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Relaxation in $BV \times L^{\infty}$ of Functionals Depending on Strain and Composition

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Relaxation in $BV \times L^\infty$ of functionals depending on strain and composition

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Abstract. We show that if $\psi(A,m)$ is a quasiconvex function with linear growth then the relaxed functional in $BV(\Omega,\mathbb{R}^n)\times L^\infty(\Omega,\mathbb{R}^d)$ of the energy

$$\int_\Omega \psi(\nabla u,m)\,dx$$

with respect to the $L^1 \times L^\infty$(weak*) topology has an integral representation of the form

$$\mathcal{F}(u,m) = \int_\Omega \psi(\nabla u,m)dx + \int_{\Sigma(u)} \psi^\infty((u^- - u^+)\otimes\nu)\, dH_{N-1}(x) + \int_\Omega \psi^\infty(dC(u))$$

where $Du = \nabla u\, dx + (u^+ - u^-)\otimes\nu\, dH_{N-1}\lfloor\Sigma(u) + C(u)$. The proof relies on a blow up argument and on a recent result obtained by Alberti showing that the Cantor part $C(u)$ is rank-one valued.

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1. Introduction

In this paper we obtain an integral representation in $BV(\Omega,\mathbb{R}^n) \times L^\infty(\Omega,\mathbb{R}^d)$ for the relaxation $\mathcal{F}(u,m)$ of an energy functional

$$E(u,m) = \int_\Omega \psi(\nabla u(x),m(x))\,dx$$

with respect to the $L^1 \times L^\infty$(weak*) topology.

One motivation for this question is the analysis of coherent thermochemical equilibria among multiphase and multicomponent solids (see [AJ], [JA], Larché and Cahn [LC1,2]). This is explained in detail in [FKP]. For example, in the case of two species in equilibrium in a matrix and a precipitate, the pertinent functional has the form

$$I(u,c) = \int_\Omega \psi(\nabla u, c)\, dx$$

subject to the constraint

$$\int_\Omega c\, dx = \theta\, |\Omega|,$$

where u denotes the deformation of the material and c the concentration of one of the two species.

Kohn [K] obtained a formula for the relaxation of I in the case where composition is uniform, i. e. $\psi(F,c) =: \psi^*(F)$, and for two linearly elastic phases with identical elastic moduli. In more general situations, the composition is not uniform (see [LC2]) and so we must address the problem of finding the effective energy in the case where it depends on the chemical composition c. When linear growth in the deformation is admitted, functionals of the sort considered here then arise.

In the scalar case $n = 1$, Ioffe [I] studied the lower semicontinuity of E in $W^{1,1}$(weak) $\times L^1_{loc}$ (see also [Am] for a new proof of this result). Here, generalizing E to the case where c may take vector values m and assuming that $N, n > 1$, we want to obtain an integral representation for the relaxed functional F in $BV(\Omega,\mathbb{R}^n) \times L^\infty(\Omega,\mathbb{R}^d)$ of the energy E, where

$$F(u,m) := \inf_{\{u_k\},\{m_k\}} \Big\{ \liminf_{k\to\infty} \int_\Omega \psi(\nabla u_k, m_k)\, dx : (u_k,m_k) \in W^{1,1} \times L^\infty,$$
$$u_k \to u \text{ in } L^1 \text{ and } m_k \overset{*}{\rightharpoonup} m \text{ in } L^\infty \Big\}.$$

Throughout this work we will assume that ψ is jointly quasiconvex in ∇u and convex in m, namely

(H1) $\psi: \mathbb{M} \times \mathbb{R}^d \to [0,+\infty)$ is a Borel measurable function such that

$$\psi(A,\lambda) \le \frac{1}{|\Omega|} \int_\Omega \psi(A + \nabla\zeta, \lambda + m)\, dx$$

for all $(A,\lambda) \in \mathbb{M} \times \mathbb{R}^d$ and $(\zeta,m) \in W_0^{1,\infty}(\Omega,\mathbb{R}^n) \times L^\infty(\Omega,\mathbb{R}^d)$ with $\int_\Omega m\, dx = 0$.

In addition, ψ grows at most linearly,

(H2) $c_1|A| - c_2 \leq \psi(A,\lambda) \leq g(\lambda)(1+|A|)$ where $c_1, c_2 > 0$ and $g \in L^{\infty}_{loc}(\mathbb{R}^d)$.

So, for example, under these hypotheses the functional determined by ψ is weakly sequentially lower semicontinuous in $W^{1,\infty} \times L^{\infty}$, cf. [FKP]. Indeed, relaxation in $W^{1,p} \times L^q$ under the hypotheses (H1) was obtained in [FKP]. Our objective here is to determined the relaxed functional when the admissible functions come from BV $\times L^{\infty}$.

Although most of the results and proofs in this work are inspired by those in [FM1,2], we note that the relaxations of $\psi(\nabla u,m)$ and $\psi(\nabla u,u)$ present several different features. In particular, in the support of the singular part of Du, the function m, being only Lebesgue measurable and not necessarily related to u in any way, may not be well defined. We recall that the distributional derivative Du is represented by

$$Du = \nabla u\, dx + (u^+ - u^-)\otimes\nu\, dH_{N-1} \llcorner \Sigma(u) + C(u).$$

Here ∇u is the density of the absolutely continuous part of Du with respect to the Lebesgue measure dx, H_{N-1} is the N-1 dimensional Hausdorff measure, $(u^+ - u^-)$ is the jump of u across the interface $\Sigma(u)$ with "generalized normal" ν and C(u) is the Cantor part of Du. For details we refer the reader to [EG], [Z].

We expect, as usual, that the integral representation of F will involve the integration of the recession function, (2.1) below, on $\Sigma(u) \cup \text{supp } C(u)$. However, if m is not well defined on this set what kind of representation are we to expect? This question is naturally solved by the convexity and growth assumptions imposed on ψ. Indeed, we will show on Lemma 2.2 that

$$\lambda \rightarrow \psi^{\infty}(A,\lambda) \text{ is constant}$$

whenever rank $A \leq 1$, and due to Alberti's [Al] result we know that

$$\text{rank } \frac{d(Du)}{d|D(u)|} \leq 1$$

on $\Sigma(u) \cup \text{supp } C(u)$. Denoting by $\psi^{\infty}(a\otimes b)$ the constant value of this function of λ, we will obtain (see (2.2) and (6.1))

$$F(u,m) = \int_{\Omega} \psi(\nabla u,m)\, dx + \int_{\Sigma(u)} \psi^{\infty}((u^- - u^+)\otimes\nu)\, dH_{N-1}(x) + \int_{\Omega} \psi^{\infty}(dC(u)) \tag{1.1}$$

where $(u,m) \in BV(\Omega, \mathbb{R}^n) \times L^{\infty}(\Omega, \mathbb{R}^d)$.

2. Preliminaries The recession function

We start by studying some properties of the recession function (see [FM2])

$$\psi^\infty(A,m) := \limsup_{t\to\infty} \frac{\psi(tA,m)}{t}. \tag{2.1}$$

Lemma 2.1.

a) $c_1|A| \le \psi^\infty(A,\lambda) \le g(\lambda)|A|$ *and* $\psi^\infty(A,\lambda)$ *is positively homogeneous of degree one in* λ;

b) ψ^∞ *satisfies the quasiconvexity/convexity condition* (H1).

Proof. a) is clear. To prove b) let $(A,\lambda) \in \mathbb{M} \times \mathbb{R}^d$, $(\varphi,m) \in W_0^{1,\infty}(\Omega,\mathbb{R}^n) \times L^\infty(\Omega,\mathbb{R}^d)$ with $\int_\Omega m\,dx = 0$, and let

$$\psi^\infty(A,\lambda) = \lim_{k\to\infty} \frac{\psi(t_kA,m)}{t_k} \quad \text{for some } t_k \to +\infty.$$

By (H1)

$$\frac{\psi(t_kA,m)}{t_k} \le \frac{1}{|\Omega|t_k}\int_\Omega \psi(t_kA + \nabla(t_k\varphi), \lambda + m)\,dx$$

$$= \frac{1}{|\Omega|t_k}\int_\Omega \psi(t_k(A+\nabla\varphi),\lambda + m)\,dx.$$

Defining

$$H(x) := g(|\lambda| + \|m\|_\infty)(1 + \|A + \nabla\varphi(x)\|),$$

we deduce that

$$\psi^\infty(A,\lambda) \le \limsup_{k\to\infty} \frac{1}{|\Omega|t_k}\int_\Omega \psi(t_k(A+\nabla\varphi),\lambda + m)\,dx$$

$$= \frac{1}{|\Omega|}\int_\Omega H(x)\,dx - \liminf_{k\to\infty} \frac{1}{|\Omega|}\int_\Omega [H - \frac{1}{t_k}\psi(t_k(A+\nabla\varphi),\lambda+m)]\,dx$$

which, by Fatou's Lemma, yields

$$\psi^\infty(A,\lambda) \le \frac{1}{|\Omega|}\int_\Omega \limsup_{k\to\infty} \frac{1}{t_k}\psi(t_k(A+\nabla\varphi),\lambda+m)dx$$

$$\leq \frac{1}{|D|}\int_D \psi^\infty(A+\nabla\varphi,\lambda+m)\,dx. \qquad \text{QED}$$

Lemma 2.2. *If* rank $A = 1$ *then the function* $\lambda \to \psi^\infty(A,\lambda)$ *is constant.*

We divide the proof of this result into two lemmas.

Lemma 2.3. *Fix* $\nu \in S^{N-1}$. *Then the function* $f: \mathbb{R}^n \times \mathbb{R}^d \to [0,+\infty)$ *defined by*

$$f(a,\lambda) := \psi^\infty(a\otimes\nu,\lambda)$$

is convex.

Proof. Let $(a,\lambda) = \theta(a_1,\lambda_1) + (1-\theta)(a_2,\lambda_2)$ for some $\theta \in (0,1)$. Let Q be a unit cube centered at the origin with two faces perpendicular to ν and let $\{\eta_j\}$ be a family of cut-off functions such that

i) $\eta_j = 1$ in $Q_j := \{x \in Q \mid \text{dist}(x,\partial Q) \geq 1/j\}$;
ii) $\eta_j = 0$ on ∂Q;
iii) $\|\nabla\eta_j\|_\infty \leq Cj$.

Define

$$\lambda_k(x) := \lambda_2 + \chi(kx.\nu)(\lambda_1-\lambda_2) - \lambda,$$

$$\varphi_k(x) := (a_2-a)\otimes\nu\cdot x + \frac{1}{k}\int_0^{kx\cdot\nu}\chi(t)dt\cdot(a_1-a_2),$$

$$\varphi_k^j(x) := \varphi_k(x)\eta_j(x)$$

where χ is the characteristic function of the interval $(0,\theta)$ extended to $\mathbb{R}$ periodically with period one. Notice that

1. $\lambda_k \overset{*}{\rightharpoonup} 0$ in $L^\infty(Q)$;
2. $\int_Q \lambda_k(x)\,dx = 0$;
3. $\nabla\varphi_k(x) = (a_2-a)\otimes\nu + \chi(kx.\nu)(a_1-a_2)\otimes\nu \overset{*}{\rightharpoonup} 0$ in $L^\infty(Q)$ and $\int_Q \varphi_k(x)dx \to 0$;
4. $\varphi_k^j \in W_0^{1,\infty}(Q,\mathbb{R}^n)$;
5. $\nabla\varphi_k^j = \eta_j\nabla\varphi_k + \varphi_k\otimes\nabla\eta_j$.

By Lemma 2.1 b) the function ψ^∞ satisfies the convexity condition (H1) and so

$$f(a,\lambda) = \psi^\infty(a\otimes v,\lambda) \leq \int_Q \psi^\infty(a\otimes v + \nabla\varphi_k^j, \lambda+\lambda_k)\,dx$$

$$\leq \int_Q \psi^\infty(a\otimes v+\nabla\varphi_k,\lambda+\lambda_k)\,dx + \int_{Q\setminus Q_j} \psi^\infty(a\otimes v+\nabla\varphi_k^j,\lambda+\lambda_k)\,dx$$

$$- \int_{Q\setminus Q_j} \psi^\infty(a\otimes v+\nabla\varphi_k, \lambda+\lambda_k)\,dx$$

$$=: I_k + II_{k,j} + III_{k,j}.$$

As $\{\|\lambda_k\|_\infty + \|\varphi_k\|_{1,\infty}\}$ is bounded, by Lemma 2.1 a) we have

$$\sup_k | III_{k,j} | \leq C\,\mathrm{meas}(Q\setminus Q_j) \to 0.$$

Fix j . From 3) it follows that $\varphi_k \to 0$ in L^∞ and so choose k(j) large enough so that

$$\| \varphi_k \|_\infty \leq \frac{1}{j^2 | Q\setminus Q_j |}$$

for $k \geq k(j)$. Then, by Lemma 2.1 a)

$$| II_{k(j),j} | \leq C |Q\setminus Q_j| + j^{-1} \text{ and } \|\varphi_{k(j)}\|_\infty | Q\setminus Q_j | \to 0 \text{ as } j \to +\infty.$$

The convexity of f follows from the fact that

$$\lim_{k\to\infty} I_k = \theta\psi^\infty(a_1\otimes v,\lambda_1) + (1-\theta)\psi^\infty(a_2\otimes v,\lambda_2)$$

$$= \theta\, f(a_1,\lambda_1) + (1-\theta)\, f(a_2,\lambda_2). \qquad \text{QED}$$

Lemma 2.4. *Let* $\xi: \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$, $\xi = \xi(a,\lambda)$, *be a convex function such that* $\xi(a_0,.)$ *is constant for some* $a_0 \in \mathbb{R}^n$. *Then the function* ξ *is independent of* λ.

Proof. Suppose that $m_0 = \xi(a_0,\lambda)$ for all λ. Given (a,λ') we have

$$m_0 = \xi(a_0,\lambda) \geq \xi(a,\lambda') + \alpha(a,\lambda')\cdot(a_0 - a) + \beta(a,\lambda')\cdot(\lambda - \lambda')$$

where $(\alpha(a,\lambda'), \beta(a,\lambda'))$ belongs to the subdifferential of ξ at (a,λ'). Letting $| \lambda | \to +\infty$ we conclude that $\beta(a,\lambda') = 0$ and so we may deduce that

$$\xi(a,\lambda) \geq \xi(a,\lambda')$$

for all λ, λ' and thus they must be equal. QED

Proof of Lemma 2.2. As $\psi^{\infty}(.,\lambda)$ is positively homogeneous of degree one,

$$\psi^{\infty}(0,\lambda) = 0 \quad \text{for all } \lambda.$$

The result now follows from Lemmas 2.3 and 2.4.

The proof of (1.1) is divided into two parts. In the first part, carried out on Sections 3, 4 and 5, we show that the representation in (1.1) is a lower bound for F i. e. if $u_k \in W^{1,1}(\Omega;\mathbb{R}^n)$ are such that $u_k \to u$ in $L^1(\Omega;\mathbb{R}^n)$, with $u \in BV(\Omega,\mathbb{R}^n)$, and if $m_k \overset{*}{\rightharpoonup} m$ in $L^{\infty}(\Omega,\mathbb{R}^d)$ then

$$\liminf_{k\to\infty} \int_{\Omega} \psi(\nabla u_k, m_k)\,dx \;\geq\; \int_{\Omega} \psi(\nabla u, m)\,dx + \int_{\Sigma(u)} \psi^{\infty}((u^- - u^+)\otimes\nu)\,dH_{N-1}(x) + \int_{C(u)} \psi^{\infty}(dC(u)). \tag{2.2}$$

Finally, in Section 6 we assert equality in (2.2) using the same reasoning as in [FM2] (see also Ambrosio, Mortola and Tortorelli [AMT]).

To prove (2.2) we use the blow up argument introduced in [FM1]. It is then reduced to verifying the pointwise inequalities (2.3), (2.4) and (2.5) below. Assume, without loss of generality, that

$$\liminf_{k\to\infty} \int_{\Omega} \psi(\nabla u_k, m_k)\,dx = \lim_{k\to\infty} \int_{\Omega} \psi(\nabla u_k, m_k)\,dx < +\infty$$

and $u_k \in C_0^{\infty}(\mathbb{R}^N;\mathbb{R}^n)$ (see Proposition 2.6 in [FM1] and also Acerbi and Fusco [AF]). As ψ is nonnegative there exists a subsequence, which for convenience of notation is still labelled $\{u_k, m_k\}$, and a nonnegative finite Radon measure μ such that

$$\psi(\nabla u_k, m_k) \overset{*}{\rightharpoonup} \mu .$$

Using the Radon-Nikodym Theorem, we can write μ as a sum of four mutually singular nonnegative measures

$$\mu = \mu_a\,dx + \zeta\,|u^+ - u^-|H_{N-1}\llcorner\Sigma(u) + \eta\,|C(u)| + \mu_s.$$

We claim that

$$\mu_a(x) \geq \psi(\nabla u(x), m(x)) \quad \text{for } dx \text{ a. e. } x \in \Omega, \tag{2.3}$$

$$\zeta(x) \geq \frac{\psi^\infty((u^-(x) - u^+(x))\otimes\nu(x))}{|u^+(x) - u^-(x)|} \quad \text{for } |u^+ - u^-| H_{N-1} \llcorner \Sigma(u) \text{ a.e. } x \in \Sigma(u) \quad (2.4)$$

and

$$\eta(x) \geq \psi^\infty(A(x)) \quad \text{for} \quad |C(u)| \text{ a. e. } x \in \Omega, \tag{2.5}$$

where (see [Al] and [ADM]) for $|C(u)|$ a. e. $x \in \Omega$ and open, convex neighborhood G of the origin,

$$A(x) := \lim_{\varepsilon\to 0} \frac{D(u)(x+\varepsilon G)}{|D(u)|(x+\varepsilon G)} = \lim_{\varepsilon\to 0} \frac{C(u)(x+\varepsilon G)}{|C(u)|(x+\varepsilon G)} = a(x)\otimes\nu(x).$$

Then, considering an increasing sequence of smooth cut-off functions η_j, with $0 \leq \eta_j \leq 1$ and $\sup_j \eta_j(x) = 1$ in Ω, we conclude that

$$\lim_{k \to \infty} \int_\Omega \psi(\nabla u_k, m_k)\, dx \geq \liminf_{k \to \infty} \int_\Omega \eta_j\, \psi(\nabla u_k, m_k)\, dx$$

$$= \int_\Omega \eta_j\, d\mu(x)$$

$$\geq \int_\Omega \eta_j\, \mu_a(x)\, dx + \int_{\Sigma(u)} \eta_j\, \zeta\, |u^+ - u^-|\, dH_{N-1}(x)) + \int_\Omega \eta_j\, \eta\, d|C(u)|(x)$$

$$\geq \int_\Omega \eta_j\, \psi(\nabla u, m)\, dx + \int_{\Sigma(u)} \eta_j\, \psi^\infty((u^- - u^+)\otimes\nu)\, dH_{N-1}(x) + \int_\Omega \eta_j\, \psi^\infty(dC(u)).$$

Letting $j \to +\infty$, (2.2) follows from the Monotone Convergence Theorem.

3. The density of the absolutely continuous part

Using the technique developed in [FM1] we prove (2.3), namely

$$\mu_a(x_0) \geq \psi(\nabla u(x_0), m(x_0)) \quad \text{for } dx \text{ a.e. } x_0 \in \Omega.$$

By the Besicovitch Differentiation Theorem (see [EG]) the limit

$$\mu_a(x_0) := \lim_{\varepsilon\to 0} \frac{\mu(B(x_0,\varepsilon))}{|\, B(x_0,\varepsilon)\, |} \quad dx \text{ a.e., } x_0 \in \Omega,$$

exists and is finite and by standard results of the theory of BV functions

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left\{ \frac{1}{|B(x_0,\varepsilon)|} \int_{B(x_0,\varepsilon)} |u(y) - u(x_0) - \nabla u(x_0)\cdot(x_0 - y)|^{N/(N-1)}\, dy \right\}^{(N-1)/N} = 0. \quad (3.1)$$

Here, and in what follows, we denote the N-dimensional measure of a set E by $|E|$. Choosing one such x_0 which is also a Lebesgue point for m, define the homogeneous function

$$u_0(x) := \nabla u(x_0)\, x.$$

We abbreviate $B = B(0, 1)$, we consider a subdomain $B' \subset\subset B$. Let $\varphi \in C_0(B)$ be a cut-off function such that $0 \le \varphi \le 1$ and $\varphi(x) = 1$ if $x \in B'$. Then

$$\begin{aligned}
\mu_a(x_0) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N |B|} \mu(B(x_0,\varepsilon)) \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N |B|} \int_{B(x_0,\varepsilon)} \varphi\left(\frac{x - x_0}{\varepsilon}\right) d\mu(x) \\
&= \limsup_{\varepsilon\to 0} \lim_{k\to\infty} \frac{1}{\varepsilon^N |B|} \int_{B(x_0,\varepsilon)} \varphi\left(\frac{x - x_0}{\varepsilon}\right) \psi(\nabla u_k(x), m_k(x))\, dx \\
&= \limsup_{\varepsilon\to 0} \lim_{k\to\infty} \frac{1}{|B|} \int_B \varphi(x)\, \psi(\nabla u_k(x_0+\varepsilon x), m_k(x_0+\varepsilon x))\, dx \\
&\ge \limsup_{\varepsilon\to 0} \limsup_{k\to\infty} \frac{1}{|B|} \int_{B'} \psi(\nabla w_{k,\varepsilon}(x), m_k(x_0+\varepsilon x))\, dx \quad (3.2)
\end{aligned}$$

where

$$w_{k,\varepsilon}(x) := \frac{u_k(x_0+\varepsilon x) - u(x_0)}{\varepsilon}.$$

By (3.1) and by Hölder's inequality

$$\lim_{\varepsilon\to 0} \lim_{k\to\infty} \|w_{k,\varepsilon} - u_0\|_{L^1(B)} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N+1}} \int_{B(x_0,\varepsilon)} |u(x) - u(x_0) - \nabla u(x_0)(x-x_0)|\, dx = 0,$$

and if $\{\varphi_j\}_{j=1}^{+\infty}$ is a countable set dense in $L^1(\Omega,\mathbb{R}^d)$, for fixed m

$$\lim_{\varepsilon\to 0} \lim_{k\to\infty} \left| \int_B (m_k(x_0 + \varepsilon x) - m(x_0))\varphi_j(x)\, dx \right| =$$

$$\lim_{\varepsilon\to 0} \left| \int_B (m(x_0 + \varepsilon x) - m(x_0))\varphi_j(x)\, dx \right| = 0.$$

Using a diagonalization procedure we will show that

$$\mu_a(x_0) \geq \limsup_{j\to\infty} \frac{1}{|B|} \int_{B'} \psi(\nabla v_j, \lambda_j)\, dx \quad \text{where} \tag{3.3}$$

$$v_j \to u_0 \text{ in } L^1(B;\mathbb{R}^n) \quad \text{and} \quad \lambda_j \overset{*}{\rightharpoonup} m(x_0) \text{ in } L^\infty(B;\mathbb{R}^d).$$

Indeed, assume that

$$\limsup_{\varepsilon\to 0} \limsup_{k\to\infty} \frac{1}{|B|} \int_{B'} \psi(\nabla w_{k,\varepsilon}(x), m_k(x_0+\varepsilon x))\, dx$$

$$= \lim_{\varepsilon_i\to 0} \limsup_{k\to\infty} \frac{1}{|B|} \int_{B'} \psi(\nabla w_{k,\varepsilon_i}(x), m_k(x_0+\varepsilon_i x))\, dx.$$

For $j = 1$ and for all i choose $k_i(1)$ so that for all $k \geq k_i(1)$ one has

$$\|w_{k,\varepsilon_i} - u_0\|_{L^1(B)} \leq \lim_{\kappa\to\infty} \|w_{\kappa,\varepsilon_i} - u_0\|_{L^1(B)} + 1/i$$

$$\left| \int_\Omega (m_k(x_0+\varepsilon_i x) - m(x_0))\cdot\varphi_1(x)\, dx \right| \leq$$

$$\lim_{\kappa\to\infty} \left| \int_\Omega (m_\kappa(x_0+\varepsilon_i x) - m(x_0))\cdot\varphi_1(x)\, dx \right| + 1/i$$

and

$$\frac{1}{|B|} \int_{B'} \psi(\nabla w_{k,\varepsilon_i}(x), m_k(x_0+\varepsilon_i x))\, dx$$

$$\leq \limsup_{\kappa\to\infty} \frac{1}{|B|} \int_{B'} \psi(\nabla w_{\kappa,\varepsilon_i}(x), m_\kappa(x_0+\varepsilon_i x))\, dx) + 1/i.$$

Recursively, for all $j \geq 2$ and for all i choose $k_i(j) > k_i(j-1)$ so that for all $k \geq k_i(j)$

$$\left| \int_\Omega (m_k(x_0+\varepsilon_i x) - m(x_0))\cdot\varphi_j(x)\, dx \right|$$

$$\leq \lim_{\kappa\to\infty} \left| \int_\Omega (m_\kappa(x_0+\varepsilon_i x) - m(x_0))\cdot\varphi_j(x)\, dx \right| + 1/i$$

$$= \left| \int_\Omega (m(x_0+\varepsilon_i x) - m(x_0))\cdot\varphi_j(x)\, dx \right| + 1/i$$

Now consider the diagonal subsequence $k_i(i)$ and define

$$\lambda_i(x) := m_{k_i(i)}(x_0 + \varepsilon_i x), \quad v_i(x) := w_{k_i(i),\varepsilon_i}(x).$$

Then

$$\| v_i - u_0 \|_{L^1(B)} \le \lim_{\kappa\to\infty} \| w_{\kappa,\varepsilon_i} - u_0 \|_{L^1(B)} + 1/i$$

and so

$$\lim_{i\to\infty} \| v_i - u_0 \|_{L^1(B)} = 0. \tag{3.4}$$

Also, since x_0 is a Lebesgue point of m,

$$\lambda_i \overset{*}{\rightharpoonup} m(x_0) \text{ in } L^\infty. \tag{3.5}$$

By (3.2) and as $k_i(i) \ge k_i(1)$,

$$\mu_a(x_0) \ge \limsup_{i\to\infty} \Big[\frac{1}{|B|} \int_{B'} \psi(\nabla v_i, \lambda_i)\, dx - 1/i \Big]$$

$$= \limsup_{i\to\infty} \frac{1}{|B|} \int_{B'} \psi(\nabla v_i, \lambda_i)\, dx$$

proving (3.3).

Next, using the "slicing method" we are going to modify $\{ \lambda_i \}$ and $\{ v_i \}$ near $\partial B'$ so that we can apply the convexity hypothesis (H1).

By (3.3) and the growth condition (H2) the L^1 norms of $\{| \nabla v_i |\}$ are uniformly bounded in B', i. e.

$$\sup \int_{B'} |\nabla v_i(x)|\, dx \le C.$$

Let $B_j = \{x \in B' : \mathrm{dist}(x,\partial B') < 1/j\}$ and divide B_2 into two annuli $S^1_{(2)}$ and $S^2_{(2)}$. It is clear that for each i there exists an annulus $S \in \{S^1_{(2)}, S^2_{(2)}\}$ so that

$$\int_S |\nabla v_i(x)|\, dx \le C/2$$

and as there are only two annuli and infinitely many indices, we conclude that one of the annuli, $S_2 = \{x \in B' \mid \alpha_2 < \mathrm{dist}(x,\partial B') < \beta_2\}$, satisfies

$$\int_{S_2} |\nabla v_{i_2}(x)|\, dx \le C/2$$

for a subsequence $\{i_2\}$. Let η_2 be a smooth cut-off function, $0 \leq \eta_2 \leq 1$, $\eta_2 = 0$ in the complement of $\{x\in B' \mid dist(x,\partial B') < \beta_2\}$, $\eta_2 = 1$ in $\{x\in B': dist(x,\partial B') < \alpha_2\}$ and $\|\nabla\eta_2\| = O(1/|S_2|)$. By (3.5)

$$\lim_{i_2 \to +\infty} \left| m(x_0) - \frac{1}{|B'|}\int_{B'} \eta_2 \lambda_{i_2}\, dx \right| = |m(x_0)| \left| 1 - \frac{1}{|B'|}\int_{B'} \eta_2\, dx \right|$$

and so, by (3.4) choose $i(2) \in \{i_2\}$ large enough so that

$$\frac{1}{|S_2|}\int_{S_2} | v_{i(2)} - u_0 |\, dx < \frac{1}{2} \quad \text{and}$$

$$\frac{\left| m(x_0) - \frac{1}{|B'|}\int_{B'} \eta_2 \lambda_{i(2)}\, dx \right|}{\left| 1 - \frac{1}{|B'|}\int_{B'} \eta_2\, dx \right|} \leq |m(x_0)| + 1.$$

Next, divide B_3 into three annuli $S^1_{(3)}, S^2_{(3)}, S^3_{(3)}$. For each i_2 there exists an annulus $S \in \{S^1_{(3)}, S^2_{(3)}, S^3_{(3)}\}$ so that

$$\int_S |\nabla v_{i_2}|\, dx \leq C/3$$

and as there are only three annuli and infinitely many indices i_2, we conclude that one of the annuli $S_3 = \{x\in B': \alpha_3 < dist(x,\partial B') < \beta_3\}$ satisfies

$$\int_{S_3} |\nabla v_{i_3}|\, dx \leq C/3$$

for a subsequence $\{i_3\}$ of $\{i_2\}$. Let η_3 be a smooth cut-off function, $0 \leq \eta_3 \leq 1$, $\eta_3 = 0$ in the complement of $\{x\in B': dist(x,\partial B') < \beta_3\}$, $\eta_3 = 1$ in $\{x\in B': dist(x,\partial B') < \alpha_3\}$ and $\| \nabla\eta_3 \| = O(1/|S_3|)$. By (3.5)

$$\lim_{i_3 \to +\infty} \left| m(x_0) - \frac{1}{|B'|}\int_{B'} \eta_3 \lambda_{i_3}\, dx \right| = |m(x_0)| \left| 1 - \frac{1}{|B'|}\int_{B'} \eta_3\, dx \right|$$

and so, by (3.4) choose $i(3) \in \{i_3\}$, $i(3) > i(2)$, large enough so that

$$\frac{1}{|S_3|}\int_{S_3} |v_{i(3)} - u_0|\,dx < \frac{1}{3} \quad \text{and}$$

$$\frac{\left| m(x_0) - \frac{1}{|B'|}\int_{B'} \eta_3\, \lambda_{i(3)}\,dx \right|}{\left| 1 - \frac{1}{|B'|}\int_{B'} \eta_3\,dx \right|} \le |m(x_0)| + 1.$$

Recursively, we construct a sequence i(j) such that

$$\int_{S_j} |\nabla v_{i(j)}|\,dx \le \frac{C}{j}, \quad \frac{1}{|S_j|}\int_{S_j} |v_{i(j)} - u_0|\,dx < \frac{1}{j}, \text{ and} \tag{3.6}$$

$$\frac{\left| m(x_0) - \frac{1}{|B'|}\int_{B'} \eta_j\, \lambda_{i(j)}\,dx \right|}{\left| 1 - \frac{1}{|B'|}\int_{B'} \eta_j\,dx \right|} \le |m(x_0)| + 1.$$

We set

$$\overline{\lambda}_j(x) := (1-\eta_j(x))\frac{m(x_0) - \frac{1}{|B'|}\int_{B'} \eta_j\, \lambda_{i(j)}\,dy}{1 - \frac{1}{|B'|}\int_{B'} \eta_j\,dy} + \eta_j(x)\, \lambda_{i(j)}(x)$$

$$\overline{v}_j(x) := (1-\eta_j(x))u_0(x) + \eta_j(x)v_{i(j)}(x).$$

Clearly

$$\int_{B'} \overline{\lambda}_j(x)\,dx = |B'|\,m(x_0), \quad \|\overline{\lambda}_j\|_\infty \le |m(x_0)| + 1 + M \quad \text{and} \quad \overline{v}_j\big|_{\partial B'} = u_0.$$

Thus, by (3.3), (H1) and (H2)

$$\mu_a(x_0) \ge \limsup_{i\to+\infty} \frac{1}{|B|}\int_{B'} \psi(\nabla v_i,\lambda_i)\,dx$$

$$\ge \limsup_{j\to+\infty} \frac{1}{|B|}\left[\int_{B'} \psi(\nabla \overline{v}_j,\overline{\lambda}_j)\,dx - \int_{S_j} \psi(\nabla \overline{v}_j,\overline{\lambda}_j)\,dx - \int_{B_j} \psi(\nabla \overline{v}_j,\overline{\lambda}_j)\,dx \right]$$

$$\ge \frac{|B'|}{|B|}\,\psi(\nabla u(x_0),m(x_0))$$

$$- Cg(|m(x_0)|+1+M)\int_{S_j}(1+|\nabla u(x_0)|+|\nabla v_{i(j)}|+|\nabla\eta_j|\,|v_{i(j)}-u_0|)\,dx$$

$$- C\,g(|m(x_0|+1)|B_j|(1+|\nabla u(x_0)|)$$

and from (3.6) we conclude that

$$\mu_a(x_0)\geq\frac{|B'|}{|B|}\psi(\nabla u(x_0),m(x_0))+O(1/j).$$

The result follows once we let $j\to+\infty$ and $|B\backslash B'|\to 0$.

4. The density of the jump part

Here we prove (2.4), precisely, that

$$\zeta(x_0)\;\geq\;\frac{\psi^\infty(u^-(x_0)-u^+(x_0))\otimes\nu(x_0)}{|u^+(x_0)-u^-(x_0)|}\quad\text{for } |u^+-u|\,H_{N-1}\llcorner\Sigma(u)\text{ a. e. } x_0\in\Sigma(u).$$

It is well known that (see [EG], [FM2], [Z]) for H_{N-1} a.e. $x_0\in\Sigma(u)$ we have

(i) $$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{N-1}}\int_{\Sigma(u)\cap(x_0+\varepsilon Q_{\nu(x_0)})}|u^+(x)-u^-(x)|\,dH_{N-1}(x)\;=\;|u^+(x_0)-u^-(x_0)|,$$

(ii) $$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^N}\int_{\{y\in B(x_0,\varepsilon):(y-x_0)\cdot\nu(x_0)>0\}}|u(y)-u^+(x_0)|^{N/(N-1)}\,dy\;=\;0,$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^N}\int_{\{y\in B(x_0,\varepsilon):(y-x_0)\cdot\nu(x_0)<0\}}|u(y)-u^-(x_0)|^{N/(N-1)}\,dy\;=\;0,\text{ and}$$

(iii) $$\zeta(x_0)\;=\;\lim_{\varepsilon\to 0}\frac{\mu(x_0+\varepsilon Q_{\nu(x_0)})}{|u^+-u^-|\,H_{N-1}\llcorner\Sigma(u)(x_0+\varepsilon Q_{\nu(x_0)})}$$

exists and is finite, where $Q_{\nu(x_0)}$ denotes a unit cube centered at the origin with two faces perpendicular to the unit vector $\nu(x_0)$.

Writting $Q=Q_{\nu(x_0)}$, $Q^*=\dfrac{1}{1+\delta}Q$, with $0<\delta<1$, let $\varphi\in C_0^\infty(Q)$ be such that $0\leq\varphi\leq 1$, $\varphi=1$ on Q^*, and let $\varepsilon_k\to 0$ be such that

$$y_0\;=\;\lim_{k\to\infty}\frac{1}{|x_0+\varepsilon_k Q|}\int_{x_0+\varepsilon_k Q} m\;dy$$

exists. By (i) and (iii),

$$\zeta(x_0) = \lim_{k\to\infty} \frac{\mu(x_0+\varepsilon_k Q)}{|u^+ - u^-| \, H_{N-1}\llcorner\Sigma(u)(x_0+\varepsilon_k Q)}$$

$$= \frac{1}{|u^+(x_0) - u^-(x_0)|} \lim_{k\to\infty} \frac{1}{\varepsilon_k^{N-1}} \int_{x_0+\varepsilon_k Q} d\mu(x)$$

$$\geq \frac{1}{|u^+(x_0) - u^-(x_0)|} \limsup_{k\to\infty} \lim_{n\to\infty} \frac{1}{\varepsilon_k^{N-1}} \int_{x_0+\varepsilon_k Q} \varphi\left(\frac{x-x_0}{\varepsilon_k}\right) \psi(\nabla u_n(x), m_n(x))dx$$

$$= \frac{1}{|u^+(x_0) - u^-(x_0)|} \limsup_{k\to\infty} \lim_{n\to\infty} \int_Q \varepsilon_k \varphi(y)\psi(\nabla u_n(x_0+\varepsilon_k y), m_n(x_0+\varepsilon_k y))\, dy$$

$$\geq \frac{1}{|u^+(x_0) - u^-(x_0)|} \limsup_{k\to\infty} \limsup_{n\to\infty} \int_{Q^*} \varepsilon_k \psi(\nabla u_n(x_0+\varepsilon_k y), m_n(x_0+\varepsilon_k y))\, dy \tag{4.1}$$

We define

$$u_{n,k}(y) := u_n(x_0+\varepsilon_k y) \quad \text{and} \quad u_0(y) := \begin{cases} u^+(x_0) & \text{if } y.\nu(x_0) > 0 \\ u^-(x_0) & \text{if } y.\nu(x_0) \leq 0 \end{cases}.$$

As $u_n \to u$ in L^1, by (ii) we obtain

$$\lim_{k\to\infty} \lim_{n\to\infty} \int_Q | u_{n,k}(y) - u_0(y) | \, dy = \lim_{k\to\infty} \int_{Q^+} | u(x_0+\varepsilon_k y) - u^+(x_0) | \, dy$$

$$+ \lim_{k\to\infty} \int_{Q^-} | u(x_0+\varepsilon y) - u^-(x_0)| \, dy = 0. \tag{4.2}$$

On the other hand, by (4.1)

$$\zeta(x_0) \geq \frac{1}{|u^+(x_0) - u^-(x_0)|} \limsup_{k\to\infty} \limsup_{n\to\infty} \varepsilon_k \int_{Q^*} \psi\left(\frac{1}{\varepsilon_k}\nabla u_{n,k}(y), m_n(x_0+\varepsilon_k y)\right) dy$$

and, as in Section 3, by (4.2) and (H2) we use a diagonalizing argument to construct sequences

$$\lambda_k \overset{*}{\rightharpoonup} y_0 \text{ in } L^\infty \quad \text{and} \quad \| v_k - u_0 \|_{L^1(Q)} \to 0, \int_{Q^*} | \nabla v_k | \, dx \leq C$$

such that

$$\zeta(x_0) \geq \frac{1}{|u^+(x_0)-u^-(x_0)|}\limsup_{k\to\infty}\varepsilon_k\int_{Q^*}\psi(\frac{1}{\varepsilon_k}\nabla v_k,\lambda_k)\,dy\,.$$

Let $w_k = \rho_k * u_0$, where $\{\rho_k\}$ is a mollifiying sequence. Then

$$\|\nabla w_k\|_\infty = O(k) \quad \text{if} \quad |x\cdot\nu(x_0)| \leq 1/k \quad \text{and} \quad \|v_k - w_k\|_{L^1(Q)} \to 0.$$

As in Section 3 we use the "slicing method" to obtain sequences

$$\overline{\lambda}_j(x) := (1-\eta_j(x))\frac{y_0 - \frac{1}{|Q^*|}\int_{Q^*}\eta_j\,\lambda_{i(j)}\,dy}{1-\frac{1}{|Q^*|}\int_{Q^*}\eta_j\,dy} + \eta_j(x)\,\lambda_{i(j)}(x)$$

$$\overline{v}_j(x) := (1-\eta_j(x))w_{i(j)}(x) + \eta_j(x)v_{i(j)}(x).$$

where

$$Q_j := \{x\in Q^*: \text{dist}(x,\partial Q^*) < 1/j\}, \quad \int_{S_j}|\nabla v_{k(j)}|\,dx \leq C/j,$$

$$\frac{1}{|S_j|}\int_{S_j}|v_{k(j)} - w_{k(j)}|\,dx \leq 1/j, \quad \frac{\left|y_0 - \frac{1}{|Q^*|}\int_{Q^*}\eta_j\,\lambda_{i(j)}\,dy\right|}{\left|1-\frac{1}{|Q^*|}\int_{Q^*}\eta_j\,dy\right|} \leq |y_0| + 1,$$

and

$$\zeta(x_0) \geq \frac{1}{|u^+(x_0)-u^-(x_0)|}\limsup_{j\to\infty}\varepsilon_j\int_{Q^*}\psi(\frac{1}{\varepsilon_j}\nabla\overline{v}_j,\overline{\lambda}_j)\,dx\,. \qquad (4.3)$$

Note that

$$\int_{Q^*}\overline{\lambda}_j\,dx = |Q^*|\,y_0, \quad \overline{\lambda}_j|_{\partial Q^*}(x) = a_j := \frac{y_0 - \frac{1}{|Q^*|}\int_{Q^*}\eta_j\,\lambda_{i(j)}\,dy}{1-\frac{1}{|Q^*|}\int_{Q^*}\eta_j\,dy}$$

and so

$$\overline{\lambda}_j(x) = a_j + \theta_j(x) \quad \text{where}$$

$$\int_{Q^*}\theta_j(x)\,dx = -|Q^*|(a_j - y_0) \quad \text{and} \quad \theta_j|_{\partial Q^*}(x) = 0.$$

Also $\nabla\overline{v}_j = \nabla w_{i(j)}$ on ∂Q^* and so it is periodic. From the Q^*- periodicity of θ_j and η_j we deduce that

$$\int_{Q^*} \psi(\frac{1}{\varepsilon_j}\nabla\bar{v}_j,\bar{\lambda}_j)\, dx$$

$$= \int_{Q^*} \psi(\frac{1}{\varepsilon_j}\nabla\bar{v}_j, a_j + \theta_j)\, dx \quad = \lim_{i\to\infty} \int_{Q^*} \psi(\frac{1}{\varepsilon_j}\nabla\bar{v}_j(ix), a_j+\theta_j(ix))\, dx$$

and since

$$\theta_j(ix) \to \frac{1}{|Q^*|}\int_{Q^*}\theta_j\, dy = y_0 - a_j \quad \text{in} \quad L^\infty \text{ weak* as } i \to +\infty \text{ and}$$

$$\bar{v}_j(ix) \to \frac{1}{|Q^*|}(\int_{Q^*}\nabla\bar{v}_j\, dy)x \quad \text{in } W^{1,1} \text{ as } i \to +\infty,$$

by (2.3) we conclude that

$$\int_{Q^*} \psi(\frac{1}{\varepsilon_j}\nabla\bar{v}_j,\bar{\lambda}_j)\, dx \;\geq\; \int_{Q^*} \psi(\frac{1}{\varepsilon_j}\frac{1}{|Q^*|}\int_{Q^*}\nabla\bar{v}_j\, dy,\, a_j + \frac{1}{|Q^*|}\int_{Q^*}\theta_j(y)\, dy))\, dx$$

$$= |Q^*|\, \psi(\frac{1}{\varepsilon_j}\frac{1}{|Q^*|}(u^+(x_0) - u^-(x_0))\otimes\nu(x_0)\,|Q^*|^{(N-1)/N}, y_0).$$

Finally, from (4.3) and Lemma 2.2 we have

$\zeta(x_0) \;\geq$

$$\limsup_{j\to\infty} \frac{\varepsilon_j}{|u^+(x_0) - u^-(x_0)|}|Q^*|\psi(\frac{1}{\varepsilon_j|Q^*|}(u^+(x_0) - u^-(x_0))\otimes\nu(x_0)|Q^*|^{(N-1)/N}, y_0)dx$$

$$= \frac{1}{|u^+(x_0) - u^-(x_0)|}\,|Q^*|^{-1/N}\;\psi^\infty(u^+(x_0) - u^-(x_0))\otimes\nu(x_0)).$$

Now it suffices to let $|Q^*| \to 1$.

5. The density of the Cantor part

We prove (2.5), that is, for $|C(u)|$ a. e. $x_0 \in \Omega$

$$\eta(x_0) \;\geq\; \psi^\infty(A(x_0))$$

where $A(.)$ is the rank-one matrix $a\otimes\nu$ (see [Al]). Let $Q = (-1/2,1/2)^N$ and $Q(x_0,\varepsilon) = x_0 + \varepsilon Q$. For $|C(u)|$ a. e. $x_0 \in \Omega$

$$A(x_0) \;:=\; \lim_{\varepsilon\to 0}\frac{D(u)Q(x_0,\varepsilon)}{|D(u)|Q(x_0,\varepsilon)} \;=\; \lim_{\varepsilon\to 0}\frac{C(u)Q(x_0,\varepsilon)}{|C(u)|Q(x_0,\varepsilon)},$$

$$\text{am}\frac{|Du|(Q(x_0,\varepsilon))}{} = !,$$

e->0 IC(u)l(Q(xo,e))

and (see [FM2]) the following hold:

$$\eta(x_0) = \lim \frac{\mu(Q(x_0,\varepsilon))}{}$$

t

lim —/ e->0 IDul(Q(xo,e)) |lu - u(x$_o$)l dx = 0,

e-»01 Q(x$_o$,e) I Q(xo,e)

IA(xo)l - l,A(xo) = a⊗v,

,. ≥DulfQ(xn.E». |Du|(Q(xo,e))

hm ' ,, lhV '' = o and hm' —— = +00. (5.1)

e-»0 eN''^{1} e-»0 e^N

Also, by [FM2], Lemma 2.13, we may assume that

to liniIlf 'D»'(Q(xo,e)\Q(x<,,,te)) — Q. (5.2)

t→l- e-»0 IDu l(Q(xo,e))

We suppose that A(xo) = a®eN- Let t € (0,1), *ye* (t, 1) and let £k -4 0 be such that

Tl(x$_0$) ^ lim sup lim sup $\frac{1}{\wedge \text{ IUl rn } v}$ $\int$ v(Vu$_n$,m$_n$)dx and
kx- n>oo I U u l ^ ^x$_0$+ Q $_{\varepsilon_k}$

$$y_0 := \lim_{k\to\infty} \frac{1}{\text{meas}(Q_{\varepsilon_k})} \int_{x_0+\gamma Q_{\varepsilon_k}} m \, dx$$

exists. Since

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{1}{|Q_\varepsilon|} \int_{x_0+\gamma Q_\varepsilon} | u_n(x) - U(XQ)| \, dx = 0,$$

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{1}{\varepsilon^{N} |Du|(Q_\varepsilon)} \int_{Q_\varepsilon} \Big| u_n(x) - u(x) - \frac{1}{|Q_\varepsilon|} \int_{Q_\varepsilon} (u_n(y) - u(y))\,dy \Big| \, dx = 0,$$

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \Big| \int_Q (m_k(x_0 + \varepsilon x) - m(x_0))\varphi_j(x)\, dx \Big| = 0,$$

writing $v_{n,k}(z) := u_n(x_0 + \varepsilon_k z)$ and using a diagonalization procedure as in Section 2, we construct sequences A* *^ yo in L^∞ and II v^ - u(xo) "L^Q) —* 0 such that

$$\eta(x_0) \geq \limsup_{k\to+\infty} \frac{\varepsilon_k^N}{|Du|(Q_{\varepsilon_k})} \int_{\gamma Q} \psi\left(\frac{1}{\varepsilon_k}\nabla v_k, \lambda_k\right) dz, \tag{5.3}$$

$$\lim_{k\to+\infty} \frac{\varepsilon_k^{N-1}}{|Du|(Q_{\varepsilon_k})} \int_Q \left| v_k(z) - a_k - \left[u(x_0+\varepsilon_k z) - \frac{1}{|Q_{\varepsilon_k}|} \int_{x_0+Q_{\varepsilon_k}} u\, dy\right]\right| dz = 0 \tag{5.4}$$

where $a_k := \int_Q v_k(z)\, dz$.

After extracting a subsequence, we may assume in (5.3) that lim sup is limit. We set

$$\bar{u}_k(z) := \frac{\varepsilon_k^{N-1}}{|Du|(Q_{\varepsilon_k})} \left[u(x_0 + \varepsilon_k z) - \frac{1}{|Q_{\varepsilon_k}|} \int_{x_0+Q_{\varepsilon_k}} u\, dy\right] \text{ and}$$

$$w_k(z) := \frac{\varepsilon_k^{N-1}}{|Du|(Q_{\varepsilon_k})} [v_k(z) - a_k].$$

Then

$$\int_Q \bar{u}_k(z)\, dz = 0, \quad |D\bar{u}_k|(Q) = 1,$$

and so $\{ \bar{u}_k \}$ is equi-integrable and by (5.4) we conclude that

$$\| \bar{u}_k - w_k \|_{L^1(Q)} \to 0 \text{ as } k \to +\infty.$$

By (5.1),

$$\mu_k := \frac{|Du|(x_0 + \varepsilon_k Q)}{\varepsilon_N^k} \to +\infty,$$

and (5.3) reduces to

$$\eta(x_0) \geq \lim_{k\to+\infty} \frac{1}{\mu_k} \int_{\gamma Q} \psi(\mu_k \nabla w_k, \lambda_k)\, dz. \tag{5.5}$$

On the other hand we have that

$$D\bar{u}_k(Q) = \frac{Du(x_0 + \varepsilon_k Q)}{|Du|(x_0 + \varepsilon_k Q)} \to a\otimes e_N \text{ and } |D\bar{u}_k - (D\bar{u}_k \cdot A_0)A_0|(Q) \to 0,$$

the latter from [FM2], Proposition A.1, and this implies that

$$|D\bar{u}_k \cdot e_i|(Q) \;\to\; 0 \text{ for all } i = 1, ..., N\text{-}1.$$

Thus, it is possible to find a sequence of smooth functions $\xi_k(x)$, which are functions $f_k(x_N)$, such that

$$\| \xi_k - \bar{u}_k \|_{L^1(Q)} \;\to\; 0, \quad \text{as } k \to +\infty,$$

and for a.e. $\tau \in (0, 1)$

$$\nabla \xi_k(\tau Q) - D\bar{u}_k(\tau Q) \to 0. \tag{5.6}$$

Fix $\tau \in (t, \gamma)$ for which (5.6) holds. Choose $\delta > 0$ such that $(1 - \delta)\tau > t$ and we may assume that

$$|D\xi_k|(\tau Q \backslash \tau(1-\delta)Q) \;\le\; |D\bar{u}_k|\,(Q \backslash tQ) \;=\; \frac{|Du|(Q(x_0,\varepsilon_k)\backslash Q(x_0,t\varepsilon_k))}{|Du|(Q(x_0,\varepsilon_k))}. \tag{5.7}$$

Note that

$$\frac{1}{\tau^N}\nabla\xi_k(\tau Q) \;=\; \frac{1}{\tau^N}\int_{\tau Q} \nabla\xi_k \, dy \;=\; \int_Q \nabla\xi_k(\tau z)\, dz \;=\; \frac{f_k(\tau/2) - f_k(-\tau/2)}{\tau} \otimes e_N. \tag{5.8}$$

As $\lambda_k \overset{*}{\rightharpoonup} y_0$ in L^∞ and $w_k - \xi_k \to 0$ in L^1, by (5.5) and using the "slicing method" will modify w_k and λ_k on the layer $\tau Q \setminus \tau(1-\delta)Q$ so that

$$\eta(x_0) \;\ge\; \limsup_{k\to\infty} \frac{1}{\mu_k} \int_{\tau Q} \psi(\mu_k \nabla \bar{v}_k, \bar{\lambda}_k)\, dz \;+\; O(1-t) \tag{5.9}$$

where $\bar{\lambda}_k \overset{*}{\rightharpoonup} y_0$ in L^∞, $\frac{1}{|\tau Q|}\int_{\tau Q} \bar{\lambda}_k \, dz = y_0$, $\bar{\lambda}_k|_{\partial(\tau Q)}$ is constant and $\bar{v}_k = \xi_{k(i)}$, for some $k(i)$, on $\partial(\tau Q)$.

We partition $\tau Q \backslash \tau(1 - \delta)Q$ into two layers $S^1_{(2)}$, $S^2_{(2)}$ with

$$|S^j_{(2)}| \;=\; \frac{|\tau Q \backslash \tau(1-\delta)Q|}{2}$$

and due to (H2) and (5.9) we choose
$S_2 = \{x \in \tau Q\backslash\tau(1-\delta)Q : \alpha_2 < \mathrm{dist}(x, \partial(\tau Q\backslash\tau(1-\delta)Q)) < \beta_2\} \in \{S^1_{(2)}, S^2_{(2)}\}$ such that, for a subsequence,

$$\int_{S_2} |\nabla w_k(z)(x)|\,dx \leq C/2.$$

Let η_2 be a smooth cut-off function, $0 \leq \eta_2 \leq 1$, $\eta_2 = 0$ in the complement of $\{x\in \tau Q: \mathrm{dist}(x,\partial(\tau Q\setminus\tau(1-\delta)Q)) < \beta_2\}$, $\eta_2 = 1$ in $\{x\in \tau Q: \mathrm{dist}(x,\partial(\tau Q\setminus\tau(1-\delta)Q)) < \alpha_2\}$ and $\|\nabla\eta_2\| = O(1/|S_2|)$. As

$$\lim_{k\to\infty} \left| y_0 - \frac{1}{|\tau Q|}\int_{\tau Q} \eta_2\, \lambda_k\, dx \right| = |y_0|\left|1 - \frac{1}{|\tau Q|}\int_{\tau Q} \eta_2\, dx\right|$$

choose k(2) large enough so that

$$\frac{1}{|S_2|}\int_{S_2} |w_{k(2)} - \xi_{k(2)}|\,dx < \frac{1}{2} \text{ and}$$

$$\frac{\left| y_0 - \frac{1}{|\tau Q|}\int_{\tau Q} \eta_2\, \lambda_{k(2)}\, dx \right|}{\left|1 - \frac{1}{|\tau Q|}\int_{\tau Q} \eta_2\, dx\right|} \leq |y_0| + 1.$$

Next, divide $\tau Q \setminus \tau(1-\delta)Q$ into $S^1_{(3)}, S^2_{(3)}, S^3_{(3)}$, with $|S^j_{(3)}| = \frac{|\tau Q\setminus\tau(1-\delta)Q|}{3}$.
One of these, S_3, must verify

$$\int_{S_3} |\nabla w_k|\,dx \leq C/3$$

for a subsequence of the previous one. Let η_3 be a smooth cut-off function, $0 \leq \eta_3 \leq 1$, $\eta_3 = 0$ "outside" S_3, $\eta_3 = 1$ "inside" S_3 and $\|\nabla\eta_3\| = O(1/|S_3|)$. Choose $k(3) > k(2)$, large enough so that

$$\frac{1}{|S_3|}\int_{S_3} |w_{k(3)} - \xi_{k(3)}|\,dx < \frac{1}{3} \text{ and}$$

$$\frac{\left| y_0 - \frac{1}{|\tau Q|}\int_{\tau Q} \eta_3\, \lambda_{k(3)}\, dx \right|}{\left|1 - \frac{1}{|\tau Q|}\int_{\tau Q} \eta_3\, dx\right|} \leq |y_0| + 1.$$

Recursively, we construct a sequence k(j) such that

$$\int_{S_j} |\nabla w_{k(j)}|\,dx \leq \frac{C}{j},\quad \frac{1}{|S_j|}\int_{S_j} |w_{k(j)} - \xi_{k(j)}|\,dx < \frac{1}{j} \text{ and}$$

$$\frac{\left| y_0 - \frac{1}{|\tau Q|}\int_{\tau Q} \eta_j \lambda_{k(j)}\, dx \right|}{\left| 1 - \frac{1}{|\tau Q|}\int_{\tau Q} \eta_j\, dx \right|} \le |y_0| + 1.$$

We set

$$\bar{\lambda}_j(x) := (1-\eta_j(x))\frac{y_0 - \frac{1}{|\tau Q|}\int_{\tau Q} \eta_j \lambda_{k(j)}\, dy}{1 - \frac{1}{|\tau Q|}\int_{\tau Q} \eta_j\, dy} + \eta_j(x)\,\lambda_{k(j)}(x),$$

VJ(X) := (1 — T]j(x)) $\xi_{k(j)}(x)$ + $\eta_j(x) v_{i(j)}(x)$.

Clearly

Xj i^ yo in L~ I f — J Aj dz = yo , ^j'd(xQ) *s constant, II Aj ll°o ≤
'^tQ I ^

lm(xo)l + 1 + M and vk = ^k on 9(xQ). By (5.9) and (H2) '

T|(xo) £ lim If J \|/(M*VwkAk) dz ^ lim If" V(|XkVwk,Ak) dz
μ μk τQ

^ lim 1/ [J V([ikO)^v v̄j,Xj) dz

- Cg(ly$_o$l + 1 + M) J (1 + IV$$_k$(j)l + IVw$_{k(j)}$)l + IVT]jl lw$_{k(j)}$ - 4kO)0 dx
S_j

-Cg(ly$_o$l + 1) (1 + |∇$\xi_{k(j)}$|) dx].
TQ\T(1 - 6)Q

By (5.7) and (5.2) we conclude that

$\eta(x_0) \ge \lim_{j\to\infty} \frac{1}{\mu_{k(j)}}$ [$\int_{\tau Q}$ v(HkG)VV̄J,XJ) dz - Cg(ly$_o$l + 1 + M)(l xQ\x(l - 8)Q l

+ $\dfrac{\text{IDul(Q(xo,ek)\textbackslash Q(x}_{o>}\text{tek))}}{\text{IDu l(Q(xo,ek))}}$ $- C_g(|y_0| + 1)(|\tau Q \setminus \tau(1-\delta)Q|$

$$+ \frac{|Du|(Q(x_0,\varepsilon_k) \setminus Q(x_0,t\varepsilon_k))}{|Du|(Q(x_0,\varepsilon_k))})]$$

$$= \lim_{j\to\infty} \frac{1}{\mu_{k(j)}} \int_{\tau Q} v(HkO)Vv\bar{j},Xj)\, dz + O(1-1).$$

Note that

$$\xi_k(x) = (\frac{f_k(\tau/2) - f_k(-\tau/2)}{\tau} \otimes e_N)x + \varphi(x),$$

with <p a xQ-periodic function, and so by (HI) and (5.9),

$$Tl(x_0) \quad L> \quad \limsup_{k\to\infty} \frac{1}{\wedge k} TN\, v(n_k \frac{fk(T/2) \; fk(T/2)}{x} \otimes e_N, y_0) + O(l-t).$$

As \|/(., yo) is quasiconvex, then (see [D]) it is Lipschitz continuous hence by (5.6) and (5.8)

$$\limsup_{k\to\infty} \left| \pm_k v(\wedge k \frac{(x/2)-f_k(-t/2)}{\tau} \otimes e_N\, o) - \pm\, \psi(M\, MXQ) \,, y_0) \right|$$

$$\leq \limsup_{k\to\infty} \frac{-\wedge}{x^{N}} \; IA(XO)-V\wedge(TQ)I$$

$$= \frac{C}{x^N} \limsup_{k\to\infty} \; IA(xo) - D\bar{u}k(xQ)l$$

$$= \frac{C}{\tau^N} \limsup_{t\to\infty} \; IA(xo) - \frac{DU(XQ + e_k xQ)}{IDul(xo + ekQ)} I$$

$$\leq \frac{C}{\tau^N} \limsup_{k\to\infty} \frac{|Du|(Q(x_0,\varepsilon_k) \setminus Q(x_0,t\varepsilon_k))}{Du|(Q(x_0,\varepsilon_k))} = O(1-t)/\tau_N.$$

We conclude that

$$Tl(xo) \wedge \limsup \frac{1}{-} xN \; v(\wedge A(xo),yo) + 0(1-t),$$

which by Lemma 2.2 yields

$$Tl(xo) \wedge V^{\circ\circ}(A(x_0)) + O(l-t)$$

and the result now follows by letting t -» **1.**

6. Relaxation

We want to show that

F(u,m) £ f \|/(Vu,m) dx + f \|f~((u~ - u^{+})®v) dH_N-i(x) + f V°°(dC(u))
Ω **Σ(u)** **Ω**

(6.1)

We will follow the proof of the relaxation section on [FM2] (see also Ambrosio, Mortola and Tortorelli [AMT]) making the necessary adaptations. It is divided into four steps and we begin by considering

F(u,m;A) := inf { lim inf J \|/(Vuk,mk) dx : (uk,mk) ∈
$\{u_k\},\{m_k\}$ **k-* o** **X**

WU(A;R^n)xL~(A;R^d),Uk -> u in L^1 and **mk** **⇀ m in L^∞** }

whenever A c *Cl* is an open set.

Stepl. By (H2)

F(u,m;A) ≤ g(llclUo) (I A I + Dul(A)). (6.2)

Also we claim that F(u,m;A) is a variational functional with respect to the L^1 topology. We recall that F(u,m;A) is said to be a *variational functional with* ***respect to the L^1 topology*** **if**

(i)F(u,m;A) is local, i. e.

F(u,m;A) = F(v,h;A)

for every u, v € BV(A; R^n) verifying u = v a.e. in A and m, h € L°°(A;R^d) such that m = h a.e. in A.

(ii) F(u,m;A) is sequentially lower semicontinuous, i. e. if Uk, u € BV(A; R^n), Uk -4 u in L^x(A;R^n), mk, m € L°°(A;R^d) and mk ^ m in L°°, then

F(u,m;A) ≤ lim inf F(uk,n^;A)
k→∞

(iii) F(u,m;A) is the trace on {A c 12: A is open} of a Borel measure on the set *B(Q)* of all Borel subsets of *Q*. De Giorgi and Letta [DGL] introduced the following criterion to assert (iii). A set function **a**: {A c *Q: A* is open} —> [0, +°o] is the trace of a Borel measure if

(a) a(B) £ a(A) forallA,B€X := {U eft: Uisopen} withB c A;

(b) $\alpha(A \cup B) \geq \alpha(A) + \alpha(B)$ for all $A, B \in X$ such that $A \cap B = \emptyset$;

(c) $\alpha(A \cup B) \leq \alpha(A) + \alpha(B)$ for all $A, B \in X$;

(d) $\alpha(A) = \sup\{\alpha(B): B \subset\subset A\}$ for all $A \in X$.

The proof of (i) is trivial.

To show (ii) one needs to use a standard diagonalization procedure. Indeed, suppose that $u_k \to u$ in L^1, $m_k \overset{*}{\rightharpoonup} m$ in L^∞ and let $\{\varphi_i\}$ be a countable set dense in $L^1(\Omega)$. Assume that

$$\mathcal{F}(u_k,m_k;A) = \lim_{j\to\infty} \int_A \psi(\nabla u_j^k, m_j^k)\, dx$$

where $u_j^k \to u_k$ in L^1 and $m_j^k \overset{*}{\rightharpoonup} m_k$ in L^∞ as $j \to +\infty$. For every k, i, choose j(k,i) such that for all $j \geq j(k,i)$

$$\left| \int_\Omega (m_j^k - m_k)\cdot\varphi_i\, dx \right| \leq 1/k.$$

We may assume that j(k,.) is increasing.

Next, for all k let p(k) be such that for all $j \geq p(k)$

$$\| u_j^k - u_k \|_{L^1} \leq 1/k.$$

Choose $s(k) \geq p(k), j(k,k)$ such that

$$\left| \mathcal{F}(u_k,m_k;A) - \int_A \psi(\nabla u_{s(k)}^k, m_{s(k)}^k)\, dx) \right| \leq 1/k.$$

Clearly

$$u_{s(k)}^k \to u \quad \text{in } L^1,$$

and for all i and $k \geq i$

$$\left| \int_\Omega (m_{s(k)}^k - m)\cdot\varphi_i\, dx \right| \leq \left| \int_\Omega (m_k - m)\cdot\varphi_i\, dx \right| + 1/k \to 0.$$

Hence

$$\mathcal{F}(u,m;A) \leq \liminf_{k\to\infty} \int_A \psi(\nabla u_{s(k)}^k, m_{s(k)}^k)\, dx) = \liminf_{k\to\infty} \mathcal{F}(u_k,m_k;A).$$

We prove (iii) using an idea developed by [AMT] in Theorem 4.3. Parts (a) and (b) are trivial. To obtain (c) and (d) we prove that if A, B, C are open subsets of Ω with $B \subset\subset C \subset\subset A$ then

$$\mathcal{F}(u,m;A) \leq \mathcal{F}(u,m;C) + \mathcal{F}(u,m;A\setminus\bar{B}). \tag{6.3}$$

Suppose that (6.3) holds. To show (d) fix $\varepsilon > 0$ and let $B \subset\subset A$ be such that

$$|A\setminus\bar{B}| + |Du|(A\setminus\bar{B}) < \frac{\varepsilon}{g(\|m\|_\infty)}.$$

By (H2) we have

$$\mathcal{F}(u,m;A\setminus\bar{B}) < \varepsilon$$

and so, if C is such that $B \subset\subset C \subset\subset A$, by (6.3) we conclude that

$$\mathcal{F}(u,m;A) \leq \mathcal{F}(u,m;C) + \varepsilon$$

proving (d). In order to obtain (c), for $t \in (0, 1)$ we define the sets

$$A_t := \{x \in A \cup B : t\,\mathrm{dist}(x,A\setminus B) < (1-t)\,\mathrm{dist}(x,B\setminus A)\},$$
$$B_t := \{x \in A \cup B : t\,\mathrm{dist}(x,A\setminus B) > (1-t)\,\mathrm{dist}(x,B\setminus A)\}$$

and

$$S_t := \{x \in A \cup B : t\,\mathrm{dist}(x,A\setminus B) = (1-t)\,\mathrm{dist}(x,B\setminus A)\}.$$

Since $(\mathcal{L}_N + |Du|)(\cup S_t) < +\infty$, where $\mathcal{L}_N$ denotes Lebesgue measure, and the sets $\{S_t\}$ are pairwise disjoint, there exists $t_0 \in (0, 1)$ such that $(\mathcal{L}_N + |Du|)(S_{t_0}) = 0$. Given $\varepsilon > 0$, by (H2) choose $K_1 \subset\subset A_{t_0}$, $K_2 \subset\subset B_{t_0}$ such that

$$\mathcal{F}(u,m; (A \cup B)\setminus(\bar{K}_1\cup\bar{K}_2)) < \varepsilon$$

and let $K_1 \subset\subset H_1 \subset\subset A_{t_0}$, $K_2 \subset\subset H_2 \subset\subset B_{t_0}$. By (6.3), (a) and (b) we deduce that

$$\begin{aligned}\mathcal{F}(u,m; A \cup B) &\leq \mathcal{F}(u,m; H_1 \cup H_2) + \mathcal{F}(u,m; (A\cup B)\setminus(\bar{K}_1\cup\bar{K}_2)) \\ &\leq \mathcal{F}(u,m; A) + \mathcal{F}(u,m;B) + \varepsilon.\end{aligned}$$

We prove (6.3). Let

$$\mathcal{F}(u,m;A\setminus\bar{B}) = \lim_{k\to\infty} \int_{A\setminus\bar{B}} \psi(\nabla u_k^1, m_k^1)\,dx\,,\ \mathcal{F}(u,m;C) = \lim_{k\to\infty} \int_C \psi(\nabla u_k^2, m_k^2)\,dx$$

where $u_k^1 \to u$ in $L^1(A\setminus\bar{B})$, $u_k^2 \to u$ in $L^1(C)$, $m_k^1 \overset{*}{\rightharpoonup} m$ in $L^\infty(A\setminus\bar{B})$ and

$m_k^2 \overset{*}{\rightharpoonup} m$ in $L^\infty(C)$.

In order to obtain admissible sequences for (u,m) in $A \cup B$, using the slicing method we are going to connect m_k^1 to m_k^2 and u_k^1 to u_k^2 across $C \setminus \bar{B}$. We partition $C \setminus \bar{B}$ into two layers $S^1_{(2)}, S^2_{(2)}$ with $|S^j_{(2)}| = |C \setminus \bar{B}|/2$ and due to (H2) and the fact that $\{ \psi(\nabla u_k^2, m_k^2) \}$ is bounded in $L^1(C)$, we choose $S_2 = \{x \in C \setminus \bar{B}: \alpha_2 < \mathrm{dist}(x, \partial(C \setminus \bar{B}) < \beta_2\} \in \{S^1_{(2)}, S^2_{(2)}\}$ such that, for a subsequence,

$$\int_{S_2} |\nabla u_k^1(x)|\, dx \;\le\; \text{const.}/2, \qquad \int_{S_2} |\nabla u_k^2(x)|\, dx \;\le\; \text{const.}/2.$$

Let η_2 be a smooth cut-off function, $0 \le \eta_2 \le 1$, $\eta_2 = 0$ in the complement of $\{x \in C: \mathrm{dist}(x,\partial(C \setminus \bar{B})) < \beta_2\}$, $\eta_2 = 1$ in $\{x \in C \mid \mathrm{dist}(x,\partial(C\setminus\bar{B})) < \alpha_2\}$ and $\|\nabla \eta_2\| = O(1/|S_2|)$. Choose k(2) large enough so that

$$\frac{1}{|S_2|} \int_{S_2} |u_k^1 - u_k^2|\, dx \;<\; 1/2.$$

Recursively, we construct a sequence k(j) such that

$$\int_{S_j} |\nabla u^1_{k(j)}|\, dx \le C/k, \quad \int_{S_j} |\nabla u^2_{k(j)}|\, dx \le C/k, \quad \frac{1}{|S_j|}\int_{S_j} |u^1_{k(j)} - u^2_{k(j)}|\, dx < 1/j.$$

We set

$$\bar{\lambda}_j := (1-\eta_j)\, m^1_{k(j)} + \eta_j\, m^2_{k(j)}, \quad \bar{v}_j := (1-\eta_j)\, u^1_{k(j)} + \eta_j\, u^2_{k(j)}.$$

Clearly $\bar{\lambda}_j \overset{*}{\rightharpoonup} m$ in $L^\infty(A \cup B)$, $\bar{v}_j \to u$ in $L^1(A \cup B)$. Let $M := \sup\{\|m_k^1\|_\infty, \|m_k^2\|_\infty\}$. By (H2)

$$\mathcal{F}(u,m; A \cup B) \;\le\; \liminf_{j \to \infty} \int_{A \cup B} \psi(\nabla \bar{v}_j(x), \bar{\lambda}_j(x))\, dx$$

$$\leq \lim_{j\to\infty} \int_{A\setminus\bar{B}} \psi(\nabla u^1_{k(j)}, m^1_{k(j)})\, dx + \lim_{j\to\infty} \int_C \psi(\nabla u^2_{k(j)}, m^2_{k(j)})\, dx +$$

$$+ Cg(M) \limsup_{j\to\infty} \int_{S_j} (1 + |\nabla u^1_{k(j)}| + |\nabla u^2_{k(j)}| + |\nabla \eta_j|\, |u^1_{k(j)} - u^2_{k(j)}|)\, dx$$

$$= \mathcal{F}(u,m;A\setminus\bar{B}) + \mathcal{F}(u,m;C).$$

Step 2. We claim that if $u \in BV(\Omega;\mathbb{R}^n)$, $m \in L^\infty(\Omega;\mathbb{R}^d)$ then

$$\mathcal{F}(u,m;\Omega\setminus\Sigma(u)) \leq \int_{\Omega\setminus\Sigma(u)} \psi(\nabla u,m)\, dx + \int_{\Omega\setminus\Sigma(u)} \psi^\infty(A(x))\, d|C(u)|(x). \tag{6.4}$$

By Step 1, $\mathcal{F}(u,m;.)$ is a Radon measure, absolutely continuous with respect to $\mathcal{L}_N$ $+|Du|$. Thus (6.4) holds if and only if

$$\frac{d\mathcal{F}(u,m;.)}{dx}(x_0) \leq \psi(\nabla u(x_0), m(x_0)) \quad \text{for } dx \text{ a.e. } x_0 \in \Omega, \text{ and} \tag{6.5}$$

$$\frac{d\mathcal{F}(u,m;.)}{d|C(u)|}(x_0) \leq \psi^\infty(A(x_0)) \quad \text{for } |C(u)| \text{ a.e. } x_0 \in \Omega. \tag{6.6}$$

We start by showing (6.6). Let $\{u_k\}$ be the regularized sequence defined in the following way. Let $\rho_k \in C_0^\infty(\mathbb{R}^N)$ be an approximation of the identity and $u_k(x) = (u*\rho_k)(x)$. Writing

$$Du = \nabla u\, dx + D_s u, \tag{6.7}$$

for $\mathcal{L}_N$ a.e. $x_0 \in \Omega$ we have

$$\lim_{\varepsilon\to 0} \frac{1}{|B(x_0,\varepsilon)|} \int_{B(x_0,\varepsilon)} |m(x) - m(x_0)|\, (1 + |\nabla u(x)|)\, dx = 0, \tag{6.8}$$

$$\lim_{\varepsilon\to 0} \frac{|D_s u|(B(x_0,\varepsilon))}{|B(x_0,\varepsilon)|} = 0, \quad \lim_{\varepsilon\to 0} \frac{|Du|(B(x_0,\varepsilon))}{|B(x_0,\varepsilon)|} \text{ exists and is finite,} \tag{6.9}$$

$$\frac{1}{|B(x_0,\varepsilon)|} \int_\Omega \psi(\nabla u(x), m(x_0))\, dx \to \psi(\nabla u(x_0), m(x_0)), \text{ and} \tag{6.10}$$

$$\frac{d\mathcal{F}(u,m;\,.)}{dx}(x_0) \text{ exists and is finite.}$$

Choose a sequence of numbers $\varepsilon \in (0, \text{dist}(x_0,\partial\Omega))$. Then

$$\frac{dF(u,m;\,.)}{dx}(x_0) = \lim_{\varepsilon\to 0}\frac{F(u,m;\,B(x_0,\varepsilon))}{|B(x_0,\varepsilon)|}$$

$$\leq \liminf_{\varepsilon\to 0}\ \liminf_{k\to\infty}\frac{1}{|B(x_0,\varepsilon)|}\int_{B(x_0,\varepsilon)}\psi(\nabla u_k,m)\,dx. \qquad (6.11)$$

Following [AMT], Proposition 4.6, we introduce the Yosida transforms of ψ, given by

$$\psi_\lambda(m,A) := \sup\{\psi(A,m') - \lambda|m - m'|(1+|A|) : m' \in \mathbb{R}^d\},\ \lambda > 0.$$

Then

(i) $\psi_\lambda(A,m) \geq \psi(A,m)$ and $\psi_\lambda(A.m)$ decreases to $\psi(m,A)$ as $\lambda \to +\infty$;

(ii) $\psi_\lambda(A,m) \geq \psi_\eta(A,m)$ if $\lambda \leq \eta$, $(A,m) \in \mathbb{M}\times\mathbb{R}^d$;

(iii) $|\psi_\lambda(A,m) - \psi_\lambda(A,m')| \leq \lambda|m - m'|(1+|A|)$, $(A,m) \in \mathbb{M}\times\mathbb{R}^d$;

(iv) the approximation is uniform on compact sets. Precisely, let K be a compact subset of $\mathbb{R}^d$ and let $\delta > 0$. There exists $\lambda > 0$ such that

$$\psi(A,m) \leq \psi_\lambda(A,m) \leq \psi(A,m) + \delta(1+|A|),\ (A,m) \in \mathbb{M}\times K.$$

Fix $\delta > 0$ and let $K = \bar{B}(0,\|m\|_\infty)$. By (i), (ii) and (iv)

$$\psi(\nabla u_k(x),m(x)) \leq \psi_\lambda(\nabla u_k(x),m(x))$$

$$\leq \psi_\lambda(\nabla u_k(x),m(x_0)) + \lambda|m(x) - m(x_0)|(1+|\nabla u_k(x)|)$$

$$\leq \psi(\nabla u_k(x),m(x_0)) + \delta(1+|\nabla u_k(x)|) + \lambda(|m(x) - m(x_0)|(1+|\nabla u_k(x)|). \qquad (6.12)$$

Taking into account that $\nabla u_k = \rho_k * \nabla u + \rho_k * D_s u$ and that $\psi(m(x_0),.)$ is a Lipschitz function, by (H2) and (6.11) we have

$$\frac{dF(u,m;\,.)}{dx}(x_0) \leq \liminf_{\varepsilon\to 0}\ \liminf_{k\to\infty}\frac{1}{|B(x_0,\varepsilon)|}\Big[\int_{B(x_0,\varepsilon)}\psi((\rho_k*\nabla u)(x),m(x_0))\,dx$$

$$+ C\,|D_s u|(B(x_0,\varepsilon+1/k)) + (\lambda\varepsilon+\delta)\,|B(x_0,\varepsilon)| + (\lambda\varepsilon+\delta)\,|Du|(B(x_0,\varepsilon+1/k))$$

$$+ \lambda\int_{B(x_0,\varepsilon)}|m(x) - m(x_0)|(1+|\nabla u_k(x)|)\,dx\Big].$$

Since

$$\lim_{k\to\infty} \int_{B(x_0,\varepsilon)} \psi((\rho_k * \nabla u)(x), m(x_0))\, dx \;=\; \int_{B(x_0,\varepsilon)} \psi(\nabla u(x), m(x_0))\, dx,$$

$$|Du|\,(B(x_0,\varepsilon + 1/k)) \;\to\; |Du|\,(\bar{B}(x_0,\varepsilon)) \;=\; |Du|\,(B(x_0,\varepsilon))$$

for a.e. ε, invoking (6.9) and (6.10) one deduces

$$\frac{dF(u,m;\,.)}{dx}(x_0) \;\le\; \psi(\nabla u(x_0), m(x_0)) \;+\; C\delta$$

$$+\; \lambda \liminf_{\varepsilon\to 0} \liminf_{k\to\infty} \frac{1}{|\,B(x_0,\varepsilon)\,|} \int_{B(x_0,\varepsilon)} |m(x) - m(x_0)|\,(1 + |\nabla u_k(x)|)\, dx \;. \tag{6.13}$$

To prove (6.6) it remains to show that the last term converges to zero. By (6.8)

$$\lim_{\varepsilon\to 0} \frac{1}{|\,B(x_0,\varepsilon)\,|} \int_{B(x_0,\varepsilon)} |m(x) - m(x_0)|\, dx \;=\; 0$$

and by the dominated convergence theorem (with respect to the measure $|Du|$)

$$\limsup_{k\to\infty} \int_{B(x_0,\varepsilon)} |m - m(x_0)|\,|\,\nabla u_k\,|\, dx \;\le\; \limsup_{k\to\infty} \int_{B(x_0,\varepsilon+1/n)} (|m - m(x_0)| * \rho_k)\, |Du|(x)$$

$$\le\; \limsup_{k\to\infty} \int_{B(x_0,\varepsilon+1/k)\setminus\Sigma(u)} |m - m(x_0)| * \rho_k(x)\, |Du|(x) \;+\; 4\|m\|_\infty\, |Du|(B(x_0,\varepsilon+1/k)\cap\Sigma(u))$$

$$\le\; \limsup_{k\to\infty} \int_{\bar{B}(x_0,\varepsilon+1/k)\setminus\Sigma(u)} |m - m(x_0)|\, |Du|(x) + \;\; 4\; \|m\|_\infty |Du|\, (\bar{B}(x_0,\varepsilon)\cap\Sigma(u))$$

$$\le\; \int_{\bar{B}(x_0,\varepsilon)\setminus\Sigma(u)} |m - m(x_0)|\, |Du|(x) \;+\;\; 4\;\; \|m\|_\infty\, |D_s u|(B(x_0,\varepsilon)). \tag{6.14}$$

Taking into account that $|Du|(\partial B(x_0,\varepsilon)) \;=\; 0$ for a.e. ε and that

$$\int_{B(x_0,\varepsilon)} |m - m(x_0)|\, |Du|(x) \;\;\le\; \int_{B(x_0,\varepsilon)} |m - m(x_0)||\nabla u(x)|\, dx + 2\, \|m\|_\infty\, |D_s u|(B(x_0,\varepsilon)),$$

we obtain from (6.8) and (6.9) that

$$\limsup_{\varepsilon\to\infty} \limsup_{k\to\infty} \frac{1}{|\,B(x_0,\varepsilon)\,|} \int_{B(x_0,\varepsilon)} |m(x) - m(x_0)|\, |\nabla u_k(x)|\, dx \;=\; 0,$$

and (6.6) follows from (6.13).

Next we prove (6.7), where using Radon-Nikodym Theorem we write IDul = IC(u)l + |x, where \i and IC(u) I are mutually singular Radon measures. As m is bounded and measurable, consider a Borel measurable function mi such that mi = m for dx a. e. in *il*. Let m2 be the projection of mi onto B(0,llmlloo). Then m2 is a Borel measurable function which is bounded by llmlloo. In particular m2 e L°°(ft,IC(u)l). For xo€ Q IC(u)l a.e., we have that

$$\lim_{e\to 0} \frac{H(B(xo,e))}{IC(u)l(B(xo,e))} = 0, \quad \lim_{e\to 0} \frac{Dul(B(xo,e))}{IC(u)l(B(xo,e))} \text{ exists and is finite,} \tag{6.15}$$

$$\lim_{\varepsilon \to 0} \frac{e^N}{IC(u)l(B(xo,e))} = 0, \tag{6.16}$$

$$\lim_{e \to 0} \frac{1}{IC(u)l(B(xo,e))} \int_{B(xo,e)} Im_2(x) - m_2(x_0)l \; IC(u)l(x) = 0, \tag{6.17}$$

$$A(x) := \lim_{e\to 0} \frac{C(u)(B(x,e))}{IC(u)l(B(x,e))} \text{ exists and is a rank-one matrix of norm one,} \tag{6.18}$$

$$\liminf_{e\to 0} \frac{1}{IC(u)l(B(xo,e))} \int_{B(xo,e)} \backslash K^{\infty}(A(x))\, dlC(u)l = \backslash|^{\infty}(A(x_0)), \text{ and} \tag{6.19}$$

$\frac{dF(u;.)}{dlC(u)l}(xo)$ exists and is finite.

As before, using (6.12) and (6.14) one sees that

$$\frac{dF(u;.)}{dlC(u)l}(xo) = \lim_{e\to 0} \frac{F(u; B(xo,e))}{IC(u)l(B(xo,e))}$$

$$\leq \liminf_{e\to 0} \liminf_{k\to\infty} \frac{1}{IC(u)l(B(xo,e))} \int_{B(xo,e)} \backslash u(Vuk,m)\, dx$$

$$= \liminf_{e\to 0} \liminf_{k\to\infty} \frac{1}{IC(u)l(B(xo,e))} \int_{B(xo,e)} \backslash i/(Vu_k,m2)\, dx$$

$$\leq \liminf_{e\to 0} \liminf_{k\to\infty} \Big[\frac{1}{IC(u)l(B(xo,e))} \int_{B(xo,e)} \backslash ir(Vuk,m2(xo))\, dx$$

$$+ (\delta + \lambda\varepsilon) \int_{B(xo,e)} IVu_kl\, dx + (8 + XE)\, I B(x_o,e) I$$

$$+ X \int_{B(xo,e)} lm_2 - m_2(xo)l\, (1 + IVu_kl)dx \Big]$$

$$\leq \liminf_{\varepsilon\to 0}\liminf_{k\to\infty} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} \psi(\nabla u_k(x),m_2(x_0))\,dx$$

$$+ \limsup_{\varepsilon\to 0} \frac{1}{|C(u)|(B(x_0,\varepsilon))}(\delta+\lambda\varepsilon)\,[\,|Du|(B(x_0,\varepsilon)) + |B(x_0,\varepsilon)|\,]$$

$$+ \lambda \limsup_{\varepsilon\to 0} \frac{1}{|C(u)|(B(x_0,\varepsilon))}\,[\int_{\bar B(x_0,\varepsilon)\setminus\Sigma(u)} |m_2(x)-m_2(x_0)|\,|Du|(x) +$$

$$+ \int_{B(x_0,\varepsilon)} |m_2(x)-m_2(x_0)|\,dx + 4\,\|m\|_\infty |Du|\,(\bar B(x_0,\varepsilon)\cap\Sigma(u))].$$

By (6.15) - (6.17) and, due to the rectifiability of the jump set, as $|C(u)|(B(x_0,\varepsilon)\cap\Sigma(u)) = 0$ we conclude that

$$\frac{dF(u;\,.)}{d\,|C(u)|}(x_0) \leq \liminf_{\varepsilon\to 0}\liminf_{k\to\infty} \frac{1}{|C(u)|(B(x_0,\varepsilon))}\Big[\int_{B(x_0,\varepsilon)} \psi(\nabla u_k(x),m_2(x_0))\,dx$$

$$+ \lambda \int_{B(x_0,\varepsilon)} |m_2(x)-m_2(x_0)|\,|C(u)|(x) + 2\lambda\,\|m\|_\infty\,\mu(B(x_0,\varepsilon))$$

$$+ 4\lambda\,\|m\|_\infty\,|Du|(B(x_0,\varepsilon)\cap\Sigma(u))\Big] + C\delta$$

$$\leq \liminf_{\varepsilon\to 0}\liminf_{k\to\infty} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} \psi(\nabla u_k(x),m_2(x_0))\,dx + C\delta. \tag{6.20}$$

Now we use Ambrosio and DalMaso's argument in [ADM], Proposition 4.2. Define

$$g(A) := \sup_{t>0} \frac{\psi(tA,m_2(x_0)) - \psi(0,m_2(x_0))}{t}.$$

Then g is Lipschitz continuous, positively homogeneous of degree one and the rank-one convexity of $\psi(.,m_2(x_0))$ implies that

$$g(A) = \psi^\infty(A,m_2(x_0)) \text{ whenever rank } A \leq 1.$$

Thus, by (6.20), (6.16) we have

$$\frac{dF(u;\,.)}{d\,|C(u)|}(x_0) \le \liminf_{\varepsilon \to 0} \liminf_{k \to \infty} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} [\psi(0,m_2(x_0)) + g(\nabla u_k)]dx + C\delta$$

$$= \liminf_{\varepsilon \to 0} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} g(Du) + C\delta$$

$$= \liminf_{\varepsilon \to 0} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} [g(A(x))\, d|C(u)| + g(d\mu)] + C\delta$$

and so, by (6.15), (6.18), (6.19), by Alberti's Theorem 2.11 and by Lemma 2.2 we conclude that

$$\frac{dF(u;\,.)}{d\,|C(u)|}(x_0) \le \liminf_{\varepsilon \to 0} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \Big[\int_{B(x_0,\varepsilon)} \psi(x_0,u(x_0),A)\, d|C(u)| + C\mu(B(x_0,\varepsilon))\Big] + C\delta$$

$$= \psi^\infty(A(x_0)) + C\delta.$$

It suffices to let $\delta \to 0^+$.

Step 3. We show that

$$F(u,m;\Sigma(u)) \le \int_{\Sigma(u)} \psi^\infty((u^-(x) - u^+(x)) \otimes \nu(x))\, dH_{N-1}(x) \qquad (6.21)$$

for every $u \in BV(\Omega;\mathbb{R}^n)$, $m \in L^\infty(\Omega;\mathbb{R}^d)$. The proof is divided into three parts according to the limit function u :

1. $u(x) = a\chi_E(x) + b(1 - \chi_E(x))$ with $Per_\Omega(E) < +\infty$;

2. $u(x) = \sum a_i \chi_{E_i}(x)$ where $\{E_i\}_{i=1}^{+\infty}$ forms a partition of Ω into sets of finite perimeter ;

3. General case, $u \in BV(\Omega;\mathbb{R}^n)$.

1. Let $u(x) = a\chi_E(x) + b(1 - \chi_E(x))$ with $Per_\Omega(E) < +\infty$. We start by proving that for every open set $A \subset \Omega$

$$F(u,m;A) \le \int_A \psi(0,m(x))\, dx + \int_{\Sigma(u)\cap A} \psi^\infty((a-b)\otimes\nu)\, dH_{N-1}(x). \qquad (6.22)$$

a) Suppose first that

$$u(x) = \begin{cases} b & \text{if } x\cdot\nu > 0 \\ a & \text{if } x\cdot\nu < 0 \end{cases} .$$

Let A = a + A,Q$_V$ be an open cube with two faces orthogonal to v. Fix y € R^d and define

$$m_k(x) = \begin{cases} m(x) & \text{if } |x.v|>1/k \\ y & \text{if } |x\text{-}v| £ 1/k \end{cases}$$

$$u_k(x) = \begin{cases} b & \text{if } x\text{-}v > 1/k \\ & \text{if } |x\text{-}v| < 1/k \\ a & \text{if } x\text{-}v < - 1/k \end{cases}$$

| [(a - b)®v]x + |(a + b) if |x-v| < 1/k

As uk —> u in L^1 and m^ *^ m in L°% we conclude that (6.22) holds since

F(u,m;A) £ lim inf J y(Vuk.mfc) dx

= J V(0,m) dx + lim inf J [\|/(z(a - b)®v,y) dx
  X k→∞ |x·v|<1/k

= J \|/(0,m) dx + ^((a - b)®v) HN-I(APII(U)) .
  A

b) Consider u as in a) and let A c *Q* be an arbitrary open set in R^N. Let *n* be the plane *n* = { x-v = 0}. It is clear that[1]

n=l
A = ∪ (∪A$_n$)
400

where A$_n$ is an increasing finite collection of non-overlapping (i. e. with disjoint interiors) cubes $\bar{Q}$ of the form a* + e$\bar{Q}$v with edge length bigger than or equal to 1/n and such that

HN-I@Q nn) = 0. (6.23)

Thus, by Step 1 (iii) and applying a) to a decreasing sequence of open cubes whose intersection is the closed cube $\bar{Q}$ one has

F(u,m; $\bar{Q}$) ≤ J V(0,m) dx + J Y~((a - b)®v) dHN-l(x)
  $\bar{Q}$ Ku)n$\bar{Q}$

and so

F(u,m;A) ≤> lim F(u,m; uAn) ≤ lim X F(u,m; $\bar{Q}$)
  n→∞ n→∞ $\bar{Q}$∈A$_n$

[1] We use the notation uA := {x: there exists Y € A such that x e Y}.

$$\leq \lim_{n\to\infty} \sum_{\bar{Q}\in A_n} \Big[\int_{\bar{Q}} \psi(0,m)\, dx + \int_{\Sigma(u)\cap\bar{Q}} \psi^\infty((a-b)\otimes\nu)\, dH_{N-1}(x)\Big].$$

By (6.23) and Lebesgue's Monotone Convergence Theorem we conclude that

$$\begin{aligned} \mathcal{F}(u,m;A) &\leq \liminf_{n\to\infty} \Big[\int_{\cup A_n} \psi(0,m)\, dx + \int_{\Sigma(u)\cap(\cup A_n)} \psi^\infty((a-b)\otimes\nu)\, dH_{N-1}(x) \\ &= \int_A \psi(0,m)\, dx + \int_{\Sigma(u)\cap A} \psi^\infty((a-b)\otimes\nu)\, dH_{N-1}(x). \end{aligned}$$

c) Now suppose that u has polygonal interface i.e. $u = \chi_E a + (1-\chi_E)b$ where E is a polyhedral set i.e. E is a bounded, strongly Lipschitz domain and $\partial E = H_1 \cup \ldots \cup H_M$, H_i are closed subsets of hyperplanes of the type $\{x\cdot\nu_i = \alpha_i\}$. Let A be an open set contained in Ω and let $I = \{i \in \{1,\ldots,M\}: H_{N-1}(H_i\cap A) > 0\}$. If $A \cap \Sigma(u) = \varnothing$, i. e. if card $I = 0$ then $u \in W^{1,1}(A;\mathbb{R}^n)$ and it suffices to consider $u_k = u \in W^{1,1}(A;\mathbb{R}^n)$, $m_k = m$, with (6.22) reducing to

$$\mathcal{F}(u;A) \leq \int_A \psi(0,m)\, dx.$$

The case card $I = 1$ was studied in part b) where E is a large cube so that $\Sigma(u)\cap\Omega$ reduces to the flat interface $\{x\cdot\nu = 0\}$. Using an induction procedure, assume that (6.22) is true if card $I = k$, $k \leq M-1$. We prove it is still true if card $I = M$. Assume that

$$\partial E \cap A = (H_1 \cap \Omega) \cup \ldots \cup (H_M \cap \Omega)$$

and consider $S := \{x \in \mathbb{R}^N: \text{dist}(x, H_1) = \text{dist}(x, H_2 \cup \ldots \cup H_M)\}$. Note that $H_{N-1}(S \cap \Sigma(u)) = 0$ because $H_{N-1}(H_i \cap H_j) = 0$ for $i \neq j$. Fix $\delta > 0$ and let

$$U_\delta = \{x \in \mathbb{R}^N: \text{dist}(x,S) < \delta\},$$

$$U_\delta^- = \{x \in \mathbb{R}^N: \text{dist}(x,S) < \delta, \text{dist}(x, H_1) < \text{dist}(x, H_2 \cup \ldots \cup H_M)\},$$

$$U_\delta^+ = \{x \in \mathbb{R}^N: \text{dist}(x,S) < \delta, \text{dist}(x, H_1) > \text{dist}(x, H_2 \cup \ldots \cup H_M)\}.$$

Let

$$A_1 = \{x \in A : \text{dist}(x, H_1) < \text{dist}(x, H_2 \cup \ldots \cup H_M)\}.$$

Clearly Ai is open and Ai n (H2 u ... *KJ* HM) = 0. We apply the induction hypothesis to Ai and to A\Aī := A2 to obtain sequences Uk e W^CA^R"), Vk € WU(A_2;R^n)m_ke L-(Ai;Rd)Ak€ L~(A_2;R^d) such that u_k -» u inLl(Ai;R"), v_k-* u inLHA^R"), 1% **• m in L~(Ai;R^d), A* *± m inL~(A2;R^d) and

lim J y(Vuk,m_k) dx £ f \|/(0,m) dx + f v~((a - b)0v) dH_N-i(x) + |,
k-^~Ai Ai Ku)nAi

lim I \|f(Vv_k,X_k) dx £ J \|/(0,m) dx+ f v~((a - b)®v) dHN-i(x) + |.
k-»~ A_2 A_2 Ku)7^A_2

We will use the "slicing method" to connect u_k to v_k. Let pk be mollifiers and define

wk(x) := (Pk*u)(x) = Jpk(x-y) u(y) dy.
B(x,l/k)

As p ^ 0, supp p = 5(0,1) and

fpdx = 1,
B((Ū)

we have

HVwklloo ≤ Ck, supp Vw_k c {x € R^N: dist(x,E(u)) ≤ 1/k}. (5.23)

Let

a_k := -^llw_k-$v_k$ll$_L$i$_{(Ai)}$, L_k := k [1 + ll$w_k$lli,i + llvklli.i], Sk := $\frac{\alpha_k}{K_k}$

where [n] denotes the largest integer less than or equal to n, set Uj" = U_{g_i}, where 5i

= (1 - ak + i Sk), i = 1,..., Lk, and consider a family of cut-off functions
9i € W{,'~(U7$_i$), 0 ≤ <pi ≤ 1, <pi= 1 in U_iT_r llVq>i IL = O(^) for i = 1. . . . L_k.

Define

uj^Cx) := (1 - 9i(x))wk(x) + <pi(x)u_k(x),x€ A_L

Then

$u_k^{(i)}$ = w_k on $\partial A_1 \cap S$,

$\nabla u_k^{(i)} = \nabla u_k$ in U_{i-1}, $\nabla u_k^{(i)} = \nabla w_k$ in $A_1 \setminus U_i$ and

$\nabla u_k^{(i)} = \nabla w_k + \varphi_i(\nabla(u_k - w_k)) + (u_k - w_k) \otimes \nabla\varphi_i$ in $U_i \setminus U_{i-1}$.

Due to the growth condition (H2) we deduce that

$$\int_{A_1} \psi(\nabla u_k^{(i)}, m_k)\, dx \leq \int_{A_1} \psi(\nabla u_k, m_k) dx)$$

$$+ C \int_{U_i \setminus U_{i-1}} (1 + |w_k - u_k| \frac{l}{\delta} + |\nabla w_k| + |\nabla u_k|) dx + C \int (1 + |\nabla w_k|) dx$$

and averaging this inequality among all the layers $U_i \setminus U_{i-1}$ and by (5.23) we obtain

$$\frac{1}{M} \sum_{i=1}^{M} \int_{A_1} \psi(\nabla u_k^{(i)}, m_k) dx \leq \int_{A_1} \psi(\nabla u_k, m_k) dx$$

$$+ \frac{C}{M} \int_Q (1 + |\nabla w_k| + |\nabla v|) dx$$

$$+ \frac{C}{M\delta} \int_Q |w_k - v_k| dx + C (1 + n) |\{x \in U_\delta \cap A_1 : dist(x, \Sigma(u)) \leq 1/k\}|.$$

Thus, there must exist an index $i(k) \in \{1, \dots, M\}$ for which

$$\bar{u}_k := u_k^{(i(k))} \to u \text{ in } L^1(A_1; R^n),$$

and taking into account that $\Sigma(u)$ is a union of finitely many closed subsets of hyperplanes

$$\limsup_{k \to \infty} \int_{A_1} \psi(\nabla \bar{u}_k, m_k)\, dx \leq \int_{A_1} \psi(0, m)\, dx$$

$$+ \int_{\Sigma(u) \cap A_1} \psi^{\infty}((a - b) \otimes \nu)\, dH_{N-1}(x) + \frac{C}{M} + C H_{N-1}(\bar{U}_\delta \cap A_1 \cap \Sigma(u)).$$

Similarly, we may construct a sequence $\bar{v}_k$ such that

$$\bar{v}_k = w_k \text{ on } \partial A_2 \cap S, \qquad \bar{v}_k \to u \text{ in } L^1(A_2;\mathbb{R}^n),$$

$$\limsup_{k\to\infty} \int_{A_2} \psi(\nabla \bar{v}_k, \lambda_k)\, dx \le \int_{A_2} \psi(0,m)\, dx$$

$$+ \int_{\Sigma(u)\cap A_2} \psi^\infty((a-b)\otimes\nu)\, dH_{N-1}(x) + \frac{\delta}{2} + CH_{N-1}\,(U^-_\delta \cap A_2 \cap \Sigma(u)).$$

We set

$$\xi_k := \chi_{A_1}\bar{u}_k(x) + \chi_{A_2}(x)\,\bar{v}_k, \quad s_k := \chi_{A_1} m_k + \chi_{A_2}\lambda_k.$$

Clearly $\xi_k \in W^{1,1}(A;\mathbb{R}^n)$, $\xi_k \to u$ in $L^1(A;\mathbb{R}^n)$ and so

$$\mathcal{F}(u,m;A) \le \liminf_{k\to\infty} \int_A \psi(\nabla\xi_k, s_k)\, dx$$

$$\le \limsup_{k\to\infty} \int_{A_1} \psi(\nabla \bar{u}_k, m_k)\, dx) + \limsup_{k\to\infty} \int_{A_2} \psi(\nabla \bar{v}_k, \lambda_k)\, dx$$

$$\le \int_A \psi(0,m)\, dx + \int_{\Sigma(u)\cap A} \psi^\infty((a-b)\otimes\nu)dH_{N-1}(x) + \delta + CH_{N-1}(U_\delta \cap A_1 \cap \Sigma(u)).$$

As $H_{N-1}(S\cap\Sigma(u)) = 0$, letting $\delta \to 0$ we obtain (6.22)

f) Finally, if E is an arbitrary set of finite perimeter in Ω, by De Giorgi's approximating lemma there exists a sequence of polyhedral sets E_k such that

$$| E_k \Delta E | \to 0, \ \text{Per}_\Omega(E_k) \to \text{Per}_\Omega(E).$$

On the other hand, $y \to \psi^\infty((a-b)\otimes y)$ is a convex function (and so continuous) and positively homogeneous of degree one. Setting

$$u_k := a\chi_{E_k} + b(1 - \chi_{E_k}),$$

by Step 1, (i), (iii)

$$\mathcal{F}(u,m;A) \le \liminf_{k\to\infty} \mathcal{F}(u_k,m; A)$$

$$\le \liminf_{k\to\infty} \Big[\int_A \psi(0,m)\, dx + \int_{\Sigma(u_k)\cap A} \psi^\infty((a-b)\otimes\nu)\, dH_{N-1}(x) \Big]$$

$$= \int_A \psi(0,m)\, dx + \int_{\Sigma(u)\cap A} \psi^\infty((a-b)\otimes\nu)\, dH_{N-1}(x) .$$

This inequality together with Step 1, (iii) yields

$$F(u,m;\Sigma(u)) \leq \inf\{F(u,m;A): A \subset \Omega, A \text{ is open}, \Sigma(u) \subset A\}$$

$$\leq \inf\{ \int_A \psi(0,m)\, dx + \int_{\Sigma(u)\cap A} \psi^\infty((a-b)\otimes\nu)\, dH_{N-1}(x) : A \subset \Omega, A \text{ is open}, \Sigma(u) \subset A\}$$

$$= \int_{\Sigma(u)} \psi^\infty((a-b)\otimes\nu)\, dH_{N-1}(x)$$

and we conclude (6.21). The cases 2 and 3 are now obtained as in [AMT] Proposition 4.8, Steps 1 and 2, respectively.

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