NAMT 93-012

Semi-Local Classification of Geometric Singularities for Hamilton-Jacobi Equations

Shyuichi Izumiya
Hokkaido University
Georgios T. Kossioris
Carnegie Mellon University

Research Report No. 93-NA-012

March 1993

Sponsors

U.S. Army Research Office
Research Triangle Park
NC 27709

National Science Foundation
1800 G Street, N.W.
Washington, DC 20550

SEMI-LOCAL CLASSIFICATION OF GEOMETRIC SINGULARITIES FOR HAMILTON-JACOBI EQUATIONS

SHYUICHI IZUMIYA* AND GEORGIOS T. KOSSIORIS**

0. INTRODUCTION

In this paper we describe the geometric framework for the study of generation and propagation of shock waves in $\mathbb{R}^n$ appearing in solutions of Hamilton-Jacobi equations

$$
\text{(P)} \qquad \begin{cases} \dfrac{\partial y}{\partial t} + H(t, x_1, \cdots, x_n, \dfrac{\partial y}{\partial x_1}, \cdots, \dfrac{\partial y}{\partial x_n}) = 0 \\ y(0, x_1, \cdots, x_n) = \phi(x_1, \cdots, x_n), \end{cases}
$$

where H and ϕ are C^∞-functions. Hamilton-Jacobi equations play an important role in various fields e.g., calculus of variations (see e.g., [30]), optimal control theory (see e.g., [13]) and differential games (see e.g., [12] and references cited therein).

The *geometric solution* y of (P) has been defined in [18], [20] in the framework of *one-parameter Legendrian unfoldings* and it is constructed by the method of characteristics. Although y is initially smooth there is in general a critical time beyond which characteristics cross. The geometric solution past the critical time is multi-valued, that is singularities appear. The classification of singularities of y has been studied in [18] (see also [20]).

The theory of *viscosity solutions* (see [7]) has provided the right weak setting for the study of (P). Existence and uniqueness of the solution of (P) in the viscosity sense have been established in [8]. The single-valued viscosity solution is continuous and coincides with the smooth geometric solution until the first critical time. After the characteristics cross, the viscosity solution develops *shock waves* i.e., surfaces across which the gradient of the viscosity solution is discontinuous. The shock surfaces are referred to as *singular surfaces* in the literature of optimal control and differential games (see e.g., [5], [16]).

The viscosity solution of (P) in a neighborhood of the first critical time has been constructed in [25] (see also [27], [24]) by selecting a continuous single-valued branch of the graph of the geometric solution. The shock surface of the weak solution corresponds to the intersection of the branches of the graph of the multi-valued geometric solution. In order to study the evolution of the shock surface we

*Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060, Japan
**Department of Mathematics, Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213, USA

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

follow the evolution of the intersections of the branches defining the shock. The case $n = 1$ has been studied in [24] (see also [25]) where the global structure of the shock waves has been described.

The goal of the present work is to study the bifurcations of the branches of the graph of the geometric solution in case $n \geq 1$. We present the correct topological setting. We formulate the problem in terms of *multi-Legendrian unfoldings* and we obtain the generic list of the bifurcations of the branches of the multi-valued graph. In this work we only present a discussion on how to obtain the generic pattern of shock waves from the obtained classification list. The constructions of the shock surfaces will be presented in a future paper [21].

The geometric interpretation of a smooth solution to a first order equation was introduced by S. Lie as the maximal integral submanifolfd of contact hyperplane fields (see [14]). The notion of the multi-valued geometric solution to first order equations has appeared e.g., in Lychagin [26] in the context of R-manifolds and in Oshima [29] in the context of Lagrangian submanifolds. Evolution equations have been considered by Izumiya in [18], [20] where the geometric solution has been defined in terms of one-parameter Legendrian unfoldings, i.e., smoothly parametrized Legendrian submanifolds in t that satisfy the Hamilton-Jacobi equation. For the definition of Legendrian submanifolds see [2], [3]; cf. Section 1.

The method of constructing the weak solution by selecting the proper single-valued branch was introduced by Tsuji ([32], [33]) for Hamilton-Jacobi equations and by Guckenheimer ([15]) for conservation laws. Nakane in [27] has constructed the weak semi-concave solution past the first critical time in case that H is convex with respect to $\nabla y = (\frac{\partial y}{\partial x_1}, \cdots, \frac{\partial y}{\partial x_n})$. The case of scalar conservation laws in $\mathbb{R}^n$ past the first critical time has been studied by Nakane in [28].

The geometric framework we introduce herein for the study of shock waves for viscosity solutions of (P) apply to several types of equations e.g., geometric Hamilton-Jacobi equations (see [6], [23]), conservation laws (see [22], [28], [15]) and Monge-Ampère equations (see [34]).

Geometric Hamilton-Jacobi equations

$$(\mathrm{P}_g) \qquad \frac{\partial y}{\partial t} - |\nabla y| a(t, x, \frac{\nabla y}{|\nabla y|}) = 0,$$

describe the evolution of the interface $\Gamma_t = \{x : y(t,x) = 0\}$ (see e.g., [4]). The interface moves with normal velocity depending on its normal vector $\nabla y / |\nabla y|$. Equation (P_g) arises in different contexts; geometric optics (see e.g., [11]), flame front propagation (see e.g., [31]) and crystal growth (see e.g., [6]). Cahn, Taylor and Handwerker in [6] study the evolution of a polyhedral interface of a crystal described by (P_g) by studying how the characteristics cross.

In [23] we study the singularities for the geometric solutions of the equation

$$|\nabla y| a(x, \frac{\nabla y}{|\nabla y|}) = 1$$

introduced in [6]. In [22] we study the singularities for the geometric solutions of single conservation laws

$$\frac{\partial y}{\partial t} + \sum_{i=1}^{n} \frac{\partial f_i(y)}{\partial x_i} = 0,$$

where f_i's are C^∞-functions.

In Section 1 we describe the framework of the geometric theory for a general first order partial differential equation introduced in [26]. In Section 2 we describe the geometric theory for (P) where we consider the time t as a parameter. A geometric solution of (P) is defined as an one-parameter Legendrian unfolding (see [18], [20]) that lies on the hypersurface

$$E(H) = \left\{(t, x, y, s, p) \in J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) | s + H(t, x, p) = 0\right\}.$$

We describe the representation of the geometric solution in terms of generating families according to Arnol'd-Zakalyukin theory ([1], [36]). In order to study the singularities of the geometric solution we have to prove that Legendrian unfoldings is the correct class of solutions. In Section 3 we present the realization theorems that associate to any Legendrian unfolding a Hamiltonian H. According to Theorem 3.1 to any Legendrian unfolding there corresponds a Hamiltonian H. Such correspondence of Legendrian unfoldings to Hamiltonians permits the use of classification techniques for Legendrian unfoldings in the context of singularities of the geometric solution of (P). Theorem 3.5 establishes the correspondence of non-degenerate Hamiltonians (strictly convex or concave Hamiltonians) to P-stable Legendrian unfoldings.

Generic lists of the singularities of the geometric solution of (P) have been given in [18]. In Section 4, in order to describe the evolution of intersections of branches of the geometric solution we formulate the problem in terms of multi-Legendrian unfoldings which we describe in terms of multi-generating families. In Section 5 we obtain the classification theorem for multi-Legendrian unfoldings. Finally, in Section 6 we discuss how we construct the shock waves for viscosity solutions from the obtained generic list of bifurcations. These constructions are undertaken in [21].

Acknowledgment. This work was partially supported by the Army Research Office and the National Science Foundation through the Center for Nonlinear Analysis and by the Japan Association for Mathematical Science. The authors are grateful to the Director of the Center for Nonlinear Analysis M. Gurtin for his support.

1. Geometric framework for Hamilton-Jacobi equations

In the present paper we treat Hamilton-Jacobi equations in the framework of the geometric theory of first order partial differential equations described e.g., in [26]. In this section we briefly describe the geometric framework and present the necessary notation.

Let $J^1(R^n,R)$ be the 1-jet bundle of functions of n-variables which may be considered as R^{2n+1} with a natural coordinate system $(xi,\ldots,x_n,2/,pi,\ldots,p_n)$, where $(xi,\ldots,x_n)$ is a coordinate system of R^n. We also have a natural projection 7T: $J^x(R^n,R)$ -• R^n x R given by 7r(x,j/,p) = (x,y).

An immersion germ *i* : (Lo,uo) -+ J^R^R) is said to be a *Legendrian immersion germ* (i.e., Legendrian submanifold germ) if dimL = n and *i*6* = 0, where *0 = dy* — X)?=i *Pi* • *dxi.* The image of x o t is called *ihe wave front set* of *i* and it is denoted by *W(i).*

We also consider the 1-jet bundle $J^a(R \times R^n,R)$ and the canonical 1-form 0 on that space. Let $(t, xi,\ldots,x_n)$ be a canonical coordinate system on R x R^n and

$$(t, xi,\ldots,x_n, y, a, pi,\ldots,p_n)$$

the corresponding coordinate system on $J^a(R \times R^n,R)$. Then, the canonical 1-form is given by

$$Q = dy - \sum^{n} Pt \cdot dx^* - 5 \cdot dt = 0 - s \cdot dt.$$

We define the natural projection II : $J^l(R \times R^n,R)$ -4 $(R \times R^n) \times R$ by II(t,x,y,5,p) = (t,x,y). We call the above 1-jet bundle *an unfolded 1-jet bundle.*

A Hamilton-Jacobi equation is defined to be a hypersurface

$$E(H) = \{(t,x,y,s,p) \in J^l(R \times R^n,R) | s + ff(t,x,p) = 0\}$$

in $J^a(R \times R^n,R)$. *A geometric solution* of *E(H)* is a Legendrian submanifold *L* in $J^1(R\times R^n,R)$ lying in *E(H).*

We say that *a generalized Cauchy problem (GCP) (with initial condition V) is given for an equation E(H)* if there is given an n-dimensional submanifold *i: V* C *E(H)* such that i*0 = 0 and X_H $ *T(L')* at any point of *V* where X_H is the characteristic vector field given by

$$X_{r} - \frac{d}{\alpha t} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x_i} + \sum_{i=1}^{n} p_i \frac{\partial H}{\partial p_i}\frac{\partial}{\partial y} - \frac{\partial H}{\partial t}\frac{\partial}{\partial s} - \sum_{i=1}^{n} \frac{\partial H}{\partial x_i}\frac{\partial}{\partial p_i}.$$

We have the following existence theorem:

Theorem 1.1. (Classical existence theorem [26]). *A GCP i: V* C *E(H) has a unique solution, that is, there is a Legendrian submanifold L* C *E(H), V* C *L and any two such Legendrian submanifolds coincide in a neighbourhood of V*

In order to study (P) we need a more restricted framework. For any c 6 (R,0), we define

$$E(H)_C = \{(c,x,y,-13^r(c,x,p),p) | (x,y,p) \in J^a(R^n,\mathbb{R})\}.$$

Then, $E(H)_c$ is a $(2n+1)$-dimensional submanifold of $J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and $\Theta_c = \Theta|E(H)_c = dz - \sum_{i=1}^n p_i dx_i$ gives a contact structure on $E(H)_c$. We define a mapping $\iota_c : J^1(\mathbb{R}^n,\mathbb{R}) \to E(H)_c$ by $\iota_c(x,y,p) = (c,x,y,-H(c,x,p),p)$. The mapping i_c is a contact diffeomorphism and the following diagram is commutative:

$$\begin{array}{ccc} J^1(\mathbb{R}^n,\mathbb{R}) & \xrightarrow{\iota_c} & E(H)_c \\ \pi\downarrow & & \downarrow\pi_c \\ \mathbb{R}^n\times\mathbb{R} & = & \mathbb{R}^n\times\mathbb{R}. \end{array}$$

We say that *a generalized Cauchy problem (with initial condition L') associated with the time parameter(GCPT) is given for an equation $E(H)$* if a GCP $i : L' \subset E(H)$ with $i(L') \subset E(H)_c$ for some $c \in (\mathbb{R},0)$ is given.

Remark. The Cauchy problem (P) is a GCPT. The initial submanifold is given by

$$L_{\phi,0} = \left\{(0,x,\phi(x),-H(0,x,\frac{\partial\phi}{\partial x}),\frac{\partial\phi}{\partial x})|x\in\mathbb{R}^n\right\} \subset E(H)_0.$$

The problem of studying the singularities of the graph of the geometric solution is formulated as follows:

Geometric problem. *Classify the generic bifurcations of wave fronts of*

$$\pi_t| : L\cap E(H)_t \to \mathbb{R}^n\times\mathbb{R}$$

with respect to the parameter t (i.e., the generic bifurcations of wave fronts of geometric solutions along the time parameter).

Following [18], in order to study the singularities of the geometric solution we identify geometric solutions with one-parameter Legendrian unfoldings. Such a characterization, which is given in Section 3 permits the use of the available singularity theory of one-parameter Legendrian unfoldings. In the next section we present the necessary background material that we use in Section 3.

2. One parameter Legendrian unfoldings

We now describe the notion of one-parameter Legendrian unfoldings. Let R be an $(n+1)$-dimensional smooth manifold, $\mu : (R,u_0) \to (\mathbb{R},t_0)$ be a submersion germ and $\ell : (R,u_0) \to J^1(\mathbb{R}^n,\mathbb{R})$ be a smooth map germ. We say that the pair (μ,ℓ) is *a Legendrian family* if $\ell_t = \ell|\mu^{-1}(t)$ is a Legendrian immersion germ for any $t\in(\mathbb{R},t_0)$. Then we have the following simple but very important lemma.

Lemma 2.1. *Let (μ,ℓ) be a Legendrian family. Then there exist a unique element $h \in C^\infty_{u_0}(R)$ such that $\ell^*\theta = h\cdot d\mu$, where $C^\infty_{u_0}(R)$ is the ring of smooth function germs at u_0.*

Define a map germ $\mathcal{L} : (R,u_0) \to J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ by

$$\mathcal{L}(u) = (\mu(u), x\circ\ell(u), y\circ\ell(u), h(u), p\circ\ell(u)).$$

We can easily show that $\mathcal{L}$ is a Legendrian immersion germ. If we fix 1-forms Θ and θ, the Legendrian immersion germ $\mathcal{L}$ is uniquely determined by the Legendrian family (μ, ℓ). We call $\mathcal{L}$ *a Legendrian unfolding associated with the Legendrian family* (μ, ℓ). In order to study bifurcations of wave fronts of Legendrian unfoldings, we introduce the following equivalence relation. Let $\mathcal{L}_i : (R, u_i) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ $(i = 0, 1)$ be Legendrian unfoldings. We say that $\mathcal{L}_0$ and $\mathcal{L}_1$ are *P-Legendrian equivalent* if there exist a contact diffeomorphism germ

$$K : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_1)$$

of the form

$$K(t, x, y, s, p) = (\phi_1(t), \phi_2(t, x, y), \phi_3(t, x, y), \phi_4(t, x, y, s, p), \phi_5(t, x, y, s, p))$$

and a diffeomorphism germ $\Psi : (R, u_o) \to (R, u_1)$ such that $K \circ \mathcal{L}_0 = \mathcal{L}_1 \circ \Psi$.

In order to understand the meaning of P-Legendrian equivalence, we introduce the following equivalence: We say that *two wave front sets* $W(\mathcal{L}_0)$ *and* $W(\mathcal{L}_1)$ *have diffeomorphic bifurcations* if there exists a diffeomorphism germ

$$\Phi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), \Pi(z_0)) \to (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), \Pi(z_1))$$

of the form $\Phi(t, x, y) = (\phi_1(t), \phi_2(t, x, y), \phi_3(t, x, y))$ such that $\Phi(W(\mathcal{L})) = W(\mathcal{L}')$. If $\mathcal{L}_0$ and $\mathcal{L}_1$ are P-Legendrian equivalent then these wavefronts have diffeomorphic bifurcations. By the theorem of Zakalyukin [36; Assertion , Section 1.1], the converse is also true for generic Legendrian unfoldings. We can define the notion of stability with respect to the P-Legendrian equivalence in the same way as for the ordinary Legendrian stability (see [1],[36]).

Motivated by Arnol'd-Zakalyukin's theory ([1],[36]), we can construct generating families of Legendrian unfoldings. A function germ $F : ((\mathbb{R} \times \mathbb{R}^n) \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0)$ is called a *a generalized phase function germ* if $d_2F|0 \times \mathbb{R}^n \times \mathbb{R}^k$ is non-singular, where $d_2F(t, x, q) = (\frac{\partial F}{\partial q_1}(t, x, q), \ldots, \frac{\partial F}{\partial q_k}(t, x, q))$. Then $C(F) = d_2F^{-1}(0)$ is a smooth $(n+1)$-manifold germ and $\pi_F : (C(F), 0) \to \mathbb{R}$ is a submersion germ, where $\pi_F(t, x, q) = t$.

Define map germs $\tilde{\Phi}_F : (C(F), 0) \to J^1(\mathbb{R}^n, \mathbb{R})$ by

$$\tilde{\Phi}_F(t, x, q) = (x, F(t, x, q), \frac{\partial F}{\partial x}(t, x, q))$$

and $\Phi_F : (C(F), 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ by

$$\Phi_F(t, x, q) = (t, x, F(t, x, q), \frac{\partial F}{\partial t}(t, x, q), \frac{\partial F}{\partial x}(t, x, q)).$$

Since $\frac{\partial F}{\partial q_i} = 0$ on $C(F)$, we can easily show that $(\tilde{\Phi}_F)^*\theta = \frac{\partial F}{\partial t}|C(F) \cdot dt|C(F)$.

By definition, Φ_F is a Legendrian unfolding associated with the Legendrian family $(\pi_F, \tilde{\Phi}_F)$. Following the lines of Arnol'd-Zakalyukin ([1],[36]), we can show the following proposition.

Proposition 2.2. *All Legendrian unfolding germs are constructed by the above method.*

We define a function germ $\tilde{F} : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0)$ by $\tilde{F}(t,x,y,q) = F(t,x,q) - y$. We call $\tilde{F}$ *a generating family of* Φ_F. We also consider an equivalence relation among generating families of Legendrian unfoldings. Let

$$\tilde{F}_i : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0) \ (i = 0, 1)$$

be generating families of Φ_{F_i}. We say that $\tilde{F}_0$ and $\tilde{F}_1$ are *t-P-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ

$$\Phi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k, 0) \to (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k, 0)$$

of the form

$$\Phi(t,x,y,q) = (\phi_1(t), \phi_2(t,x,y), \phi_3(t,x,y), \phi_4(t,x,y,q))$$

such that

$$\langle \tilde{F}_1 \circ \Phi \rangle_{\mathcal{E}_{(t,x,y,q)}} = \langle \tilde{F}_0 \rangle_{\mathcal{E}_{(t,x,y,q)}},$$

where $\langle \tilde{F}_0 \rangle_{\mathcal{E}_{(t,x,y,q)}}$ denotes the ideal generated by $\tilde{F}_0$ in the ring $\mathcal{E}_{(t,x,y,q)}$ of function germs of (t,x,y,q) -variables at the origin. The definition of *the stable t-P-$\mathcal{K}$-equivalence* is given in the usual way (see [1],[36]).

For a generating family $\tilde{F}$ of Φ_F, we define

$$T_e(\text{P-}\mathcal{K})(\tilde{f}) = \langle \frac{\partial \tilde{f}}{\partial q_1}, \dots, \frac{\partial \tilde{f}}{\partial q_k}, \tilde{f} \rangle_{\mathcal{E}_{(x,y,q)}} + \langle \frac{\partial \tilde{f}}{\partial x_1}, \dots, \frac{\partial \tilde{f}}{\partial x_n}, 1 \rangle_{\mathcal{E}_{(x,y)}}$$

and P-$\mathcal{K}$-cod $\tilde{f} = \dim_{\mathbb{R}} \mathcal{E}_{(x,y,q)} / T_e(\text{P-}\mathcal{K})(\tilde{f})$, where $\tilde{f} = \tilde{F}|0 \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k$. We also say that $\tilde{F}$ is *a P-$\mathcal{K}$-versal deformation of* $\tilde{f}$ if

$$\mathcal{E}_{(x,y,q)} = \langle \frac{\partial \tilde{F}}{\partial t} | 0 \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k \rangle_{\mathbb{R}} + T_e(\text{P-}\mathcal{K})(\tilde{f}).$$

Then we have the following proposition whose proof is like that of the ordinary theory of Legendrian singularities ([1],[36]).

Proposition 2.3. (1) *Let $\tilde{F}_i$ $(i = 0, 1)$ be generating families of Φ_{F_i}. Then Φ_{F_0} and Φ_{F_1} are P-Legendrian equivalent if and only if $\tilde{F}_0$ and $\tilde{F}_1$ are stably t-P-$\mathcal{K}$-equivalent.*
(2) *Let $\tilde{F}$ be a generating family of Φ_F, then Φ_F is stable with respect to the P-Legendrian equivalence if and only if $\tilde{F}$ is a P-$\mathcal{K}$-versal deformation of $\tilde{f}$.*

We can classify generic Legendrian unfoldings under the P-Legendrian equivalence for $n \leq 4$ by means of the classification of one parameter perestroikas of wave fronts. In [36] Zakalyukin has given a generic classification of function germs

$\tilde{F} : (\mathbb{R} \times \mathbb{R}^{(n+1)} \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0)$ under the stable t-P-$\mathcal{K}$-equivalence. Since the set of function germs $\tilde{F}(t, x, y, q)$ which satisfy $\frac{\partial \tilde{F}}{\partial y} \neq 0$ is an open subset, then such a function germ is stably t-P-$\mathcal{K}$-equivalent to one of the germs in the following list :

$$({}^0A_r) \qquad q_1^{r+1} \pm q_2^2 + \sum_{i=1}^{r-1} x_i q^i - y \ (1 \leq r \leq n)$$

$$({}^0D_r) \qquad q_1^2 q_2 \pm q_2^{r-1} + \sum_{i=2}^{r-1} x_i q_2^{i-1} + x_1 q_1 - y \ (4 \leq r \leq n)$$

$$({}^1A_r) \qquad q_1^{r+1} \pm q_2^2 + q_1^{r-1}(t \pm x_r^2 \pm \cdots \pm x_n^2) + \sum_{i=1}^{r-2} x_i q_1^i - y \ (2 \leq r \leq n+1)$$

$$({}^1D_r) \quad q_1^2 q_2 \pm q_2^{r-1} + q_2^{r-2}(t \pm x_r^2 \pm \cdots \pm x_n^2) + \sum_{i=2}^{r-2} x_i q_2^{i-1} + x_1 q_1 - y \ (4 \leq r \leq n+1)$$

$$({}^1E_6) \qquad q_1^3 + q_2^4 + q_1 q_2^2 t + x_4 q_1 q_2 + x_3 q_2^2 + x_1 q_1 + x_2 q_2 - y.$$

Since for the germs ${}^1A_1$ and ${}^1A_2$ the corresponding map $d_2\tilde{F}|0 \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k$ is not submersive, these germs can be removed from the list of generating families of Legendrian unfoldings. Thus we have the following :

Theorem 2.4. *For $n \leq 5$, the generic Legendrian unfolding is P-Legendrian equivalent to one of germs of the following type :*

$${}^0A_r \ (1 \leq r \leq n),\ {}^0D_r \ (4 \leq r \leq n),\ {}^1A_r \ (3 \leq r \leq n),\ {}^1D_r \ (4 \leq r \leq n),\ {}^1E_6.$$

We can explicitly list all generating families in case $n \leq 2$ as follows :

Corollary 2.5. *For $n \leq 2$, the generic Legendrian unfolding is P-Legendrian equivalent to one of germs of the following type:*

$n = 1$:
${}^0A_1$: $q_1^2 - y$;
${}^0A_2$: $q_1^3 + x_1 q_1 - y$;
${}^1A_3$: $q_1^4 + q_1^2 t + x_1 q_1 - y$.
$n = 2$:
${}^0A_1$: $q_1^2 - y$;
${}^0A_2$: $q_1^3 + x_1 q_1 - y$;
${}^0A_3$: $q_1^4 + x_1 q_1 + x_2 q_1^2 - y$;
${}^1A_3$: $q_1^4 + q_1^2(t \pm x_2^2) + x_1 q_1 - y$;
${}^1A_4$: $q_1^5 + q_1^3 t + x_1 q_1 + x_2 q_1^2 - y$;
${}^1D_4$: $q_1^2 \pm q_2^3 + q_2^2 t + x_2 q_2 + x_1 q_1 - y$.

3. Realization theorems

In this section we identify the geometric solution of a (GCPT) introduced in Section 3 with the notion of one-parameter Legendrian unfoldings. Let $i : L' \subset E(H)_0 \subset E(H)$ be the initial condition of a (GCPT) and let L be the unique solution. Since $X_H \notin TE(H)_c$, then L is transverse to $E(H)_c$ in $E(H)$ for any $c \in (\mathbb{R}, 0)$. It follows that $L_c = L \cap E(H)_c$ is an n-dimensional submanifold of $E(H)_c$ and it satisfies $\Theta_c|L_c = 0$ (i.e., L_c is a Legendrian submanifold of $E(H)_c$). If we consider the local parametrization of L, we may assume that L is the image of an immersion germ $\mathcal{L} : (\mathbb{R}\times\mathbb{R}^n, 0) \to E(H)$ such that $\mathcal{L}|(c\times\mathbb{R}^n)$ is a Legendrian immersion germ of $E(H)_c$. Hence the coordinate representation of $\mathcal{L}$ is given by $\mathcal{L}(t,u) = (t, x(t,u), y(t,u), -H(t, x(t,u), p(t,u)), p(t,u))$.

Let $\tilde{\pi} : J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \to J^1(\mathbb{R}^n, \mathbb{R})$ be the canonical projection defined by $\tilde{\pi}(t, x, y, s, p) = (x, y, p)$. Then the map germ $\ell = \tilde{\pi} \circ \mathcal{L}$ satisfies that $\ell_t = \ell|\pi_1^{-1}(t)$ is a Legendrian immersion germ for any $t \in (\mathbb{R}, 0)$. Hence $\mathcal{L}$ is a Legendrian unfolding associated with $(\pi_1, \tilde{\pi} \circ \mathcal{L})$, where π_1 is the canonical projection $\pi_1 : (\mathbb{R} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$. This completes the proof of the first part of Theorem 3.1. The proof of the second part is given in [18].

Theorem 3.1. (1) *The local solution of the generalized Cauchy problem associated with the time parameter for the Hamilton-Jacobi equation*

$$s + H(t, x, p) = 0$$

is a Legendrian unfolding

$$\mathcal{L} : (\mathbb{R} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}).$$

(2) *Let*

$$\mathcal{L} : (\mathbb{R} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$$

be a Legendrian unfolding associated with (π_1, ℓ). *Then there exists a* C^∞*-function germ* $H(t, x_1, \ldots, x_n, p_1, \ldots, p_n)$ *such that* $\mathcal{L}$ *is a local solution of the generalized Cauchy problem associated with the time parameter for Hamilton-Jacobi equation*

$$s + H(t, x, p) = 0,$$

where the initial condition is given by $\ell(0, u)$.

The above theorem guarantees that the class of Legendrian unfoldings supplies the correct class to describe the geometric solutions of (GCPT) for Hamilton-Jacobi equations. Thus, generic results for the singularities of Legendrian unfoldings can be translated to generic results in the class of all Hamiltonians and all initial conditions. However, we must also concern ourselves with what are the types of singularities that the geometric solution to a given Hamilton-Jacobi equation might exhibit. Representation Theorems 3.1-3.2 address this question.

Let $i : (L, u_0) \to J^1(\mathbb{R}^n, \mathbb{R})$ be a Legendrian immersion germ. We define an immersion germ $\tilde{i} : (\mathbb{R} \times L, (0, u_0)) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ by $\tilde{i}(t, u) = (t, x(u), y(u), 0, p(u))$, where $i(u) = (x(u), y(u), p(u))$. Since $\tilde{i}^*\Theta = i^*\theta - 0 = 0$, $\tilde{i}$ is a Legendrian unfolding associated with $(\pi_1, \bar{\pi} \circ \tilde{i})$. We call $\tilde{i}$ *a trivial Legendrian unfolding induced by i*. Let $\mathcal{L} : (\mathbb{R} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ be a Legendrian unfolding. We say that $\mathcal{L}$ *has a trivial bifurcation* if $\mathcal{L}$ is P-Legendrian equivalent to a trivial Legendrian unfolding $\tilde{\mathcal{L}}|0 \times \mathbb{R}^n$. Then we have the following theorem.

Theorem 3.2. *Let $s + H(t, x, p) = 0$ be a Hamilton-Jacobi equation and $\mathcal{L}$: $(\mathbb{R} \times \mathbb{R}^n, (0, 0)) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ be a P-Legendrian stable Legendrian unfolding associated with (π_1, ℓ). If $\mathcal{L}$ has a trivial bifurcation, then there exists a Legendrian unfolding $\mathcal{L}'$ such that $\mathcal{L}'$ is a geometric solution of $s + H(t, x, p) = 0$ and $\mathcal{L}$, $\mathcal{L}'$ are P-Legendrian equivalent.*

Proof. Let $\tilde{G} : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0)$ be a generating family of the Legendrian unfolding $\mathcal{L}$. Since $d_2G|0 \times \mathbb{R}^n \times \mathbb{R}^k$ is non-singular, the set

$$\phi_g = \{(0, x, -H(0, x, \frac{\partial g}{\partial x}(x, q)), \frac{\partial g}{\partial x}(x, q)) | \frac{\partial g}{\partial q_i}(x, q) = 0 \; i = 1, \ldots, k\}$$

is an initial condition for the (GCPT), where $g = G|0 \times \mathbb{R}^n \times \mathbb{R}^k$. By the arguments of the proof of the first part of Theorem 3.1 we can construct a Legendrian unfolding $\mathcal{L}'$ which is the local unique geometric solution around ϕ_g.

We now choose a generating family $\tilde{F} : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^{k'}, 0) \to (\mathbb{R}, 0)$ of $\mathcal{L}'$. By definition, Image $\Phi_F|t = 0$ is equal to ϕ_g, so we may assume that $k = k'$ and $\tilde{f}$, $\tilde{g}$ are P-$\mathcal{K}$-equivalent, where $f = F|0 \times \mathbb{R}^n \times \mathbb{R}^k$, $\tilde{f}(x, y, q) = f(x, q) - y$ and $\tilde{g}(x, y, q) = g(x, q) - y$.

By Proposition 2.3, $\tilde{F}$ is a P-$\mathcal{K}$-versal deformation of $\tilde{f}$. Since $\mathcal{L}$ has a trivial bifurcation, $\frac{\partial \tilde{F}}{\partial t}|0 \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k \in T_e(\text{P-}\mathcal{K})(\tilde{f})$, and hence $\mathcal{E}_{(x,y,q)} = T_e(P\text{-}\mathcal{K})(\tilde{f})$. Therefore, it follows that $\mathcal{E}_{(x,y,q)} = T_e(P\text{-}\mathcal{K})(\tilde{g})$ and $\tilde{G}$ is also a P-$\mathcal{K}$-versal deformation of $\tilde{g}$. By the uniqueness theorem of P-$\mathcal{K}$-versal deformations (see [10], [17]), $\tilde{F}$ and $\tilde{G}$ are t-P-$\mathcal{K}$-equivalent. This completes the proof.

We remark that ${}^0A_r$, ${}^0D_r$-type germs in Theorem 2.4 can be realized as geometric solutions for any Hamilton-Jacobi equation. However, if a Legendrian unfolding has a non-trivial bifurcation, the situation is different as follows:

Example 3.3. Consider the equation : $s + p_n = 0$ (i.e. $H(t, x, p) = p_n$).

Let $\tilde{F}(t, x, y, q)$ be the generating family of the Legendrian unfolding Φ_F. Suppose that $\Phi_F|t = 0$ is an initial submanifold of the (GCPT) for $s + p_n = 0$. Then we have

$$\frac{\partial F}{\partial t}|0 \times \mathbb{R}^n \times \mathbb{R}^k + \frac{\partial F}{\partial x_n}|0 \times \mathbb{R}^n \times \mathbb{R}^k \equiv 0 \mod \langle \frac{\partial f}{\partial q_1}, \ldots, \frac{\partial f}{\partial q_k} \rangle_{\mathcal{E}_{(x,q)}}.$$

It follows that $\frac{\partial F}{\partial t}|0 \times \mathbb{R}^n \times \mathbb{R}^k \in T_e(\text{P-}\mathcal{K})(\tilde{f})$, where $f = F|0 \times \mathbb{R}^n \times \mathbb{R}^k$. Thus $\tilde{F}$ cannot be a P-$\mathcal{K}$-versal deformation of $\tilde{f}$, so that, for example, ${}^1A_r$, ${}^1D_r$-type germs in Theorem 2.4 cannot be realized.

Of course, since the above example is a linear equation, the characteristics never cross, so there are no shocks if the initial condition is a smooth function.

Example 3.4. Consider the equation : $s+p^3=0$ (i.e. $n=1$ and $H(t,x,p)=p^3$).

We now consider the function germ $f(x,q)=q^4+xq$, then $\frac{\partial f}{\partial q}=4q^3+x$, so that $\phi_f=\{(-4q^3,-3q^4,q)|q\in(\mathbb{R},0)\}$ is a Legendrian submanifold of $J^1(\mathbb{R},\mathbb{R})$ whose generating family is given by $\tilde{f}(x,y,q)=f(x,q)-y=q^4+xq-y$. Since $\ell_0=i_0\circ\phi_f=\{(0,-4q^3,-3q^4,-q^3,q)|q\in(\mathbb{R},0)\}\subset E(H)_0$ is the initial condition for the (GCPT), we can get a Legendrian unfolding $(\mathcal{L},0)\subset E(H)$ by the method of characteristics. Let $\tilde{G}(t,x,y,q)=G(t,x,q)-y$ be a generating family of $(\mathcal{L},0)$, then $\tilde{g}(x,y,q)=g(x,q)-y=\tilde{G}(0,x,y,q)$ is a generating family of ℓ_0. By the proof of Theorems 19.4 and 20.8 in [1], there exists a diffeomorphism germ $\Psi:(\mathbb{R}\times\mathbb{R},0)\to(\mathbb{R}\times\mathbb{R},0)$ of the form $\Psi(x,q)=(x,\psi(x,q))$ such that $g\circ\Psi(x,q)=f(x,q)$. Define a function germ $F:(\mathbb{R}\times\mathbb{R}\times\mathbb{R},0)\to(\mathbb{R},0)$ by $F(t,x,q)=G(t,x,\psi(x,q))$, then the Legendrian unfolding Φ_F is equal to $\Phi_G=\mathcal{L}$. It follows that Φ_F is the geometric solution of $E(H)$ and $\frac{\partial F}{\partial t}|_{t=0}+(\frac{\partial F}{\partial x}|_{t=0})^3\equiv 0 \bmod \langle\frac{\partial F}{\partial q}|_{t=0}\rangle_{\mathcal{E}_{(x,y,q)}}$. Since $F|_{t=0}=f$, we have $\frac{\partial F}{\partial t}|_{t=0}+q^3\in\langle 4q^3+x\rangle_{\mathcal{E}_{(x,y,q)}}$. It is easy to show that $\frac{\partial F}{\partial t}|_{t=0}\in T_e(P\text{-}\mathcal{K})(\tilde{f})$. Since $\tilde{F}$ and $\tilde{G}$ are t-P-$\mathcal{K}$-equivalent, $\frac{\partial G}{\partial t}|_{t=0}=\frac{\partial\tilde{G}}{\partial t}|_{t=0}\in T_e(P\text{-}\mathcal{K})(\tilde{g})$. This formula shows that $\tilde{G}$ cannot be a P-$\mathcal{K}$-versal deformation of $\tilde{g}$. However the generating family of type ${}^1A_3$: $q^4+xq+tq^2-y$ is a P-$\mathcal{K}$-versal deformation of $\tilde{f}(x,y,q)$, so that $\tilde{G}$ is not t-P-$\mathcal{K}$-equivalent to the germ of type ${}^1A_3$. Thus the Legendrian unfolding of type ${}^1A_3$ cannot be realized as a geometric solution of $s+p^3=0$.

Hence, we assume a kind of non-degeneracy condition on the Hamiltonian function germ. We say that a Hamiltonian function germ $H(t,x,p)$ at (t_0,x_0,p_0) is *non-degenerate* if the matrix $(\frac{\partial^2 H}{\partial p_i\partial p_j}(t_0,x_0,p_0))$ is positive (or negative) definite. Then we have the following realization theorem.

Theorem 3.5. *Let $H(t,x,p)$ be a non-degenerate Hamiltonian function germ at (t_0,x_0,p_0) and $\mathcal{L}:(R,u_0)\to(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),(t_0,x_0,y_0,s_0,p_0))$ be a P-Legendrian stable Legendrian unfolding associated with (μ,ℓ). Then there exists a Legendrian unfolding $\mathcal{L}'$ which is a geometric solution of the Hamilton-Jacobi equation $s+H(t,x,p)=0$ such that $\mathcal{L}$ and $\mathcal{L}'$ are P-Legendrian equivalent.*

Proof. Without loss of generality, we assume that $(t_0,x_0,y_0)=(0,0,0)$. Let $\tilde{G}:(\mathbb{R}\times(\mathbb{R}^n\times\mathbb{R})\times\mathbb{R}^k,0)\to(\mathbb{R},0)$ be a generating family of the Legendrian unfolding $\mathcal{L}$. Since $d_2G|0\times\mathbb{R}^n\times\mathbb{R}^k$ is non-singular, the set

$$\phi_g=\{(0,x,-H(0,x,\frac{\partial g}{\partial x}(x,q)),\frac{\partial g}{\partial x}(x,q))|\frac{\partial g}{\partial q_i}(x,q)=0\ i=1,\dots,k\}$$

is the initial condition for the corresponding (GCPT), where $g=G|0\times\mathbb{R}^n\times\mathbb{R}^k$. By the arguments of the last part of §1, we can construct a Legendrian unfolding $\mathcal{L}'$ which is the local unique geometric solution around ϕ_g.

We now choose a generating family $\tilde{F}:(\mathbb{R}\times(\mathbb{R}^n\times\mathbb{R})\times\mathbb{R}^{k'},0)\to(\mathbb{R},0)$ of $\mathcal{L}'$. By definition, Image $\Phi_F|t=0$ is equal to ϕ_g, so that we may assume that $k=k'$

and /, $\tilde{g}$ are P-JC-equivalent, where / = F|0 x R^n x R*, /(x, y, *q*) = /(x, *q*) — *y* and $\tilde{g}(x,y,q)=g(x,q)-y$.

If P-/C-cod $\tilde{p}$ = 0, then P-JC-cod / = 0, so that / is already a *P-KL*-versal deformation of itself. Hence, for the same reason as in the proof of Theorem 3.2, $\tilde{F}$ is t-P-AC-equivalent to $\tilde{G}$.

We now assume that *P-K-cod*$\tilde{g}$ = 1, so that P-JC-cod/ = 1. If $\frac{\partial F}{\partial t}|_{t=0} \notin T_e(P\text{-}K)(\tilde{f})$, then we can get the required assertion by the uniqueness of the *P-K*-versal deformation as in the previous case.

Suppose that ^jp|t=o € $T_e(P\text{-}lC)(\tilde{f})$ for any generating family $\tilde{F}$ of *C!*. Since Φ_F is a geometric solution of *s* + H(t,x,p) = 0, we have a relation

$$-\frac{\partial F}{\partial t} \equiv H(t,x,\frac{\partial F}{\partial x}) \bmod \langle\frac{\partial F}{\partial q_1},\dots,\frac{dF.}{\partial q_k}\ {}_{,x,q)},$$

so that

$$-\frac{\partial F}{\partial t}|_{t=0} \equiv H(0,x,\frac{\partial f}{\partial x}) \bmod \langle\frac{}{\partial q_1},\dots,\frac{}{\partial q_k}\rangle\varepsilon_{(x,q)}.$$

Therefore

$$-\frac{\partial F}{\partial t}|_{t=x=0} \equiv H(0,0,\frac{\partial f}{\partial x}(0,q)) \bmod \langle\frac{\partial f_0}{\partial q_1},\dots,\frac{\partial f_0}{\partial q_k}\rangle\varepsilon_q,$$

where /o(?) = /(0??)- We may assume that /o € SDI^, where $9Jl_g$ is the unique maximal ideal of $£_q$.

We now consider the Taylor polynomial of fT(t,x,p) of degree 2 at (t,x,po) with respect to *p* = (pi,... ,p_n)-variables as follows :

$$H(t,x,p) = H(t,x,p_0)+\sum_{i=1}^{n}\frac{8H}{\partial p_i}(t,x,p_0)(p_i-p_{0,i})$$
$$+\frac{1}{2}\sum_{i,j}\frac{\partial^2 H}{\partial p_i\partial p_j}\text{-(t,z,}p_o\text{)(Pi - Po,»)(Pj - Po,i) + higher term.}$$

If we set *t* = 0, *x* = 0, and p* = 4£(0, *q)*, then

$$H(0,0,\frac{\partial f}{\partial x_i}(0,q)) = H(0,0,p_0)+|\sum_{i=1}\frac{\partial H}{\partial p_i}(x,p_0)(\frac{\partial f}{\partial x_i}(0,q)-p_{0,i})$$

+1/2 5,2 ^ a^{2}T/ap-j(xpo)(flf/aX(o,9)" ^ ^ ^ ^ " ^ Qf + higher term_

Since F(t,x,g) is a generalized phase function germ, rank(af~^-lt=o) = fc- Then we may assume that ^J*x*(0) = £y for *i*, j = 1,..., *k* and ^|(i,x)=(o,o) € SXJ for

$\ell = k+1,\dots,n$. It follows that $\frac{\partial F}{\partial x_i}(0,0,q) - p_{0i} = q_i + \psi(q)$, where $\psi(q) \in \mathfrak{M}_q^2$. Since H is non-degenerate, the quadratic form $\sum_{i=1}^k \frac{\partial^2 F}{\partial p_i \partial p_j}(0,0,p_0)q_i q_j$ never vanishes.

On the other hand

$$-\frac{\partial F}{\partial t}|_{t=0} \in \langle f(x,q) - y, \frac{\partial f}{\partial q}(x,q)\rangle_{\mathcal{E}_{(x,y,q)}} + \langle 1, \frac{\partial f}{\partial x}(x,q)\rangle_{\mathcal{E}_{(x,y)}}.$$

It follows that

$$H(0,0,\frac{\partial f}{\partial x}(0,q)) \in \langle f(x,q) - y, \frac{\partial f_0}{\partial q}(q)\rangle_{\mathcal{E}_q} + \langle 1, \frac{\partial f_0}{\partial x}(q)\rangle_{\mathbf{R}},$$

and

$$\begin{aligned}
&\frac{1}{2}\sum_{1\le i,j\le n} \frac{\partial^2 H}{\partial p_i \partial p_j}(0,0,p_0)(\frac{\partial f}{\partial x_i}(0,q) - p_{0,i})(\frac{\partial f}{\partial x_j}(0,q) - p_{0,j}) \\
\equiv &H(0,0,\frac{\partial f}{\partial x}(0,q)) - H(0,0,p_0) - \sum_{i=1}^n \frac{\partial H}{\partial p_i}(0,0,p_0)(\frac{\partial f}{\partial x_i}(0,q) - p_{0,i}) \bmod \mathfrak{M}_q^3 \\
= &H(0,\frac{\partial f}{\partial x}(0,q)) - H(0,0,p_0) - \sum_{i=1}^n \frac{\partial H}{\partial p_i}(0,0,p_0)\frac{\partial f}{\partial x_i}(0,q) + \sum_{i=1}^n p_{0,i}\frac{\partial H}{\partial p_i}(0,0,p_0) \\
\in &\langle 1, \frac{\partial f}{\partial x_1}(0,q),\dots,\frac{\partial f}{\partial x_n}(0,q)\rangle_{\mathbf{R}} + \langle f_0, \frac{\partial f_0}{\partial q}\rangle_{\mathcal{E}_q} \bmod \mathfrak{M}_q^3.
\end{aligned}$$

For any linear isomorphism $A : \mathbb{R}^n \to \mathbb{R}^n$, we have a relation

$$\frac{\partial f(Ax,q)}{\partial x_i} = \sum_{j=1}^n A_{ij}\frac{\partial f}{\partial x_j}(Ax,q).$$

Since

$$\sum_{1\le i,j\le n} \frac{\partial^2 H}{\partial p_i \partial p_j}(0,0,p_0)(\frac{\partial f}{\partial x_i}(0,q) - p_{0,i})(\frac{\partial f}{\partial x_j}(0,q) - p_{0,j}) \equiv \sum_{1\le i,j\le k} \frac{\partial^2 H}{\partial p_i \partial p_j}(0,0,p_0)q_i q_j \bmod \mathfrak{M}_q^3$$

and the vector space

$$\langle f_0, \frac{\partial f_0}{\partial q}\rangle_{\mathcal{E}_q} + \langle 1, \frac{\partial f}{\partial x_1}(0,q),\dots,\frac{\partial f}{\partial x_1}(0,q)\rangle_{\mathbf{R}} \bmod \mathfrak{M}_q^3$$

is an invariant under the action of the linear isomorphism A, then any quadratic form of $q = (q_1,\dots,q_k)$-variables is contained in the above vector space. If there exists a quadratic form of q-variables that it is contained in $\langle f_0, \frac{\partial f_0}{\partial q}\rangle_{\mathcal{E}_q} \bmod \mathfrak{M}_q^3$, then all quadratic forms are contained in it for the same reason as above. In this case, since the vector space $\langle f_0, \frac{\partial f_0}{\partial q}\rangle_{\mathcal{E}_q} \bmod \mathfrak{M}_q^3$ has at most dimension k, then k should be 1 and f_0 is an A_2-type function germ. It follows from Theorem 2.4 that $F(t,x,q)$ is of ${}^0A_2$-type, so that this case is contained in the case of P-$\mathcal{K}$-cod$(\tilde{f}) = 0$.

We may assume that any quadratic form of q-variables is contained in

$$\langle 1, \frac{\partial f}{\partial x_1}(0,q), \ldots, \frac{\partial f}{\partial x_n}(0,q)\rangle_{\mathbf{R}} \bmod \mathfrak{M}_q^3.$$

Since $\mathcal{K}$-cod(f_0) is finite (for the definition of $\mathcal{K}$-finiteness, see [35]), then there exists $r \in \mathbf{N}$ such that $\mathfrak{M}_q^r \subset \langle f_0, \frac{\partial f_0}{\partial q}\rangle_{\mathcal{E}_q}$. By the same arguments as those of the previous paragraph, we can assert that every monomial of q-variables of degree 3 is contained in the vector space

$$\langle 1, \frac{\partial f}{\partial x_1}(0,q), \ldots, \frac{\partial f}{\partial x_n}(0,q)\rangle_{\mathbf{R}} + \langle f_0, \frac{\partial f_0}{\partial q}\rangle_{\mathcal{E}_q} \bmod \mathfrak{M}_q^4.$$

If there exists a monomial of degree 3 which is contained in $\langle f_0, \frac{\partial f_0}{\partial q}\rangle_{\mathcal{E}_q} \bmod \mathfrak{M}_q^4$, then k should be 1 and f_0 is an A_3-type function germ. It follows that

$$\dim_{\mathbf{R}}\langle 1, \frac{\partial f}{\partial x_1}(0,q), \ldots, \frac{\partial f}{\partial x_n}(0,q)\rangle_{\mathbf{R}} \geq 3 \bmod \mathfrak{M}_q^4.$$

By Theorem 2.4, $F(t,x,q)$ should be of ${}^0A_3$-type, then this case is contained in the case of P-$\mathcal{K}$-cod$(\tilde{f}) = 0$.

For ${}^0A_\ell$ or ${}^1A_\ell$-type germs, we get same normal forms as in Theorem 2.4 without the assumption $n \leq 5$ (cf. Theorem 2.2 in [36]). We can continue this procedure up to degree $r-1$. Eventually, it remains the case that every polynomial of degree $r-1$ is contained in the vector space

$$\langle 1, \frac{\partial f}{\partial x_1}(0,q), \ldots, \frac{\partial f}{\partial x_n}(0,q)\rangle_{\mathbf{R}} + \langle f_0, \frac{\partial f_0}{\partial q}\rangle_{\mathcal{E}_q} \bmod \mathfrak{M}_q^r.$$

Since $\mathfrak{M}_q^r \subset \langle f_0, \frac{\partial f_0}{\partial q}\rangle_{\mathcal{E}_q}$, then

$$\mathcal{E}_q = \langle 1, \frac{\partial f}{\partial x_1}(0,q), \ldots, \frac{\partial f}{\partial x_n}(0,q)\rangle_{\mathbf{R}} + \langle f_0, \frac{\partial f_0}{\partial q}\rangle_{\mathcal{E}_q}.$$

It follows that

$$\mathcal{E}_{(x,q)} = T_e(P\text{-}\mathcal{K})(\tilde{f}) + \mathfrak{M}_q\mathcal{E}_{(x,q)},$$

so that we have $\mathcal{E}_{(x,q)} = T_e(P\text{-}\mathcal{K})(f)$ by the Malgrange preparation theorem. This contradicts the fact that P-$\mathcal{K}$-cod$(\tilde{f}) = 1$. This completes the proof.

Remark. The method of the proof is analogous to that of Theorem 4.2 in [19].

4. Multi-Legendrian unfoldings

As we discussed in the introduction, in order to study how the shock waves for the viscosity solution of (P) evolve, we study the evolution of intersections of branches of the graph of the corresponding geometric solution. We classify the bifurcations of the branches of the graph by classifying the bifurcations of singularities of multi-Legendrian unfoldings which are expressed in terms of multi-germs.

Let $\mathcal{L}_i : (R, u_0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_i)$ $(i = 1, \ldots, r)$ be Legendrian unfoldings with $\Pi(z_i) = 0$ where $z_1, \ldots, z_r$ are distinct. We call $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ *a multi-Legendrian unfolding.* Let $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ and $(\mathcal{L}'_1, \ldots, \mathcal{L}'_r)$ be multi-Legendrian unfoldings. We say that these are $P_{(r)}$*-Legendrian equivalent* if there exist contact diffeomorphism germs

$$K_i : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_i) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z'_i) \quad (i = 1, \ldots, r)$$

of the form

$$K_i(t, x, y, s, p) = (\phi_1(t), \phi_2(t, x, y), \phi_3(t, x, y), \phi_4^i(t, x, y, s, p), \phi_5^i(t, x, y, s, p))$$

and a diffeomorphism germ $\Psi : (R, u_0) \to (R, u'_0)$ such that $K_i \circ \mathcal{L}_i = \mathcal{L}'_i \circ \Psi$ for any $i = 1, \ldots, r$. It is clear that if two multi-Legendrian unfoldings are $P_{(r)}$-Legendrian equivalent, then there exists a diffeomorphism germ

$$\Phi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0)$$

of the form $\Phi(t, x, y) = (\phi_1(t), \phi_2(t, x, y), \phi_3(t, x, y))$ such that $\Phi(\cup_{i=1}^r W(\mathcal{L}_i)) = \cup_{i=1}^r W(\mathcal{L}_i)$. Thus the above equivalence describes how bifurcations of wavefronts (i.e. graphs of solutions) interact.

By Proposition 2.2, there exist generating families

$$\tilde{F}_i : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^{k_i}, 0) \to (\mathbb{R}, 0)$$

of $\mathcal{L}_i$, $i = 1, \ldots, r$. We may choose those generating families for which $k = k_1 = \cdots = k_r$. We call $\tilde{F} = (\tilde{F}_1, \ldots, \tilde{F}_r)$ *a multi-generating family of* the multi-Legendrian unfolding $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$. We also consider an equivalence relation among multi-generating families of multi-Legendrian unfoldings. Multi-generating families $\tilde{F} = (\tilde{F}_1, \ldots, \tilde{F}_r)$ and $\tilde{F}' = (\tilde{F}'_1, \ldots, \tilde{F}'_r)$ are t*-*$(P\text{-}\mathcal{K})_{(r)}$*-equivalent* if there exists a diffeomorphism germ

$$\Phi_i : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k, 0) \to (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k, 0) \quad (i = 1, \ldots, r)$$

of the form

$$\Phi_i(t, x, y, q) = (\phi_1(t), \phi_2(t, x, y), \phi_3(t, x, y), \phi_4^i(t, x, y, q))$$

such that

$$\langle \tilde{F}'_i \circ \Phi \rangle_{\mathcal{E}_{(t,x,y,q)}} = \langle \tilde{F}_i \rangle_{\mathcal{E}_{(t,x,y,q)}}.$$

We also define the notion of *stable* t*-*$(P\text{-}\mathcal{K})_{(r)}$*-equivalence* in the same way as the *stable* t*-*$(P\text{-}\mathcal{K})$*-equivalence* (see Section 2). We have the following theorem which is a corollary of Proposition 2.3.

Proposition 4.1. *Let $\tilde{F} = (\tilde{F}_1, \dots, \tilde{F}_r)$ and $\tilde{F}' = (\tilde{F}'_1, \dots, \tilde{F}'_r)$ be multi-generating families of multi-Legendrian unfoldings $(\Phi_{F_1}, \dots, \Phi_{F_r})$ and $(\Phi_{F'_1}, \dots, \Phi_{F'_r})$ respectively. Then $(\Phi_{F_1}, \dots, \Phi_{F_r})$ and $(\Phi_{F'_1}, \dots, \Phi_{F'_r})$ are $P_{(r)}$-Legendrian equivalent if and only if $\tilde{F}$ and $\tilde{F}'$ are stably t-$(P\text{-}\mathcal{K})_{(r)}$-equivalent.*

According to the above proposition it is enough to give a classification of generic multi-generating families under the $(P\text{-}\mathcal{K})_{(r)}$-equivalence. For this we need to extend the results of the previous section to multi-generating families.

For generating families $\tilde{F}_i$ of Φ_{F_i}, $i = 1, \dots, r$, we define a subspace of $\mathcal{E}^r_{(x,y,q)}$ by

$$T_e(\mathrm{P}_{(r)}\text{-}\mathcal{K})(\tilde{f}) = \langle \frac{\partial \tilde{f}_1}{\partial q}, \tilde{f}_1 \rangle_{\mathcal{E}_{(x,y,q)}} \times \cdots \times \langle \frac{\partial \tilde{f}_r}{\partial q}, \tilde{f}_r \rangle_{\mathcal{E}_{(x,y,q)}} + \langle \frac{\partial \tilde{f}}{\partial x_1}, \dots, \frac{\partial \tilde{f}}{\partial x_n}, \frac{\partial \tilde{f}}{\partial y} \rangle_{\mathcal{E}_{(x,y)}}$$

and $(P\text{-}\mathcal{K})_{(r)}\text{-cod}\, \tilde{f} = \dim_{\mathbf{R}} \mathcal{E}^r_{(x,y,q)} / T_e((P\text{-}\mathcal{K})_{(r)})(\tilde{f})$, where $\tilde{f}_i = \tilde{F}_i | 0 \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k$. We also say that $\tilde{F}$ is *a $(P\text{-}\mathcal{K})_{(r)}$-versal deformation of $\tilde{f}$* if

$$\mathcal{E}^r_{(x,y,q)} = \langle \frac{\partial \tilde{F}}{\partial t} | 0 \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k \rangle_{\mathbf{R}} + T_e((P\text{-}\mathcal{K})_{(r)})(\tilde{f}).$$

By the versality theorem in [10], we have the following uniqueness result.

Theorem 4.2. *Let $\tilde{F}$ and $\tilde{G}$ be $(P\text{-}\mathcal{K})_{(r)}$-versal deformations of $\tilde{f}$ and $\tilde{g}$ respectively. Then $\tilde{F}$ and $\tilde{G}$ are t-$(P\text{-}\mathcal{K})_{(r)}$-equivalent if and only if $\tilde{f}$ and $\tilde{g}$ are $(P\text{-}\mathcal{K})_{(r)}$-equivalent.*

Our objective is to classify multi-generating families $\tilde{F} = (\tilde{F}_1, \dots, \tilde{F}_r)$ which are $(P\text{-}\mathcal{K})_{(r)}$-versal deformations of $\tilde{f} = \tilde{F} | 0 \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k$. Then we need a classification of multi-germs $\tilde{f}$ of $(P\text{-}\mathcal{K})_{(r)}\text{-cod}\, \tilde{f} \leq 1$. The following estimate of codimensions is useful for such a classification.

Lemma 4.3. $\sum_{i=1}^r \mathcal{K}$-*cod*$(f_{0,i}) \leq (P\text{-}\mathcal{K})_{(r)}$-*cod*$(f) + n + 1$. *Here,* $\mathcal{K}$-*cod*$(f_{0,i}) = $ *dim*$_{\mathbf{R}} \mathcal{E}_q / \langle \frac{\partial f_{0,i}}{\partial q}, f_{0,i} \rangle_{\mathcal{E}_q}$.

Proof. We have $\mathcal{K}\text{-cod}\,(f_{0,i}) = \dim_{\mathbf{R}} \mathcal{E}_{(x,y,q)} / \langle \frac{\partial \tilde{f}_i}{\partial q}, f_i \rangle_{\mathcal{E}_{(x,y,q)}} + \mathfrak{M}_{(x,y)} \mathcal{E}_{(x,y,q)}$, so that

$$\begin{aligned}
&\sum_{i=1}^r \mathcal{K}\text{-cod}\,(f_{0,i}) \\
=&\dim_{\mathbf{R}} \mathcal{E}^r_{(x,y,q)} / \langle \frac{\partial \tilde{f}_1}{\partial q}, f_1 \rangle_{\mathcal{E}_{(x,y,q)}} \times \cdots \times \langle \frac{\partial \tilde{f}_r}{\partial q}, f_r \rangle_{\mathcal{E}_{(x,y,q)}} + \mathfrak{M}_{(x,y)} \mathcal{E}^r_{(x,y,q)} \\
\leq&\dim_{\mathbf{R}} \mathcal{E}^r_{(x,y,q)} / T_e(\mathrm{P}_{(r)}\text{-}\mathcal{K})(\tilde{f}) \\
&\quad + \dim_{\mathbf{R}} \frac{T_e((P\text{-}\mathcal{K})_{(r)})(f)}{\langle \frac{\partial \tilde{f}_1}{\partial q}, f_1 \rangle_{\mathcal{E}_{(x,y,q)}} \times \cdots \times \langle \frac{\partial \tilde{f}_r}{\partial q}, f_r \rangle_{\mathcal{E}_{(x,y,q)}} + \mathfrak{M}_{(x,y)} \mathcal{E}^r_{(x,y,q)}} \\
\leq&\dim_{\mathbf{R}} \mathcal{E}^r_{(x,y,q)} / T_e((P\text{-}\mathcal{K})_{(r)})(\tilde{f}) + \dim_{\mathbf{R}} \langle \frac{\partial \tilde{f}}{\partial x_1}, \dots, \frac{\partial \tilde{f}}{\partial x_n}, \frac{\partial \tilde{f}}{\partial y} \rangle_{\mathbf{R}} \\
\leq&(P\text{-}\mathcal{K})_{(r)}\text{-cod}\,(f) + n + 1.
\end{aligned}$$

We say that multi-function germs $f_0 = (f_{0,1}, \dots, f_{0,r})$ and $g_0 = (g_{0,1}, \dots, g_{0,r})$ are $\mathcal{K}_{(r)}$*-equivalent* if $f_{0,i}$ and $g_{0,i}$ are $\mathcal{K}$-equivalent for $i = 1, \dots, r$. For the definition and properties of the $\mathcal{K}$-equivalence, see [35]. We now define

$$T_e(\mathcal{K})_{(r)}(f_0) = \langle \frac{\partial f_{0,1}}{\partial q}, f_{0,1} \rangle_{\mathcal{E}_q} \times \cdots \times \langle \frac{\partial f_{0,r}}{\partial q}, f_{0,r} \rangle_{\mathcal{E}_q}.$$

We also define the notion of $\mathcal{K}_{(r)}$-versal deformation of f_0 as follows : Let $\bar{F} = (\bar{F}_1, \dots, \bar{F}_r)$ be a s-parameter deformation of f_0 (i.e. $\bar{F}_i : (\mathbb{R}^s \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0)$ is a function germ such that $\bar{F}_i | 0 \times \mathbb{R}^k = f_{0,i}$ for any $i = 1, \dots, r$.) We say that $\bar{F} = (\bar{F}_1, \dots, \bar{F}_r)$ is *a* $\mathcal{K}_{(r)}$*-versal deformation* of a multi-germ f_0 if

$$\mathcal{E}_q = \langle \frac{\partial \bar{F}}{\partial u_1} | 0 \times \mathbb{R}^k, \dots, \frac{\partial \bar{F}}{\partial u_s} |_0 \times \mathbb{R}^k \rangle_{\mathbb{R}} + T_e(\mathcal{K})_{(r)}(f_0).$$

We also define *the discriminant set of* $\bar{F}$ as follows :

$$D_{\bar{F}} = \cup_{i=1}^r D_{\bar{F}_i},$$

where

$$D_{\bar{F}_i} = \{ u \in \mathbb{R}^s | \, \bar{F}_i(u, q) = \frac{\partial \bar{F}_i}{\partial u_1}(u, q) = \cdots = \frac{\partial \bar{F}_i}{\partial u_s}(u, q) = 0 \text{ for some } q \in (\mathbb{R}^k, 0) \}.$$

We have the following lemma.

Lemma 4.4. *If a multi-generating family $\tilde{F}$ is an one-parameter $(P\text{-}\mathcal{K})_{(r)}$-versal deformation of $f = \tilde{F} | 0 \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^k$, then it is an $(n+2)$-parameter $(\mathcal{K})_{(r)}$-versal deformation of $f_0 = \tilde{F} | 0 \times (0 \times 0) \times \mathbb{R}^k$.*

We summarize the strategy for the classification, which will be presented in the next section, as follows :

Step 1. We classify multi-germs $f_0 = (f_{0,1}, \dots, f_{0,r})$ with $(\mathcal{K})_{(r)}\text{-cod}\,(f_0) \le n + 2$ under the $(\mathcal{K})_{(r)}$-equivalence, where $(\mathcal{K})_{(r)}\text{-cod}\,(f_0) = \sum_{i=1}^r \mathcal{K}\text{-cod}\,(f_{0,i})$.

Step 2. We construct $(n+2)$-parameter $(\mathcal{K})_{(r)}$-versal deformations $\bar{F}$ of the normal forms $f_0 = (f_{0,1}, \dots, f_{0,r})$ obtained by Step 1. We fix each germ $\bar{F}$ and we consider a smooth function germ $t : (\mathbb{R}^{n+2}, 0) \to (\mathbb{R}, 0)$. By the theorem in Zakalyukin [36, part 2.2], we can classify t under coordinate changes of $\mathbb{R}^{n+2}$ which preserve the discriminant set $D_{\bar{F}}$. This classification generically corresponds to the classification of F under the $(P\text{-}\mathcal{K})_{(r)}$-equivalence which preserves the projection on the t-space (i.e. the t-$(P\text{-}\mathcal{K})_{(r)}$-equivalence).

Step 3. We can check that each normal form obtained by Step 2 is a $(P\text{-}\mathcal{K})_{(r)}$-versal deformation when we consider t as a parameter. Thus we can detect normal forms of generic multi-Legendrian unfoldings by the $P_{(r)}$-Legendrian equivalence.

5. Classifications

In this section we pursue the strategy we referred to at the end of the last section in case n=1 or 2. Since a generalized phase function germ F satisfies $d_2F(0) = 0$, we may assume that $\mathcal{K}\text{-cod}\,(f_{0,i}) \geq 1$ for a multi-germ $f_0 = (f_{0,1}, \dots, f_{0,r})$. It follows from Lemma 4.3 that $r \leq n+2$. By Corollary 2.5, we have the following classification.

Lemma 5.1. *For $n \leq 2$ and generic multi-generating families $\tilde{F} = (\tilde{F}_1, \dots, \tilde{F}_r)$ of multi-Legendrian unfoldings, the corresponding multi-germs $f_0 = (f_{0,1} \dots, f_{0,r})$ are $(\mathcal{K})_{(r)}$-equivalent to one of the multi-germs in the following list :*
$n = 1$;
  $r = 1$; q_1^2, q_1^3, q_1^4.
  $r = 2$; (q_1^2, q_1^2), (q_1^2, q_1^3).
  $r = 3$; (q_1^2, q_1^2, q_1^2).
$n = 2$;
  $r = 1$; q_1^2, q_1^3, q_1^4, q_1^5, $q_1^2 \pm q_2^3$.
  $r = 2$; (q_1^2, q_1^2), (q_1^2, q_1^3), (q_1^2, q_1^4), (q_1^3, q_1^3).
  $r = 3$; (q_1^2, q_1^2, q_1^2), (q_1^2, q_1^2, q_1^3).
  $r = 4$; $(q_1^2, q_1^2, q_1^2, q_1^2)$.

The case of $r = 1$ has already been classified in Corollary 2.5, so we do not consider this case. For the case $r \geq 2$, we can easily construct a $\mathcal{K}_{(r)}$-versal deformation for each multi-germ by the usual method. Then the corresponding list is as follows :
$n = 1$;

$$(q_1^2 + u_{1,1}, q_1^2 + u_{2,1}) \tag{1}$$
$$(q_1^2 + u_{1,1}, q_1^3 + u_{2,1} + u_{2,2}q_1) \tag{2}$$
$$(q_1^2 + u_{1,1}, q_1^2 + u_{2,1} + u_{3,1}, q_1^2 + u_{3,1}) \tag{3}$$

$n = 2$;

$$(q_1^2 + u_{1,1}, q_1^2 + u_{2,1}) \tag{4}$$
$$(q_1^2 + u_{1,1}, q_1^3 + u_{2,1} + u_{2,2}q_1) \tag{5}$$
$$(q_1^2 + u_{1,1}, q_1^4 + u_{2,1} + u_{2,2}q_1 + u_{2,3}q_1^2) \tag{6}$$
$$(q_1^3 + u_{1,1} + u_{1,2}q_1, q_1^3 + u_{2,1} + u_{2,2}q_1) \tag{7}$$
$$(q_1^2 + u_{1,1}, q_1^2 + u_{2,1}, q_1^2 + u_{3,1}) \tag{8}$$
$$(q_1^2 + u_{1,1}, q_1^2 + u_{2,1}, q_1^3 + u_{3,1} + u_{3,2}q_1) \tag{9}$$
$$(q_1^2 + u_{1,1}, q_1^2 + u_{2,1}, q_1^2 + u_{3,1}, q_1^2 + u_{4,1}) \tag{10}$$

Let $G(u_{1,1}, \dots, u_{1,\mu_1}, \dots, u_{r,1}, \dots, u_{r,\mu_r}, q_1, \dots, q_k)$ be a $\mathcal{K}_{(r)}$-versal deformation of a multi-germ $g = (g_1, \dots, g_r)$, where $\mu_i = \mathcal{K}\text{-cod}\,f_i$ for $i = 1, \dots, r$. Define a multi-germ $\bar{G}$ by

$$\bar{G}(u_{1,1},\ldots,u_{1,\mu_1},\ldots,u_{r,1},\ldots,u_{r,\mu_r},u_1,\ldots,u_\mu,q_1,\ldots,q_k)$$
$$=G(u_{1,1},\ldots,u_{1,\mu_1},\ldots,u_{r,1},\ldots,u_{r,\mu_r},q_1,\ldots,q_k)$$

for $\mu = n+2-\sum_{i=1}^{r}\mu_i$. We now consider a function germ $t:(\mathbb{R}^{n+2},0)\to(\mathbb{R},0)$ on the $(u_{1,1},\ldots,u_{1,\mu_1},\ldots,u_{r,1},\ldots,u_{r,\mu_r},u_1,\ldots,u_\mu)$-space. We can apply the theorem of part 2.2 in [36], so we get the following :

Proposition 5.2. (Zakalyukin [36]) *Suppose that $n\leq 4$,*

$$\frac{\partial t}{\partial u_{1,\mu_1}}\neq 0,\ldots,\frac{\partial t}{\partial u_{r,\mu_r}}\neq 0$$

and

$$t|_{u_{1,1}=\cdots=u_{r,\mu_r}=0} \text{ is a Morse function germ.}$$

Then there exists a diffeomorphism germ $\phi:(\mathbb{R}^{n+2},0)\to(\mathbb{R}^{n+2},0)$ preserving the discriminant set $D_{\bar{G}}$ such that $t\circ\phi$ is equal to u_1 or $\pm u_{1,\mu_1}\pm\cdots\pm u_{r,\mu_r}\pm(u_1)^2\pm\cdots\pm(u_\mu)^2$.

We notice that a submersion germ $t:(\mathbb{R}^{n+2},0)\to(\mathbb{R},0)$ which satisfies the assumption of the proposition is generic. We can detect generic normal forms of multi-Legendrian unfoldings as follows.

Theorem 5.3. *Suppose that $n\leq 2$. Then a generic multi-Legendrian unfolding is $P_{(r)}$-Legendrian equivalent to one of the multi-Legendrian unfoldings defined by multi-generating families in the following list :*

$n=1$;
  $r=1$;
  ${}^0A_1\ :\ q_1^2-y$;
  ${}^0A_2\ :\ q_1^3+x_1q_1-y$;
  ${}^1A_3\ :\ q_1^4+q_1^2t+x_1q_1-y$.
  $r=2$;
  ${}^0({}^0A_1\,{}^0A_1)\ :\ (q_1^2-x-y,q_1^2+x-y)$;
  ${}^1({}^0A_1\,{}^0A_1)\ :\ (q_1^2+t\pm x^2-y,q_1^2-y)$;
  ${}^1A_2\,{}^0A_1\ :\ (q_1^3+(x-t)q_1-y,q_1^2-x-y)$;
  $r=3$;
  ${}^0A_1\,{}^0A_1\,{}^0A_1\ :\ (q_1^2+t-x-y,q_1^2-y,q_1^2+x-y)$;
$n=2$;
  $r=1$;
  ${}^0A_1\ :\ q_1^2-y$;
  ${}^0A_2\ :\ q_1^3+x_1q_1-y$;
  ${}^0A_3\ :\ q_1^4+x_1q_1+x_2q_1^2-y$;
  ${}^1A_3\ :\ q_1^4+q_1^2(t\pm x_2^2)+x_1q_1-y$;
  ${}^1A_4\ :\ q_1^5+q_1^3t+x_1q_1+x_2q_1^2-y$;

$^{*}D_4 \ : \ q_1^2 q_2 \pm q_2^3 + t q_2^2 + x_2 q_2 + x_1 q_1 - y$;

$r = 2$;

${}^0({}^0A_1\,{}^0A_1) \ : \ (q_1^2+x_1+x_2-y,\ q_1^2+x_1-x_2-y)$;

${}^1({}^0A_1\,{}^0A_1) \ : \ (q_1^2+t\pm x_1^2\pm x_2^2-y,\ q_1^2-y)$;

${}^0({}^0A_1$ [illegible] $+ x_1q_1 - y)$;

${}^1A_1\,{}^0A_2 \ : \ (q_1^2+t+x_1\pm x_2^2-y,\ q_1^3+x_1q_1-y)$;

${}^0A_1\,{}^0A_3 \ : \ (q_1^2+x_1+x_2+t-y,\ q_1^4+x_1q_1+x_2q_1^2-y)$;

${}^0A_2\,{}^0A_2 \ : \ (q_1^3+t+(x_2-x_1)q_1-x_2-y,\ q_1^3+x_1q_1-y)$;

$r = 3$;

${}^0({}^0A_1\,{}^0A_1\,{}^0A_1) \ : \ (q_1^2+x_1-y,\ q_1^2-y,\ q_1^2+x_2-y)$;

${}^1({}^0A_1\,{}^0A_1\,{}^0A_1) \ : \ (q_1^2+t+x_1\pm x_2^2-y,\ q_1^2-y,\ q_1^2-x_1-y)$;

${}^0A_1\,{}^0A_1\,{}^1A_2 \ : \ (q_1^2+x_1-y,\ q_1^2+x_2-y,\ q_1^3+(t-x_1-x_2)q_1-y)$;

$r = 4$;

${}^0A_1\,{}^0A_1\,{}^0A_1\,{}^0A_1 \ : \ (q_1^2+t+x_1+x_2-y,\ q_1^2-y,\ q_1^2+x_1-y,\ q_1^2+x_2-y)$.

Proof. Let $G(u_{1,1},\dots,u_{1,\mu_1},\dots,u_{r,1},\dots,u_{r,\mu_r},q_1,\dots,q_k)$ be a $\mathcal{K}_{(r)}$-versal deformation of a multi-germ $g=(g_1,\dots,g_r)$, where $\mu_i = \mathcal{K}\text{-cod}\,g_i$ for $i=1,\dots,r$. Define a multi-germ $\bar{G}$ by

$$\bar{G}(u_{1,1},\dots,u_{1,\mu_1},\dots,u_{r,1},\dots,u_{r,\mu_r},u_1,\dots,u_\mu,q_1,\dots,q_k) = G(u_{1,1},\dots,u_{1,\mu_1},\dots,u_{r,1},\dots,u_{r,\mu_r},q_1,\dots,q_k)$$

for $\mu = n+2-\sum_{i=1}^{r}\mu_i$.

Let $F(t,x,y,q)$ be a multi-generating family of a $\mathcal{P}_{(r)}$-Legendrian stable multi-Legendrian unfolding. Then $\bar{F}$ is a $(\mathcal{P}\text{-}\mathcal{K})_{(r)}$-versal deformation of a multi-germ F, so that $\bar{F}$ is a $\mathcal{K}_{(r)}$-versal deformation of a multi-germ f. If f is $\mathcal{K}_{(r)}$-equivalent to g, then $\bar{F}$ and $\bar{G}$ are $(\mathcal{P}\text{-}\mathcal{K})_{(r)}$-equivalent (i.e., there exist diffeomorphism germs

$$\Psi_i : (\mathbb{R}^{n+2}\times\mathbb{R}^k,0)\to((\mathbb{R}\times\mathbb{R}^n\times\mathbb{R})\times\mathbb{R}^k,0)\qquad (i=1,\dots,r)$$

of the form

$$\Psi(u,q) = (\psi_1(u),\psi_2(u),\psi_3(u),\psi_4^i(u,q))$$

such that $\Psi_i^*(\bar{F})\mathcal{E}_{(u,q)} = (\bar{G})\mathcal{E}_{(u,q)}$). By the remark after Proposition 5.2, we may assume that Ψ satisfies the assumption of Proposition 5.2, so that there exists a diffeomorphism germ $\phi : (\mathbb{R}^{n+2},0)\to(\mathbb{R}^{n+2},0)$ preserving the discriminant set $\bigcup_{i=1}^{r} D_{G_i}$ such that $t\circ\phi$ is equal to u_1 or $\pm u_{1,\mu_1}\pm\cdots\pm u_{r,\mu_r}\pm(u_1)^2\pm\cdots\pm(u_\mu)^2$. Here, the discriminant set $\bigcup_{i=1}^{r} D_{G_i}$ is the wave front set of a multi-germ of a Legendrian submanifold in $J^1(\mathbb{R}^{n+1},\mathbb{R})\subset PT^*\mathbb{R}^{n+2}$, where $PT^*\mathbb{R}^{n+2}$ is the projectivization of the cotangent bundle $T^*\mathbb{R}^{n+2}$. (See [36]). Then we can construct the unique contact lift $\hat{\phi}$ of ϕ preserving the multi-germ of the Legendrian submanifold. That is, $(\phi\times 1_{\mathbb{R}^k})^*\bar{G}(u,q) = \bar{G}(\phi(u),q)$ gives the same multi-germ of the Legendrian submanifold given by $\bar{G}$. We can choose the coordinates of $(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R},0)$ as follows

$$\begin{cases} t = \psi_1\circ\phi(u) \\ (x,y) = (\psi_2\circ\phi(u),\ \psi_3\circ\phi(u)), \end{cases}$$

where $u = (u_{1,1}, \ldots, u_{1,\mu_1}, \ldots, u_{r,1}, \ldots, u_{r,\mu_r}, u_1, \ldots, u_\mu)$. By the above equality we can represent the coordinates u by the coordinates (t, x, y). This procedure gives the normal forms as follows : Let G be a germ of the form

$$(q_1^2 + u_{1,1}, q_1^2 + u_{2,1}). \tag{1}$$

In this case we have $r = 2$ and $n = 1$, so that $\mu_1 = 1$, $\mu_2 = 1$ and $\mu = 1$. It follows from Proposition 5.2 that $t = u_1$ or $t = \pm u_{1,1} \pm u_{2,1} \pm u_1^2$. If $t = u_1$, we can adopt the coordinates $u_{1,1} = x - y$ and $u_{2,1} = -x - y$. If $t = \pm u_{1,1} \pm u_{2,1} \pm u_1^2$, then we can adopt the coordinates $u_{1,1} = \pm t - u_{2,1} - u_1^2$, $u_1 = x$ and $u_{2,1} = y$. Finally, we get the normal forms of type ${}^0({}^0A_1\,{}^0A_1)$ and ${}^1({}^0A_1\,{}^0A_1)$. For other germs, we can get the normal forms by repeating the steps in the above calculations. Since these calculations are straightforward we omit the details.

Finally, we can easily check that each normal form is $(P\text{-}\mathcal{K})_{(r)}$-versal.

6. Discussion

In this section we discuss how we can construct the viscosity solution by using the classification Theorem 5.3 and hence study the geometric structure of the shock waves. The first time the characteristics cross the geometric solution becomes singular and a singularity of type ${}^1A_3$ appears. See Figure 1. We construct the unique viscosity solution past the first critical time by selecting a continuous single-valued branch of the geometric solution. The proof that the so constructed function is the viscosity solution was given in [25] without making any assumption on the convexity or concavity of the Hamiltonian H. The case of a convex Hamiltonian, where the weak solution is defined in the class of the semi-concave functions, was studied in [27].

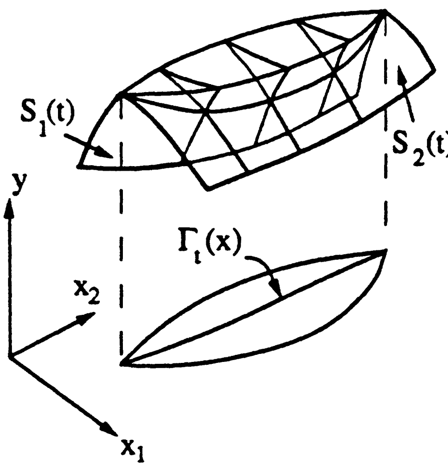

FIGURE 1

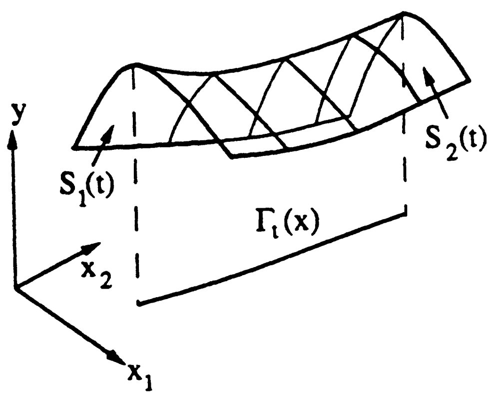

FIGURE 2

The graph of the viscosity solution for a given time t past the first critical time is shown in Figure 2. The projection of the intersection of the two branches $S_1(t)$

and $S_2(t)$ on the x-plane is the shock surface Γ_t. In order to follow the evolution of Γ_t we study the evolution of the intersection $S_1(t) \cap S_2(t)$ which is described by the semi-local classification Theorem 5.3. The global structure of shock waves in case $n = 1$ for $H = H(p)$ was studied in [24] by using classical techniques. Here, we only discuss how to construct the viscosity solution from the geometric solution in case $n = 1$, $H = H(t, x, p)$ using the obtained classification list. These constructions as well as the constructions in case $n \geq 1$ for H convex will be presented in [21]. A shock is a piece-wise smooth curve χ in the (x, t)-plane parametrized by t across which $\partial y / \partial x$ is discontinuous. The speed of the shock wave is given in terms of the Rankine-Hugoniot condition

$$\frac{d\chi}{dt} = \frac{H(t, x, y_x(\chi(t)+, t)) - H(t, x, y_x(\chi(t)-, t))}{y_x(\chi(t)+, t) - y_x(\chi(t)-, t)}$$

and the viscosity solution satisfies the *viscosity criterion* ([8, Theorem I.10], [24, Appendix]) across the shock curve. This is the analogue of the *shock-entropy condition* for the distributional solution of conservation laws (see [9], [27]).

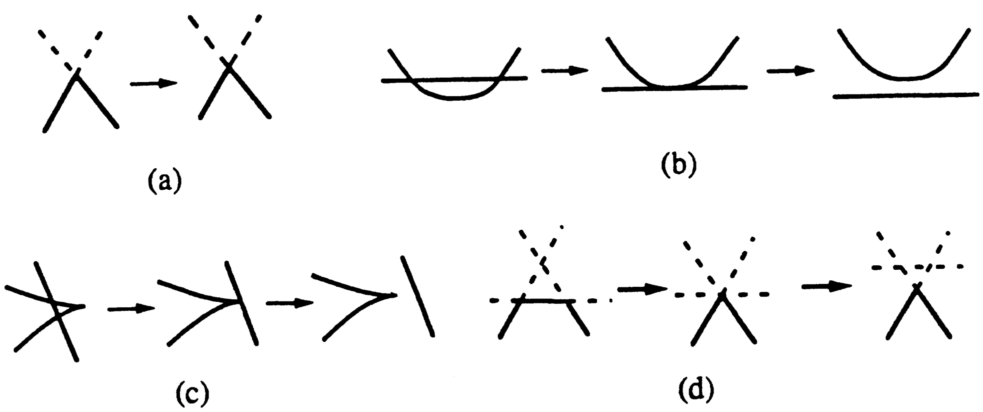

FIGURE 3

Figure 3 depicts the bifurcations of the graph for $r = 2, 3$ that involve the interaction of more than two branches. In case ${}^0({}^0A_1\, {}^0A_1)$ (Figure 3a) the intersection of the two branches corresponds to the shock point and the graph of the viscosity solution is either the minimum or the maximum value of the two branches depending on whether $y_x(\chi(t)\pm, t)$ are in a neighborhood where H is convex or concave. The corresponding shock is called *genuine* shock and it is defined by two incoming waves. See [24], cf. [9]. In Figure 3a the solution that corresponds to the minimum value is shown by a full line. Figure 3a excludes the possibility that a singularity ${}^1A_3$ might appear in one of the branches. This implies that a shock does not split into two forward shocks.

In case of ${}^1({}^0A_1\, {}^0A_1)$ (Figure 3b) the intersection points do not correspond to shock points. E.g., when $H = H(p)$ this would imply that the two incoming waves defining the shock will have the same speed at the separation time and the shock speed will be zero. The shock curve will degenerate to a line restricted between the two incoming waves.

Case ${}^1A_2\,{}^0A_1$ (Figure 3c) is more complicated as a point of intersection corresponds to a shock point until the time the two branches separate. The graph of the viscosity solution until the separation time is either the minimum or the maximum of the two branches. After the two branches separate we continue the viscosity solution by connecting the two branches by a *rarefaction-wave*-type solution. The time the two branches separate the genuine shock turns into a *contact discontinuity* (see [24], cf. [9]). The contact discontinuity is defined from one side by an incoming wave while from the other side it emits outgoing rarefaction waves.

Case ${}^0A_1\,{}^0A_1\,{}^0A_1$ describes the interaction of two shocks. The viscosity solution is either the minimum or the maximum of the three branches. The graph of the viscosity solution in case of a minimum is shown in Figure 3d by a full line. The two shocks that the viscosity solution exhibits meet at one point and they continue as a single forward shock.

REFERENCES

1. V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps*, Birkhauser, 1986.
2. V. I. Arnol'd, Monographie de l' Enseignement. Math **34** (1989).
3. V. I. Arnol'd, *Singularities of Caustics and Wave fronts, Mathematics and its Applications* (Soviet Series), vol. 62, Kluwer Academic Publisher, 1990.
4. G. Barles, H. M. Soner and P.E. Souganidis, *Front Propagation And Phase Field Theory*, Center for Nonlinear Analysis, Research Report No. 92-NA-020, Department of Mathematics, Carnegie Mellon University, June 1992.
5. P. Bernhard, *Singular surfaces in differential games, an introduction;* in Differential Games and Applications, Lecture Notes in Control and Information Sciences (P. Hagedorn et al., eds.), vol. 3, Springer Verlag, 1977, pp. 1–33.
6. J. W. Cahn, J. E. Taylor and C. A. Handwerker, *Evolving Crystal Forms: Frank's Characteristics Revisited, Sir Charles Frank, OBE, FRS, An eightieth birthday tribute,* (1991), Adam Hilger, New York.
7. M. G. Crandall, H. Ishii and P.-Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. **27** (1992), 1–67.
8. G. Crandall and P. -Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **277** (1983), 1–42.
9. C. M. Dafermos, *Regularity and large time behavior of solutions of a conservation law without convexity*, Proc. Royal Soc. Edinb. **99A** (1985), 201–239.
10. J. Damon, *The unfolding and determinacy theorems for subgroups of $\mathcal{A}$ and $\mathcal{K}$*, Memoirs of Amer. Math. Soc. **50-306** (1984).
11. J. J. Duistermaat, *Oscillatory integrals, Lagrange immersions and Unfolding of singularities*, Commu. pure and applied Math. **27** (1974), 207–281.
12. L. C. Evans and P. E. Souganides, *Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations*, Indiana Univ. Math. J. **33** (1984), 773–797.
13. W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, 1993.
14. A. R. Forsyth, *Theory of differential equations, Part III partial differential equations*, Cambridge Univ. Press, London, 1906.
15. J. Guckenheimer, *Solving a single conservation law*, Lecture notes in Mathematics 468, Springer Verlag, New York, 1975, pp. 108–134.

16. R. Isaacs, *Differential Games*, John Wiley, New York, 1965.
17. S. Izumiya, *Generic bifurcations of varieties*, Manuscripta Math. **46** (1984), 137–164.
18. S. Izumiya, *Geometric singularities for Hamilton-Jacobi equation*, to appear in Advanced Studies in Pure Math.
19. S. Izumiya, *Perestroikas of optical wave fronts and graphlike Legendrian unfoldings*, to appear in J. of Differential Geometry.
20. S. Izumiya, *The theory of Legendrian unfoldings and first order differential equations*, to appear in Proc. Royal Soc. Edinburgh.
21. S. Izumiya and G. T. Kossioris, *Geometric singularities for solutions of Hamilton-Jacobi equations and construction of shock waves I.*, in preparation.
22. S. Izumiya and G. T. Kossioris, *Geometric singularities for solutions of single conservation laws*, in preparation.
23. S. Izumiya and G. T. Kossioris, *Geometric singularities for solutions of Hamilton-Jacobi equations of geometric type*, in preparation.
24. G. T. Kossioris, *Propagation of singularities for viscosity solutions of Hamilton-Jacobi equations in one space variable*, to appear in Comm. P.D.E.
25. G. T. Kossioris, *Formation of singularities for viscosity solutions of Hamilton-Jacobi equations in higher dimensions*, preprint.
26. V. V. Lychagin, *Local classification of non-linear first order partial differential equations*, Russian Math. Surveys **30** (1975), 105–175.
27. S. Nakane, *Formation of singularities for Hamilton-Jacobi equations in several space variables*, J. Math. Soc. Japan **43** (1991), 89–100.
28. S. Nakane, *Formation of shocks for a single conservation law*, SIAM J. Math. Anal. **19** (1988), 1391–1408.
29. T. Oshima, *Singularities in contact geometry and degenerate pseudo-differential equations*, J. Fac. Sc. Univ. Tokyo **21** (1974), 43–83.
30. H. Rund, *The Hamilton-Jacobi theory in the calculus of variations*, D. Van Nostrand, London, 1966.
31. J. A. Sethian, *An analysis of flame propagation*, Ph.D Thesis, Lawrence Berkeley Laboratory, University of California Berkeley, California, June 1982; CPAM Rep. 79.
32. M. Tsuji, *Solution globale et propagation des singularites pour l'equation de Hamilton-Jacobi*, C. R. Acad. Sc. Paris **289** (1979), 397–400.
33. M. Tsuji, *Formation of singularities for Hamilton-Jacobi equation II*, J. Math. Kyoto Univ. **26** (1986), 299–308.
34. M. Tsuji, *Singularities for Monge-Ampère equations*, preprint, 1991.
35. C. T. C. Wall, *Finite determinacy of smooth map germs*, Bull. London Math. Soc. **14** (1981), 481–539.
36. V. M. Zakalyukin, *Reconstructions of fronts and caustics depending on a parameter and versality of mappings*, Itogi Nauki, Contemporary Problems in Mathematics **22** (1983), 53–93.

Center for Nonlinear Analysis
Report Series • Complete List

Nonlinear Analysis Series

No.

91-NA-001 [ ] Lions, P.L., **Jacobians and Hardy spaces**, June 1991

91-NA-002 [ ] Giga, Y. and Soto, M.-H., **Generalized interface evolution with the Neumann boundary condition**, July 1991

91-NA-003 [ ] Soner, H.M. and Souganidis, P.E., **Uniqueness and singularities of cylindrically symmetric surfaces moving by mean curvature**, July 1991

91-NA-004 [ ] Coleman, B.D., Marcus, M. and Mizel, V.J., **On the Thermodynamics of periodic phases**, August 1991

91-NA-005 [ ] Gurtin, M.E. and Podio-Guidugli, P., **On the formulation of mechanical balance laws for structured continua**, August 1991

91-NA-006 [ ] Gurtin, M.E. and Voorhees, P., **Two-phase continuum mechanics with mass transport and stress**, August 1991

91-NA-007 [ ] Fried, E., **Non-monotonic transformation kinetics and the morphological stability of phase boundaries in thermoelastic materials**, September 1991

91-NA-008 [ ] Gurtin, M.E., **Evolving phase boundaries in deformable continua**, September 1991

91-NA-009 [ ] Di Carlo, A., Gurtin, M.E., and Podio-Guidugli, P., **A regularized equation for anisotropic motion-by-curvature**, September 1991

91-NA-010 [ ] Kinderlehrer, D. and Ou, B., **Second variation of liquid crystal energy at $x/|x|$**, August 1991

91-NA-011 [ ] Baughman, L.A. and Walkington, N., **Co-volume methods for degenerate parabolic problems**, August 1991

91-NA-012 [ ] James, R.D. and Kinderlehrer, D., **Frustration and microstructure: An example in magnetostriction**, November 1991

91-NA-013 [ ] Angenent, S.B. and Gurtin, M.E., **Anisotropic motion of a phase interface**, November 1991

92-NA-001 [ ] Nicolaides, R.A. and Walkington, N.J., **Computation of microstructure utilizing Young measure representations,** January 1992

92-NA-002 [ ] Tartar, L., **On mathematical tools for studying partial differential equations of continuum physics: H-measures and Young measures**, January 1992

92-NA-003 [ ] Bronsard, L. and Hilhorst, D., **On the slow dynamics for the Cahn-Hilliard equation in one space dimension**, February 1992

92-NA-004 [ ] Gurtin, M.E., **Thermodynamics and the supercritical Stefan equations with nucleations**, March 1992

92-NA-005 [ ] Antonic, N., **Memory effects in homogenisation linear second order equation**, February 1992

92-NA-006 [ ] Gurtin, M.E. and Voorhees, P.W., **The continuum mechanics of coherent two-phase elastic solids with mass transport,** March 1992

92-NA-007 [ ] Kinderlehrer, D. and Pedregal, P., **Remarks about gradient Young measures generated by sequences in Sobolev spaces**, March 1992

92-NA-008 [ ] **Workshop on Shear Bands**, March 23-25, 1992 (Abstracts), March 1992

92-NA-009 [ ] Armstrong, R. W., **Microstructural/Dislocation Mechanics Aspects of Shear Banding in Polycrystals,** March 1992

92-NA-010 [ ] Soner, H. M. and Souganidis, P. E., **Singularities and Uniqueness of Cylindrically Symmetric Surfaces Moving by Mean Curvature**, April 1992

92-NA-011 [ ] Owen, David R., Schaeffer, Jack, andWang, Keming, **A Gronwall Inequality for Weakly Lipschitzian Mappings**, April 1992

92-NA-012 [ ] Alama, Stanley and Li, **Yan Yan, On "Multibump" Bound States for Certain Semilinear Elliptic Equations,** April 1992

92-NA-013 [ ] Olmstead, W. E., Nemat-Nasser, S., and Li, L., **Shear Bands as Discontinuities**, April 1992

92-NA-014 [ ] Antonic, N., **H-Measures Applied to Symmetric Systems**, April 1992

92-NA-015 [ ] Barroso, Ana Cristina and Fonseca, Irene, **Anisotropic Singular Perturbations - The Vectorial Case**, April 1992

92-NA-016 [ ] Pedregal, Pablo, **Jensen's Inequality in the Calculus of Variations**, May 1992

92-NA-017 [ ] Fonseca, Irene and Muller, Stefan, **Relaxation of Quasiconvex Functionals in BV($\Omega,\Re^p$) for Integrands f($x,u,\nabla u$)**, May 1992

92-NA-018 [ ] Alama, Stanley and Tarantello, Gabriella, **On Semilinear Elliptic Equations with Indefinite Nonlinearities**, May 1992

92-NA-019 [ ] Owen, David R., **Deformations and Stresses With and Without Microslip**, June 1992

92-NA-020 [ ] Barles, G., Soner, H. M., Souganidis, P. E., **Front Propagation and Phase Field Theory**, June 1992

92-NA-021 [ ] Bruno, Oscar P. and Reitich, Fernando, **Approximation of Analytic Functions: A Method of Enhanced Convergence**, July 1992

92-NA-022 [ ] Bronsard, Lia and Reitich, Fernando, **On Three-Phase Boundary Motion and the Singular Limit of a Vector-Valued Ginzburg-Landau Equation**, July 1992

92-NA-023 [ ] Cannarsa, Piermarco, Gozzi, Fausto and Soner, H.M., **A Dynamic Programming Approach to Nonlinear Boundary Control Problems of Parabolic Type**, July 1992

92-NA-024 [ ] Fried, Eliot and Gurtin, Morton, **Continuum Phase Transitions With An Order Parameter; Accretion and Heat Conduction**, August 1992

92-NA-025 [ ] Swart, Pieter J. and Homes, Philip J., **Energy Minimization and the Formation of Microstructure in Dynamic Anti-Plane Shear**, August 1992

92-NA-026 [ ] Ambrosio, I., Cannarsa, P. and Soner, H.M., **On the Propagation of Singularities of Semi-Convex Functions**, August 1992

92-NA-027 [ ] Nicolaides, R.A. and Walkington, Noel J., **Strong Convergence of Numerical Solutions to Degenerate Varational Problems**, August 1992

92-NA-028 [ ] Tarantello, Gabriella, **Multiplicity Results for an Inhomogenous Neumann Problem with Critical Exponent**, August 1992

92-NA-029 [ ] Noll, Walter, **The Geometry of Contact, Separation, and Reformation of Continous Bodies**, August 1992

92-NA-030 [ ] Brandon, Deborah and Rogers, Robert C., **Nonlocal Superconductivity**, July 1992

92-NA-031 [ ] Yang, Yisong, **An Equivalence Theorem for String Solutions of the Einstein-Matter-Gauge Equations**, September 1992

92-NA-032 [ ] Spruck, Joel and Yang,Yisong, **Cosmic String Solutions of the Einstein-Matter-Gauge Equations,** September 1992

92-NA-033 [ ] **Workshop on Computational Methods in Materials Science (Abstracts)**, September 16-18, 1992.

92-NA-034	[ ]	Leo, Perry H. and Herng-Jeng Jou, **Shape evolution of an initially circular precipitate growing by diffusion in an applied stress field,** October 1992
92-NA-035	[ ]	Gangbo, Wilfrid, **On the weak lower semicontinuity of energies with polyconvex integrands,** October 1992
92-NA-036	[ ]	Katsoulakis, Markos, Kossioris, Georgios T. and Retich, Fernando, **Generalized motion by mean curvature with Neumann conditions and the Allen-Cahn model for phase transitions,** October 1992
92-NA-037	[ ]	Kinderlehrer, David, **Some methods of analysis in the study of microstructure,** October 1992
92-NA-038	[ ]	Yang, Yisong, **Self duality of the Gauge Field equations and the Cosmological Constant,** November 1992
92-NA-039	[ ]	Brandon, Deborah and Rogers, Robert, **Constitutive Laws for Pseudo-Elastic Materials,** November 1992
92-NA-040	[ ]	Leo, P. H., Shield, T. W., and Bruno, O. P., **Transient Heat Transfer Effects on the Pseudoelastic Behavior of Shape-Memory Wires,** November 1992
92-NA-041	[ ]	Gurtin, Morton E., **The Dynamics of Solid-Solid Phase Transitions 1. Coherent Interfaces,** November 1992
92-NA-042	[ ]	Gurtin, Morton E., Soner, H. M., and Souganidis, **P.** E., **Anisotropic Motion of an Interface Relaxed by the Formation of Infinitesimal Wrinkles,** December 1992
92-NA-043	[ ]	Bruno, Oscar P., and Fernando Reitich, **Numerical Solution of Diffraction Problems: A Method of Variation of Boundaries II. Dielectric gratings, Padé Approximants and Singularities,** December 1992
93-NA-001	[1	Mizel, Victor **J., On Distribution Functions of the Derivatives of Weakly Differentiable Mappings,** January 1993
93-NA-002	[1	Kinderlehrer, David, Ou, Biao and Walkington, Noel, **The Elementary Defects of the Oseen-Frank Energy for a Liquid Crystal,** January 1993
93-NA-003	[ ]	Bruno, Oscar P., Reitich, Fernando, **Numerical Solutions of Diffraction Problems: A Method of Variation of Boundaries m. Doubly Periodic Gratings,** January 1993
93-NA-004	[ ]	James, Richard and Kinderlehrer, David, **Theory of Magnetostriction with Applications to** $Tb_xDyi._xFe2$, February 1993
93-NA-005	[1	Del Piero, Gianpietro and Owen, David R., **Structured Deformations of Continua,** February 1993

93-NA-006 [ ] Cheng, Chih-Wen and Mizel, Victor J., **On the Lavrentiev Phenomenon for Autonomous Second Order Integrands**, February 1993

93-NA-007 [ ] Ma, Ling, **Computation of Magnetostrictive Materials**, February 1993

93-NA-008 [ ] James, Richard and Kinderlehrer, David, **Mathematical Approaches to the Study of Smart Materials**, February 1993

93-NA-009 [ ] Kinderlehrer, David, Nicolaides, Roy, and Wang, Han, **Spurious Oscillations in Computing Microstructures,** February 1993

93-NA-010 [ ] Ma, Ling and Walkington, Noel, **On Algorithms for Non-Convex Optimization,** February 1993

93-NA-011 [ ] Fonseca, Irene, Kinderlehrer, David, and Pedregal, Pablo, **Relaxation in BV x L^{∞} of Functionals Depending on Strain and Composition,** February 1993

93-NA-012 [ ] Izumiya, Shyuichi and Kossioris, Georgios T., **Semi-Local Classification of Geometric Singularities for Hamilton-Jacobi Equations**, March 1993

Stochastic Analysis Series

91-SA-001 [ ] Soner, H.M., **Singular perturbations in manufacturing**, November 1991

91-SA-002 [ ] Bridge, D.S. and Shreve, S.E., **Multi-dimensional finite-fuel singular stochastic control**, November 1991

92-SA-001 [ ] Shreve, S. E. and Soner, H. M., **Optimal Investment and Consumption with Transaction Costs**, September 1992