simons equation (2.3), we obtain

Theorem 3.2. *For $N \geq 0$, a radially symmetric solution of (2.3) is either trivial, $u \equiv 0$, or negative, $u < 0$. Corresponding to each given N, there exists a unique solution $u = u(r)$ $(r = |x|)$ satisfying*

$$\lim_{r\to\infty} u(r) = 0. \tag{3.6}$$

All other solutions observe the behavior $u(r) \to -\infty$ as $r \to \infty$ and

$$\lim_{r\to\infty} r u_r(r) = -\beta, \qquad \beta > 2N + 4. \tag{3.7}$$

More importantly, for any $\beta \in (2N + 4, \infty)$, there exists at least one solution u realizing the asymptote (3.7).

Proof. Under the transformation (3.1), the equation (2.3) becomes (3.2)–(3.3) with

$$\alpha = 2N, \qquad f(t) = \frac{4}{\kappa^2} e^{2t}, \qquad g(u) = e^u(1 - e^u).$$

If u is a solution of (3.2)–(3.3) which becomes positive at some point $t = t_0$, then the maximum principle says that $u'(t_0) > 0$. Thus $u''(t) > 0$ and $u'(t) > 0$ for all $t > t_0$. In particular, $e^{u(t)} - 1 > e^{u(t_0)} - 1 > 0$, $t > t_0$. Now the equation gives us the inequality

$$u'' > \delta e^u, \qquad t > t_0,$$

where $\delta > 0$ is a constant depending on t_0. Clearly u blows up in finite time $t > t_0$.

To obtain non-positive solutions, we can modify $g(u)$ for $u \in (1, \infty)$ such that g and g' are uniformly bounded. Then, we can directly verify that such f and g satisfy (H1)–(H8), and hence the assertions of the theorem follow from Theorem 3.1. □

For the Einstein-matter-gauge equation (2.8), the only physically interesting solutions are those verifying $u(r) \leq 0$ (see Sect. 5.2). Thus we state

Theorem 3.3. *For non-positive valued radially symmetric solutions of (2.8), the following statements hold:*

(1) When $2\pi NG < 1$, all the assertions of Theorem 3.2 are valid provided that we replace $2N+4$ by $2(1-\pi NG)/\pi G$.

(2) When $2\pi NG > 1$, there exists a desired solution if and only if $\pi NG < 1$.

(3) When $2\pi NG = 1$, there are no non-positive radially symmetric solutions, although there exists a unique (up to a scaling) radially symmetric solution, whose range fills **R**.

Proof. Under the transformation (3.1), the equation (2.8) becomes (3.2)–(3.3) with

$$\alpha = 2N, \qquad f(t) = e^{2(1-2\pi NG)t}, \qquad g(u) = e^{2\pi G(u-e^u)}(1-e^u) = \frac{1}{2\pi G}\frac{d}{du}\Big[e^{2\pi G(u-e^u)}\Big].$$

(1) Clearly, when $2\pi NG < 1$, (H1)–(H8) hold with $f_1 = 2(1-2\pi NG)$ and $g_1 = 1/2\pi G$, and therefore the first assertion of the theorem follows from Theorem 3.1.

(2) When $2\pi NG > 1$, we make a transformation from t to $-t$, obtaining a similar equation (3.2) with $f(t) = e^{2(2\pi NG-1)t}$ whereas (3.3) is replaced by $\lim_{t\to\infty} u'(t) = -2N$. In this case (H1)–(H8) are valid. Hence, if there exists a non-positive radially symmetric solution of (2.8), then $\alpha = \lim_{t\to-\infty} u'$ exists and is non-negative, where $u = u(t)$ is the corresponding solution of (3.2)–(3.3). In fact, the existence and finiteness of α is trivial from (H1)–(H8). If $\alpha < 0$, then u gives rise to a symmetric solution of (2.8) which becomes positive when $r > 0$ is sufficiently large. Hence, there exists a non-positive solution of (2.8) if and only if $2N \in (\alpha + (4\pi NG - 2)/\pi G, \infty)$ for some $\alpha \geq 0$, or equivalently, $\pi NG < 1$.

(3) When $2\pi NG = 1$, the equation (3.2) becomes

$$u'' = e^{2\pi G(u-e^u)}(e^u - 1)$$

with the boundary condition $u'(-\infty) = 2N$. Multiplying both sides of the differential equation by u' and integrating over $(-\infty, t)$ yield

$$u'^2(t) = 4N^2 - \frac{1}{\pi G}e^{2\pi G(u-e^u)} = 2N(2N - e^{(u-e^u)/N}) \equiv \zeta(N,u)$$

since $2\pi G = 1/N$. Note that $\sup_{u\in\mathbf{R}}(u - e^u) = -1$ and $\inf_{N>0}(N\ln(2N)) = -1/2e$, so that

$$\zeta(N,u) = 2N\Big[\exp\Big(\frac{1}{N}N\ln(2N)\Big) - \exp\Big(\frac{1}{N}(u-e^u)\Big)\Big] \geq 2N(e^{-1/2Ne} - e^{-1/N}) > 0$$

for all $N > 0$ and $u \in \mathbf{R}$. Therefore, there are no non-positive solutions. The only solution is implicitly given by

$$\int_0^{u(t)} \frac{1}{\sqrt{\zeta(N,w)}}\, dw = t + c,$$

where c is an additive constant. □

Remark 3.1. Theorem 3.3 (2) states that there are no non-positive radially symmetric solutions when $\pi NG \geq 1$. However, since $g(u) \equiv e^{2\pi G(u-e^u)}(1-e^u)$ is uniformly bounded, one can find solutions whose ranges are $\mathbf{R}$. Solutions of such a type do not have a finite energy as will be seen in Sect. 5.2.

For the generalized Chern–Simons equation (2.9), we have the following result concerning a sub-class of h:

Theorem 3.4. *Consider the non-positive radially symmetric solutions of (2.9) where $N \geq 0$ is a constant. Assume that the function h is smooth in a neighborhood containing $[0,1]$ and is positive in $(0,1]$. Then the assertion (1) of Theorem 3.2 holds. In addition, if*

$$h_0 \equiv \lim_{t \searrow 0} \frac{\int_0^t h(s)\, ds}{t\, h(t)} \tag{3.8}$$

exists, then for any $\beta \in (2N+4h_0, \infty)$, the equation (2.9) admits at least one negative radially symmetric solution satisfying (3.7). Furthermore, if $h(t)$ satisfies

$$\frac{\int_0^t h(s)\, ds}{t\, h(t)} \cdot \frac{\int_0^1 h(s)\, ds + \int_t^1 h(s)\, ds}{2\int_t^1 h(s)\, ds} > h_0, \qquad \forall t \in (0,1), \tag{3.9}$$

then the assertion (2) of Theorem 3.2 is true provided that we replace $2N+4$ by $2N+4h_0$.

Consequently, when h is given by (2.10) or (2.11), (3.8) and (3.9) hold with $h_0 = 1/(1+m)$ (in the case (2.10)) or $h_0 = 1$ (in the case (2.11)). Therefore all the assertions of Theorem 3.2 are valid if we replace $2N+4$ by $2N+4h_0$.

The proof follows the same lines as that of Theorem 3.2.

Remark 3.2. For general h satisfying only (h1) and (h2) in Sect. 2.3, our method can be applied to obtain existence of solutions of various kinds. In particular, if $h(e^u) = 0$ has roots in $(-\infty, 0)$, then there are rich structures of the solutions, such as solutions connecting $-\infty$ with any of the roots, besides the topological one and the non-topological ones. For brevity, we shall not pursue this here.

Similarly, we may obtain various classes of solutions for the equation (2.12).

4 The Mathematical Analysis

In this section, we shall study (3.2)-(3.3) under the assumption (Hl)-(H8).

First we establish the existence of the initial value problem for the equation (3.2).

Lemma 4.1. *Assume that (HI) holds. Then for any constants a € R and a € R, the equation (3.2) admits a unique solution U such that when t —* —oo,*

$$U(t) = at + a + o(l). \tag{4.1}$$

Conversely, if U(t) is a solution of (3.2) in some interval, then it can be uniquely extended to a global solution of (3.2) in R *so that (4.1) holds for some a, a €* R.

Proof. One can directly verify that U is a solution of (3.2) satisfying (4.1) if and only if U verifies the integral equation

$$U\{t) = at + a - f \quad (t- s)f(s)g(U(s))\, ds, \qquad t \in R. \tag{4.2}$$

Let T € R be a constant such that

$$\int^T (T-s)\backslash f(s)\backslash ds \;(= f \quad I'^1 \; \backslash f(s)\backslash ds d_{Sl}) < \frac{}{2\sup_{u\in\mathbf{R}}\{1 + \backslash g(u)\backslash + g'(u)|\}}.$$

Then one can use the Picard successive iteration method (with the initial iteration £/(°) = at + a) to establish a solution in the interval (—00, T]. Since g is bounded, we can extend U to a solution of (3.2) in R.

Next we prove the uniqueness. Assume that U^1 and U^2 are two solutions of (4.2) in the interval (-00, T]. Then their difference $U = U^1 - U^2$ satisfies

$$\backslash U(t)\backslash = \backslash f\_Jt-s)n_S)(g(U\backslash s))-g(U^2(s)))ds\backslash$$

$$\leq \sup\backslash g'(u)\backslash \int^T (T-s)\backslash f(s)\backslash ds \quad \sup \quad |t/(-)|$$

$$\leq \tfrac{1}{2} \sup_{(-\infty,T]} |t/(-)|, \qquad t < T,$$

by the assumption on T. Since the first equation implies that sup^.^jj $\backslash U\{')\backslash < 00$, we obtain, upon taking the superum on the left-hand side of the above inequality, that $\sup_{(-\infty,T]}$ I^OI = 0; namely, $U^1 = U^2$ in (—oo, T]. Hence, by the unique continuation, $U^1 = U^2$ in R.

Finally we prove the last assertion of the lemma. Assume that $U(t)$ is a solution of (3.2) in some interval. Then since $g(\cdot)$ is Lipschitz and bounded, U can be uniquely extended to a solution of (3.2) in $\mathbf{R}$. Noting that $\int_{-\infty}^{0} |f(s)g(U(s))|\,ds < \infty$ and for any $t < 0$, $U'(t) = U'(0) + \int_t^0 f(s)g(U(s))\,ds$, we know that $\alpha \equiv \lim_{t\to-\infty} U'(t)$ exists and $\alpha = U'(0) + \int_{-\infty}^{0} f(s)g(U(s))\,ds$. Consequently, for any $t \in \mathbf{R}$, $U'(t) = \alpha - \int_{-\infty}^{t} f(s)g(U(s))\,ds$ and

$$U(t) = U(0) + \alpha t - \int_0^t \int_{-\infty}^{s_1} f(s)g(U(s))\,ds\,ds_1. \tag{4.3}$$

Since

$$\int_{-\infty}^{t} \int_{-\infty}^{s_1} |f(s)g(U(s))|\,ds = \int_{-\infty}^{t} (t-s)|f(s)g(U(s))|\,ds < \infty,$$

we can write (4.3) as

$$U(t) = \alpha t + \Big(U(0) + \int_{-\infty}^{0} \int_{-\infty}^{s_1} f(s)g(U(s))\,dsds_1\Big) - \int_{-\infty}^{t} \int_{-\infty}^{s_1} f(s)g(U(s))\,dsds_1;$$

i.e., U satisfies (4.1) with $a = U(0) + \int_{-\infty}^{0} \int_{-\infty}^{s_1} f(s)g(U(s))\,dsds_1$. This completes the proof of the lemma. □

In the sequel, we shall study the behavior of the solution as $t \to \infty$. To do this, we shall fix the constant $\alpha \geq 0$, and vary the parameter $a \in \mathbf{R}$. For convenience, we denote by $u(t, a)$ the solution given by Lemma 4.1 and denote by $'$ the derivative with respect to t and by a subscript $_a$ the derivative with respect to a.

Define

$$\begin{aligned}
\mathcal{A}^+ &= \{a \in \mathbf{R} \mid \text{ there exists } t \in \mathbf{R} \text{ such that } u(t,a) > 0\}, \\
\mathcal{A}^0 &= \{a \in \mathbf{R} \mid u(t,a) \leq 0, \ \ u'(t,a) \geq 0 \ \ \forall t \in \mathbf{R}\}, \\
\mathcal{A}^- &= \{a \in \mathbf{R} \mid u(t,a) \leq 0 \ \forall t \in \mathbf{R}, \quad u'(t_0,a) < 0 \text{ for some } \ t_0 \in \mathbf{R}\}.
\end{aligned}$$

Clearly, the following relations hold:

$$\mathcal{A}^+ \cup \mathcal{A}^0 \cup \mathcal{A}^- = \mathbf{R}, \qquad \mathcal{A}^+ \cap \mathcal{A}^0 = \mathcal{A}^0 \cap \mathcal{A}^- = \mathcal{A}^+ \cap \mathcal{A}^- = \emptyset.$$

Lemma 4.2. *Assume (H1) and (H2). Then, the following holds:*

(1) If $a \in \mathcal{A}^+$, then $u' > 0$ in the set $\{t \mid u(\tau, a) < 0 \ \forall \tau \in (-\infty, t)\}$.

(2) If $a \in \mathcal{A}^0$, then $u'' \leq 0$ and $u' \geq 0$ in $\mathbf{R}$ and $\lim_{t\to\infty} u(t,a) = 0$.

(3) If $a \in \mathcal{A}^-$, then $u'' < 0$, $u < 0$ in $\mathbf{R}$ and $\lim_{t\to\infty} u(t,a) = -\infty$.

(4) $\mathcal{A}^+$ is open and if $a > M_1 \equiv \sup_{u\in\mathbf{R}} |g(u)| \int_{-\infty}^{0} |sf(s)|\,ds$, then $a \in \mathcal{A}^+$.

(5) $\mathcal{A}^-$ is open.

(6) Let T be a positive constant such that

$$\left(\inf_{u\in[-2,-1]} g(u)\right) \inf_{t>T} \int_t^{t+1/\alpha} f(s)\,ds > 1+\alpha.$$

Then $(-\infty, -M_1 - 2 - \alpha T) \subset \mathcal{A}^-$.

(7) $\mathcal{A}^0$ is non-empty, closed, and bounded.

Proof. (1) Let $a \in \mathcal{A}^+$ and t_0 be the first time at which $u(t,a)$ hits the t axis from below. Then $u(t,a) < 0$ for all $t \in (-\infty, t_0)$. Hence, by the assumption (H2) and the equation (3.2), $u'' < 0$ in $(-\infty, t_0)$, which implies that $u'(t,a) > 0$ in $(-\infty, t_0)$. The first assertion of the lemma thus follows.

(2) If $a \in \mathcal{A}^0$, then by the assumption (H2), the equation (3.2) and the definition of $\mathcal{A}^0$, $u'' \le 0$ in $\mathbf{R}$. In addition, $b \equiv \lim_{t\to\infty} u(t,a)$ exists and is non-positive. If $b < 0$, then $\lim_{t\to\infty} u''(t,a) = -g(b)\lim_{t\to\infty} f(t) = -\infty$, which is impossible. Hence, $b = 0$.

(3) Since the only solution of (3.2) with $U(t_0) = U'(t_0) = 0$ is $U \equiv 0$, it follows that if $a \in \mathcal{A}^-$ then $u(\cdot,a) < 0$ in $\mathbf{R}$, and therefore $u''(\cdot,a) < 0$ in $\mathbf{R}$; that is, $u'(t,a)$ strictly decreases. Hence, $\limsup_{t\to\infty} u'(t,a) < 0$. Assertion (3) of the lemma thus follows.

(4) Since $u(t,a)$ is continuous in a (Cf. the uniqueness proof of Lemma 4.1), if $u(t_0,a_0) > 0$, then $u(t_0,a) > 0$ when a is close to a_0; that is, $\mathcal{A}^+$ is open. From (4.2), $u(0,a) > a - M_1 > 0$ if $a > M_1$, so that $(M_1,\infty) \subset \mathcal{A}^+$.

(5) Assume that $a_0 \in \mathcal{A}^-$. Then there exists $t_0 \in \mathbf{R}$ such that $u'(t_0,a_0) < 0$, and consequently, $u'(t_0,a) < 0$ when a is close to a_0. In addition, by the third assertion of the lemma, $u(t,a_0) < 0$ for all $t \le t_0$ which also implies that $u(t,a) < 0$ for all $t \le t_0$ and a close to a_0. (When t is negatively very large, use (4.1); when t is in a compact subset, use the continuity of the solution in a.) Furthermore, since the assumption (H2) implies that any solution of (3.2) cannot take a local negative minimum, $u'(t,a) \le 0$ for all $t > t_0$ as long as $u(t_0,a) < 0$ and $u'(t_0,a) < 0$. Therefore, $u(t,a) < 0$ for all $t > t_0$ when a is close to a_0. That is, $\mathcal{A}^-$ is open.

(6) We need only consider the case $\alpha > 0$ since when $\alpha = 0$, $\mathcal{A}^- = (-\infty, 0)$. Let $a < -M_1 - 2 - \alpha T$ be any constant. From (4.2), $u(a,t) \le \alpha t + a + M_1 < -2$ for all $t \in (-\infty, 0]$. Since u cannot take a local negative minimum, it follows that if $a \notin \mathcal{A}^-$, then there exist positive constants T_1 and T_2 such that $T_2 < T_1$, $u(t,a) \le -2$ in $(-\infty, T_2]$, $u(T_2,a) = -2$, $u'(T_2,a) \ge 0$, $u(t,a) \in [-2,-1]$ for all $t \in [T_2,T_1]$, $u(T_1,a) = -1$, and $u'(T_1,a) \ge 0$. It then follows that $u''(t,a) = -f(t)g(u(t)) \le 0$ for all $t \le T_1$, which implies that $u'(t,a) \le \alpha$ for all $t \in (-\infty, T_1]$. Therefore,

$T_2 \geq [u(T_2,a) - u(0,a)]/\alpha \geq [-2 - a - M_1]/\alpha > T$ and $T_1 - T_2 \geq 1/\alpha$. Consequently,

$$\begin{aligned} u'(T_1,a) &= u'(T_2,a) - \int_{T_2}^{T_1} f(s)g(u(s))\,ds \\ &\leq \alpha - \left(\inf_{t\geq T}\int_t^{t+1/\alpha} f(s)\,ds\right)\left(\inf_{u\in[-2,-1]} g(u)\right) < -1, \end{aligned}$$

by the definition of T, which contradicts the assumption that $u'(T_1,a) \geq 0$. Hence $a \in \mathcal{A}^-$.

(7) Since $\mathbf{R}$ cannot be decomposed into two disjoint non-empty open sets, the assertion follows from the conclusions (4)–(6). □

The following lemma deals with the monotonicity of the solution with respect to the parameter a and will play an essential role in analyzing the three sets $\mathcal{A}^+$, $\mathcal{A}^0$, and $\mathcal{A}^-$.

Lemma 4.3. *Assume (H1)–(H3) and let $T_0(a) \in [-\infty,\infty]$ be the first time such that either $u'(t,a) > 0$ or $u(t,a) < 0$ is violated, namely,*

$$T_0(a) = \sup\{T \in [-\infty,\infty] \mid u(t,a) < 0, u'(t,a) > 0 \quad \forall t \in (-\infty,T)\}.$$

Then

$$u_a(t,a) \geq \frac{1}{\alpha}u'(t,a) > 0 \quad \forall t \in (-\infty, T_0(a)).$$

Proof. We need only consider the case $T_0(a) > -\infty$. From (4.2) and the standard ODE techniques on the continuous dependence of solutions on the the parameters, one can show that $v(t,a) \equiv u_a(t,a)$ exists, is smooth and satisfies

$$\begin{aligned} v''(a,t) &= -f(t)g'(u(t,a))v(t,a), & -\infty < t < \infty, \\ \lim_{t\to-\infty} v(t,a) &= 1, & \lim_{t\to-\infty} v'(t,a) = 0. \end{aligned}$$

Define $T_1(a) = \sup\{\tau \in \mathbf{R} \mid v(\cdot,a) > 0 \text{ in } (-\infty,\tau)\}$. Then, by the last two equations, $T_1(a) > -\infty$.

Set $w = u'$. Then $\lim_{t\to-\infty} w(t,a) = \alpha$ and by (3.2), $\lim_{t\to-\infty} w'(t,a) = 0$. It follows that the function $C(t,a) \equiv w(t,a)/v(t,a)$, $t \in (-\infty, T_1(a))$, satisfies $\lim_{t\to-\infty} C(t,a) = \alpha$ and $\lim_{t\to-\infty} C'(t,a) = 0$. Since the function w satisfies the equation $w'' = -f(t)g'(u)w - f'(t)g(u)$, the method of variation of parameters yields

$$C'(t,a) = -\frac{1}{v^2(t,a)}\int_{-\infty}^{t} f'(s)u_a(t,a)g(u(s,a))\,ds \qquad \forall t \in (-\infty, T_1(a)). \tag{4.4}$$

Since $f' \geq 0$, it follows that $C' \leq 0$ and therefore $C \leq \alpha$ in the set $(-\infty, T_1(a))$; that is, $v(t,a) \geq \frac{1}{\alpha}w(t,a)$ in $(-\infty, T_1(a))$. Clearly this implies that $T_0(a) \leq T_1(a)$. The lemma thus follows. □

The following statements characterize the sets $\mathcal{A}^+$, $\mathcal{A}^0$, and $\mathcal{A}^-$.

Lemma 4.4. *Assume (H1)-(H3). Then there exist constants a_x and a_2 satisfying $a\ \leq a_2$ such that*

(1) A+ = (a_2, oo);

(2) $A^- = (-\infty, a_1)$;

(3) $A^0 = [a_1, a_2]$;

(4) if in addition (H8) holds, then a\ = a_2.

Proof. We need only consider the case *a* > 0 since in case a = 0, one can directly verify that *A*+ = (0,oo), A^0 = {0}, and *A"* = (-oo,0).

(1) Since *A*+ is open, it suffices to show that if (61,62) C >1^+, then 62 € *A. For any *a* € (61,62), let $z_o(a)$ be the first time the solution crosses the *t* axis. (Since *a* > 0, limt-^_oott(<,a) = —00, so $z_o(a)$ is well defined.) Clearly, *u(zo(a),a)* = 0, *u'(zo(a),a)* > 0, and by Lemma 4.2 (1), u' > 0 in (—00, z_o{a)]. By Lemma 4.3, $u_a \geq$ \u' > 0 in (—00, $z_o(a)$]. Applying the Implicit Function Theorem to the equation $u(z_o(a),a)$ = 0 then yields that z_o{a) is a differentiate function of *a* in the set (61,62) and ^$z_o(a)$ = —$u_a(z_o(a),a)$/u'(zo(a),a) < 0. Noting that (4.2) implies that *u* ≤ *at* + *a* in (—00, $z_o(a)$], we then know that $z_o(a) \geq$ —*a/a* for every *a* 6 (61,62). Thus 20(62) = $\lim_{a/}6_2$ $z_o(a)$ exists and is finite. By continuity, $u(z_o(b_2), b_2)$ = 0. Since u'($2_0(6_2)$, ^2) = 0 would result in tt(J, 6_2) = 0, we also find that *u\zo{b_2), b_2)* ^ 0, which implies *u(t,b_2)* > 0 for *t* near 2:0(62). That is, 62 € -4^+. The first assertion of the lemma thus follows.**

(2) It is sufficient to show that (61,6_2) € *A'* implies 61 € *A~*. For every *a* € *A~*, let *z\(a)* be the point where *u'(zi(a),a)* = 0 and let *m(a)* = u(zi(a),a) be the maximum of t/(-,a) in R. Since *\i"(z\(a),a)* < 0, the Implicit Function Theorem implies that *Z\(a)* is a differentiate function on *A'*. Hence,

$$\frac{d}{da}m\{a) = u'(zi(a),a)\frac{d}{da}zi(a) + u_a\{zi\{a),a) = u_a(zi(a),a) > \underline{0} \qquad \text{Va € (61,62).}$$

Consequently

$$m(a) = \sup_{tcR} u(a,t) \leq m\left(\frac{6i + \&2}{2}\right) \qquad \text{for all a € } \left(6_1, \frac{b_1 + b_2}{2}\right).$$

By continuity, m(6i) = $\sup_{t€R}$u(t,6i) ≤ *m{^^)* < 0. This implies that 6_X € *A°UA~*. However, by Lemma 4.2 (2), we can easily conclude that 6 £ A^0. Therefore, 61 € *A~*. This completes the proof of the second assertion.

(3) Since ,4° = R \ *(A U,4~), the assertion follows from the first two conclusions.**

(4) For every *a* £ A^0 = [a_u a_2], we have *u'* > 0 in R and, by Lemma4.3, u_a{t, a) > 0 in R. In addition, since u is monotonic and $\lim_{t\text{-}MX>}$t/(f,a) = 0, for each *6* > 0 there

exists a continuous function $T_\delta(a)$ such that $u(T_\delta(a),a) = -\delta$ and $u(t,a) > -\delta$ in $(T_\delta(a),\infty)$. By the assumption (H8), $g'(u) \le 0$ when $u \in [-\delta, 0]$. Therefore

$$u_a'' = -f(t)g'(u)u_a \ge 0 \qquad \forall t \ge T_\delta(a),\ a \in [a_1,a_2].$$

Hence u_a is a non-negative convex function on $[T_\delta(a),\infty)$, so that

$$u_a(\infty,a) \equiv \lim_{t\to\infty} u_a(t,a)$$

exists and $u_a(\infty,a) \in [0,\infty]$. Suppose that we have shown $u_a(\infty,a) > 0$ for all $a \in [a_1,a_2]$. Then, by Fatou's lemma,

$$0 = \lim_{t\to\infty}(u(t,a_2) - u(t,a_1)) = \lim_{t\to\infty}\int_{a_1}^{a_2} u_a(t,a)\,da \ge \int_{a_1}^{a_2} u_a(\infty,a)\,da$$

which implies that $a_1 = a_2$. It remains to show that $u_a(\infty,a) > 0$ for all $a \in [a_1,a_2]$.

Suppose, on the contrary, that $u_a(\infty,a) = 0$ for some $a \in [a_1,a_2]$. Then by (4.4), the function $C = u'/u_a$ satisfies

$$\begin{aligned} C'(t,a) &= -\frac{1}{u_a^2(t,a)}\int_{-\infty}^{t} f'(s)u_a(s,a)g(u(s,a))ds \\ &\le -\frac{1}{u_a^2(t,a)}\int_{-\infty}^{0} f'(s)u_a(s,a)g(u(s,a))ds \to -\infty \qquad \text{as } t\to\infty, \end{aligned}$$

which implies that $C(t,a) < 0$ when t is large enough. However, this is impossible since $C = u'/u_a > 0$ for all $t \in \mathbf{R}$. This proves that $u_a(\infty,a) > 0$ for all $a \in [a_1,a_2]$ and thus the last assertion of the lemma follows. □

Now we want to find more detailed behavior of the solution $u(t,a)$ when $t \to \infty$ and $a \in \mathcal{A}^-$.

Lemma 4.5. *Assume (H1)-(H4). Then for any $a \in \mathcal{A}^-$,*

$$\beta(a) \equiv -\lim_{t\to\infty} u'(t,a)$$

exists and is positive and finite.

Proof. Since for any $a \in \mathcal{A}^-$, $u'' < 0$ in $\mathbf{R}$, it follows that $\beta(a)$ exists and belongs to the interval $(0,\infty]$. We want to show that $\beta(a) < \infty$.

Assume that $\beta(a) > M$ where M is the constant in the assumption (H4). Then there exists a constant $T > 1$ such that $u(t,a) \le -Mt$ for all $t > T$. Since $g'(u) > 0$

when $u < -M$, it follows that $g(u(t,a)) \le g(-Mt)$ when $t \ge T$. Consequently, for all $t > T$,

$$\begin{aligned} u'(t,a) &= u'(T,a) - \int_T^t f(s)g(u(s,a))\,ds \\ &\ge u'(T,a) - \int_T^t f(s)g(-Ms)\,ds \ge u'(T,a) - \int_0^\infty f(s)g(-Ms)\,ds. \end{aligned}$$

Therefore, $\beta(a) \le -u'(T,a) + \int_0^\infty f(s)g(-Ms)\,ds < \infty$. The assertion of the lemma thus follows. □

Finally, we would like to find the range of $\beta(a)$ when a runs over the set $\mathcal{A}^-$. Note that the assumptions (H2)–(H4) imply that $\int_{-\infty}^0 g(u)\,du$ is finite, so that we can define

$$G_0(u) = \int_{-\infty}^u g(w)\,dw. \tag{4.5}$$

Lemma 4.6. *Assume (H1)–(H4) and let G_0 be defined as in (4.5). Then for any $a \in \mathcal{A}^-$, both the function $f(\cdot)g(u(\cdot,a))$ and the function $f'(\cdot)G_0(u(\cdot,a))$ are in $L^1(\mathbf{R})$ and there hold the relations*

$$\beta(a) + \alpha = \int_{\mathbf{R}} f(t)g(u(t,a))\,dt, \tag{4.6}$$

$$\frac{1}{2}\beta^2(a) - \frac{1}{2}\alpha^2 = \int_{\mathbf{R}} f'(t)G_0(u(t,a))\,dt. \tag{4.7}$$

Proof. Since $u'(t,a) = \alpha - \int_{-\infty}^t f(t)g(u(t,a))\,dt$ and $fg \ge 0$, the identity (4.6) follows from Lemma 4.5.

To show (4.7), we use the identity

$$\frac{d}{dt}\Big(\frac{u'^2(t,a)}{2} + f(t)G_0(u(t,a))\Big) = f'(t)G_0(u(t,a))$$

which follows by multiplying (3.2) by u'. Integrating both sides over $[-T,T]$ yields

$$\int_{-T}^T f'(t)G_0(u(t,a))\,dt = \frac{u'^2(t,a)}{2}\Big|_{t=-T}^{t=T} + f(t)G_0(u(t,a))\Big|_{t=-T}^{t=T}.$$

Since the integrand on the left-hand side is positive, to finish the proof, we need only show that $\lim_{t\to\pm\infty} f(t)G_0(u(t,a)) = 0$. Since $f(t) \to 0$ as $t \to -\infty$ and $G_0(u)$ is bounded, we have $f(t)G_0(u(t,a)) \to 0$ as $t \to -\infty$. It remains to show that $f(t)G_0(u(t,a)) \to 0$ as $t \to \infty$.

Since $u' > -\beta(a)$ in $\mathbf{R}$, for T sufficiently large so that $u' < 0$ when $t > T$, we have

$$\begin{aligned} G_0(u(t,a)) &= \int_{-\infty}^{u(t,a)} g(u(s,a))\,du(s,a) \\ &= \int_\infty^t g(u(s,a))u'(s,a)ds \le \beta(a)\int_t^\infty g(u(s,a))\,ds. \end{aligned}$$

Hence, using the facts that f monotonically increases and that $f(t)g(u(t,a)) \in L^1(\mathbf{R})$, we obtain that $0 \leq f(t)G_0(u(t,a)) \leq \beta(a) \int_t^\infty f(s)g(u(s,a))\,ds \to 0$ as $t \to \infty$, thereby completing the proof of the lemma.□

Lemma 4.7. *Assume that (H1)-(H5) hold. Then the function $\beta(a)$ is continuous in $\mathcal{A}^-$.*

Proof. Let $a_0 \in \mathcal{A}^-$ be any point and M, M_0 be the constants in the assumptions (H4) and (H5). First we claim that $\beta(a_0) > M_0$. In fact, if $M_0 = 0$, there is nothing to prove. Thus it suffices to assume $M_0 > 0$. If the claim is not true, then since $u''(t,a) < 0$, $u'(t,a) > -M_0$ in $\mathbf{R}$, which implies that there exists a positive constant C such that $u(t,a) \geq -C - M_0 t$ for all $t > 0$. Let $T > 0$ be a time such that $u(t,a_0) < -M$ for all $t > T$. Then

$$\begin{aligned} \int_T^\infty f(t)g(u(t,a_0))\,dt &\geq \int_T^\infty f(t)g(-C - M_0 t) \\ &\geq \inf_{t>0} \frac{f(t - C/M_0)}{f(t)} \int_{T+\frac{C}{M_0}}^\infty f(t)g(-M_0 t)\,dt = \infty \end{aligned}$$

by the assumption (H5), contradicting the finiteness of $\beta(a_0)$ and (4.6). This shows that $\beta(a_0) > M_0$.

Let $\delta = (\beta(a_0) - M_0)/4$. Then there exists a positive constant T_1 such that $(M_0+\delta)T_1 \geq M$, $u'(T_1,a_0) < -(M_0+2\delta)$, and $u(T_1,a_0) \leq -(M_0+2\delta)T_1$. Since $u(a,t)$ and $u'(a,t)$ are continuous in a, $u'(T_1,a) \leq -(M_0+\delta)$ and $u(T_1,a) \leq -(M_0+\delta)T_1$ when a is close to a_0. It then follows that, since $u'' < 0$, $u(t,a) \leq -(M_0+\delta)t$ in $[T_1,\infty)$ for all a close to a_0. Let $W(t)$ be the function defined by $W(t) = \sup_{u\in\mathbf{R}} |g(u)| f(t)$ for $t \leq T_1$ and $W(t) = f(t)g(-(M_0+\delta)t)$ for $t > T_1$. Then by the definition of M_0, $W \in L^1(\mathbf{R})$. In addition, when a is close to a_0, $f(t)g(u(t,a)) \leq W(t)$ for all $t \in \mathbf{R}$. The assertion of the lemma then follows from the Lebesgue Dominated Convergence Theorem and the formula (4.6). □

In order to find the range of $\beta(a)$, we need find the behavior of $\beta(a)$ as $a \nearrow a_0 \equiv \sup_{a\in\mathcal{A}^-}\{a\}$ and as $a \to -\infty$.

Lemma 4.8. *Assume (H1)-(H4) and let $a_0 = \sup_{a\in\mathcal{A}^-}\{a\}$. Then*

$$\lim_{a\nearrow a_0} \beta(a) = \infty.$$

Proof. Using the identity (4.7), we have

$$\begin{aligned}
\liminf_{a\nearrow a_0} \frac{\beta^2(a)}{2} - \frac{\alpha^2}{2} &= \liminf_{a\nearrow a_0} \int_{\mathbf{R}} f'(t) G_0(u(t,a))\, dt \\
&\geq \liminf_{T\to\infty} \liminf_{a\nearrow a_0} \int_0^T f'(t) G_0(u(t,a))\, dt \\
&= \lim_{T\to\infty} \int_0^T f'(t) G_0(u(t,a_0))\, dt \\
&\geq \lim_{T\to\infty} G_0(u(0,a_0)) \int_0^T f'(t)\, dt = \infty
\end{aligned}$$

where in the last inequality we have used the property that both $G_0(\cdot)$ and $u(\cdot, a_0)$ are monotonic so that $G_0(u(t,a_0)) \geq G_0(u(0,a_0))$ for all $t > 0$. Since $\beta(a) > 0$, the assertion of the lemma thus follows. □

To study the behavior of $\beta(a)$ as $a \to -\infty$, we need the following property of the solutions.

Lemma 4.9. *Assume (H1)-(H4) and for any $a \in \mathcal{A}^-$, let $m(a) = \sup_{t\in\mathbf{R}} u(t,a)$. Then*

$$\lim_{a\to-\infty} m(a) = -\infty.$$

Proof. Since when $\alpha = 0$, $m(a) = a$, the assertion of the lemma is obviously true, so that we need only consider the case $\alpha > 0$.

Let $a \in \mathcal{A}^-$ be any constant and let $z_1(a)$ be the point such that $u'(z_1(a), a) = 0$. Then (4.2) implies that $u(t,a) < \alpha t + a$ which, in turn, implies that $m(a) = u(z_1(a), a) < \alpha z_1(a) + a$; that is,

$$z_1(a) \geq \frac{m(a) - a}{\alpha}. \tag{4.8}$$

Since $0 \leq u' \leq \alpha$ in $(-\infty, z_1(a))$, there holds the inequality

$$m(a) - 1 < u(t,a) < m(a) \qquad \forall t \in (z_1(a) - 1/\alpha, z_1(a)),$$

so that

$$\begin{aligned}
0 = u'(z_1(a), a) &= \alpha - \int_{-\infty}^{z_1(a)} f(s) g(u(s,a))\, ds \\
&\leq \alpha - \Big(\inf_{u\in[m(a)-1, m(a)]} g(u)\Big) \int_{z_1(a)-1/\alpha}^{z_1(a)} f(s)\, ds.
\end{aligned}$$

Therefore, by (4.8) and the monotonicity of f,

$$\alpha \geq \Big(\inf_{u\in[m(a)-1,m(a)]} g(u)\Big) \int_{z_1(a)-1/\alpha}^{z_1(a)} f(s)\, ds \geq \Big(\inf_{u\in[m(a)-1,m(a)]} g(u)\Big) \int_{\frac{m(a)-a-1}{\alpha}}^{\frac{m(a)-a}{\alpha}} f(s)\, ds.$$

Since $f(t) \to \infty$ as $t \to \infty$, the assertion of the lemma must hold. □

Now we are in a position to find the behavior of $\beta(a)$ as $a \to -\infty$.

Lemma 4.10. *Assume (H1)-(H4) and (H6). Then*

$$\lim_{a\to-\infty} \beta(a) = \alpha + 2f_1g_1. \tag{4.9}$$

If in addition (H5) holds, then $(\alpha + 2f_1g_1, \infty) \subset \{\beta(a) \mid a \in \mathcal{A}^-\}$.

Proof. Let $a \ll -1$ and $T \gg 1$ be any fixed constants. Then the identity (4.7) implies that

$$\begin{aligned}
\frac{\beta^2(a)}{2} - \frac{\alpha^2}{2} &= \int_{\mathbf{R}} f'(t)G_0(u(t,a))\,dt \\
&= \int_{-\infty}^{T} f'(t)G_0(u(t,a))\,dt + \int_T^{\infty} F_1(t)G_1(u(t,a))f(t)g(u(t,a))\,dt \\
&= \int_{-\infty}^{T} f'(t)G_0(u(t,a))\,dt + F_1(T^*)G_1(u(T^*,a)) \int_T^{\infty} f(t)g(u(t,a))\,dt
\end{aligned}$$

by the Mean Value Theorem, where $T^* \in [T, \infty)$. Using the identity (4.6) we have that

$$\frac{\beta^2(a)}{2} - \frac{\alpha^2}{2} = F_1(T^*)G_1(u(T^*,a))(\alpha + \beta(a)) + \Delta(T,a) \tag{4.10}$$

where

$$\Delta(T,a) = \int_{-\infty}^{T} \left[f'(t)G_0(u(t,a)) - F_1(T^*)G_1(u(T^*,a))f(t)g(u(t,a))\right] dt.$$

By Lemma 4.9, $\lim_{a\to-\infty} \Delta(T,a) = 0$. Solving $\beta(a)$ from the algebraic equation (4.10) yields that

$$\beta(a) = F_1(T^*)G_1(u(T^*,a)) + \sqrt{[\alpha + F_1(T^*)G_1(u(T^*,a))]^2 + 2\Delta(T,a)}.$$

Therefore,

$$\begin{aligned}
\lim_{a\to-\infty} \beta(a) &= \lim_{T\to\infty}\lim_{a\to-\infty} \Big(F_1(T^*)G_1(u(T^*,a)) \\
&\qquad + \sqrt{[\alpha + F_1(T^*)G_1(u(T^*,a))]^2 + 2\Delta(T,a)}\,\Big) \\
&= \lim_{T\to\infty}\lim_{a\to-\infty} \big(\alpha + 2F_1(T^*)G_1(u(T^*,a))\big) = \alpha + 2f_1g_1
\end{aligned}$$

by the assumption (H6) and Lemma 4.9. This proves (4.9).

Since when (H5) holds, $\beta(a)$ is continuous in $\mathcal{A}^-$, so that the range of $\beta(a)$ when a runs over $\mathcal{A}^-$ contains the set $(\alpha + 2f_1g_1, \infty)$. □

Lemma 4.11. *If (H1)-(H7) hold then*

$$\{\beta(a) \mid a \in \mathcal{A}^-\} = (\alpha + 2f_1g_1, \infty).$$

Proof. We need only show that $\beta(a) > \alpha + 2f_1g_1$ for all $a \in \mathcal{A}^-$. In fact, if (H7) holds, then

$$\beta^2(a) - \alpha^2 = 2\int_{\mathbf{R}} f'(t)G_0(u(t,a)) > 2f_1g_1 \int_{\mathbf{R}} f(t)g(u(t,a)) = 2f_1g_1(\beta(a)+\alpha),$$

which implies that $\beta(a) > \alpha + 2f_1g_1$. □

Clearly, Theorem 3.1 follows from Lemmas 4.1–4.11.

Remark 4.1. There is another way to prove part of Theorem 3.1 which in some sense is easier than what we have presented. We outline it here. For each $m > 0$ and $T \in \mathbf{R}$, let $u = u(t,m,T)$ be the solution of (3.2) with the initial condition $u|_{t=T} = -m$ and $u'|_{t=T} = 0$. Then, one immediately knows that u takes its maximum at $t = T$, $u'' < 0$, and $\alpha^{\pm}(m,T) = \lim_{t\to\pm\infty} u'$ exists. In addition, $0 < \alpha^- = \int_{-\infty}^T f(t)g(u) \leq \sup g \int_{-\infty}^T f(t)$, which implies that $\lim_{T\to-\infty} \alpha^-(m,T) = 0$ uniformly for all $m > 0$. The proof of Lemma 4.5 yields that α^+ is finite and a similar proof as that for part (6) of Lemma 4.2 yields that $\lim_{T\to\infty} |\alpha^{\pm}(m,T)| = \infty$ for each $m > 0$. Thus for any given positive constant α_0, a topological argument shows that, for any $M > \varepsilon > 0$, there exists a continuum γ in the (m,T) space intersecting the lines $m = \varepsilon$ and $m = M$, such that any pair (m,T) on γ satisfies $\alpha^-(m,T) = \alpha_0$. The same proof as Lemma 4.10 shows that $\alpha^+(m,T) \to -(\alpha_0 + 2f_1g_1)$ when (m,T) is on γ and $m \to -\infty$. Using the explicit bound of α^-, one knows that T on γ is uniformly bounded from below, which, together with the identity (4.7) and the fact that the solution can be close to the t axis for an arbitrarily long period of time if m is small enough, (m,T) varies on γ, the value of α^+ varies from $-(\alpha_0 + 2f_1g_2)$ to $-\infty$. This furnishes a proof of the second statement in Theorem 3.1.

5 Direct Physical Implications

Let u be an arbitrary radially symmetric solution produced in either Theorem 3.2 or 3.3. It will be convenient to use $z = x^1 + \mathrm{i}x^2$ to denote a point in $\mathbf{R}^2$. We understand

$$\partial = \partial_1 - \mathrm{i}\partial_2, \qquad \partial^* = \partial_1 + \mathrm{i}\partial_2.$$

Set

$$\begin{aligned} \phi(z) &= \exp\left(\tfrac{1}{2}u(z) + \mathrm{i}N \arg z\right), \\ A_1(z) &= -\mathrm{Re}\{\mathrm{i}\partial^* \ln \phi\}, \quad A_2(z) = -\mathrm{Im}\{\mathrm{i}\partial^* \ln \phi\}. \end{aligned} \tag{5.1}$$

5.1. The Chern–Simons Model.

From a solution u obtained in Theorem 1, we can use (5.1) to construct an N-vortex solution of the self-dual Chern–Simons equation (2.2). For any given $\beta > 2N+4$, let u be such a solution that (3.7) is fulfilled. Then it follows from (2.2), (2.3), and (5.1) that the magnetic flux is

$$\begin{aligned} \Phi &= \int_{\mathbf{R}^2} F_{12}\, dx = \frac{2}{\kappa^2}\int_{\mathbf{R}^2} e^u(1-e^u)\, dx \\ &= \frac{4\pi}{\kappa^2}\int_0^\infty r e^{u(r)}(1-e^{u(r)})\, dr \\ &= \pi[\lim_{r\to 0} r u_r(r) - \lim_{r\to\infty} r u_r(r)] = 2\pi N + \pi\beta. \end{aligned}$$

The first term on the right-hand side equals to the flux of a topological N-vortex solution. The electric charge is just $\kappa\Phi$. Furthermore, the energy density (2.1) has the Bogomol'nyi decomposition (Jackiw & Weinberg 1990)

$$\begin{aligned} \mathcal{E} = &\ \frac{1}{4}\Big[\frac{\kappa F_{12}}{|\phi|} + \frac{2}{\kappa}|\phi|(|\phi|^2 - 1)\Big]^2 + |D_1\phi + \mathrm{i}D_2\phi|^2 \\ &+ F_{12} + \mathrm{Im}\{\partial_j \varepsilon_{jk}\phi^*(D_k\phi)\}, \end{aligned}$$

where ε_{jk} is skew-symmetric with $\varepsilon_{12} = 1$. On the other hand, in view of (5.1), we have

$$|D_j\phi|^2 = 2u_r^2 e^u, \qquad r = |x|. \tag{5.2}$$

Thus (3.7) gives us $|D_j\phi|^2 = \mathrm{O}(r^{-(2+\beta)})$ for large $r > 0$ and the integral over $\mathbf{R}^2$ of the last term in the expansion of $\mathcal{E}$ vanishes. Therefore, by virtue of (2.2), we have the total energy

$$E = \int_{\mathbf{R}^2} \mathcal{E}\, dx = \int_{\mathbf{R}^2} F_{12}\, dx = \Phi.$$

In summary, we can state

Theorem 5.1. *For given integer $N \geq 0$ and any $\beta > 2N+4$, the Chern–Simons system allows a non-topological N-vortex solution which realizes the following prescribed asymptotic decay properties*

$$|\phi|^2 = O(r^{-\beta}), \quad |D_j\phi|^2 = O(r^{-(2+\beta)}), \quad F_{jk} = O(r^{-\beta}) \qquad \textit{for large } |x| = r > 0$$

and the corresponding values of energy, electric charge, and magnetic flux

$$E = \Phi, \qquad Q = \kappa\Phi, \qquad \Phi = 2\pi N + \pi\beta.$$

Moreover, the radially symmetric topological N-vortex solution is uniquely determined by the vortex location.

5.2. Finite-Energy Cosmic Strings.

In Sect. 3, we remarked that (2.8) also permits solutions with positive values. Here we show that such solutions cannot carry finite energy. In fact, we shall prove that the original self-dual Einstein-matter-gauge equations (2.6)–(2.7) have a finite-energy radial cosmic string solution with the winding number $N \geq 1$ if and only if (2.8) has a negative radial solution (with the same N). As a consequence, we are able to arrive at the important conclusion that the existence of a symmetric self-dual cosmic string with winding number N is equivalent to the condition

$$2\pi NG \neq 1, \qquad \pi NG < 1. \tag{5.3}$$

This result is stated as follows.

Theorem 5.2. *Consider the radially symmetric solutions of the equations (2.6)–(2.7) so that the 2-surface is conformally flat, $(M, \{g_{jk}\}) = (\mathbf{R}^2, \{e^\eta \delta_{jk}\})$, and that the zero of ϕ is the origin of $\mathbf{R}^2$ with multiplicity $N \geq 1$. Let K be the Gaussian curvature of the surface. Then the finite energy condition*

$$\int_{\mathbf{R}^2} \mathcal{E}\, e^\eta\, dx < \infty, \qquad \int_{\mathbf{R}^2} K\, e^\eta\, dx < \infty \tag{5.4}$$

is equivalent to the bound $|\phi| < 1$. In other words, there exists a symmetric cosmic string solution with winding number $N \geq 1$ if and only if N satisfies (5.3).

Proof. Since a radially symmetric solution verifies the representation (5.1) with $u = u(r)\,(r = |z|)$ and u satisfies

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r &= r^{-4\pi NG} e^{2\pi G(u - e^u)}(e^u - 1), \qquad r > 0, \\ \lim_{r\to 0} r u_r(r) &= 2N \end{aligned} \tag{5.5}$$

(see (2.8)), where $u = \ln|\phi|^2$, it suffices to establish the equivalence of (5.4) with

$$u(r) < 0, \qquad r > 0. \tag{5.6}$$

Assume first that (5.4) is true. Let us verify (5.6).

Suppose otherwise that there is some $r_0 > 0$ to make $u(r_0) \geq 0$. Since $u(r) < 0$ for $r > 0$ small, we may assume r_0 to be the smallest such number at which $u(r_0) \geq 0$. Obviously $u(r_0) = 0$. Because $u(r_0)$ cannot be a local minimum of u and $r = r_0$ is an isolated zero of u, we see that there exists some $\delta > 0$ so that $u(r) > 0$ for $r \in (r_0, r_0 + \delta)$. The maximum principle prohibits the existence of an $r_1 > r_0$ to make $u(r_1) = 0$. Thus $u(r) > 0$ for all $r > r_0$.

As a consequence, we can strengthen the above observation by the statement $u_r(r) > 0$ $(r > 0)$. In fact, if there were some $r_1 > 0$ so that $u_r(r_1) = 0$, then $r_1 \neq r_0$. Thus $u(r_1) < 0$ if $r_1 < r_0$ or $u(r_1) > 0$ if $r_1 > r_0$. However, either case would violate the maximum principle applied to (5.5).

Thus the equation (5.5) says that $(ru_r(r))_r > 0$ when $r > r_0$. Therefore $ru_r(r) > r_0 u_r(r_0) \equiv \sigma > 0$ for all $r > r_0$, which implies that

$$u(r) > \sigma[\ln r - \ln r_0], \qquad r > r_0.$$

Using the expression (5.2), we find

$$|D_j\phi|^2 > 2\sigma^2 r_0^{-\sigma} r^{\sigma-2}, \qquad |x| = r > r_0.$$

Recall the definition of $\mathcal{E}$ in Sect. 2.2. Since $\mathcal{E} > \frac{1}{2}e^{-\eta}|D_j\phi|^2$, we arrive at

$$\int_{\mathbf{R}^2} \mathcal{E}\, e^{\eta}\, dx > 2\pi \int_{r_0}^{\infty} \sigma^2 r_0^{-\sigma} r^{\sigma-1}\, dr = \infty,$$

namely, the solution does not carry a finite energy.

Suppose next that (5.6) holds. Then, according to Theorem 3.3, either $2\pi NG < 1$ or $\pi NG < 1 < 2\pi NG$. Let us first deal with the former case. In this situation, a negative solution will satisfy either (3.6) or (3.7) with $\beta > 2(1 - \pi NG)/\pi G$. We shall now concentrate on (3.6) because the latter case has been worked out in (Spruck & Yang 1992b).

Let u be a solution of (5.5) satisfying (3.6). Then (3.7) holds with $\beta = 0$ and $u_r(r) > 0$, $r > 0$.

Consider the comparison function

$$v(r) = Cr^{-b}, \qquad r > 0, \tag{5.7}$$

where $C > 0, b > 0$ are constants. Set $w = u + v$. Then

$$\begin{aligned} w_{rr} + \frac{1}{r}w_r &= r^{-4\pi NG} e^{2\pi G(u - e^u)}(e^u - 1) + b^2 r^{-2} v \\ &= r^{-4\pi NG} e^{2\pi G(u - e^u) + \lambda u} u + b^2 r^{-2} v, \end{aligned} \tag{5.8}$$

where A = A(u) € [0,1]. Using the facts that $2vNG < 1$ and u(r) -» 0 as r-» oo, we can find some ro > 0 so that

$$r^{-4\pi NG}e^{2\pi G(u-e^u)+\lambda u} > b^2 r^{-2}, \qquad r > r_0.$$

Inserting this information into (5.8) gives the elliptic inequality

$$w_{rr} + \frac{1}{r}w_r < 6^2 r\sim^2 u;, \qquad r > r_0. \tag{5.9}$$

Choose $C > 0$ in (5.7) sufficiently large so that

$$w(r_0) = ti(r_0) + Cr\bar{o}^b \geq 0. \tag{5.10}$$

Applying (5.10) to (5.9) and using the boundary behavior $w(r)$ —> 0 (as r —• oo), we get $w(r) \geq 0$, r $\geq$ ro. In summary, there holds the decay rate estimate

$$- C(b)r\sim^b \leq u(r) < 0, \qquad r > 1. \tag{5.11}$$

Note that, in (5.11), $b > 0$ is arbitrary. Thus u vanishes at infinity faster than any power function of the type r"° (a > 0). Therefore, for any $a > 0$, e"$^{(r)}$ - 1 =O(r~°) for large r as well. Inserting this information into (5.5), using (3.7) (with /? = 0), and integrating, we obtain the same type of decay rate for u_r:

Ur(r) = --1 r p*-**NGe2wGlu-e*)(eu _ j) df> = Q^-a) for any fl > Q . (5 .12)
r Jr

In terms of u, we easily obtain from Sect. 2.2 the expressions

rf = -47riVGlnr + 2TTG(U - c^v), /i: = 7rG[u^{2}e^ + (e^u - I)^2],

$$|\phi|^2 = e^u, \qquad |D_j\phi|^2 = 2u_r^2 e^u, \qquad F_{12} = \tfrac{1}{2}e^\eta(1-e^u).$$

Thus, in view of $u < 0$, we see that the 2-surface where the strings reside *always* has a *positive* Gaussian curvature and, as r —> oo, there hold the following estimates for the physical fields

eV = 0(r-$^{4*"G}$), M^2 - 1, |£>^|^2, F_{jk}, K = O(r-), (5.13)

where $a > 0$ is arbitrary. Consequently (5.4) is verified.

Finally, we assume $vNG < 1 < 2xNG$. Then a negative solution u of (5.5) will satisfy (3.7) with 0>Q. If $0 > 0$, we easily derive the estimates as r —• oo:

e\ F_{jk} = CXr-^{2}^^{2}"-"^{3})), M^2 = O(r-"), \Dj<j>\$^2$, KJ> = O(r-<$^{2+}$">). (5.14)

However, $K > \pi G/2$ (say) at points far away from the location of the string. This fact implies that the 2-surface is not asymptotically Euclidean.

On the other hand, if $\beta = 0$, then $u_r(r) > 0$ for all $r > 0$ because $ru_r(r)$ is a decreasing function in view of the property $u < 0$ and (5.5). Thus $\lim_{r\to\infty} u(r)$ exists and is non-positive. Applying this fact to the integral in (5.12) yields directly the bound

$$ru_r(r) = \mathrm{O}(r^{2(1-2\pi NG)}) \qquad \text{for large } r > 0.$$

Therefore we arrive at the estimates at $|x| = r = \infty$:

$$e^\eta,\;\; F_{jk},\;\; Ke^\eta = \mathrm{O}(r^{-4\pi NG}), \quad |\phi|^2 = \mathrm{O}(1), \quad |D_j\phi|^2 = \mathrm{O}(r^{-2(4\pi NG-1)}). \qquad (5.15)$$

Both (5.14) and (5.15) lead to (5.4). □

Remark 5.1. The expressions (5.13)–(5.15) give us the asymptotic behavior of the physical fields and the geometry of the conformally flat 2-surface $(\mathbf{R}^2, \{e^\eta\delta_{jk}\})$. In particular, since the canonical volume $\int_{\mathbf{R}^2} e^\eta\, dx < \infty$ when $\pi NG < 1 < 2\pi NG$, we find that the finite-energy condition implies that the space cannot be asymptotically Euclidean at infinity.

Remark 5.2. In terms of β given in (3.7) and the winding number N, the values of the energies of the matter-gauge sector and the gravity sector may all be obtained explicitly. Here we choose not to pursue these calculations.

Remark 5.3. The condition (5.3) is a rather peculiar obstruction to the existence of the $U(1)$ self-dual cosmic strings which are produced from gravitational and electromagnetic interactions. On the other hand, when nuclear forces are put into the coupling, cosmic strings with an arbitrary winding number may exist. This drastic distinction has been observed in the Einstein–Weinberg–Salam system where the interactions of gravitational, electromagnetic, and weak forces are considered (Yang 1992).

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