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FINITE ELEMENT METHODS FOR THE TIME-DEPENDENT GINZBURG-LANDAU MODEL OF SUPERCONDUCTIVITY

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Abstract. The initial-boundary value problem for the time-dependent Ginzburg-Landau equations that model the macroscopic behavior of superconductors is considered. The convergence of finite-dimensional, semi-discrete Galerkin approximations is studied as is a fully-discrete scheme. The results of some computational experiments are presented.

Key words. superconductivity, time-dependent Ginzburg-Landau equations, finite element methods

AMS(MOS) subject classifications. 82D55, 35A05

1. The time-dependent Ginzburg-Landau equations.

The steady state Ginzburg-Landau model for superconductivity (see, e.g., [6] or [18]) was extended to the time-dependent case by Gor'kov and Eliashberg in [13]. The latter model is defined by the differential equations

$$\eta\frac{\partial\psi}{\partial t} + i\eta\kappa\Phi\psi + \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2\psi - \psi + |\psi|^2\psi = 0 \quad \text{in } \Omega\times[0,T] \tag{1.1}$$

and

$$\frac{\partial\mathbf{A}}{\partial t} + \operatorname{curl}\operatorname{curl}\mathbf{A} + \nabla\Phi + \frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) + |\psi|^2\mathbf{A} = \operatorname{curl}\mathbf{H} \quad \text{in } \Omega\times[0,T], \tag{1.2}$$

the boundary conditions

$$(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi)\cdot\mathbf{n} = 0 \quad \text{on } \Gamma\times[0,T], \tag{1.3}$$

$$\operatorname{curl}\mathbf{A}\times\mathbf{n} = \mathbf{H}\times\mathbf{n} \quad \text{on } \Gamma\times[0,T], \tag{1.4}$$

and

$$\mathbf{E}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma\times[0,T], \tag{1.5}$$

and the initial conditions

$$\psi(\mathbf{x},0) = \psi_0(\mathbf{x}) \qquad \text{and} \qquad \mathbf{A}(\mathbf{x},0) = \mathbf{A}_0(\mathbf{x}) \qquad \text{in } \Omega. \tag{1.6}$$

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In (1.1)-(1.6), all variables have been nondimensionalized following standard practices; see, e.g., [6] or [18]. Ω denotes an open bounded set in $\mathbf{R}^d$, $d = 2$ or 3, with boundary Γ, T a positive constant, $\psi = |\psi|e^{i\phi}$ the complex, scalar-valued order parameter, $\mathbf{A}$ the real, vector-valued magnetic potential, and Φ the real, scalar-valued electric potential. Also, κ and η are positive material constants and $\mathbf{H}$ is the vector-valued external magnetic field; these, along with geometric information, serve to specify the model. It has become customary to refer to (1.1) and (1.2) as the *time-dependent Ginzburg-Landau equations.*

The existence and uniqueness of solutions of the time-dependent Ginzburg-Landau equations have been considered in [2], [4], and [9]. Numerical studies using the model are given in [8], [10], [16], and [17]. Studies connected with these equations have also appeared in the theoretical physics literature; see, e.g., [14]. In this paper, we consider numerical methods and their analysis for the approximate solution of the time-dependent Ginzburg-Landau equations; spatial discretization is effected by finite element methods; a backward Euler scheme is used for the temporal discretization. Many of the results given below have been previously reported on in [5]; here we provide details and proofs.

Physical variables of interest are related to the dependent variables ψ, $\mathbf{A}$, and Φ of the model by the relations:

$$\begin{aligned} |\psi|^2 &= \text{density of superconducting charge carriers;} \\ \mathbf{h} &= \operatorname{curl}\mathbf{A} = \text{magnetic field;} \\ \mathbf{E} &= \operatorname{grad}\Phi + \partial\mathbf{A}/\partial t = \text{electric field; and} \\ \mathbf{j} &= |\psi|^2[\mathbf{A} - (1/\kappa)\operatorname{grad}\phi] = \text{current.} \end{aligned}$$

In particular, in the nondimensionalization being used, $\psi = 0$ represents the non-superconducting state, $|\psi| = 1$ a superconducting state, and $0 < |\psi| < 1$ a mixed, or intermediate state.

We assume that $|\psi_0(x)| \le 1, a.e.$, which implies that the magnitude of the initial order parameter does not exceed the value at the superconducting state. The external field $\mathbf{H}$ is assumed to be time-independent.

In the remainder of this section, we introduce some notation that will be used in the sequel and give a brief discussion of gauge choices. In §2, we discuss semi-discrete Galerkin approximations, first in the context of general finite-dimensional approximations, and then, specifically, in the finite-element context. In §3 we discuss fully-discrete approximations, and then, in §4, we provide the results of some numerical experiments.

Throughout, for any non-negative integer s, $H^s(\Omega)$ will denote the Sobolev space of real-valued functions having square integrable spatial derivatives of order up to s in the domain Ω. The corresponding spaces of complex-valued functions whose real and imaginary parts belong to $H^s(\Omega)$ will be denoted by $\mathcal{H}^s(\Omega)$. Corresponding spaces of vector-valued functions, each of whose d components belong to $H^s(\Omega)$, will be denoted by $\mathbf{H}^s(\Omega)$, i.e., $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^d$. Norms of functions belonging to $H^s(\Omega)$, $\mathbf{H}^s(\Omega)$, and $\mathcal{H}^s(\Omega)$ will all be denoted, without any possible ambiguity, by $\|\cdot\|_s$. For details concerning these spaces, one may consult [1]. A similar notational convention will hold for the Lebesgue spaces $L^p(\Omega)$ and their complex and vector-valued counterparts $\mathcal{L}^p(\Omega)$ and $\mathbf{L}^p(\Omega)$, respectively. We will sometimes use $\|\cdot\|_B$ to denote the norm defined on the space B. We will use the convention that $(\cdot,\cdot)$ denotes the standard L^2 inner-product in real function spaces, while for complex valued functions

$$(\psi,\tilde{\psi}) = \int_\Omega \psi\tilde{\psi}^*\,d\Omega\,, \quad \forall\,\psi,\tilde{\psi}\in\mathcal{L}^2(\Omega)\,.$$

We will also make use of the following subspaces of $\mathbf{H}^1(\Omega)$:

$$\mathbf{H}^1_n(\Omega) = \{\,\mathbf{Q}\in\mathbf{H}^1(\Omega)\ :\ \mathbf{Q}\cdot\mathbf{n} = 0 \text{ on } \Gamma\,\}$$

and

$$\mathbf{H}^1_n(\mathrm{div}\,;\Omega) = \{\,\mathbf{Q}\in\mathbf{H}^1(\Omega)\ :\ \mathrm{div}\,\mathbf{Q} = 0 \text{ in } \Omega \text{ and } \mathbf{Q}\cdot\mathbf{n} = 0 \text{ on } \Gamma\,\}\,.$$

We note that $(\|\mathrm{div}\,\mathbf{Q}\|_0^2 + \|\mathrm{curl}\,\mathbf{Q}\|_0^2)^{1/2}$ and $\|\mathrm{curl}\,\mathbf{Q}\|_0$ define norms on $\mathbf{H}^1_n(\Omega)$ and $\mathbf{H}^1_n(\mathrm{div}\,;\Omega)$, respectively, that are equivalent to the standard $\mathbf{H}^1(\Omega)$-norm $\|\mathbf{Q}\|_1$; see, e.g., [11].

To take into account the time-dependence, we define the following spaces: for any given $T>0$ and given Hilbert space B,

$$L^p(0,T;B) = \left\{ f\ :\ f(\cdot,t)\in B\,, \forall\,t\in(0,T)\ a.e.\,,\ \int_0^T \|f(\cdot,t)\|_B^p\,dt < \infty \right\}.$$

Spaces such as $L^\infty(0,T;B)$ and $H^m(0,T;B)$ are defined in a similar manner. In particular, we let $\mathbf{S} = \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))$ and

$$\mathbf{V} = \mathbf{L}^\infty(0,T;\mathbf{H}^1_n(\Omega)) \cap \mathbf{H}^1(0,T;\mathbf{L}^2(\Omega))\,.$$

Also, we let $\mathcal{S} = \mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))$ and

$$\mathcal{V} = \mathcal{L}^\infty(0,T;\mathcal{H}^1(\Omega)) \cap \mathcal{H}^1(0,T;\mathcal{L}^2(\Omega))\,.$$

For convenience when considering finite element approximations, we assume that Ω is a bounded convex polygon or convex polyhedron in $\mathbf{R}^d$, where $d=2$ or 3. Results may be extended to domains with smooth boundary if curved finite element spaces are used.

The time-dependent Ginzburg-Landau model (1.1)-(1.6) lack uniqueness and thus are not well-posed. However, they possess a gauge invariance property, see, e.g., [4], which, among other things, implies that the physical variables of interest are indeed uniquely determined from (1.1)-(1.6). Also, one can choose a gauge in order to obtain mathematically well-posed equations. Such a procedure was thoroughly discussed in [4] where several possible gauge choices were given. Here, we focus our attention to the gauge that eliminates the electric potential Φ. This is one of most frequently used gauge choice in numerical simulations; see, e.g., [8], [10] and [17]. In this gauge, the time-dependent Ginzburg-Landau equations are given by

(1.7) , ^ + f$\left(\frac{i}{\kappa}\right.$v + A) ^{2} ^ - ^ + 1 ^ = 0 in ft

and

(1.8) $\frac{dA}{}$ -^- + curl curl A + $\frac{i}{}$—(^ W> - *tpVrp**) + M^{2}A = curl H in Ω.

We also have the boundary conditions

(1.9) V ^ n = 0 onT,

(1.9) curlAxn = H x n on *T*,

and

(1.10) A-n = 0 onT,

and the initial conditions

(1.11) V(x,0) = ^o(x), A(x,0) = A_o(x), and divA(x,0) = 0 in Ω.

Again, see [4] for details. In the gauge currently being used, the vector potential A need not be divergence free, though for the steady state solution, we do have div A = 0; see [6].

2. Semi-discrete in space finite element approximations. We now study semi-discrete Galerkin finite element approximations of the time-dependent Ginzburg-Landau equations in the zero electric potential gauge. The global existence and uniqueness of strong solutions in this gauge has been proved in [4]. By semi-discrete, we mean that discretization is effected only with respect to the spatial variables.

2.1. Weak formulation. The solution (^, A) € V x V of equations (1.7)-(1.11) satisfies the following weak formulation:

$$d \quad + \left((|\psi|^2 - 1)\psi, \tilde{\psi}\right) = 0 \quad \forall \tilde{\psi} \in \mathcal{H}^1(\Omega)$$

and

(2-2) **-£(A, Ã)+(curl A, curl Ã) + (V>I^{2}A, Ã)**

$$+ \left(\frac{i}{2\kappa}(\psi^* \nabla\psi - \psi\nabla\psi^*), \tilde{\mathbf{A}}\right) = (\mathrm{H}, \mathrm{curl}\tilde{\mathrm{A}}) \quad \forall \tilde{\mathbf{A}} \in \mathbf{H}^1_n(\Omega).$$

together with initial condition $\psi_0 \in \mathcal{H}^1(\Omega)$ and $\mathbf{A}_0 \in \mathbf{H}^1_n(\mathrm{div}\,;\Omega)$. Such initial conditions make sense for functions belonging to $\mathcal{V} \times \mathbf{V}$ that satisfy the weak equations. For convenience, we assume that the applied field $\mathbf{H} \in \mathbf{H}^1(\Omega)$ and is independent of t. It was shown in [4] that for any $T > 0$, (2.1)-(2.2) have a unique solution in $\mathcal{V} \times \mathbf{V}$.

To study the existence and uniqueness of the solutions of the above system, the following modified problem was introduced: find $(\psi^\epsilon, \mathbf{A}^\epsilon) \in \mathcal{V} \times \mathbf{V}$ such that

$$\begin{aligned}\eta\frac{d}{dt}(\psi^\epsilon, \tilde{\psi}) + &\left(\frac{i}{\kappa}\nabla\psi^\epsilon + \mathbf{A}^\epsilon\psi^\epsilon\,,\ \frac{i}{\kappa}\nabla\tilde{\psi} + \mathbf{A}^\epsilon\tilde{\psi}\right)\\ &+ \left((|\psi^\epsilon|^2 - 1)\psi^\epsilon, \tilde{\psi}\right) = 0 \quad \forall\, \tilde{\psi} \in \mathcal{H}^1(\Omega)\end{aligned} \tag{2.3}$$

$$\begin{aligned}\frac{d}{dt}(\mathbf{A}^\epsilon, \tilde{\mathbf{A}}) + &(\mathrm{curl}\,\mathbf{A}^\epsilon, \mathrm{curl}\,\tilde{\mathbf{A}}) + \epsilon(\mathrm{div}\,\mathbf{A}^\epsilon, \mathrm{div}\,\tilde{\mathbf{A}}) + (|\psi^\epsilon|^2\mathbf{A}^\epsilon, \tilde{\mathbf{A}})\\ &+ \Re\left\{\left(\frac{i}{\kappa}\nabla\psi^\epsilon, \psi^\epsilon\tilde{\mathbf{A}}\right)\right\} = (\mathbf{H}, \mathrm{curl}\,\tilde{\mathbf{A}}) \quad \forall\, \tilde{\mathbf{A}} \in \mathbf{H}^1_n(\Omega)\end{aligned} \tag{2.4}$$

with the same initial conditions are those for (2.1)-(2.2), so that the initial conditions are independent of ϵ. Here, $\epsilon > 0$ is an arbitrary parameter. Note that the modified system (2.3)-(2.4) reduces to the original system (2.1)-(2.2) when $\epsilon = 0$. It was also shown in [4] that, for any $T > 0$ and $\epsilon > 0$, (2.3)-(2.4) have a unique solution in $\mathcal{V} \times \mathbf{V}$. Moreover, for any $T > 0$, as $\epsilon \to 0$, solutions of (2.3)-(2.4) converge (weakly in $\mathcal{V} \times \mathbf{V}$) to the unique solution of (2.1)-(2.2).

2.2. Finite-dimensional Galerkin approximations. In [4], we studied abstract finite-dimensional Galerkin approximations of the system (2.3)-(2.4). Let $\mathbf{\Lambda}_n$ and $\mathcal{Z}_n$ be n-dimensional subspaces of $\mathbf{H}^1_n(\Omega)$ and $\mathcal{H}^1(\Omega)$ respectively such that

$$\bigcup \mathbf{\Lambda}_n \quad \text{is dense in } \mathbf{H}^1_n(\Omega) \qquad \text{and} \qquad \bigcup \mathcal{Z}_n \quad \text{is dense in } \mathcal{H}^1(\Omega)\,.$$

A standard Galerkin-finite dimensional approximation is defined as follows: find $(\psi^\epsilon_n(t), \mathbf{A}^\epsilon_n(t)) \in \mathcal{Z}_n \times \mathbf{\Lambda}_n$ such that

$$(\nabla\psi^\epsilon_n(0), \nabla\tilde{\psi}_n) + (\psi^\epsilon_n(0), \tilde{\psi}_n) = (\nabla\psi(0), \nabla\tilde{\psi}_n) + (\psi(0), \tilde{\psi}_n) \quad \forall\tilde{\psi}_n \in \mathcal{Z}_n\,, \tag{2.5}$$

$$(\nabla\mathbf{A}^\epsilon_n(0), \nabla\tilde{\mathbf{A}}_n) + (\mathbf{A}^\epsilon_n(0), \tilde{\mathbf{A}}_n) = (\nabla\mathbf{A}(0), \nabla\tilde{\mathbf{A}}_n) + (\mathbf{A}(0), \tilde{\mathbf{A}}_n) \quad \forall\tilde{\mathbf{A}}_n \in \mathbf{\Lambda}_n\,, \tag{2.6}$$

$$\begin{aligned}\eta\frac{d}{dt}(\psi^\epsilon_n, \tilde{\psi}_n) + &\left(\frac{i}{\kappa}\nabla\psi^\epsilon_n + \mathbf{A}^\epsilon_n\psi^\epsilon_n\,,\ \frac{i}{\kappa}\nabla\tilde{\psi}_n + \mathbf{A}^\epsilon_n\tilde{\psi}_n\right)\\ &+ \left((|\psi^\epsilon_n|^2 - 1)\psi^\epsilon_n, \tilde{\psi}_n\right) = 0 \quad \forall\, \tilde{\psi}_n \in \mathcal{Z}_n\,,\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\frac{d}{dt}(\mathbf{A}^\epsilon_n, \tilde{\mathbf{A}}_n) + &(\mathrm{curl}\,\mathbf{A}^\epsilon_n, \mathrm{curl}\,\tilde{\mathbf{A}}_n) + \epsilon(\mathrm{div}\,\mathbf{A}^\epsilon_n, \mathrm{div}\,\tilde{\mathbf{A}}_n) + (|\psi^\epsilon_n|^2\mathbf{A}^\epsilon_n, \tilde{\mathbf{A}}_n)\\ &+ \Re\left\{\left(\frac{i}{\kappa}\nabla\psi^\epsilon_n, \psi^\epsilon_n\tilde{\mathbf{A}}_n\right)\right\} = (\mathbf{H}, \mathrm{curl}\,\tilde{\mathbf{A}}_n) \quad \forall\, \tilde{\mathbf{A}}_n \in \mathbf{\Lambda}_n\,.\end{aligned} \tag{2.8}$$

Note that we are defining discrete initial conditions by H^1-projections.

We now quote a result of [4] concerning the solutions of (2.5)-(2.8).

THEOREM 2.1. *Given $T > 0$, if $\psi_0 \in \mathcal{H}^1(\Omega)$, $|\psi_0(x)| \leq 1$ a.e., and $\mathbf{A}_0 \in \mathbf{H}^1_n(\mathrm{div}\,;\Omega)$, for any $\epsilon > 0$ and $n > 0$, there exists a unique solution $(\psi^\epsilon_n, \mathbf{A}^\epsilon_n)$ to* (2.5)-(2.8) *in $[0,T]$. Moreover, $(\psi^\epsilon_n, \mathbf{A}^\epsilon_n)$ is uniformly bounded in $\mathcal{V} \times \mathbf{V}$, independent of n and ϵ and, for any $\epsilon > 0$, the sequence $(\psi^\epsilon_n, \mathbf{A}^\epsilon_n)$ converges weakly in $\mathcal{V} \times \mathbf{V}$ (and therefore strongly in $\mathcal{S} \times \mathbf{S}$) to the unique solution $(\psi^\epsilon, \mathbf{A}^\epsilon)$ of* (2.3)-(2.4) *as $n \to \infty$. In addition, for any $\epsilon > 0$, the sequence $(\psi^\epsilon_n, \mathbf{A}^\epsilon_n)$ converges strongly in $\mathcal{L}^2(0,T;\mathcal{H}^1(\Omega)) \times \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega))$ to $(\psi^\epsilon, \mathbf{A}^\epsilon)$ as $n \to \infty$.* □

As was mentioned above, in [4], it was also shown that solutions of the modified problem (2.3)-(2.4) converge to the solution of the original system (2.1)-(2.2) as $\epsilon \to 0$. Furthermore, once can easily check that the steady state equations of both problems are identical.

2.3. Semi-discrete Galerkin finite element approximation.

Let Λ_h and $\mathcal{Z}_h$ be C^0 finite element subspaces of $\mathbf{H}^1_n(\Omega)$ and $\mathcal{H}^1(\Omega)$, respectively, defined on a regular quasi-uniform mesh, parametrized by a parameter h that tends to zero. These spaces are constructed in a standard way and h is some measure of the size of the finite elements in the mesh. We assume that the subspaces satisfy the following approximation properties:

$$\inf_{\psi_h \in \mathcal{Z}_h} \|\psi - \psi_h\|_1 \to 0 \quad \text{as } h \to 0 \quad \forall\, \psi \in \mathcal{H}^1(\Omega) \tag{2.9}$$

and

$$\inf_{\mathbf{A}_h \in \Lambda_h} \|\mathbf{A} - \mathbf{A}_h\|_1 \to 0 \quad \text{as } h \to 0 \quad \forall\, \mathbf{A} \in \mathbf{H}^1_n(\Omega)\,. \tag{2.10}$$

One may consult [3] for conditions on the finite element partitions such that (2.9)-(2.10) are satisfied.

Therefore, by the Theorem 2.1, we have:

COROLLARY 2.2. *Assume that the approximation properties* (2.9)-(2.10) *and the hypothseses of Theorem 2.1 hold. Then, given $T > 0$, for any $\epsilon > 0$, the semi-discrete finite element approximation $(\psi^\epsilon_h, \mathbf{A}^\epsilon_h)$ exists in $[0,T]$. Moreover, $(\psi^\epsilon_h, \mathbf{A}^\epsilon_h)$ is uniformly bounded in $\mathcal{V} \times \mathbf{V}$, independent of h and ϵ. Furthermore, for any $\epsilon > 0$, the sequence $(\psi^\epsilon_h, \mathbf{A}^\epsilon_h)$ converges weakly in $\mathcal{V} \times \mathbf{V}$ (and therefore strongly in $\mathcal{S} \times \mathbf{S}$) to the unique solution $(\psi^\epsilon, \mathbf{A}^\epsilon)$ of* (2.3)-(2.4) *as $n \to \infty$. In addition, for any $\epsilon > 0$, the sequence $(\psi^\epsilon_h, \mathbf{A}^\epsilon_h)$ converges strongly in $\mathcal{L}^2(0,T;\mathcal{H}^1(\Omega)) \times \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega))$ to $(\psi^\epsilon, \mathbf{A}^\epsilon)$ as $n \to \infty$.* □

2.4. Asymptotic behavior of the finite element approximations.

We now examine, for given $h > 0$ and $\epsilon > 0$, the asymptotic behavior of the semi-discrete finite element solution $(\psi^\epsilon_h, \mathbf{A}^\epsilon_h)$.

A nondimensionalized form of the Ginzburg-Landau free energy functional is given by

$$\mathcal{G}(\psi, \mathbf{A}) = \int_\Omega \left(\left| \frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi \right|^2 + \frac{1}{2}(|\psi|^2 - 1)^2 + |\mathrm{curl}\,\mathbf{A} - \mathbf{H}|^2 \right) d\Omega\,.$$

Now, let

$$\mathcal{G}_\epsilon(\psi, \mathbf{A}) = \mathcal{G}(\psi, \mathbf{A}) + \epsilon \int_\Omega |\operatorname{div} \mathbf{A}|^2 \, d\Omega . \tag{2.11}$$

The dynamical system (2.7)-(2.8) is a gradient system by the definition in [15], since the functional $\mathcal{G}_\epsilon$ serves as a Lyapunov functional. Hence, it is straightforward to obtain the following result.

LEMMA 2.3. *The ω-limit set of the system (2.7)-(2.8) is a subset of the equilibrium points which consists of solutions of the following equations:*

$$\left(\frac{i}{\kappa}\nabla\psi_h^\epsilon + \mathbf{A}_h^\epsilon\psi_h^\epsilon \, , \, \frac{i}{\kappa}\nabla\tilde{\psi}^h + \mathbf{A}_h^\epsilon\tilde{\psi}^h\right) + \left((|\psi_h^\epsilon|^2 - 1)\psi_h^\epsilon, \tilde{\psi}^h\right) = 0 \quad \forall\, \tilde{\psi}^h \in \mathcal{Z}^h \tag{2.12}$$

and

$$\begin{aligned}(\operatorname{curl} \mathbf{A}_h^\epsilon - \mathbf{H}, \operatorname{curl} \tilde{\mathbf{A}}^h) + \epsilon(\operatorname{div} \mathbf{A}_h^\epsilon, \operatorname{div} \tilde{\mathbf{A}}^h) + (|\psi_h^\epsilon|^2 \mathbf{A}_h^\epsilon, \tilde{\mathbf{A}}^h) \\ + \Re\left\{\left(\frac{i}{\kappa}\nabla\psi_h^\epsilon, \psi_h^\epsilon\tilde{\mathbf{A}}^h\right)\right\} = 0 \quad \forall\, \tilde{\mathbf{A}}^h \in \mathbf{\Lambda}^h . \quad \square\end{aligned} \tag{2.13}$$

3. Fully-discrete approximations. Semi-discrete approximations only deal with spatial discretization and the resulting equations form a system of ordinary differential equations. Fully discrete approximations involve a discretization of these ordinary differential equations. Here, we will study the implicit Euler method. An interesting feature of this full discretizaiton is the existence of a discrete Lyapunov like functional that may be very useful for long time integration.

3.1. The implicit Euler method. Let $t_0 = 0$, and $t_{n+1} = t_n + \Delta t$ where Δt is the step size in time. The initial approximation is given by the H^1-projection of the given initial data, i.e., define $(\psi_0^h, \mathbf{A}_0^h) \in \mathcal{Z}^h \times \mathbf{\Lambda}^h$ by

$$(\nabla\psi_0^h, \nabla\tilde{\psi}^h) + (\psi_0^h, \tilde{\psi}^h) = (\nabla\psi(0), \nabla\tilde{\psi}^h) + (\psi(0), \tilde{\psi}^h) \quad \forall\tilde{\psi}^h \in \mathcal{Z}_h \tag{3.1}$$

and

$$(\nabla\mathbf{A}_0^h, \nabla\tilde{\mathbf{A}}^h) + (\mathbf{A}_0^h, \tilde{\mathbf{A}}^h) = (\nabla\mathbf{A}(0), \nabla\tilde{\mathbf{A}}^h) + (\mathbf{A}(0), \tilde{\mathbf{A}}^h) \quad \forall\tilde{\mathbf{A}}^h \in \mathbf{\Lambda}^h . \tag{3.2}$$

Then, for $n = 0, 1, \ldots$, we let

$$\begin{aligned}\eta(\frac{\psi_{n+1}^h - \psi_n^h}{\Delta t}, \tilde{\psi}^h) + \left(\frac{i}{\kappa}\nabla\psi_{n+1}^h + \mathbf{A}_{n+1}^h\psi_{n+1}^h \, , \, \frac{i}{\kappa}\nabla\tilde{\psi}^h + \mathbf{A}_{n+1}^h\tilde{\psi}^h\right) \\ + \left((|\psi_{n+1}^h|^2 - 1)\psi_{n+1}^h, \tilde{\psi}^h\right) = 0 \quad \forall\, \tilde{\psi}^h \in \mathcal{Z}^h\end{aligned} \tag{3.3}$$

and

$$(3.4)\qquad \begin{aligned}&(\frac{\mathbf{A}_{n+1}^h - \mathbf{A}_n^h}{\Delta t}, \tilde{\mathbf{A}}^h) + (\operatorname{curl} \mathbf{A}_{n+1}^h - \mathbf{H}, \operatorname{curl} \tilde{\mathbf{A}}^h) \\ &+ \epsilon(\operatorname{div} \mathbf{A}_{n+1}^h, \operatorname{div} \tilde{\mathbf{A}}^h) + (|\psi_{n+1}^h|^2 \mathbf{A}_{n+1}^h, \tilde{\mathbf{A}}^h) \\ &+ \left(\frac{i}{2\kappa}(\psi_{n+1}^{h*} \nabla \psi_{n+1}^h - \psi_{n+1}^h \nabla \psi_{n+1}^{h*}), \tilde{\mathbf{A}}^h\right) = 0 \quad \forall\, \tilde{\mathbf{A}}^h \in \mathbf{A}^h .\end{aligned}$$

THEOREM 3.1. *For any $h > 0$, $\Delta t > 0$, and $\epsilon \geq 0$, there exists a solution to the system* (3.3)-(3.4) *for any n. Moreover, for all $n = 0, 1, ...,$*

$$\mathcal{G}_\epsilon(\psi_{n+1}^h, \mathbf{A}_{n+1}^h) + \int_\Omega \left(\frac{(\psi_{n+1}^h - \psi_n^h)^2}{\Delta t} + \frac{(\mathbf{A}_{n+1}^h - \mathbf{A}_n^h)^2}{\Delta t} \right) d\Omega \leq \mathcal{G}_\epsilon(\psi_n^h, \mathbf{A}_n^h) .$$

Proof. The solution of (3.3)-(3.4) is a critical point of the following minimization problem:

$$\min \mathcal{J}_n^h(\psi^h, \mathbf{A}^h) \quad \text{over} \quad (\psi^h, \mathbf{A}^h) \in \mathcal{Z}^h \times \mathbf{A}^h ,$$

where

$$\mathcal{J}_n^h(\psi^h, \mathbf{A}^h) = \mathcal{G}_\epsilon(\psi^h, \mathbf{A}^h) + \int_\Omega \left(\frac{(\psi^h - \psi_n^h)^2}{\Delta t} + \frac{(\mathbf{A}^h - \mathbf{A}_n^h)^2}{\Delta t} \right) d\Omega .$$

Obviously, there exists a minimizer for this finite dimensional minimization problem. Hence, the solution to (3.1)-(3.2) exists. The inequality in the theorem follows from

$$\mathcal{J}_n^h(\psi_{n+1}^h, \mathbf{A}_{n+1}^h) \leq \mathcal{J}_n^h(\psi_n^h, \mathbf{A}_n^h) \leq \mathcal{G}_\epsilon(\psi_n^h, \mathbf{A}_n^h) \quad \forall\, n = 0, 1, \ldots . \quad \square$$

COROLLARY 3.2. *Given initial data and $T > 0$, there exists a constant $C > 0$ such that, for any $h > 0$, $\Delta t > 0$ and $(n+1)\Delta t \leq T$, any solution $(\psi_{n+1}^h, \mathbf{A}_{n+1}^h)$ to the system* (3.3)-(3.4) *satisfies*

$$(3.5)\qquad ||\mathbf{A}_{n+1}^h - \mathbf{A}_n^h||_0 \leq C \Delta t^{1/2} ,$$

$$(3.6)\qquad ||\psi_{n+1}^h - \psi_n^h||_0 \leq C \Delta t^{1/2} ,$$

$$(3.7)\qquad ||\psi_{n+1}^h||_{0,4} \leq C ,$$

$$(3.8)\qquad ||\mathbf{A}_{n+1}^h||_0 \leq C ,$$

$$(3.9)\qquad ||\mathbf{A}_{n+1}^h||_1 \leq C \min\{\epsilon^{-1/2}, h^{-1}\} ,$$

$$\|\frac{i}{\kappa}\nabla\psi^h_{n+1} + \mathbf{A}^h_{n+1}\psi^h_{n+1}\|_0 \le C\,, \tag{3.10}$$

$$\|\mathbf{A}^h_{n+1}\|_{0,4} \le C\min\{h^{-1/2}, (log|h|)^{1/4}\epsilon^{-1/4}\} \qquad (d=2)\,, \tag{3.11}$$

$$\|\mathbf{A}^h_{n+1}\|_{0,4} \le C\min\{h^{-3/4}, \epsilon^{-3/8}\} \qquad (d=3)\,, \tag{3.12}$$

$$\|\mathbf{A}^h_{n+1}\|_{0,\infty} \le C\min\{h^{-1}, (log|h|)^{1/2}\epsilon^{-1/2}\} \qquad (d=2)\,, \tag{3.13}$$

$$\|\mathbf{A}^h_{n+1}\|_{0,\infty} \le C\min\{h^{-3/2}, h^{-1/2}\epsilon^{-1/2}\} \qquad (d=3)\,, \tag{3.14}$$

and

$$\|\psi^h_{n+1}\|_1 \le C\min\{h^{-n/4}, \epsilon^{-1/2}\}\,. \tag{3.15}$$

Proof. Inequalities (3.5)-(3.7) and (3.10) follows immediately from the previous theorem. (3.5) implies (3.8), which in turn, implies (3.9) by the inverse inequality

$$\|u^h\|_1 \le ch^{-1}\|u^h\|_0\,.$$

Similarly, one gets (3.11)-(3.15) from the previous estimates, the inverse inequality

$$\|u^h\|_{0,r} \le ch^{d/r-d/s}\|u^h\|_{0,s}\,,$$

and the discrete imbedding inequalities:

$$\|u^h\|_{0,\infty} \le c|logh|^{1/2}\|u^h\|_1 \qquad (d=2)\,,$$

$$\|u^h\|_{0,r} \le \left(\|u^h\|_{0,\infty}\right)^{1-2/r}\left(\|u^h\|_0\right)^{2/r}\,,$$

and for $d=3$

$$\|u^h\|_{0,4} \le (\|u^h\|_{0,6})^{3/4}(\|u^h\|_{0,2})^{1/4}\,. \qquad \square$$

3.2. Uniqueness of the approximate solution. Next, we discuss the uniqueness of the solution to (3.3)-(3.4) for a given value of n. In general, the solution may not be unique; however, if one seeks a solution that actually minimizes the functional $\mathcal{J}^h_n$, then some uniqueness results may be obtained. First, we have the following result.

LEMMA 3.2. *Let $C>0$ be a constant. If Δt and $\Delta t h^{-d/2}$ are sufficiently small, then for any $\epsilon \ge 0$, the functional $\mathcal{J}^h_n$ is convex for any $(\psi^h, \mathbf{A}^h)$ in the set*

$$\mathcal{M} = \left\{ (\psi^h, \mathbf{A}^h) \in \mathcal{Z}^h \times \mathbf{A}^h \;\middle|\; \|\psi^h\|_{0,4} \le C, \quad \|\mathbf{A}^h\|_0 \le C, \right. \\ \left. \textit{and} \quad \|\frac{i}{\kappa}\nabla\psi^h_{n+1} + \mathbf{A}^h_{n+1}\psi^h_{n+1}\|_0 \le C \right\}.$$

Proof. Let $(\psi^h, \mathbf{A}^h)$ be in the set $\mathcal{M}$. Then for any $(\tilde{\psi}^h, \tilde{\mathbf{A}}^h)$, we have, for Δt sufficiently small,

$$\begin{aligned}
&\frac{d^2}{d\mu^2}\mathcal{J}_n^h(\psi^h + \mu\tilde{\psi}^h, \mathbf{A}^h + \nu\tilde{\mathbf{A}}^h)|_{(0,0)} \\
&= \int_\Omega \left[2|\frac{i}{\kappa}\nabla\tilde{\psi}^h + \mathbf{A}^h\tilde{\psi}^h|^2 + (12|\psi^h|^2 - 2 + \frac{2}{\Delta t})|\tilde{\psi}^h|^2\right] d\Omega \\
&\geq \int_\Omega \left[2|\frac{i}{\kappa}\nabla\tilde{\psi}^h + \mathbf{A}^h\tilde{\psi}^h|^2 + \frac{1}{\Delta t}|\tilde{\psi}^h|^2\right] d\Omega \geq 2\|\frac{i}{\kappa}\nabla\tilde{\psi}^h + \mathbf{A}^h\tilde{\psi}^h\|_0^2 + \frac{1}{\Delta t}\|\tilde{\psi}^h\|_0^2, \\
&\frac{d^2}{d\nu^2}\mathcal{J}_n^h(\psi^h + \mu\tilde{\psi}^h, \mathbf{A}^h + \nu\tilde{\mathbf{A}}^h)|_{(0,0)} \\
&= \int_\Omega [(2|\psi^h|^2 + \frac{2}{\Delta t})|\tilde{\mathbf{A}}^h|^2 + 2\epsilon|\operatorname{div}\tilde{\mathbf{A}}^h|^2 + 2|\operatorname{curl}\tilde{\mathbf{A}}^h|^2]\, d\Omega \geq \frac{2}{\Delta t}\|\tilde{\mathbf{A}}^h\|_0^2,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d^2}{d\mu d\nu}\mathcal{J}_n^h(\psi^h + \mu\tilde{\psi}^h, \mathbf{A}^h + \nu\tilde{\mathbf{A}}^h)|_{(0,0)} \\
&= \int_\Omega 2\Re\{(\frac{i}{\kappa}\nabla\tilde{\psi}^h + \mathbf{A}^h\tilde{\psi}^h)(\psi^h)^*\tilde{\mathbf{A}}^h + (\frac{i}{\kappa}\nabla\psi^h + \mathbf{A}^h\psi^h)(\tilde{\psi}^h)^*\tilde{\mathbf{A}}^h\}\, d\Omega \\
&\leq 2\|(\frac{i}{\kappa}\nabla\tilde{\psi}^h + \mathbf{A}^h\tilde{\psi}^h)\|_0\|\psi^h\|_{0,4}\|\tilde{\mathbf{A}}^h\|_{0,4} + 2\|(\frac{i}{\kappa}\nabla\psi^h + \mathbf{A}^h\psi^h)\|_0\|\tilde{\psi}^h\|_{0,4}\|\tilde{\mathbf{A}}^h\|_{0,4} \\
&\leq c\|(\frac{i}{\kappa}\nabla\tilde{\psi}^h + \mathbf{A}^h\tilde{\psi}^h)\|_0 \cdot h^{-d/4}\|\tilde{\mathbf{A}}^h\|_0 + ch^{-d/4}\|\tilde{\psi}^h\|_0 \cdot h^{-d/4}\|\tilde{\mathbf{A}}^h\|_0 \\
&\leq c\left(\|(\frac{i}{\kappa}\nabla\tilde{\psi}^h + \mathbf{A}^h\tilde{\psi}^h)\|_0^2 + h^{-d/2}\|\tilde{\psi}^h\|_0^2\right)^{1/2}\left(h^{-d/2}\|\tilde{\mathbf{A}}^h\|_0^2\right)^{1/2} \\
&\leq c(\Delta t h^{-d/2})^{1/2}\left(\frac{d^2}{d\mu^2}\mathcal{J}_n^h(\psi^h + \mu\tilde{\psi}^h, \mathbf{A}^h + \nu\tilde{\mathbf{A}}^h)|_{(0,0)}\right)^{1/2}\left(\Delta t^{-1}\|\tilde{\mathbf{A}}^h\|_0^2\right)^{1/2} \\
&\leq c(\Delta t h^{-d/2})^{1/2}\left(\frac{d^2}{d\mu^2}\mathcal{J}_n^h(\psi^h + \mu\tilde{\psi}^h, \mathbf{A}^h + \nu\tilde{\mathbf{A}}^h)|_{(0,0)}\right)^{1/2} \\
&\qquad \cdot\left(\frac{d^2}{d\nu^2}\mathcal{J}_n^h(\psi^h + \mu\tilde{\psi}^h, \mathbf{A}^h + \nu\tilde{\mathbf{A}}^h)|_{(0,0)}\right)^{1/2}.
\end{aligned}$$

Thus, for $\Delta t h^{-d/2}$ sufficiently small, the functional $\mathcal{J}_n^h$ is convex on the set $\mathcal{M}$. □

From the convexity of the functional and the estimates (3.9)-(3.15), we have the following result.

COROLLARY 3.3. *If $\Delta t h^{-d/2}$ is sufficiently small, then for any $\epsilon \geq 0$, the functional $\mathcal{J}_n^h$ has a unique global minimizer which is a solution of* (3.3)-(3.4). □

We see from the above proof that for any $\epsilon \geq 0$, $h > 0$, and $\Delta t > 0$,

$$\frac{d^2}{d\nu^2}\mathcal{J}_n^h(\psi^h + \mu\tilde{\psi}^h, \mathbf{A}^h + \nu\tilde{\mathbf{A}}^h)|_{(0,0)} > 0 \quad \forall\, \tilde{\mathbf{A}}^h \neq \mathbf{0}.$$

Hence, we have the following result.

COROLLARY 3.4. *There are no local maxima for the functional* $\mathcal{J}_n^h$. □

In case ϵ is taken to be a positive constant, independent of h, then, the above proof may be modified to show that if Δt is small enough, then the global minimizer of $\mathcal{J}_n^h$ is unique for any $h > 0$, i.e., we do not need to assume that $\Delta t h^{-d/2}$ is small.

LEMMA 3.5. *Let* $\epsilon > 0$ *and* $K > 0$ *be given constants. Then, for* Δt *sufficiently small, the functional* $\mathcal{J}_n^h$ *is convex for any* $(\psi^h, \mathbf{A}^h)$ *in the set* $\{(\psi^h, \mathbf{A}^h) \in \mathcal{Z}^h \times \mathbf{A}^h \mid \|\psi^h\|_1 \le K\,,\ \|\mathbf{A}^h\|_1 \le K\}$.

Proof. Let $(\psi^h, \mathbf{A}^h)$ be in the set $\{\|\psi^h\|_1 \le K\ ,\ \|\mathbf{A}^h\|_1 \le K\}$. Then, for any $(\tilde{\psi}^h, \tilde{\mathbf{A}}^h)$, we have

$$
\begin{aligned}
&\frac{d^2}{d\mu^2}\mathcal{J}_n^h(\psi^h + \mu\tilde{\psi}^h, \mathbf{A}^h + \nu\tilde{\mathbf{A}}^h)|_{(0,0)} \\
&= \int_\Omega \left[2|\frac{i}{\kappa}\nabla\tilde{\psi}^h + \mathbf{A}^h\tilde{\psi}^h|^2 + (12|\psi^h|^2 - 2 + \frac{2}{\Delta t})|\tilde{\psi}^h|^2 \right] d\Omega \\
&\ge \int_\Omega \left[|\frac{i}{\kappa}\nabla\tilde{\psi}^h + \mathbf{A}^h\tilde{\psi}^h|^2 + \frac{1}{2\kappa^2}|\nabla\tilde{\psi}^h|^2 + (12|\psi^h|^2 - 2 - 2|\mathbf{A}^h|^2 + \frac{2}{\Delta t})|\tilde{\psi}^h|^2 \right] d\Omega \\
&\ge \|\frac{i}{\kappa}\nabla\tilde{\psi}^h + \mathbf{A}^h\tilde{\psi}^h\|_0^2 + \frac{1}{2\kappa^2}\|\tilde{\psi}^h\|_1^2 + \frac{1}{\Delta t}\|\tilde{\psi}^h\|_0^2 - C\|\mathbf{A}^h\|_1^2\|\tilde{\psi}^h\|_{0,3}^2 \\
&\ge \|\frac{i}{\kappa}\nabla\tilde{\psi}^h + \mathbf{A}^h\tilde{\psi}^h\|_0^2 + \frac{1}{2\kappa^2}\|\tilde{\psi}^h\|_1^2 + \frac{1}{\Delta t}\|\tilde{\psi}^h\|_0^2 - C\|\mathbf{A}^h\|_1^2\|\tilde{\psi}^h\|_0\|\tilde{\psi}^h\|_1 \\
&\ge \|\frac{i}{\kappa}\nabla\tilde{\psi}^h + \mathbf{A}^h\tilde{\psi}^h\|_0^2 + \frac{1}{4\kappa^2}\|\tilde{\psi}^h\|_1^2 + \frac{1}{2\Delta t}\|\tilde{\psi}^h\|_0^2 .
\end{aligned}
$$

Here we have used the assumption that Δt is sufficiently small. Similarly,

$$
\begin{aligned}
&\frac{d^2}{d\nu^2}\mathcal{J}_n^h(\psi^h + \mu\tilde{\psi}^h, \mathbf{A}^h + \nu\tilde{\mathbf{A}}^h)|_{(0,0)} \\
&= \int_\Omega [(2|\psi^h|^2 + \frac{2}{\Delta t})|\tilde{\mathbf{A}}^h|^2 + 2\epsilon|\operatorname{div}\tilde{\mathbf{A}}^h|^2 + 2|\operatorname{curl}\tilde{\mathbf{A}}^h|^2]\, d\Omega \\
&\ge \frac{2}{\Delta t}\|\tilde{\mathbf{A}}^h\|_0^2 + 2C\|\tilde{\mathbf{A}}^h\|_1^2
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{d^2}{d\mu^2}\mathcal{J}_n^h(\psi^h+\mu\tilde\psi^h,\mathbf{A}^h+\mu\tilde{\mathbf{A}}^h)\big|_{(0,0)}\\
&= \int_\Omega 2\Re\{(\frac{i}{\kappa}\nabla\tilde\psi^h+\mathbf{A}^h\tilde\psi^h)(\psi^h)^*\tilde{\mathbf{A}}^h + \frac{i}{\kappa}\quad + \mathbf{A}^h\psi^h)(\tilde\psi^h)^*\tilde{\mathbf{A}}^h\}\,d\Omega\\
&\le 2\|(\frac{i}{\kappa}\nabla\tilde\psi^h+\mathbf{A}^h\tilde\psi^h)\|_0\|\psi^h\|_1\|\tilde{\mathbf{A}}^h\|_{0,3} + 2\|(\frac{i}{\kappa}\nabla\psi^h+\mathbf{A}^h\psi^h)\|_0\|\tilde\psi^h\|_1\|\tilde{\mathbf{A}}^h\|_{0,3}\\
&\le C\|(\frac{i}{\kappa}\nabla\tilde\psi^h+\mathbf{A}^h\tilde\psi^h)\|_0\|\tilde{\mathbf{A}}^h\|_0^{1/2}\|\tilde{\mathbf{A}}^h\|_1^{1/2} + c\|\tilde\psi^h\|_1\|\tilde{\mathbf{A}}^h\|_0^{1/2}\|\tilde{\mathbf{A}}^h\|_1^{1/2}\\
&\le C\left(\|(\frac{i}{\kappa}\nabla\tilde\psi^h+\mathbf{A}^h\tilde\psi^h)\|_0^2 + \|\tilde\psi^h\|_1^2\right)^{1/2}\left(\|\tilde{\mathbf{A}}^h\|_0\|\tilde{\mathbf{A}}^h\|_1\right)^{1/2}\\
&\le C\left(\frac{d^2}{d\mu^2}\mathcal{J}_n^h(\psi^h+\mu\tilde\psi^h,\mathbf{A}^h+\mu\tilde{\mathbf{A}}^h)\big|_{(0,0)}\right)^{1/2}\left(\Delta t^{-1/2}\|\tilde{\mathbf{A}}^h\|_0^2+\Delta t^{1/2}\|\tilde{\mathbf{A}}^h\|_1^2\right)^{1/2}\\
&\le C\Delta t^{1/4}\left(\frac{d^2}{d\mu^2}\mathcal{J}_n^h(\psi^h+\mu\tilde\psi^h,\mathbf{A}^h+\mu\tilde{\mathbf{A}}^h)\frac{d^2}{d\mu^2}\mathcal{J}_n^h(\psi^h+\mu\tilde\psi^h,\mathbf{A}^h+\mu\tilde{\mathbf{A}}^h)\right)^{1/2}\Big|_{(0,0)}.
\end{aligned}
$$

Above, C is a generic constant, independent of $\mathbf{A}^h$ and μ. Thus, for Δt small, the functional $\mathcal{J}_n^h$ is convex on the set $\{\|\psi^h\|_1 \le K,\ \|\mathbf{A}^h\|_1 \le K\}$ when κ is a given constant. □

Similarly, we have the following result.

COROLLARY 3.5. *Let $\kappa > 0$ be a given constant. Then, for Δt sufficiently small, the global minimizer of the functional $\mathcal{J}_n^h$ is unique.* □

3.3. Discrete-in-time approximation. In the proof of the above lemma, no use of any inverse inequality [3] was made. In fact, the same proof is valid for the solution of the following problem which, by itself, is a time-discretized version of the original time-dependent Ginzburg-Landau equations:

$$
\eta\left(\frac{\psi_{n+1}-\psi_n}{\Delta t},\tilde\psi\right) + \quad \frac{i}{\kappa} \quad \frac{i}{\kappa} \quad \forall\tilde\psi\in\mathcal{Z} \tag{3.16}
$$

and

$$
\begin{aligned}
&\left(\frac{\mathbf{A}_{n+1}-\mathbf{A}_n}{\Delta t},\tilde{\mathbf{A}}\right) + (\operatorname{curl}\mathbf{A}_{n+1} - \mathbf{H}, \operatorname{curl}\tilde{\mathbf{A}})\\
&\quad + \epsilon(\operatorname{div}\mathbf{A}_{n+1}, \operatorname{div}\tilde{\mathbf{A}}) + (|\psi_{n+1}|^2\mathbf{A}_{n+1}, \tilde{\mathbf{A}}^h)\\
&\quad + (\Re(\ldots\psi_{n+1}\nabla\psi_{n+1}\ldots\psi_{n+1}\nabla\ldots_{+1}), \tilde{\mathbf{A}}) = 0 \quad \forall\tilde{\mathbf{A}}\in\mathcal{A}.
\end{aligned} \tag{3.17}
$$

Let

$$
\mathcal{J}_n(\psi,\mathbf{A}) = \mathcal{G}_\epsilon(\psi,\mathbf{A}) + \int_\Omega\left(\frac{(\psi-\psi_n)^2}{\Delta t} + \frac{(\mathbf{A}-\mathbf{A}_n)^2}{\Delta t}\right)d\Omega\,.
$$

Then, we have the following result.

PROPOSITION 3.6. *For any* $\Delta t > 0$ *and* $\epsilon \geq 0$, *there exists a solution to the system* (3.16)-(3.17) *for any* n. *Moreover, for all* $n = 0, 1, \ldots$,

$$\mathcal{G}_\epsilon(\psi_{n+1}, \mathbf{A}_{n+1}) + \int_\Omega \left(\frac{(\psi_{n+1} - \psi_n)^2}{\Delta t} + \frac{(\mathbf{A}_{n+1} - \mathbf{A}_n)^2}{\Delta t} \right) d\Omega \leq \mathcal{G}_\epsilon(\psi_n, \mathbf{A}_n) .$$

Proof. The solution is a critical point of the following minimization problem:

$$\min \mathcal{J}_n(\psi, \mathbf{A}) \quad \text{over} \quad (\psi, \mathbf{A}) \in \mathcal{Z} \times \mathbf{\Lambda} \quad \Box$$

Similarly, we have the following results.

LEMMA 3.7. *Let* $\epsilon > 0$ *and* $K > 0$ *be given constants. Then, for* Δt *sufficently small, the functional* $\mathcal{J}_n$ *is convex for any* $(\psi, \mathbf{A})$ *in the set* $\{(\psi, \mathbf{A}) \in \mathcal{Z} \times \mathbf{\Lambda} \mid \|\psi\|_1 \leq K\,,\ \|\mathbf{A}\|_1 \leq K\}$. □

COROLLARY 3.8. *Let* $\epsilon > 0$ *be a given constant. Then, for* Δt *sufficiently small, the global minimizer of the functional* $\mathcal{J}_n$ *is unique.* □

3.4. Asymptotic behavior. We now examine, for given $h > 0, \Delta t > 0$, and $\epsilon > 0$, the asymptotic behavior of the finite element solution $(\psi_n^h, \mathbf{A}_n^h)$. By compactness, it is straightforward to deduce the following result.

LEMMA 3.9. *If* Δt *is sufficiently small, the limit set of the sequence* $\{(\psi_n^h, \mathbf{A}_n^h)\}$ *is a subset of the solution set of* (2.12)-(2.13). □

Unfortunately, the solution set of (2.12)-(2.13) does not consist of only isolated points, even for $\epsilon > 0$. The reason is that if $(\psi^h, \mathbf{A}^h)$ is in the set, so is $(\lambda\psi^h, \mathbf{A}^h)$ for any complex constant λ such that $|\lambda| = 1$. This corresponds to the $U(1)$ symmetry of the solution space of (2.12)-(2.13). One can show, however, for almost all κ, there are only finite number of isolated solutions to (2.12)-(2.13), modulus the $U(1)$ symmetry. It remains to be seen whether this will imply that the sequence $\{(\psi_n^h, \mathbf{A}_n^h)\}$ is convergent for almost all κ.

3.5. Error estimates for the backward Euler scheme. Here, we give an error estimates for the backward Euler scheme (3.1)-(3.4). We assume that the solutions to continuous problem (2.3)-(2.4) as well as the semi-discrete in time scheme (3.16)-(3.17) have enough regularity and the finite element spaces have the best approximation property [3], i.e., for some integer m, if h is sufficiently small, then

$$\inf_{\psi^h \in \mathcal{Z}_h} \|\psi - \psi^h\|_1 \leq Ch^m \|\psi\|_{m+1} \quad \forall\, \psi \in \mathcal{H}^{m+1}(\Omega) \tag{3.18}$$

and

$$\inf_{\mathbf{A}^h \in \mathbf{\Lambda}_h} \|\mathbf{A} - \mathbf{A}^h\|_1 \leq Ch^m \|\mathbf{A}\|_{m+1} \quad \forall\, \mathbf{A} \in \mathbf{H}^{m+1}(\Omega) \cap \mathbf{H}_n^1(\Omega) . \tag{3.19}$$

THEOREM 3.10. *For any* $T > 0$ *and* $\epsilon > 0$, *if* h *and* Δt *are sufficiently small and the solution* $(\psi^\epsilon, \mathbf{A}^\epsilon)$ *to the problem* (2.3)-(2.4) *is sufficiently smooth, then there exists a constant* $C > 0$, *independent of* h *and* Δt, *such that*

$$\|\psi^\epsilon(\cdot, t_n) - \psi_n^h\|_1 \leq C\Delta t + h^m\,, \quad \forall\, n = 1, 2, \ldots, N = [T/\Delta t], \tag{3.20}$$

and

$$\|\mathbf{A}^\epsilon(\cdot,t_n) - \mathbf{A}_n^h\|_1 \le C\Delta t + h^m\,, \quad \forall\, n = 1,2,\dots,N = [T/\Delta t]. \tag{3.21}$$

Proof. First,

$$\psi^\epsilon(\cdot,t_n) - \psi_n^h = \psi^\epsilon(\cdot,t_n) - \pi^h\psi^\epsilon(t_n) + \pi^h\psi^\epsilon(t_n) - \psi_n^h$$

and

$$\mathbf{A}^\epsilon(\cdot,t_n) - \mathbf{A}_n^h = \mathbf{A}^\epsilon(\cdot,t_n) - \pi^h\mathbf{A}^\epsilon(t_n) + \pi^h\mathbf{A}^\epsilon(t_n) - \mathbf{A}_n^h\,,$$

where $\pi^h\psi^\epsilon(t_n) \in \mathcal{Z}_h$ and $\pi^h\mathbf{A}^\epsilon(t_n) \in \mathbf{A}_h$ are the standard elliptic projections of $\psi^\epsilon(\cdot,t_n)$ and $\mathbf{A}^\epsilon(\cdot,t_n)$, respectively. By the approximation properties and standard finite element theory, for given integer $k \ge 0$,

$$\|\partial_t^k(\psi^\epsilon(\cdot,t_n) - \pi^h\psi^\epsilon(t_n))\|_1 \le ch^m\,, \quad \forall\, n = 1,2,\dots,N = [T/\Delta t]$$

and

$$\|\partial_t^k(\mathbf{A}^\epsilon(\cdot,t_n) - \pi^h\mathbf{A}^\epsilon(t_n))\|_1 \le ch^m\,, \quad \forall\, n = 1,2,\dots,N = [T/\Delta t]\,.$$

Now, we consider $e_n^h = \pi^h\psi^\epsilon(t_n) - \psi_n^h$ and $\zeta_n^h = \pi^h\mathbf{A}^\epsilon(t_n) - \mathbf{A}_n^h$. Setting $\tilde{\psi}^h = e_{n+1}^h - e_n^h$ and $\tilde{\mathbf{A}}^h = \zeta_{n+1}^h - \zeta_n^h$ in (3.3) and (3.4), yields

$$\frac{\eta}{\Delta t}\|e_{n+1}^h - e_n^h\|_0^2 + \frac{1}{2\kappa^2}\|\nabla e_{n+1}^h - \nabla e_n^h\|_0^2 + \frac{1}{2\kappa^2}\left(\|\nabla e_{n+1}^h\|_0^2 - \|\nabla e_n^h\|_0^2\right) = \Delta t f_n^h$$

and

$$\frac{1}{\Delta t}\|\zeta_{n+1}^h - \zeta_n^h\|_0^2 + \frac{1}{2\kappa^2}\left(\|\operatorname{curl}\zeta_{n+1}^h - \operatorname{curl}\zeta_n^h\|_0^2 + \epsilon\|\operatorname{div}\zeta_{n+1}^h - \operatorname{div}\zeta_n^h\|_0^2\right)$$
$$+\frac{1}{2\kappa^2}\left(\|\operatorname{curl}\zeta_{n+1}^h\|_0^2 + \epsilon\|\operatorname{div}\zeta_{n+1}^h\|_0^2 - \|\operatorname{curl}\zeta_n^h\|_0^2 - \epsilon\|\operatorname{div}\zeta_n^h\|_0^2\right) = \Delta t g_n^h\,,$$

where f_n^h and g_n^h denote the remaining terms. Using Sobolev imbedding theorems, the approximation properties, and the uniform bounds on the solutions given earlier, it is not difficult to show that there exists a constant $C > 0$ such that, if Δt and h are sufficiently small, then for $n = 1,2,\dots,N = [T/\Delta t]$, we have

$$|f_n^h| \le c\left[\Delta t^2 + h^{2m} + \|e_{n+1}^h\|_1^2 + \|e_n^h\|_1^2 + \|\zeta_{n+1}^h\|_1^2 + \|\zeta_n^h\|_1^2\right]$$

and

$$|g_n^h| \le c\left[\Delta t^2 + h^{2m} + \|e_{n+1}^h\|_1^2 + \|e_n^h\|_1^2 + \|\zeta_{n+1}^h\|_1^2 + \|\zeta_n^h\|_1^2\right]\,.$$

The estimates in the theorem now follows from the discrete Gronwall inequality and the triangle inequality. □

Note that the above results can be easily extended to the case where a variable time-step is used. A similar error estimate for the time-dependent G-L equations has also been given in [7], in which a different gauge from ours was choosen.

3.6. Higher-order in time discretization. Similar to the backward Euler methods, higher order in time discretization can also be formulated and analyzed. For example, the following scheme yields a second-order in time discretization:

$$\eta\left(\frac{\psi_{n+1}^h-\psi_n^h}{2\Delta t}\,,\,\tilde{\psi}^h\right)+\left(\left(|\psi_n^a|^2-1\right)\psi_n^a\,,\,\tilde{\psi}^h\right)$$
$$+\left(\frac{i}{\kappa}\nabla\psi_n^a+\mathbf{A}_n^a\psi_n^a\,,\,\frac{i}{\kappa}\nabla\tilde{\psi}^h+\mathbf{A}_n^a\tilde{\psi}^h\right)=0\quad\forall\,\tilde{\psi}^h\in\mathcal{Z}_h \tag{3.22}$$

and

$$\left(\frac{\mathbf{A}_{n+1}^h-\mathbf{A}_n^h}{2\Delta t},\tilde{\mathbf{A}}^h\right)+(\operatorname{curl}\mathbf{A}_n^a-\mathbf{H},\operatorname{curl}\tilde{\mathbf{A}}^h)+\epsilon(\operatorname{div}\mathbf{A}_n^a,\operatorname{div}\tilde{\mathbf{A}}^h)$$
$$+(|\psi_n^a|^2\mathbf{A}_n^a,\tilde{\mathbf{A}}^h)+\left(\frac{i}{2\kappa}(\psi_n^{a*}\nabla\psi_n^a-\psi_n^a\nabla\psi_n^{a*})\,,\,\tilde{\mathbf{A}}^h\right)=0\quad\forall\,\tilde{\mathbf{A}}^h\in\mathbf{A}_h\,, \tag{3.23}$$

where

$$\psi_n^a=\frac{\psi_{n+1}^h+\psi_n^h}{2}\qquad\text{and}\qquad\mathbf{A}_n^a=\frac{\mathbf{A}_{n+1}^h+\mathbf{A}_n^h}{2}\,.$$

This scheme is similar to the one-leg multi-step method for numerical solution of ordinary differential equations. Uniform bounds on the discrete solutions and higher-order error estimates may be obtained similar to the earlier discussion. Here, let us simply state the following result.

LEMMA 3.11. *For any $h>0$, $\Delta t>0$, and $\epsilon\geq 0$, let $(\hat{\psi}_n^h,\hat{\mathbf{A}}_n^h)$ be a critical point of the following minimization problem:*

$$\min\ \mathcal{J}_n^h(\psi^h,\mathbf{A}^h)\quad over\quad(\psi^h,\mathbf{A}^h)\in\mathcal{Z}^h\times\mathbf{A}^h\,.$$

Then, $(2\hat{\psi}_n^h-\psi_n^h,2\hat{\mathbf{A}}_n^h-\mathbf{A}_n^h)$ is a solution to the scheme (3.22)-(3.23).

By the above lemma, we see that the actual implementation of the above second-order method does not involve more work than the implementation of the first-order backward Euler method. Higher order schemes may be useful in a better resolution of the initial transient period.

4. Computational example. Numerical experiments have been performed on a Sun Sparcstation using a two-dimensional finite element code. More extensive reports on the experiments will be given in future papers. Here, let us describe a simple experiment in which the time-dependent Ginzburg-Landau equations are solved using the fully discrete Backward Euler scheme on a two-dimensional square box. The code uses piecewise biquadratic polynomials on a uniform spatial mesh. A Newton linearization is used for the nonlinear algebraic equations that must be solved at each time step. The resulting linear systems are solved for by the conjugate gradient method. For the results reported on here, the Ginzburg-Landau parameter is $\kappa=3$ with an external field $H=1.5$. The solution for these values should correspond to a vortex state. For the particular experiment described here, initial conditions correspond to

$\psi_0 = 0.8 + 0.6i$, $\mathbf{A}_0 = (0, 0)$, i.e, a perfect superconducting state. Figure 1 gives contour plots of the magnitude of the order parameter. Vortices that correspond to where $\psi = 0$ first start to form near the midsides and then settle down in the interior. For comparison, Figure 2 gives a couple of plots of the computed magnetic field curl $\mathbf{A}$ with a grayscale. Lighter regions correspond to cores of the vortex, the magnetic field reaches maximum at the center of the vortices. Finally, Figure 3 gives the decay of the Free energy and the magnetization. We have performed many other numerical simulations of the vortex dynamics and "flux pinning", using the time-dependent G-L models and their variants, more details will be given in future reports.

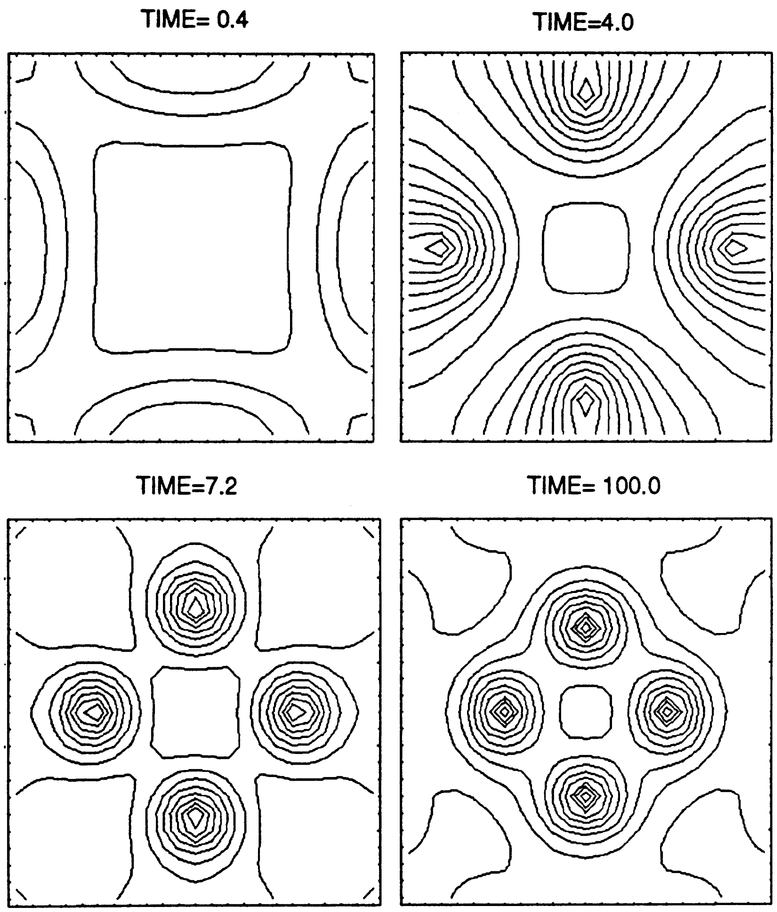

Figure 1. *Magnitude of the order parameter*

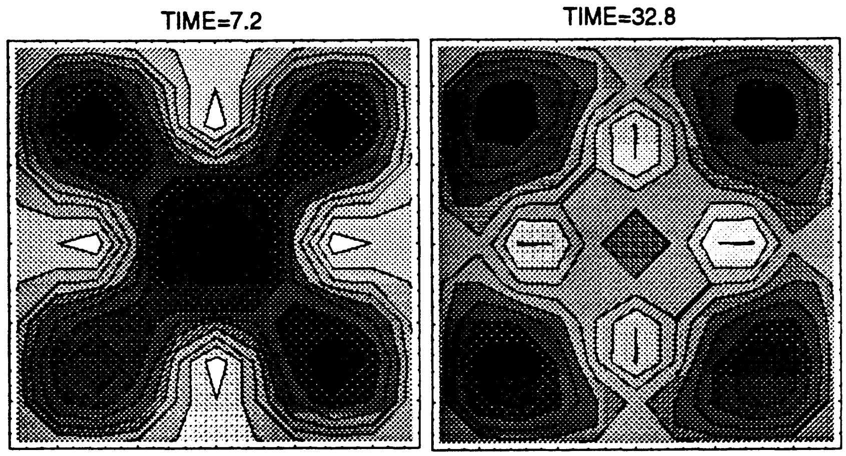

Figure 2. *Magnetic field*

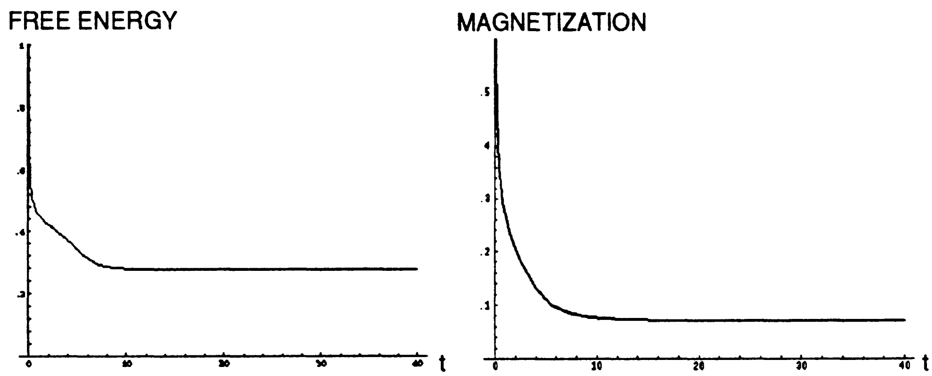

Figure 3. *Free energy vs. time and Magnetization vs. time.*

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