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Order Integrands**

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Autonomous Second Order Integrands

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1. INTRODUCTION

The so-called *Lavrentiev phenomenon* is related to the question of whether the infimum of a functional on the class of absolutely continuous functions is strictly lower than the infimum of the same functional on a dense subspace of this class. This phenomenon was first introduced by M. A. Lavrentiev [16] in 1926. B. Maniá [18, 1934] later produced a simpler example than Lavrentiev's.

Since that time there has been additional work on various aspects of this phenomenon. Ball & Mizel [4,5, 1984-5] (see also Davie [10, 1988]) came up with the first fully regular integrands in one dimension for which the Lavrentiev phenomenon occurs. Loewen [17, 1987] reexamined Maniá's example and extended Angell's [2, 1979] results (see also Cesari [7, 1983]). His work has given conditions which are sufficient to preclude the Lavrentiev phenomenon. Clarke & Vinter [9, 1985] (see also Ambrosio, Ascenzi & Buttazzo [1, 1989]) show, in particular, that even for  $\mathbb{R}^n$ -valued  $u$  the Lavrentiev phenomenon cannot occur when the variational integrand  $f = f(x, u, \xi)$  is coercive and independent of  $x$ . Heinricher & Mizel [11,12,21, 1986-7] provided examples of stochastic control problems which exhibit the Lavrentiev phenomenon. They also [13,14, 1988] discuss the deterministic Lavrentiev phenomenon in one dimension, showing that integrands endowed with a certain *homogeneity* property are in a sense the borderline between integrands exhibiting the Lavrentiev phenomenon and those for which this phenomenon is absent. Ball & Knowles [3,15, 1987] (see also [20,24, 1990,1992]) have succeeded in the development of numerical approximation schemes which detect the low-energy *singular minimizers*. It appears

[22,23,8, 1989–1993] that for elastic materials the occurrence of the Lavrentiev phenomenon may signal the beginning of a fracture in the material. Recently, Buttazzo & Mizel [6, 1992] interpret the phenomenon as a relaxation effect.

The present article clarifies the result presented in [9] (see also [1,6]) ensuring the non-existence of a Lavrentiev gap in the case of autonomous first order integrands on  $\mathbb{R}^n$ -valued  $u$ . We present theorems and related examples showing that for certain autonomous *second order* integrands a Lavrentiev gap will occur for a dense subclass  $W^{2,p_0}(0,1)$  of the function space  $W^{2,1}(0,1)$  if we focus on *nonnegative* functions ( $u \geq 0$ ). On the other hand, it typically fails to take place for smooth integrands over the full function space (i.e., if we allow functions possessing mixed sign). For more results on first order and second order integrands exhibiting the Lavrentiev phenomenon we refer to [8], where conditions for the occurrence of the Lavrentiev phenomenon for general types of both first order integrands and second order integrands have been presented; in addition the results developed there have been applied to certain models in nonlinear elasticity.

We consider the problem of minimizing a functional of the calculus of variations of the form

$$I(u) \stackrel{\text{def}}{=} \int_0^1 f(u(x), u'(x), u''(x)) dx.$$

For given  $\beta, \beta' > 0$ ,  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^+$ , where  $f = f(u, \xi, \theta)$  is assumed to satisfy the normalization condition

$$(N) \quad f(u, \xi, 0) \equiv 0,$$

we discuss the occurrence of a Lavrentiev gap between the following two problems: ( $1 < p \leq \infty$ )

$$(\mathcal{P}_1) \quad \inf_{u \in \mathcal{A}_1(S)} I(u); \quad (\mathcal{P}_p) \quad \inf_{u \in \mathcal{A}_p(S)} I(u),$$

where for a prescribed subset  $S$  of  $W^{2,1}(0,1)$

$$\mathcal{A}_q(S) \stackrel{\text{def}}{=} \{ u \in W^{2,q}(0,1) \cap S \mid u(0) = 0, u'(0) = 0; u(1) = \beta, u'(1) = \beta' \}.$$

Thus the question is whether  $m_1 < m_p$ , where  $m_q := \inf_{u \in \mathcal{A}_q(S)} I(u)$ . We denote this gap phenomenon by  $LP(1,p)$ . Hereafter minimizers of  $(\mathcal{P}_1)$  will be referred to as **absolute minimizers**, minimizers of  $(\mathcal{P}_p)$  will be referred to as **pseudominimizers**. Of course,  $m_1 \leq m_p$ .

As a simple example of the above second order problem in which the Lavrentiev phenomenon arises, consider first the integrand  $f$  defined by

$$f(u, \xi, \theta) = \xi^6 \theta^8,$$

with  $u(0) = 0$ ,  $u'(0) = 0$ ,  $u(1) = \beta > 0$ , and  $u'(1) = \beta' > 0$ . We have, by Jensen's inequality and the chain rule for absolutely continuous functions (cf. e.g., [19]),

$$\begin{aligned} \int_0^y [u'(x)]^6 [u''(x)]^8 dx &= \int_0^y \left( |u'(x)|^{3/4} u''(x) \right)^8 dx \\ &= \left( \frac{4}{7} \right)^8 \int_0^y \left( \frac{d}{dx} |u'(x)|^{7/4} \right)^8 dx \\ &\geq \left( \frac{4}{7} \right)^8 \frac{[u'(y)]^{14}}{y^7}. \end{aligned} \quad (1.1)$$

In particular it follows that  $I(u) \geq (4/7)^8 \beta'^{14} > 0$ .

Next consider the curve

$$\Gamma_0 := \{ (u, \xi) \mid \xi = u^{1/3} \text{ for } u \geq 0 \},$$

and introduce a weight function  $b = b(u, \xi)$  into the variational integrand  $f$  which vanishes along  $\Gamma_0$  and is positive everywhere else. Denoting the modified integrand by  $\tilde{f}$ , we have

$$\tilde{f}(u, \xi, \theta) = b(u, \xi) f(u, \xi, \theta).$$

If we take the drastic case

$$b(u, \xi) = \begin{cases} 0 & \text{if } (u, \xi) \in \Gamma_0; \\ 1 & \text{if } (u, \xi) \in \{ (u, \xi) \mid u, \xi \geq 0 \} \setminus \Gamma_0; \\ \infty & \text{otherwise,} \end{cases}$$

it is not hard to see that LP(1,2) occurs for this modified integrand  $\tilde{f}$  if we take  $\mathcal{S}$  to be  $W^{2,1}(0,1)$ . Indeed if the boundary data  $\beta, \beta'$  satisfy  $\beta = (2/3)^{3/2}$ ,  $\beta' = (2/3)^{1/2}$  then there is a trajectory  $u^* = u^*(x)$  in  $\mathcal{A}_1(\mathcal{S})$  which attains zero cost. We define  $u^*$  by

$$u^{*'}(x) = \sqrt{2x/3}.$$

This trajectory  $u^*$  is an element of  $W^{2,p}(0,1)$  for  $1 \leq p < 2$ . However if we restrict attention to the smaller class  $\mathcal{A}_2(\mathcal{S})$  then the absolute minimizer  $u^*$  is no longer admissible. It then follows from the previous estimate (1.1) that we have the Lavrentiev phenomenon LP(1,2) even if we consider  $W^{2,2}(0,1)$  trajectories which coincide with  $u^*$  for all  $x \in [y, 1]$  for some  $y > 0$ . The result also holds for other  $\beta, \beta'$  values, as will follow from Theorem 3.2 in Section 3. The definition of the function  $b$  is crucial to the occurrence of the above gap phenomenon, since for example if we take  $b$  given by (cf. [13,6])

$$b(u, \xi) = \begin{cases} 0 & \text{if } (u, \xi) \in \Gamma_0; \\ 1 & \text{otherwise,} \end{cases}$$

it will follow from Theorem 3.3 in Section 3 that the above gap phenomenon disappears. In the analogous first order problem where  $f = f(u, \xi)$ ,  $u(0) = 0$ ,  $u(1) = \beta > 0$ , and (N)  $f(u, 0) \equiv 0$ , it is easily seen that a minimizer  $u^*$  of  $I$  is necessarily monotone, so that in particular  $u^* \geq 0$  on  $[0, 1]$ . What we have seen here is that the gap phenomenon LP(1, 2) for the above second order problem will hold if we take  $\mathcal{S} = \{u \in W^{2,1}(0, 1) \mid u \geq 0\}$  (Actually, it follows from the proof of Theorem 3.2 that the gap persists for all  $\beta$  and  $\beta'$  satisfying  $0 < \beta'^3 \leq \beta \leq \beta' - \beta'^3/2$ ). The degenerate integrand  $\tilde{f}$  described above is not interesting in itself but it does provide insight into the mechanism behind the Lavrentiev phenomenon for autonomous second order integrands. If we take for instance  $\tilde{f}$  to be the polynomial integrand

$$\tilde{f}(u, \xi, \theta) = (u - \xi^3)^2 \theta^8,$$

corresponding to  $b(u, \xi) = (u\xi^{-3} - 1)^2$ , then it will follow from the main theorems of this article that the gap phenomenon LP(1, 2) is again present among *nonnegative* functions  $\mathcal{S} = \{u \mid u \geq 0\}$ , but the gap phenomenon disappears when we take  $\mathcal{S} = W^{2,1}(0, 1)$ . For the first class of problems which we discuss in this paper the integrand has the form  $f(u, \xi, \theta) = a(u, \xi)|\theta|^k$  and retains the following key properties of the initial example:

- There are zero-cost curves on the first quadrant of the  $(u, \xi)$  plane, curves which correspond to functions in  $W^{2,p}(0, 1)$  only for  $1 \leq p < p_0$ . The absolute minimizers follow these zero-cost curves and they are not admissible when the trajectories are restricted to  $W^{2,p_0}(0, 1)$ .
- The function  $a = a(u, \xi)$  has the following *homogeneity* property:

$$a(u\lambda^\gamma, \xi\lambda^{\gamma-1}) = \lambda^\alpha a(u, \xi) \quad \forall \lambda > 0,$$

for some  $\gamma \in (1, 2)$  and  $\alpha > 0$ .

This article is organized as follows. In Section 2 two main propositions are presented to demonstrate that there is a certain region in the first quadrant of the  $(u, \xi)$  plane over which every trajectory in the dense subclass  $\mathcal{A}_{p_0}(\mathcal{S})$  for  $\mathcal{S} = \{u \mid u \geq 0\}$  must cross, while such crossing can be avoided for the trajectories corresponding to the nonnegative absolute minimizers in the full space  $\mathcal{A}_1(\mathcal{S})$ . Section 3 is devoted to the analysis of the problem for certain types of autonomous second order integrands, by verifying that there is a minimal penalty for the above crossing by every *nonnegative* trajectory, but that this penalty can be avoided by some functions of mixed sign. In Section 4 we discuss perturbations of the original problem. By considering additive perturbations, we can easily obtain a class of *fully regular* problems exhibiting the Lavrentiev gap. In the last section some examples are given which follow from our results.

## 2. MAIN PROPOSITIONS

**Proposition 2.1.** *Let  $\gamma \in (1, 2)$  and  $p_0 = 1/(2 - \gamma)$ . Suppose that  $u = u(x) \in W^{2,p_0}(0, 1)$  satisfies the conditions  $u(x_0) = 0$  and  $u'(x_0) = 0$ . Define  $Y_0(x) = u(x)|x - x_0|^{-\gamma}$  and  $Y_1(x) = u'(x)|x - x_0|^{1-\gamma}$ . Then*

$$Y_0(x) = o(1) \quad \text{and} \quad Y_1(x) = o(1) \quad \text{as } x \downarrow x_0.$$

**PROOF.** Let  $q_0 \in (1, \infty)$  satisfy  $1/p_0 + 1/q_0 = 1$ . For  $i = 0$  or  $1$  we have, with  $K_0(t) = \int_{x_0}^t u''(s) ds$  and  $K_1(t) = u''(t)$ ,

$$\begin{aligned} |Y_i(x)| &= \left| \int_{x_0}^x K_i(t) dt \right| |x - x_0|^{i-\gamma} \\ &= \frac{|x - x_0|^{i-\gamma}}{(1-i)!} \left| \int_{x_0}^x |x-t|^{1-i} u''(t) dt \right| \quad (\text{by Fubini's theorem if } i=0) \\ &\leq \frac{|x - x_0|^{i-\gamma}}{(1-i)!} \left| \int_{x_0}^x |(x-t)^{1-i}|^{q_0} dt \right|^{\frac{1}{q_0}} \left| \int_{x_0}^x |u''(t)|^{p_0} dt \right|^{\frac{1}{p_0}} \\ &\leq |x - x_0|^{i-\gamma+1-i+1/q_0} o(1) \\ &= o(1) \quad \text{as } x \downarrow x_0. \end{aligned}$$

The next to last inequality holds because of Hölder's inequality.  $\square$

**Proposition 2.2.** *Let  $\gamma \in (1, 2)$  and  $p_0 = 1/(2 - \gamma)$ . Suppose that  $u = u(x) \in W^{2,p_0}(0, 1) \cap \{u \mid u \geq 0\}$  satisfies the boundary conditions  $u(0) = 0$ ,  $u'(0) = 0$ , and  $u(1) = \beta > 0$ . Define*

$$x_0 = \max\{x \in [0, 1] \mid u(x) = u'(x) = 0\},$$

$Y(x) = u(x)|x - x_0|^{-\gamma}$ , and  $Q(x) = u'(x)|x - x_0|^{1-\gamma}$ . Let  $C, c_3$  be two arbitrary constants satisfying  $0 < c_3 < C\gamma < \beta\gamma$ . Then for every constant  $c_1$  and  $c_2$  satisfying  $0 < c_1 < c_2 < c_3 C^{(1-\gamma)/\gamma}$ , there exist  $x_0 < x_1 = x_1(c_1) < x_2 = x_2(c_2) < 1$  such that the following condition holds.

- $0 < Y(x) \leq C$ ,  $0 < Q(x) \leq c_3$ ,  $c_1 \leq u'(x)[u(x)]^{(1-\gamma)/\gamma} \leq c_2$  for all  $x \in [x_1, x_2]$ ,  $u'(x_1) = c_1 [u(x_1)]^{(\gamma-1)/\gamma}$ , and  $u'(x_2) = c_2 [u(x_2)]^{(\gamma-1)/\gamma}$ .

**PROOF.** Define  $G_4(x) = Y(x) - C$ . Since by Proposition 2.1  $Y(x)$  is  $o(1)$  as  $x \downarrow x_0$ ,  $G_4(x_0) < 0$ . On the other hand, we have  $G_4(1) \geq \beta - C > 0$ . By the Intermediate Value Theorem applied to the continuous function  $G_4(x)$ , there exists  $\hat{x} \in (x_0, 1)$  such that  $G_4(\hat{x}) = 0$ . Define

$$x_4 = \min\{\hat{x} \in (x_0, 1) \mid Y(\hat{x}) = C\}.$$

Thus we have  $0 \leq Y(x) \leq C$  for all  $x \in [x_0, x_4]$  and  $Y(x_4) = C$ . It is now easy by differentiation of  $u(x)|x - x_0|^{-\gamma}$  at  $x_4$  to see that this implies  $Q(x_4) \geq C\gamma$ .

Next, define  $G_3(x) = Q(x) - c_3$ . Since by Proposition 2.1  $Q(x)$  is  $o(1)$  as  $x \downarrow x_0$ , we have  $G_3(x_0) < 0$ . On the other hand, we have  $G_3(x_4) = Q(x_4) - C \geq C\gamma - c_3 > 0$ . By the Intermediate Value Theorem applied to the continuous function  $G_3(x)$ , there exists  $\bar{x} \in (x_0, x_4)$  such that  $G_3(\bar{x}) = 0$ . By letting

$$x_3 = \min \{ \bar{x} \in (x_0, x_4) \mid Q(\bar{x}) = c_3 \},$$

we obtain  $0 \leq Y(x) \leq C$ ,  $Q(x) \leq c_3$  for all  $x \in [x_0, x_3]$ , and  $Q(x_3) = c_3$ .

Finally, we define  $G_1(x) = Q(x) - c_1 [Y(x)]^{(\gamma-1)/\gamma}$  and  $G_2(x) = Q(x) - c_2 [Y(x)]^{(\gamma-1)/\gamma}$ . If there exists  $\bar{t} \in (x_0, x_3)$  such that  $Q(\bar{t}) \leq 0$  then  $G_1(\bar{t}) < 0$  by the definition of  $x_0$ , otherwise we have  $u(x), u'(x) > 0$  for  $x \in (x_0, x_3)$ . In this latter case, we shall show that  $\liminf_{t \rightarrow 0} Q(t) [Y(t)]^{(1-\gamma)/\gamma} = 0$ . Otherwise, for some  $k > 0$  there exists a  $t^* \in (x_0, x_3)$  such that  $u'(x) [u(x)]^{(1-\gamma)/\gamma} > k$  for all  $x \in (x_0, t^*]$ . This implies that, for  $x \in (x_0, t^*]$ ,

$$\int_{x_0}^x u'(t) [u(t)]^{(1-\gamma)/\gamma} dt > k(x - x_0).$$

But this in turn implies (by the chain rule)

$$u(x) > \left( \frac{k}{\gamma} \right)^\gamma (x - x_0)^\gamma,$$

which is a contradiction to Proposition 2.1 that  $u(x) = o(1)|x - x_0|^\gamma$  as  $x \downarrow x_0$ .

On the other hand, we have  $G_1(x_3) \geq c_3 - c_1 C^{(\gamma-1)/\gamma} > 0$ . By the Intermediate Value Theorem applied to the continuous function  $G_1(x)$ , there exists  $x_* \in (x_0, x_3)$  such that  $G_1(x_*) = 0$ . Let

$$x_1 = \max \{ x_* \in (x_0, x_3) \mid Q(x_*) = c_1 [Y(x_*)]^{(\gamma-1)/\gamma} \}.$$

Similarly, we can define

$$x_2 = \min \{ x^* \in (x_1, x_3) \mid Q(x^*) = c_2 [Y(x^*)]^{(\gamma-1)/\gamma} \}.$$

Therefore the proof of this proposition is complete.  $\square$

**Remark.** Proposition 2.2 is the key to the proof of the Lavrentiev phenomenon for second order integrands: any positive  $\mathcal{A}_{p_0}$  trajectory must pay a penalty bounded away from 0 for crossing the zone  $c_1 \leq u'(x)u(x)^{(1-\gamma)/\gamma} \leq c_2$ , while absolute minimizers can avoid this zone and hence avoid the penalty.

### 3. AUTONOMOUS INTEGRANDS



**Lemma 3.1.** Let  $a : \mathbb{R}^2 \rightarrow \mathbb{R}^+ \cup \{\infty\}$  be a function continuous at  $(0, 0)$  which satisfies the following homogeneity property,

$$a(u\lambda^\gamma, \xi\lambda^{\gamma-1}) = \lambda^\alpha a(u, \xi) \quad \forall \lambda > 0,$$

for some  $\gamma > 1$  and  $\alpha \geq 0 \in \mathbb{R}$ . If there exists  $(\bar{u}, \bar{\xi}) \neq (0, 0)$  such that  $a(\bar{u}, \bar{\xi}) < \infty$  then  $a(0, 0) = 0$ . Moreover, if there exist  $u^* \neq 0, \xi^* \neq 0$  such that  $a(u^*, \xi^*) = 0$  then for all  $u \neq 0$  with  $\text{sign}(u) = \text{sign}(u^*)$ , we have

$$a(u, \omega|u|^{(\gamma-1)/\gamma}) = 0,$$

where  $\omega = \text{sign}(\xi^*)|\xi^*||u^*|^{(1-\gamma)/\gamma}$ .

PROOF. The proof is left to the reader.  $\square$

**Theorem 3.2.** Let  $S = \{u \mid u \geq 0\}$  and consider the integrand  $f(u, \xi, \theta) = a(u, \xi)|\theta|^k \geq 0$  with  $k > 1$ . Assume that the function  $a$  is continuous and satisfies the following homogeneity property,

$$a(u\lambda^\gamma, \xi\lambda^{\gamma-1}) = \lambda^\alpha a(u, \xi) \quad \forall \lambda > 0,$$

for some  $\gamma \in (1, 2)$  and  $\alpha \geq 0$ . Define the "free zone"  $\mathbf{F}$  by

$$\mathbf{F} \stackrel{\text{def}}{=} \{\beta^* > 0 \mid a(\beta^*, \gamma\beta^*) = 0\}.$$

Assume that  $\mathbf{F} \neq \emptyset$  is bounded and nowhere dense. Let  $\beta_{\max}^* = \max\{\beta^* \mid \beta^* \in \mathbf{F}\}$ . Suppose that the boundary data  $\beta$  and  $\beta'$  satisfy

$$\frac{1}{\gamma} \left( \frac{\beta'}{\gamma\beta_{\max}^*} \right)^{1/(\gamma-1)} \leq \frac{\beta}{\beta'} \leq 1 - \frac{\gamma-1}{\gamma} \left( \frac{\beta'}{\gamma\beta_{\max}^*} \right)^{1/(\gamma-1)}. \quad (3.1)$$

Then

$$\alpha - k(2 - \gamma) \leq -1 \implies \text{LP}(1, p_0) \text{ occurs,}$$

where  $p_0 = 1/(2 - \gamma)$ .

PROOF. We begin appraising  $I$  over  $\mathcal{A}_{p_0}(S)$ , so let  $u = u(x) \in \mathcal{A}_{p_0}(S)$  be given and suppose that  $\alpha - k(2 - \gamma) \leq -1$ . By the homogeneity of  $a$  and setting  $\lambda = |\xi|^{1/(1-\gamma)}$ , we can express the function  $f$  as

$$f(u, \xi, \theta) = a(u|\xi|^{\gamma/(1-\gamma)}, \pm 1)|\xi|^{\alpha/(\gamma-1)}|\theta|^k.$$

Next, choose positive constants  $C, c_1, c_2$  and  $c_3$  satisfying  $0 < c_3 < C\gamma < \beta\gamma$ ,  $0 < c_1 < c_2 < c_3C^{(1-\gamma)/\gamma}$  such that the interval  $(c_1, c_2)$  contains no element of the set  $\mathbf{F}$ . By Proposition 2.2, for the above choice of constants there exists  $[x_1, x_2] \subset (0, 1)$  such that

$$u'(x_1) = c_1 [u(x_1)]^{(\gamma-1)/\gamma}, \quad u'(x_2) = c_2 [u(x_2)]^{(\gamma-1)/\gamma};$$

and

$$0 < u(x) \leq C, \quad c_1 \leq u'(x)[u(x)]^{(1-\gamma)/\gamma} \leq c_2 \quad \text{for all } x \in [x_1, x_2].$$

Hence from the fact that  $a$  is continuous, we conclude that for  $x \in [x_1, x_2]$  there exists a positive constant  $\delta_1 = \delta_1(c_1, c_2) > 0$  such that

$$a\left(u(x)[u'(x)]^{\gamma/(1-\gamma)}, \pm 1\right) > \delta_1 > 0.$$

Now, since  $u(x), u'(x) > 0$  for all  $x \in [x_1, x_2]$ , we have, setting  $\eta = \alpha/(\gamma - 1)$ ,

$$\begin{aligned} I(u) &\geq \delta_1 \int_{x_1}^{x_2} [u'(x)]^\eta |u''(x)|^k dx \\ &= \delta_1 \int_{u_1}^{u_2} [u'(x)]^{\eta+k-1} \left| \frac{du'(x)}{du} \right|^k du \\ &= \delta_1 \int_{u_1}^{u_2} \left| [u'(x)]^{(\eta+k-1)/k} \frac{du'(x)}{du} \right|^k du \\ &= \delta_1 \left( \frac{k}{\eta + 2k - 1} \right)^k \int_{u_1}^{u_2} \left| \frac{d}{du} [u'(x)]^{(\eta+2k-1)/k} \right|^k du. \end{aligned}$$

Let  $\mu = (\eta + 2k - 1)(\gamma - 1)/(k\gamma) > 0$  and  $\nu = (\eta + 2k - 1)/k > 0$ . By Jensen's inequality and use of the chain rule, we have

$$\begin{aligned} I(u) &\geq \delta_1 \left( \frac{k}{\eta + 2k - 1} \right)^k \int_{u_1}^{u_2} \left| \frac{d}{du} [u'(x)]^\nu \right|^k du \\ &\geq \delta_1 \left( \frac{k}{\eta + 2k - 1} \right)^k \left| \frac{[u'(x_2)]^\nu - [u'(x_1)]^\nu}{u_2 - u_1} \right|^k (u_2 - u_1) \\ &= \delta_1 \left( \frac{k}{\eta + 2k - 1} \right)^k \left| \frac{(c_2)^\nu (u_2)^\mu - (c_1)^\nu (u_1)^\mu}{u_2 - u_1} \right|^k \left( 1 - \frac{u_1}{u_2} \right) u_2. \end{aligned}$$

Let

$$A = \left[ 1 - \left( \frac{c_1}{c_2} \right)^\nu \left( \frac{u_1}{u_2} \right)^\mu \right] / \left( 1 - \frac{u_1}{u_2} \right).$$

Then we have

$$\begin{aligned} I(u) &\geq \delta_1 \left( \frac{k}{\eta + 2k - 1} \right)^k (c_2)^{\eta+2k-1} (u_2)^{1+k(\mu-1)} \left[ 1 - \left( \frac{c_1}{c_2} \right)^\nu \left( \frac{u_1}{u_2} \right)^\mu \right] A^{k-1} \\ &\geq \delta_1 \left( \frac{k}{\eta + 2k - 1} \right)^k [(c_2)^\nu - (c_1)^\nu]^k C^{1+k(\mu-1)} =: \delta(C, c_1, c_2) > 0. \quad (3.2) \end{aligned}$$

The last inequality above holds because  $\mu, \nu > 0$ ,  $1 + k(\mu - 1) \leq 0$ , and for  $x \in [x_1, x_2]$  we have  $u(x), u'(x) > 0$  with  $u(x) \leq C$ , so  $0 < u_1 < u_2 \leq C$ . Thus we have proved that  $m_{p_0} \geq \delta > 0$ .

Next we appraise  $I$  over  $\mathcal{A}_1(\mathcal{S})$ . Since  $\beta$  and  $\beta'$  satisfy (3.1), there exists  $x^* \in [0, 1)$  such that

$$x^* = 1 - \frac{\beta}{\beta'} - \frac{\gamma - 1}{\gamma} \left( \frac{\beta'}{\gamma \beta_{\max}^*} \right)^{1/(\gamma-1)}. \quad (3.3)$$

We define a function  $u^* = u^*(x)$  by

$$u^*(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq x^*; \\ \min\{\gamma \beta_{\max}^* (x - x^*)^{\gamma-1}, \beta'\} & \text{if } x^* \leq x \leq 1. \end{cases} \quad (3.4)$$

It is not hard to verify that  $u^* \in \mathcal{A}_1(\mathcal{S})$  is an absolute minimizer which attains  $m_1 = 0$ , since on the subinterval where  $u^* = \beta'$ ,  $u^{*''} = 0$ , so the second factor in  $f$  is zero, while on the subinterval where  $u^* = \gamma \beta_{\max}^* (x - x^*)^{\gamma-1}$ , the first factor in  $f$  is

$$\begin{aligned} a(u^*, u^{*'}) &= a(\beta_{\max}^* (x - x^*)^\gamma, \gamma \beta_{\max}^* (x - x^*)^{\gamma-1}) \\ &= (x - x^*)^\alpha a(\beta_{\max}^*, \gamma \beta_{\max}^*) \\ &= 0, \end{aligned}$$

because  $\beta_{\max}^* \in \mathbf{F}$ , and on the subinterval where  $u^* = 0$ , by Lemma 3.1 the first factor in  $f$  is  $a(0, 0) = 0$ . Thus we have  $m_{p_0} \geq \delta > m_1 = 0$ . Therefore  $\text{LP}(1, p_0)$  occurs.  $\square$

The positivity assumption  $\mathcal{S} = \{u \mid u \geq 0\}$  on the admissible functions is crucial in ensuring the occurrence of the Lavrentiev phenomenon for autonomous second order integrands. The next theorem shows that under the assumptions of the previous theorem the gap phenomenon will disappear if instead we take  $\mathcal{S}$  to be  $W^{2,1}(0, 1)$ , so that we allow functions with mixed sign.

**Theorem 3.3.** *Under the assumptions of Theorem 3.2, the Lavrentiev phenomenon  $\text{LP}(1, p_0)$  disappears if we take  $\mathcal{S}$  to be  $W^{2,1}(0, 1)$ .*

**PROOF.** Let the boundary data  $\beta$  and  $\beta'$  satisfy equation (3.1) and define  $x^*$  by equation (3.3). We separate the proof into two cases  $x^* > 0$  or  $x^* = 0$ .

*Case 1:  $x^* > 0$  (i.e.,  $\beta < \beta' [1 - \frac{\gamma-1}{\gamma} (\frac{\beta'}{\gamma \beta_{\max}^*})^{1/(\gamma-1)}]$ )*

In this case, we define an absolute minimizer by equation (3.4) and let  $\varepsilon > 0$  be given. We shall construct a function  $u_\varepsilon = u_\varepsilon(x) \in W^{2,\infty}(0, 1)$  satisfying  $u_\varepsilon(0) = 0$ ,  $u'_\varepsilon(0) = 0$ ,  $u_\varepsilon(1) = \beta$ , and  $u'_\varepsilon(1) = \beta'$  via a bifurcation of the absolute minimizer  $u^*$  at the point  $x^*$  in such a way that  $u_\varepsilon$  can avoid the zone  $c_1 \leq u'(x)[u(x)]^{(1-\gamma)/\gamma} \leq c_2$  as defined in Proposition 2.2 and hence avoid the penalty  $\delta$  as defined by (3.2), i.e.,  $I(u_\varepsilon) < \varepsilon$ . Let  $\rho > 0$  be sufficiently small (its actual value will be determined later). On the part of the trajectory  $u^*$  where  $u^{*'}(x) = \gamma \beta_{\max}^* (x - x^*)^{\gamma-1}$  in the  $(x, u')$  plane, we choose a point  $(t_4, \rho)$  such

that  $\zeta < x^*/4$  where  $\zeta := t_4 - x^* > 0$ . Of course we have  $\zeta \downarrow 0$  as  $\rho \downarrow 0$ . Now, we choose three points  $0 < t_1 < t_2 < t_3 < x^*$  such that  $t_1 = \zeta$ ,  $t_3 = x^*/2 + \zeta$ , and  $t_2 - t_1 = t_3 - t_2 = x^*/4$ . Notice that we have  $t_4 - t_3 = t_3 - t_1 = x^*/2$ . Next, we define a function  $u_\varepsilon = u_\varepsilon(x)$  by

$$u_\varepsilon'(x) = \begin{cases} u^*(x) & \text{if } t_4 \leq x \leq 1; \\ 2\rho(x - t_3)/x^* & \text{if } t_3 \leq x \leq t_4; \\ 4\tau(x - t_3)/x^* & \text{if } t_2 \leq x \leq t_3; \\ -4\tau(x - t_1)/x^* & \text{if } t_1 \leq x \leq t_2; \\ 0 & \text{if } 0 \leq x \leq t_1, \end{cases} \quad (3.5)$$

where  $0 < \tau < \rho$  is chosen such that  $u_\varepsilon(t_4) = u^*(t_4)$  and so  $u_\varepsilon(1) = \beta$ , namely  $(\rho - \tau)x^*/4 = \int_{x^*}^{t_4} u^*(x) dx$ . It is easily seen that  $|u_\varepsilon''(x)| < 4\rho/x^*$ ,  $|u_\varepsilon'(x)| < \rho$ , and  $|u_\varepsilon(x)| < \rho x^*/2$  if  $x \in [0, t_4]$ . Since the function  $a$  is continuous and  $a(0, 0) = 0$ , for an appropriate choice of  $\rho$  we have  $a(u_\varepsilon(x), u_\varepsilon'(x)) < 1$  for  $x \in [0, t_4]$ . Therefore

$$\begin{aligned} I(u_\varepsilon) &= \int_0^{t_4} a(u_\varepsilon(x), u_\varepsilon'(x)) |u_\varepsilon''(x)|^k dx \\ &\leq \int_0^{t_4} |u_\varepsilon''(x)|^k dx \\ &\leq \left| \frac{4\rho}{x^*} \right|^k t_4 < \varepsilon, \end{aligned}$$

if  $\rho < \varepsilon^{1/k} x^*/4$ . Hence the gap phenomenon  $LP(1, p_0)$  does not occur.

*Case 2:  $x^* = 0$  (i.e.,  $\beta = \beta' [1 - \frac{\gamma-1}{\gamma} (\frac{\beta'}{\gamma \beta_{\max}^*})^{1/(\gamma-1)}]$ )*

In this case, the construction of the bifurcation function  $u_\varepsilon$  as in the above case for the function  $u^*$  is no longer available since  $x^* = 0$ . But nevertheless we will still apply this technique to another function  $\tilde{u} = \tilde{u}(x) \in \mathcal{A}_p(S)$  for  $1 \leq p < p_0$  which has the properties that  $I(\tilde{u}) < \varepsilon$  and there is an  $0 < \tilde{x} < 1$ , corresponding to the role of  $x^*$  for the function  $u^*$ , such that  $\tilde{u}(x) = \tilde{u}'(x) = 0$  for  $x \in [0, \tilde{x}]$ . We shall have our analysis done on the first quadrant of the  $(u, \xi)$  plane. By Lemma 3.1 and normalization condition (N), we have a curve  $\Gamma_0$  of zero cost (i.e.,  $f = 0$  on this curve) on the first quadrant of the  $(u, \xi)$  plane, namely

$$\Gamma_0 = \left\{ (u, \xi) \mid \xi = \min \left\{ \gamma (\beta_{\max}^*)^{1/\gamma} u^{(\gamma-1)/\gamma}, \beta' \right\} \right\}.$$

By our assumption  $x^* = 0$ , it is not hard to see that the free curve  $\Gamma_0$  steers from  $(\beta, \beta')$  at  $x = 1$  to  $(0, 0)$  at  $x = 0$  when we take  $u = u(x)$  and  $\xi = u'(x)$ . Notice that since  $\Gamma_0$  is concave  $d\xi/du$  is decreasing and for  $\xi < \beta'$  each tangent line to the curve  $\Gamma_0$  lies above  $\Gamma_0$ . Let  $P = P(p_1, p_2)$  be a point on that part of  $\Gamma_0$  which is not flat and let  $Q = Q(q_1, q_2)$  be the point of intersection of the

tangent line  $T$  at point  $P$  with the horizontal line  $H : \xi = \beta'$ . Next, we define a curve  $\Theta$  in the  $(u, \xi)$  plane by

$$(u, \xi) \in \begin{cases} H & \text{for } q_1 \leq u \leq \beta; \\ T & \text{for } p_1 \leq u \leq q_1; \\ \Gamma_0 & \text{for } 0 \leq u \leq p_1. \end{cases}$$

Since  $dx = du/u'$ , it is easily seen that the curve  $\Theta$  will steer  $(u, \xi)$  from  $(\beta, \beta')$  at  $x = 1$  to  $(0, 0)$  at some  $\tilde{x} > 0$  when we take  $u = \hat{u}(x)$  and  $\xi = \hat{u}'(x)$ . Now define a function  $\tilde{u} = \tilde{u}(x)$  by

$$\tilde{u}'(x) = \begin{cases} \hat{u}'(x) & \text{if } \tilde{x} \leq x \leq 1; \\ 0 & \text{if } 0 \leq x \leq \tilde{x}. \end{cases}$$

Now it is not hard to see that for an appropriate choice of  $P$  close enough to the horizontal line  $H$  in the  $(u, \xi)$  plane the function  $\tilde{u}$  satisfies  $\tilde{u} \in \mathcal{A}_p(\mathcal{S})$  for  $1 \leq p < p_0$  and  $I(\tilde{u}) < \varepsilon$ . Hence the corresponding bifurcation function  $\tilde{u}_\varepsilon$  as defined by (3.5) for the function  $\tilde{u}$  will give that  $I(\tilde{u}_\varepsilon) < 2\varepsilon$ . Therefore the gap phenomenon  $\text{LP}(1, p_0)$  does not occur.  $\square$

#### 4. PERTURBATIONS OF THE INTEGRAND

In this section we provide an analysis for additive perturbations of autonomous second order integrands. In particular, it can be easily seen that the Lavrentiev phenomenon can occur even in cases where the modified integrand  $\tilde{f}$  satisfies the strict form of Tonelli's regularity and growth conditions:

$$\tilde{f}_{\theta\theta}(u, \xi, \theta) > 0 \quad \text{and} \quad \tilde{f}(u, \xi, \theta) > \varphi(\theta), \quad \text{where} \quad \liminf_{|\theta| \rightarrow \infty} \frac{\varphi(\theta)}{|\theta|} = \infty. \quad (4.1)$$

**Corollary 4.1.** *Let  $\mathcal{S} = \{u \mid u \geq 0\}$  and suppose that  $f = f(u, \xi, \theta)$  is a function satisfying the hypotheses of Theorem 3.2. Let  $u^* = u^*(x)$  denote an absolute minimizer as defined in the proof of Theorem 3.2 and consider*

$$\tilde{f}(u, \xi, \theta) = f(u, \xi, \theta) + \varepsilon e(u, \xi, \theta),$$

where  $\varepsilon > 0$  and  $e = e(u, \xi, \theta)$  is a nonnegative function satisfying  $e_{\theta\theta} > 0$ ,  $e(u, \xi, \theta) > \varphi(\theta)$  (with  $\varphi$  as in (4.1)) such that  $e(u^*, u^{*'}, u^{*''})$  is integrable on  $x \in (0, 1)$ . Then for sufficiently small  $\varepsilon$ , the variational problem of minimizing

$$\tilde{I}(u) = \int_0^1 \tilde{f}(u(x), u'(x), u''(x)) dx$$

exhibits  $\text{LP}(1, p_0)$ .

**PROOF.** Apply Theorem 3.2 directly by choosing an appropriate  $\varepsilon > 0$ .  $\square$

## 5. EXAMPLES

**Example 5.1.** Consider the integrand  $f$ ,

$$f(u, \xi, \theta) = (u - \xi^3)^2 |\theta|^k,$$

with  $u(0) = 0$ ,  $u'(0) = 0$ ,  $u(1) = \beta > 0$ , and  $u'(1) = \beta' > 0$ . This integrand  $f$  satisfies Theorem 3.2 with  $a(u, \xi) = (u - \xi^3)^2$ ,  $\gamma = 3/2$ ,  $\alpha = 3$ ,  $\mathbf{F} = \{(2/3)^{3/2}\}$ ,  $\beta_{\max}^* = (2/3)^{3/2}$ , and  $p_0 = 2$ . Let  $\beta'^3 \leq \beta \leq \beta' - \beta'^3/2$  and  $k \geq 8$ . Hence LP(1, 2) occurs among *nonnegative* functions, but fails when  $\mathcal{S} = W^{2,1}(0, 1)$ .

**Example 5.2.** Consider the integrand  $\tilde{f}$ ,

$$\tilde{f}(u, \xi, \theta) = (|u|^{4/9} - \xi)^{18} |\theta|^k + \varepsilon |\theta|^2,$$

with  $u(0) = 0$ ,  $u'(0) = 0$ ,  $u(1) = \beta > 0$ , and  $u'(1) = \beta' > 0$ . By Corollary 4.1, this problem can be treated as an additive perturbation for the function  $f$  defined by

$$f(u, \xi, \theta) = (|u|^{4/9} - \xi)^{18} |\theta|^k,$$

with the perturbation  $e(u, \xi, \theta) = |\theta|^2$ . The function  $f$  satisfies Theorem 3.2 with  $\gamma = 9/5$ ,  $\alpha = 72/5$ ,  $\mathbf{F} = \{(5/9)^{9/5}\}$ , and  $p_0 = 5$ . The function  $e(u^*, \xi^*, \theta^*) \sim x^{-2/5}$  is indeed integrable on  $x \in (0, 1)$ . Hence for sufficiently small  $\varepsilon > 0$ , if  $\beta'^{9/4} \leq \beta \leq \beta' - (4\beta'^{9/4}/5)$  then LP(1, 5) occurs among *nonnegative* functions if  $k \geq 77$ .

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