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Computation of Magnetostrictive Materials

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2. CONTINUUM MODEL

In order to fix ideas and to facilitate computations we consider the following two dimensional model. Let Ω be a bounded region in $\mathbb{R}^2$, u the linear displacement, and $m = (m_1, m_2)$ the magnetization field with the constraint of saturation $|m| = 1$. The energy we consider is given by[3,11]

$$I(u,m) = \int_\Omega E(\epsilon, m)\, dx - \int_\Omega H_e \cdot m\, dx + \frac{1}{2}\int_{\mathbb{R}^2} |H|^2\, dx, \tag{3}$$

where H_e is the external constant magnetic field, H is the induced magnetic field, $\epsilon = \frac{1}{2}(\nabla u + \nabla u^T)$ is the linear deformation strain. $E(\epsilon, m)$ consists the sum of three parts: the elastic energy E_{el}, the magnetoelastic energy E_{me}, and the anisotropy energy E_{an},

$$\begin{aligned} E_{el}(\epsilon) &= \frac{1}{2}C_{11}(\epsilon_{11}^2 + \epsilon_{22}^2) + C_{12}\epsilon_{11}\epsilon_{22} + \frac{1}{2}C_{44}\epsilon_{12}^2, \\ E_{me}(\epsilon, m) &= B_1(\epsilon_{11}m_1^2 + \epsilon_{22}m_2^2), \\ E_{an}(m) &= \kappa(m_1^2 m_2^2), \kappa > 0, \end{aligned}$$

with $C_{11} - |C_{12}| > 0$ and $C_{44} > 0$. Maxwell's equations in dimensionless units are given by

$$\nabla \cdot B = 0 \qquad \text{and} \qquad \nabla \times H = 0$$

where $B = H + m : \mathbb{R}^2 \to \mathbb{R}^2$ is the magnetic induction. So the magnetic potential $\phi : \mathbb{R}^2 \to \mathbb{R}$ where $H = -\nabla\phi$ must satisfy the partial differential equation

$$\nabla \cdot (-\nabla\phi + m\chi_\Omega) = 0 \text{ in } \mathbb{R}^2 \tag{4}$$

where $\chi_\Omega(x)$ is the characteristic function of Ω:

$$\chi_\Omega(x) = \begin{cases} 1, & \text{if } x \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

We require that

$$\kappa \geq \frac{3B_1^2}{C_{11} - C_{12}}$$

in order for $\pm(1,0), \pm(0,1)$ to be easy directions.

Traditionally there is also the contribution to the bulk energy by the so–called exchange energy given by:

$$\varepsilon \int_\Omega |\nabla m(x)|^2\, dx$$

where ε is the exchange constant. Since our numerical experiment is concerned with the computation of the bulk magnetic properties of large bodies without resolving the finest scale in the microstructure of the materials we omit the exchange energy, we have found that the contribution of the exchange energy to the total energy will be much less than the numerical discretization error and will thus have a negligible effect on the numerical solution.

3. THE NUMERICAL APPROXIMATION

We consider for simplicity a rectangular domain $\Omega = [0, x_l] \times [0, y_l]$. We partition Ω first into $N_x \times N_y$ squares with sides of length h denoted by

$$\Omega_{ij} = \{x = (x_1, x_2) : ih < x_1 < (i+1)h,\ jh < x_2 < (j+1)h\}$$

for $i = 0, 1, \ldots, N_x - 1, j = 0, 1, \ldots, N_y - 1$, where $N_x \cdot h = x_l$ and $N_y \cdot h = y_l$. We partition then Ω_{ij} into triangles Ω_{ij}^+, Ω_{ij}^- illustrated in figure 1.

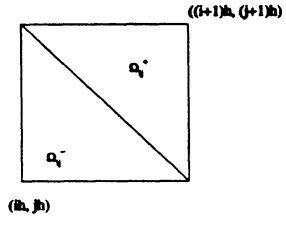

Fig. 1. Triangulation of the square Ω_{ij}.

We seek approximations to the displacement u in the space V_h, and approximations to the magnetization m in the space A_h where

$$V_h = \left\{ \begin{array}{ll} u: & u \text{ is piecewise linear on each } \Omega_{ij}^+,\ \Omega_{ij}^- \\ & \text{for } i = 0, 1, \ldots, N_x - 1, j = 0, 1, \ldots, N_y - 1 \end{array} \right\}$$

and

$$A_h = \left\{ \begin{array}{ll} m: & m \text{ is piecewise constant on each } \Omega_{ij} \\ & \text{for } i = 0, 1, \ldots, N_x - 1, j = 0, 1, \ldots, N_y - 1 \end{array} \right\}.$$

The hysteresis diagrams are computed by a method of continuation of minimizers with respect to changes in the external fields. The method of minimization that we implemented is the Polak-Ribière version of conjugate gradient method over the the approximate space $V_h \times A_h$ where we represent the constraint of saturation $|m| = 1$ by $m = (\cos\theta, \sin\theta)$ with θ piecewise constant on each Ω_{ij}. The conjugate gradient minimization algorithm requires the computation of energy and also the gradient of the energy with respect to the discrete variables for a given set of $(u, m) \in V_h \times A_h$. We remark that the most expensive part in these computations is the determination of the averages of $\nabla\phi$ on the cells Ω_{ij}, i.e.,

$$\overline{\nabla\phi}_{ij} = \frac{1}{h^2} \int_{\Omega_{ij}} \nabla\phi(x)\, dx.$$

See eg., Luskin and Ma[17]. Further details of the computation will be collected in a forthcoming paper.

We restrict to the case of $H_e = h_e \vec{n}$ where $\vec{n}$ is a fixed unit vector namely $\vec{n} = (1, 0)$. A homogeneous Dirichlet boundary condition for u was imposed for simplicity. For a given initial configuration (u, m) we

denote by $\mathcal{O}(u,m)$ the computed minimizer for the functional (3). Let H_{max} and H_{min} be the maximum and minimum external field strength respectively, n_h be a positive integer, and $\delta_h = (H_{max} - H_{min})/n_h$.

An algorithm for hysteresis

1. Initialize (u,m), set $h_e = h_0 = H_{min}$, and compute $(u_0, m_0) = \mathcal{O}(u,m)$.
2. For $k = 1, 2, \cdots, n_h$ set $h_e = h_k = H_{min} + k \cdot \delta_h$ and compute $(u_k, m_k) = \mathcal{O}(u_{k-1}, m_{k-1})$.
3. For $l = 1, 2, \cdots, n_h$ set $h_e = h_{n_h+l} = H_{max} - l \cdot \delta_h$
   compute $(u_{n_h+l}, m_{n_h+l}) = \mathcal{O}(u_{n_h+l-1}, m_{n_h+l-1})$.

The hysteresis diagram is the diagram of $(\overline{m}_k, h_k), k = 0, 1, 2, \cdots, 2n_h$ where for a given magnetization m, $\overline{m}$ is the average of m given by

$$\overline{m} = \frac{1}{|\Omega|} \int_\Omega m \cdot \vec{n}\, dx,$$

$|\Omega|$ denotes the area of the reference domain Ω.

4. COMPUTATIONAL RESULTS.

We describe in this section our experiences in the computation of the hysteresis diagrams for the linearly elastic magnetostrictive energy (3). In the following computations the parameters are taken to be $C_{11} = 1$, $C_{12} = 0.3$, $C_{44} = 1$, $B_1 = 0.1$, $H_{max} = 2.5$, $H_{min} = -2.5$, $xl = 2$, and $yl = 1$.

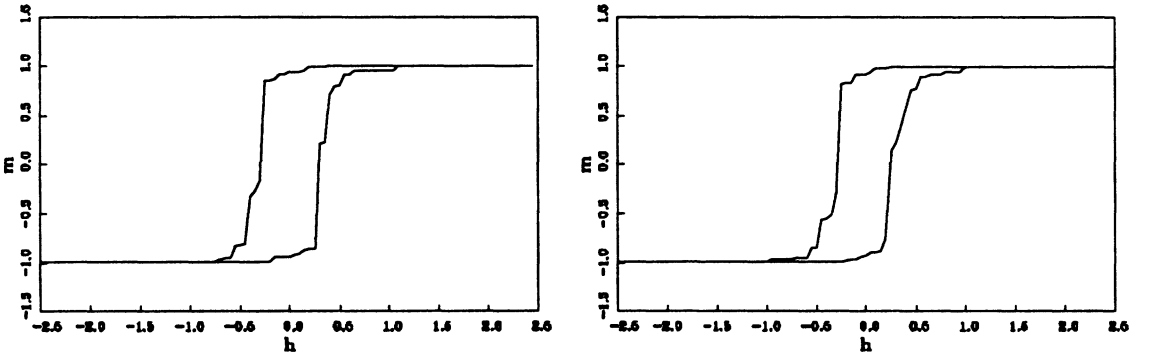

Fig. 2. hysteresis diagrams for two different N_x.

Two hysteresis curves are drawn in figures 2(a), 2(b) for $N_x \times N_y = 16 \times 8$ and 32×16 respectively. Observe that the gross features such as critical fields for the two curves are similar. The computation of the diagram in figure 2(b) takes about 37 minutes of CRAY C–90 CPU time at the speed of 168 MFLOPS. In the following figures the 32×16 grid will be used. Figure 3 is a hysteresis curve with a larger $\kappa = 0.9$ and also a minor loop. We note that the asymmetry of the hysteresis curve owes to the elastic effects. Such asymmetry can be reduced by applying the hysteresis algorithm a few times, each starting with the result of previous computations.

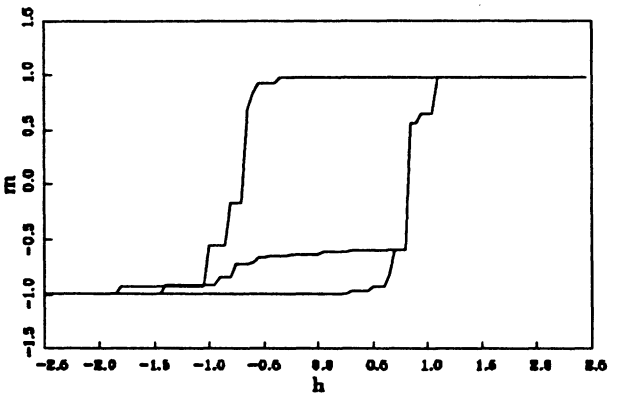

Fig. 3. Larger κ and a minor loop

Note in all the above figures the distinct stable, metastable and unstable regions. We comment here that the pure uniform field $m = (1, 0)$ is not an extremal for the functional (3) due to the boundary effects caused by the induced field. The stable states for the energy are nevertheless approximately uniform. A typical metastable region is drawn in figure 4, where we observe only the first and last columns of magnetization changes significantly from the uniform state. This we believe is the characterization of metastable states. On the other hand, unstable states are those magnetization where the interior magnetization varies from uniform field as well. A typical unstable state is drawn in figure 5.

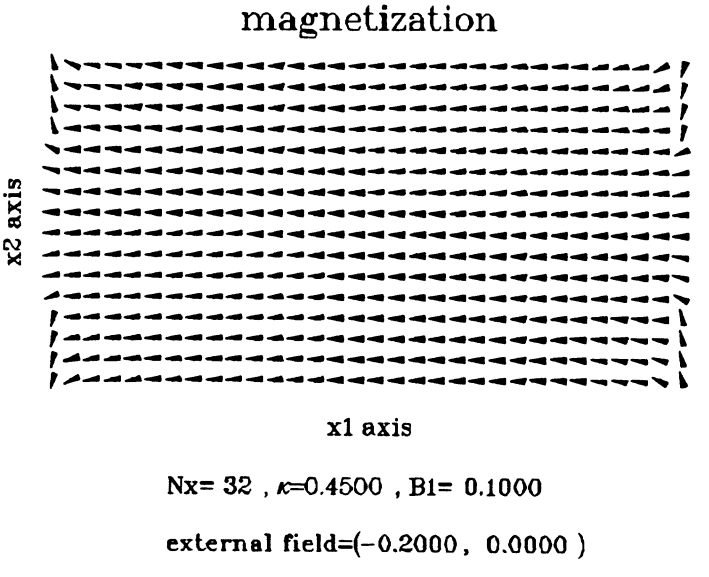

Fig. 4. A typical metastable configuration.

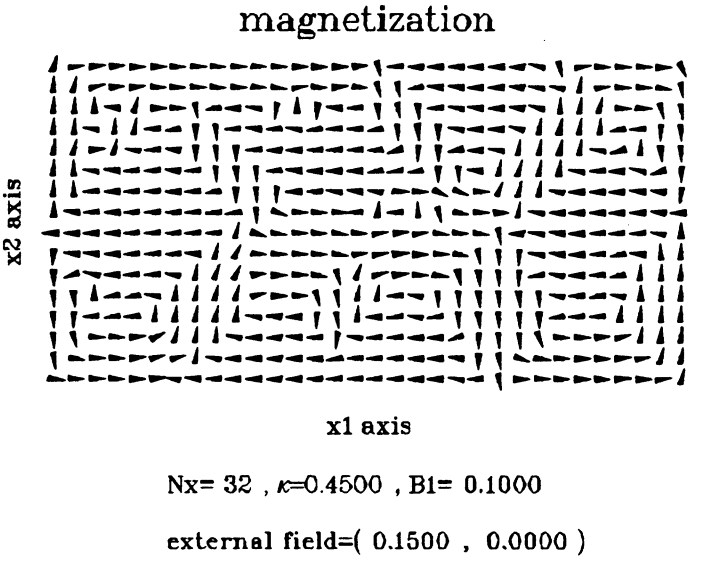

Fig. 5. A typical unstable configuration.

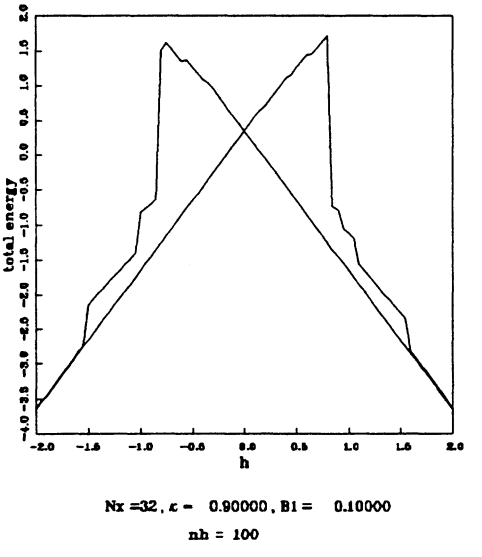

Fig. 6. The total energy vs the external field.

Figure 6 draws the total energy versus the external field.

We observe in the above hysteresis curves that for a fixed anisotropy constant κ the critical fields are essentially similar for different grid size h. Here we regard critical fields as the external fields corresponding to the metastable and the unstable regions. Precise investigation of this aspect is under way by the author and David Kinderlehrer. Preliminary results indeed suggest that the critical fields are independent of grid size h. We also note that when deformation and the induced field H are set to zero

so

$$I = I(m) = \int_\Omega E_{an}(m)\,dx - \int_\Omega H_e \cdot m\,dx,$$

the classically known Stoner–Wohlfarth instability is recovered.[14] These results suggest that the demagnetizing field governs the metastable regions and induces the unstable ones.

5. ACKNOWLEDGMENTS

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