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Structured Deformations of Continua

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by

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CONTENTS

1. Introduction

A principal goal of continuum mechanics is to describe how a continuous body will deform under prescribed applied forces. An essential initial step towards this goal is that of choosing a class of deformations for the continuum. For the description of many continua, some generally accepted requirements on the chosen class of deformations have emerged: deformations should be invertible, differentiable mappings with differentiable inverses, and compositions of two deformations in the class should again be in the class. However, such classical deformations are not adequate for the description of all continua, and in many cases alternative choices must be made. One type of choice involves the introduction of supplementary kinematical variables such as the director fields of a polar continuum. Another choice involves the introduction of supplementary fields that, although related to deformation, have the status of internal variables. For example, in theories of plasticity, the plastic deformation tensor is governed by an evolution law included in the constitutive equations of the continuum.

Our goal in this paper is to provide a methodology for both the construction of classes of deformations appropriate for continua with supplementary kinematical variables and for the construction of classes of deformations appropriate for continua with internal variables. Our initial goal was narrower: we attempted to describe deformations appropriate for continua with fractures by removing the requirement of continuity made on classical deformations. Thus, we started from a class of deformations that can exhibit jumps of limited magnitude over surfaces with prescribed regularity. The main difficulty we encountered was in the choice of a class of regions to serve as the domains of deformations, i.e., a class of regions in space that the continuum with fractures can occupy. In order to allow the continuum to have unopened cracks, we had to generalize the notion of a fit region, introduced by NOLL & VIRGA [17], to that of a piecewise fit region. This concept permitted us to define a class of deformations, called simple deformations, rich enough to describe the formation and opening of cracks of a fairly general nature and to describe the smooth deformations, also called transplacements, of regions away from the crack sites.

In order to extend the scope of our description of fracture, we found it natural to consider limits of sequences of simple deformations. Among a variety of possible notions of limits, we chose one in which the crack site for the limit deformation is the limit inferior of the sequence of crack sites. This embodies the idea that a point of the continuum is in the crack site for the limit if, from some term on, it belongs to the crack site of every term in the sequence of simple deformations. In addition, our choice of limit requires that the sequence of transplacements for the simple deformations converge in the sense of L^∞ to a mapping called the transplacement for the limit, and that the sequence of gradients of transplacements converge in L^∞. Our choice of this particular notion of limit was dictated mainly for reasons of simplicity, and different choices are necessary for the inclusion of some types of deformations not covered by our choice.

The limits of sequences of simple deformations defined in this manner form a class of deformations that we denote by LimSid. A surprising feature of LimSid that emerges from our analysis is that the fractures associated with the terms of a sequence of simple deformations can diffuse throughout the continuum and yet the crack site of the limit can be the empty set. Moreover, the manner in which the fractures diffuse leads to limit deformations that may or may not be free from the effects of fractures. Mathematically, the difference between the presence and absence of the effects of fractures in the limit is reflected by the difference between G, the L^∞– limit of the sequence $n \longmapsto \nabla f_n$ of the gradients, and ∇g, the gradient of the L^∞ – limit of the sequence of transplacements $n \longmapsto f_n$. Indeed, this difference reveals a difference between the deformation due to smooth changes away from crack sites, measured by G, and the local deformation at the macroscopic level, measured by ∇g. This observation has led us far beyond our initial goal: not only does the class LimSid describe complicated processes of fracture at the macroscopic level, but also it permits us to identify processes of microfracture that describe a continuum with structure.

The limit procedure leading to the class LimSid turns out to yield some limits that correspond to the shrinking of portions of a body to single points and to other types of deformation that have no ready interpretation in most of the applications that we consider in this paper. Moreover, that procedure does not yield a natural way of composing limits of simple deformations. For these reasons, we identify another class Std whose elements we call structured deformations. Structured deformations are defined, without reference to a limit process, as triples (κ, g, G) in which κ is the crack site and g the transplacement associated with a simple deformation, and in which G is a tensor field having regularity properties similar to those of ∇g. We define a notion of composition of structured deformations for which the composition of two structured deformations is again a structured deformation. The main mathematical result of this paper, the Approximation Theorem (Theorem 5.8), shows that every structured deformation is a limit of simple deformations, i.e., Std is a subset of LimSid. Along with this result, we have the following relations between classes of deformations introduced in this paper:

LimSid

$\cup$

Std

$\subset$ $\supset$

Inv Std Sid

$\supset$ $\subset$

InvSid

$\cup$

Cld

where Cld denotes our choice for the set of classical deformations, Sid is the set of simple deformations, and InvSid and InvStd are the sets of those elements of Sid and Std, respectively, that have an inverse in a sense that we make precise.

After a study of the mathematical properties of Std, we describe classes of deformations appropriate to specific types of continua. In this description, we do not treat concepts, such as motions, that are linked to time; nor do we discuss the notion of stress and constitutive relations. There is a useful organization of the classes of deformations considered here that is based on decompositions of structured deformations established at various points in the paper and summarized here in the relation:

$$(\kappa,g,G) = (\emptyset,g,\nabla g) \circ (\emptyset,i,U) \circ (\emptyset,i,\varphi^{1/3}\, I) \circ (\kappa,i,I). \tag{1.1}$$

This decomposition involves (in the order from right to left) a fracture without any displacement, a purely microscopic deformation that creates voids without distortion, a purely microscopic deformation that distorts without creating voids, and a simple deformation without fracture. The first three factors, taken individually and then combined with a simple deformation without fracture, define deformations appropriate to continua with macrofracture, continua with voids, and continua with purely microscopic distortions, respectively. The last class includes the Cosserat continua.

In the last section we describe the application of structured deformations to some specific continua. We use measures of local deformation due to microfracture and local deformation without fracture to give precise kinematical meaning to the concepts of elastic and plastic deformation in plasticity, to the notions of director field and degree of orientation in liquid crystals, and to the continuous distributions of defects and lattice bases in theories of defective crystals. In addition, we attempt to describe deformations of mixtures within a collection of limits of simple deformations somewhat larger than the collection of structured deformations.

We conclude this introduction with some remarks on notation. We denote by $\mathcal{E}$ a Euclidean point space whose dimension in different circumstances will vary from one to three. The associated inner product space is denoted by $\mathcal{V}$, and $\mathrm{Lin}\,\mathcal{V}$ denotes the set of all linear mappings of $\mathcal{V}$ into itself. Both $\mathcal{V}$ and $\mathrm{Lin}\,\mathcal{V}$ are made into normed spaces with the norms

$$|\boldsymbol{a}|_{\mathcal{V}} := (\boldsymbol{a}\cdot\boldsymbol{a})^{1/2}, \qquad |\,A\,|_{\mathrm{Lin}\,\mathcal{V}} := \sup_{\boldsymbol{a}\in\mathcal{V}\setminus\{0\}} \frac{|A\boldsymbol{a}\,|_{\mathcal{V}}}{|\,\boldsymbol{a}\,|_{\mathcal{V}}}, \tag{1.2}$$

where the subscripts $\mathscr{V}$, Lin $\mathscr{V}$ will be omitted for simplicity. If the dimension of $\mathscr{E}$ is one, then $\mathscr{E}$, $\mathscr{V}$ and Lin $\mathscr{V}$ will be all identified with the real line $\mathbb{R}$. The empty set will be denoted by $\emptyset$. If $\mathscr{A}$ is a subset of $\mathscr{E}$, then by int, clo, bdy we denote the interior, the closure and the boundary of $\mathscr{A}$, respectively, and by vol $\mathscr{A}$ the volume (the Lebesgue measure) of $\mathscr{A}$. $\mathscr{B}(x,\delta)$ denotes the open ball centered at x with radius δ. The identity mappings in $\mathscr{E}$ and Lin $\mathscr{V}$ are denoted by i and I, respectively, $i_{\mathscr{A}}$ denotes the restriction of i to $\mathscr{A}$, and $I_{\mathscr{A}} : \mathscr{A} \longrightarrow \text{Lin}\,\mathscr{V}$ denotes the mapping $I_{\mathscr{A}}(x) := I$, $x \in \mathscr{A}$. With some abuse of notation, we use the symbol I in place of $I_{\mathscr{E}}$.

For mappings $f_1 : \mathscr{A} \longrightarrow \mathscr{E}$ and $f_2 : f_1(\mathscr{A}) \longrightarrow \mathscr{E}$, we define

$$(f_2 \circ f_1)(x) := f_2(f_1(x)), \qquad \text{for all } x \in \mathscr{A}, \tag{1.3}$$

and, if f_1 is injective, we define

$$f_1^{-1}(f_1(x)) := x, \qquad \text{for all } x \in \mathscr{A}. \tag{1.4}$$

To within evident changes in domains and codomains of mappings, $f_2 \circ f_1$ is the composition of f_2 and f_1, and f_1^{-1} is the inverse of f_1.

2. Classical deformations

Classically, a deformation of a continuous body is a mapping whose domain is the region in space initially occupied by the body. To each point in this region, the mapping assigns the point occupied after the deformation has occurred. An example of a collection of deformations is the class of all restrictions to open sets of C^n – diffeomorphisms between Euclidean spaces [13]. However, as discussed in more detail in the article [14], the choice of open sets as domains for deformations has some disadvantages. Indeed, not all open sets enjoy properties which render their boundaries surface–like, namely, the property of having an exterior normal defined at almost every point of the boundary and the property of satisfying even a generalized version of the Gauss–Green formula. Nevertheless, imposing specific regularity requirements on the boundary leads, in general, to the loss of some fundamental algebraic properties. For instance, the set of all open sets with piecewise continuously differentiable boundaries is not closed under finite intersection [17].

A collection of open sets having surface–like boundaries and yet enjoying nice algebraic properties has been identified by NOLL & VIRGA [17]. According to their definition, a subset $\mathcal{A}$ of the Euclidean space $\mathcal{E}$ is a fit region if (i) $\mathcal{A}$ is bounded, (ii) $\mathcal{A}$ is regularly open, i.e., $\mathcal{A}$ coincides with the interior of its closure, (iii) $\mathcal{A}$ has finite perimeter, and (iv) the boundary of $\mathcal{A}$ has zero volume. Among the properties of fit regions, several are relevant here:

(F1) the intersection of finitely many fit regions is a fit region;

(F2) C^1 – diffeomorphisms of $\mathcal{E}$ map fit regions into fit regions;

(F3) for almost every line L parallel to a given direction, to within a set of one–dimensional measure zero, the intersection $L \cap \mathcal{A}$ consists of finitely many pairwise disjoint closed intervals.

(F1) and (F2) are proved in the article [17], and (F3) expresses a known property of sets of finite perimeter (see e.g. [19], Sect. 4.2.2). It is worth noting that fit regions are not necessarily connected. Indeed, when $\mathcal{E}$ is one–dimensional, every fit region is a finite union of bounded open intervals whose closures are pairwise disjoint. When dim $\mathcal{E}$ is greater than one, examples provided in the paper [17] show that there are fit regions having infinitely many connected components.

As a starting point for our study we introduce a class of deformations which we call classical and which are appropriate for many branches of continuum mechanics.

2.1 Definition: Let $\mathcal{A}$ be a fit region of $\mathcal{E}$. A classical deformation from $\mathcal{A}$ is a mapping f from $\mathcal{A}$ into $\mathcal{E}$ satisfying:

(Cld 1) f can be extended to a C^1 – diffeomorphism of $\mathcal{E}$;

(Cld 2) f is orientation preserving, i.e.,

$$\det \nabla f(x) > 0 \quad \text{for all } x \in \mathcal{A}. \tag{2.1}$$

The set of all classical deformation from $\mathcal{A}$ will be denoted by $\mathrm{Cld}(\mathcal{A})$, and Cld will denote the set

$$\mathrm{Cld} := \{f \in \mathrm{Cld}(\mathcal{A}) \mid \mathcal{A} \text{ is a fit region in } \mathcal{E}\}.$$

This set has the following properties:

(D1) each $f \in \mathrm{Cld}$ is injective;

(D2) if $f_1 \in \mathrm{Cld}(\mathcal{A})$ and $f_2 \in \mathrm{Cld}(f_1(\mathcal{A}))$, then $f_2 \circ f_1 \in \mathrm{Cld}(\mathcal{A})$;

(D3) if $f \in \mathrm{Cld}(\mathcal{A})$, then f^{-1} belongs to $\mathrm{Cld}(f(\mathcal{A}))$.

Indeed, (D1) follows from (Cld 1), and (D2) and (D3) are consequences of the following facts: (i) by the property (F_2) of the class of fit regions, the image of a fit region under a C^1 – diffeomorphism of $\mathcal{E}$ is a fit region, (ii) the composition of two C^1 – diffeomorphisms of $\mathcal{E}$ and the inverse of a C^1 – diffeomorphism of $\mathcal{E}$ are C^1 – diffeomorphisms of $\mathcal{E}$. We remark explicitly that (i) implies that the image of a fit region under a classical deformation is a fit region.

The fact that each $f \in \text{Cld}$ is the restriction to a bounded set of a C^1 – diffeomorphism of $\mathscr{E}$ and the condition (2.1) imply that the determinant of ∇f is bounded below by a positive number m. By (D3), the determinant of ∇f^{-1} is bounded below by a positive number M^{-1}. Because the determinant of $\nabla f^{-1}(f(x))$ is the reciprocal of the determinant of $\nabla f(x)$, we are led to the following statement.

2.2 Proposition: Let $\mathscr{A}$ be a fit region of $\mathscr{E}$. For every $f \in \text{Cld}(\mathscr{A})$ it is possible to find two positive numbers m, M such that

$$m \leq \det \nabla f(x) \leq M \qquad \text{for all } x \in \mathscr{A}. \tag{2.2}$$

The set $\text{Cld}(\mathscr{A})$ can be made into a metric space using, for example, the metric

$$d(f_1,f_2) := \Big[\int_{\mathscr{A}} | f_1(x) - f_2(x) |^2 \, dx + \int_{\mathscr{A}} | \nabla f_1(x) - \nabla f_2(x) |^2 \, dx \Big]^{1/2} \tag{2.3}$$

associated with the norm of the Sobolev space $H^1(\mathscr{V})$. The resulting metric space is not complete. For example, if $\mathscr{E}$ is one–dimensional and $\mathscr{A} = (-1,1)$, the function f defined by

$$f(x) := \begin{cases} 2x & \text{for } \; 0 < x < 1, \\ x & \text{for } -1 < x \leq 0, \end{cases} \tag{2.4}$$

does not belong to $\text{Cld}(\mathscr{A})$, because it is not continuously differentiable, but it can be obtained as the limit in $H^1(\mathscr{V})$ of a Cauchy sequence in $\text{Cld}(\mathscr{A})$. This circumstance is very useful when one is interested in defining "generalized deformations" to be used, for example, as weak solutions for boundary value problems. However, not all limit elements of Cauchy sequences are of interest. For example, the sequence $n \longmapsto f_n$ defined by

$$f_n(x) = x/n, \quad -1 < x < 1,$$

is a Cauchy sequence whose limit is a constant function. Constant functions do not represent desirable deformations, because they map $\mathscr{B}$ into a single point. This suggests that a desirable set of generalized deformations should be a proper subset of some completion of Cld($\mathscr{B}$); we shall develop this idea in Sections 4 and 5 in the more general context of deformations allowing for fractures in the body.

3. Simple deformations

The main concept that we introduce in this section, that of a simple deformation, describes the geometrical changes associated with the formation and growth of cracks in a continuous body. Namely, we wish to describe the formation of cracks in an initially uncracked body, as well as the growth and the opening of the existing cracks in an initially cracked body. Fit regions are not adequate for this purpose. Indeed, property (ii) of fit regions (a fit region is regularly open) excludes regions representing a body containing unopened cracks. Moreover, functions that extend to C^1 – diffeomorphisms of $\mathscr{E}$ are not adequate for describing the discontinuities in displacement associated with the opening of a crack. For these reasons, we relax the regularity requirements on deformations made in the preceding section. The following definition provides a generalization of the notion of a fit region that includes the possibility of unopened cracks.

<u>3.1 Definition</u>: A subset $\mathscr{A}$ of $\mathscr{E}$ is a <u>piecewise fit region</u> if it is a finite union of fit regions.

For example, the open set

$$\mathscr{A} := ((-1,1) \times (-1,1)) \setminus \{ (x,0) \in \mathbb{R}^2 \mid -\tfrac{1}{2} \leq x \leq \tfrac{1}{2} \} \tag{3.1}$$

is not a fit region in $\mathbb{R}^2$ because it is not regularly open. Indeed, int clo $\mathscr{A} = (-1,1) \times (-1,1) \neq \mathscr{A}$. However, $\mathscr{A}$ is the union of the two fit regions

$$\mathscr{A}_1 := \mathscr{A} \setminus \{(x,y) \in \mathbb{R}^2 \mid -\tfrac{1}{2} \leq x \leq \tfrac{1}{2},\ -1 \leq y \leq 0\}, \tag{3.2}$$

$$\mathscr{A}_2 := \mathscr{A} \setminus \{(x,y) \in \mathbb{R}^2 \mid -\tfrac{1}{2} \leq x \leq \tfrac{1}{2},\ 0 \leq y \leq 1\}, \tag{3.3}$$

and, therefore, is piecewise fit. Properties (F1) – (F3) of fit regions have the following counterparts for piecewise fit regions:

(PF1) intersections and unions of finitely many piecewise fit regions are piecewise fit;

(PF2) C^1 – diffeomorphisms of $\mathscr{E}$ map piecewise fit regions into piecewise fit regions;

(PF3) for almost every line L parallel to a given direction, to within a set of one –dimensional measure zero, the intersection $L \cap \mathscr{A}$ consists of finitely many pairwise disjoint closed intervals.

Properties (PF1) and (PF2) are direct consequences of Definition 3.1, and (PF3) will be proved as a part of the proof of Theorem 3.8. In comparing (PF1) with (F1) we see that the class of piecewise fit regions is closed both under finite unions and finite intersections, whereas fit regions are closed only under finite intersections. Let us also recall that, when we speak of a finite union of fit regions, one or more of the fit regions may consist of infinitely many connected components. However, when $\mathscr{E}$ is one–dimensional, every piecewise fit region is a finite union of bounded open intervals.

We interpret the region $\mathscr{A}$ in (3.1) as a two–dimensional body with an unopened crack. Moreover, we interpret the replacement of the region $(-1,1) \times (-1,1)$ by $\mathscr{A}$ as the creation of the unopened crack

$$\kappa := ((-1,1) \times (-1,1)) \setminus \mathscr{A}.$$

More generally, when describing the deformation of a fractured continuum, one must prescribe two elements: the crack created in the deformation, and the position occupied after the deformation by each point of the body which is not on the crack. To this end, we make the following definition.

<u>3.2 Definition</u>: Let $\mathscr{A}$ be a piecewise fit region in $\mathscr{E}$. A <u>simple</u> <u>deformation</u> <u>from</u> $\mathscr{A}$ is a pair (κ, f), where κ is a subset of $\mathscr{A}$ and f is a mapping from $\mathscr{A} \setminus \kappa$ into $\mathscr{E}$, with the

following properties:

(Sid 1) vol $\kappa = 0$;

(Sid 2) f is injective;

(Sid 3) $\mathscr{A} \setminus \kappa$ is the union of finitely many fit regions such that the restriction of f to each of the fit regions is a classical deformation.

A finite collection $\mathbf{A} := \{\mathscr{A}_j \mid j \in \{1,...J\}\}$ of fit regions satisfying (Sid 3) for a simple deformation (κ,f) from $\mathscr{A}$ will be called <u>admissible</u> for (κ,f). We may think of f as a "piecewise classical deformation" in which each $\mathscr{A}_j$ undergoes the classical deformation

$$f_j := f\,|_{\mathscr{A}_j}. \tag{3.4}$$

Notice that not only $\mathscr{A}$ but also $\mathscr{A} \setminus \kappa$ is a piecewise fit region, as is clear from (Sid 3). Moreover, (Sid 3) combined with the property (F3) of fit regions ensures that the image of $\mathscr{A} \setminus \kappa$ under f is piecewise fit. Indeed, by (F3), the image of $\mathscr{A}_j$ under f_j is a fit region, so that the set

$$f(\mathscr{A} \setminus \kappa) = f\Big(\bigcup_{j=1}^{J} \mathscr{A}_j\Big) = \bigcup_{j=1}^{J} f(\mathscr{A}_j) = \bigcup_{j=1}^{J} f_j(\mathscr{A}_j)$$

is piecewise fit. Although f need not extend to a C^1 – diffeomorphism of $\mathscr{E}$, nevertheless, (Sid 2) and (Sid 3) imply that f is a C^1 – diffeomorphism. It is also an easy consequence of (Sid 3) and Proposition 2.2 that for any simple deformation (κ,f) from $\mathscr{A}$ there are positive numbers m, M such that

$$m \le \det \nabla f(x) \le M, \quad m \le |\nabla f(x)| \le M, \qquad \text{for all } x \in \mathscr{A} \setminus \kappa. \tag{3.5}$$

Because $\mathscr{B}$ is piecewise fit, it can describe a body with unopened cracks. When $\kappa \neq \emptyset$, $\mathscr{B} \setminus \kappa$ describes a body in which new cracks have been added to the pre–existing ones.

We denote by Sid ($\mathscr{B}$) the collection of all simple deformations from $\mathscr{B}$ and by Sid the set

$$\text{(3.6)} \qquad \text{Sid} := \{(\kappa,f) \in \text{Sid}(\mathscr{B}) \mid \mathscr{B} \text{ is piecewise fit}\}.$$

Clearly, for each classical deformation f, the pair $(\emptyset,f)$ obeys (Sid 1) – (Sid 3). Therefore we can regard each element of Cld as an element of Sid, and thereby identify Cld with a subset of Sid.

An important subclass of Sid is provided by piecewise affine simple deformations. These are defined to be the simple deformations for which there exists an admissible collection $\underline{A}$ such that each restriction f_j in (3.4) is affine. When $\mathscr{E} = \mathbb{R}$ and $\mathscr{B}$ is an interval of the real line, for a piecewise affine deformation (κ,f), κ consists of a finite number of points x_k, $k \in \{1...K\}$, in $\mathscr{B}$, and the restriction f_k of f to each interval (x_k, x_{k+1}) is of the form:

$$f_k(x) := a_k + b_k x, \qquad x_k < x < x_{k+1},$$

with a_k, b_k constants chosen in such a way that f is injective and orientation preserving. The intervals (x_k, x_{k+1}) then form an admissible collection for (κ, f). Indeed, each interval is a fit region; moreover, each of the restrictions f_k can be extended to the affine function $x \longmapsto a_k + b_k x$ which is an orientation preserving C^1 – diffeomorphism of $\mathbb{R}$. As a first example of a piecewise affine deformation, take $\mathscr{B}$ to be the interval (0,1), take κ to be the set

$$\text{(3.7)} \qquad \sigma_n := \left\{ \frac{h}{n} \mid h \in \{1,2,...n-1\} \right\},$$

and take f to be the broken ramp function

$$s_n(x) := x + \frac{k}{n}, \quad \frac{k}{n} < x < \frac{k+1}{n}, \quad k \in \{0, 1, \dots n-1\}, \tag{3.8}$$

with n a given positive integer. A related three–dimensional example which we will use later is that of the deck of cards, in which we take $\mathcal{A}$ to be the unit cube $(0,1) \times (0,1) \times (0,1)$, and κ to be the set

$$\tau_n := (0,1) \times (0,1) \times \sigma_n \tag{3.9}$$

with σ_n given by (3.7). This corresponds to slicing the cube with equidistant planes perpendicular to the x_3 – direction. Finally, we take f to be the function

$$t_n(x_1, x_2, x_3) := (x_1 + s_n(x_3) - x_3, x_2, x_3), \tag{3.10}$$

with s_n given by (3.8), which assigns to the k^{th} slice a rigid translation of amount k/n in the x_1 – direction.

An example of a deformation with fracture which is not a simple deformation is supplied by cavitation. For $\mathcal{E} = \mathbb{R}^2$, let $\mathcal{A}$ be the unit disc, let κ be the singleton consisting of the center of the disc, and let f be the mapping which maps the point with polar coordinates (r,φ) into the point with polar coordinates

$$f(r,\varphi) = (h(r) + c, \varphi), \tag{3.11}$$

where c is a positive constant and h is a continuously differentiable mapping of (0,1) into the reals that is monotone increasing and has right–hand limit $h(0+) > -c$. If we compute the $\varphi\varphi$– component of the gradient we find

$$(3.12)\qquad (\nabla f)_{\varphi\varphi} = \frac{h(r)+c}{r}.$$

Thus, $(\nabla f)_{\varphi\varphi}$ tends to $+\infty$ as $r \to 0$. If (κ, f) were a simple deformation, then by (Sid 3) there would be an admissible collection A of fit regions $\mathscr{B}_j$ such that the restriction f_j of f to each $\mathscr{B}_j$ is a classical deformation. Since A is finite, the center of $\mathscr{B}$ belongs to the closure of at least one of the $\mathscr{B}_j$. For the corresponding f_j, the gradient is unbounded by (3.12), and therefore there is no extension of f_j to a C^1 – diffeormorphism of $\mathbb{R}^2$. Thus, (Sid 3) is violated and (κ, f) is not a simple deformation.

We wish now to establish for simple deformations counterparts of properties (D1) – (D3) of classical deformations. A counterpart of (D1) is supplied by (Sid 2). The properties (D2), (D3) concerning the composition and the inverse require the definition of the corresponding operations in the class of simple deformations. Let us begin with the definition of composition.

<u>3.3 Definition</u>: Let $\mathscr{B}$ be a piecewise fit region of $\mathscr{E}$, let (κ, f) be a simple deformation from $\mathscr{B}$ and let (μ, h) be a simple deformation from $f(\mathscr{B} \setminus \kappa)$. Then the <u>composition</u> of (μ, h) and (κ, f) is the pair

$$(3.13)\qquad (\mu, h) \circ (\kappa, f) := (\kappa \cup f^{-1}(\mu),\ h \circ f\,|_{\mathscr{B}\setminus(\kappa \cup f^{-1}(\mu))}).$$

In this definition we have used the fact that, in a simple deformation (κ, f), the image of $\mathscr{B} \setminus \kappa$ under f is a piecewise fit region. We are now in position to prove a counterpart of property (D2) for simple deformations.

<u>3.4 Proposition</u>: Let $\mathscr{B}$, (κ, f) and (μ, h) be as in Definition 3.3. Then the composition defined in (3.13) is a simple deformation from $\mathscr{B}$.

<u>Proof</u>: We have to prove that the pair $(\kappa \cup f^{-1}(\mu), h \circ f\,|_{\mathscr{B}\setminus(\kappa\cup f^{-1}(\mu))})$ has the properties (Sid 1) – (Sid 3). (Sid 1) is satisfied, because by assumption κ has zero volume, and because

$f^{-1}(\mu)$ has zero volume as it is the image of a region with zero volume under a mapping f^{-1} which satisfies $(3.5)_1$. To prove (Sid 2), it is sufficient to remark that $h \circ f\,|_{\mathcal{B}\setminus(\kappa\cup f^{-1}(\mu))}$ is a composition of two injective mappings. To prove (Sid 3), take collections A_1, A_2 of fit regions $\mathcal{B}_{1j}$, $j \in \{1,..,J\}$, $\mathcal{B}_{2p}$, $p \in \{1,..,P\}$, admissible for $(\mathcal{B} \setminus \kappa, f)$ and $(f(\mathcal{B} \setminus \kappa)\setminus\mu, h)$ respectively. The collection

$$A := \{ \mathcal{B}_{1j} \cap f_j^{-1}(\mathcal{B}_{2p}) \mid j \in \{1,...,J\}, p \in \{1,...,P\}\}$$

then is admissible for $(\mathcal{B} \setminus (\kappa \cup f^{-1}(\mu)), h \circ f|_{\mathcal{B}\setminus(\kappa\cup f^{-1}(\mu))})$. Indeed, A is finite and is made up of fit regions, because $\mathcal{B}_{1j}$, $f_j^{-1}(\mathcal{B}_{2p})$ are fit, and the intersection of fit regions is fit. Moreover, since f_j and h_p are classical deformations, so are their restrictions to the fit regions $\mathcal{B}_{1j} \cap f_j^{-1}(\mathcal{B}_{2p})$ and $f_j(\mathcal{B}_{1j}) \cap \mathcal{B}_{2p}$, respectively. Hence, the restriction

$$(h \circ f)\,\Big|_{\mathcal{B}_{1j} \cap f_j^{-1}(\mathcal{B}_{2p})} = h\Big|_{f_j(\mathcal{B}_{1j}) \cap \mathcal{B}_{2p}} \circ f\Big|_{\mathcal{B}_{1j} \cap f_j^{-1}(\mathcal{B}_{2p})}$$

of $h \circ f$ to each element of A is the composition of two classical deformations and, therefore, is a classical deformation. ∎

The definition of composition of simple deformations permits us to state the following decomposition theorem for simple deformations.

<u>3.5 Proposition</u>: Every simple deformation (κ, f) from $\mathcal{B}$ admits the decomposition

$$(\kappa, f) = (\emptyset, f) \circ (\kappa, i_{\mathcal{B}\setminus\kappa}). \tag{3.14}$$

Moreover, the pairs $(\emptyset, f)$ and $(\kappa, i_{\mathscr{B}\setminus\kappa})$ are simple deformations from $\mathscr{B}\setminus\kappa$ and $\mathscr{B}$, respectively.

Proof: If $(\emptyset, f)$ and $(\kappa, i_{\mathscr{B}\setminus\kappa})$ are simple deformations, then (3.14) follows from the definition (3.13). Thus, we have only to prove that $(\emptyset, f)$ and $(\kappa, i_{\mathscr{B}\setminus\kappa})$ are simple deformations. $(\kappa, i_{\mathscr{B}\setminus\kappa})$ is a simple deformation from $\mathscr{B}$ because κ satisfies (Sid 1) by assumption and the identity mapping satisfies (Sid 2) and (Sid 3). $(\emptyset, f)$ is a simple deformation from $\mathscr{B} \setminus \kappa$ because $\mathscr{B} \setminus \kappa$ is piecewise fit, the empty set has volume zero, and f satisfies (Sid 2) and (Sid 3) by assumption. ■

Because the simple deformation $(\kappa, i_{\mathscr{B}\setminus\kappa})$ describes the creation of a new crack site κ and leaves every point of the body fixed, we call it a pure cracking. In contrast, $(\emptyset, f)$ is a simple deformation which does not involve the creation of a crack, and we call it a deformation without cracking. It is not difficult to see that, for a deformation without cracking $(\emptyset, f)$, the mapping f is a classical deformation from $\mathscr{B} \setminus \kappa$ only if the domain $\mathscr{B} \setminus \kappa$ is a fit region. Using the terminology just introduced, the decomposition in (3.14) can be expressed as follows: every simple deformation is the composition of a pure cracking and a deformation without cracking. There is only one simple deformation from $\mathscr{B}$ which is both a pure cracking and a deformation without cracking, and this is the identity deformation $(\emptyset, i_{\mathscr{B}})$.

If (κ, f) is a simple deformation from $\mathscr{B}$, it is natural to define right and left inverses of (κ, f) as pairs which, when composed with (κ, f) according to (3.13), result in the identity deformation:

$$(3.15)\qquad \begin{cases} (\lambda, \ell) \circ (\kappa, f) = (\emptyset, i_{\mathscr{B}}), \\ (\kappa, f) \circ (\rho, r) = (\emptyset, i_{f(\mathscr{B}\setminus\kappa)}). \end{cases}$$

The question of the existence and uniqueness of right and left inverses for a simple deformation is answered by the following proposition.

<u>3.6 Proposition</u>: Let (κ, f) be a simple deformation from $\mathcal{A}$. A right inverse (ρ, r) and a left inverse (λ, ℓ) exist if and only if $\kappa = \emptyset$. In this case,

$$(\rho, r) = (\lambda, \ell) = (\emptyset, f^{-1}).$$

<u>Proof</u>: If $\kappa = \emptyset$, then by composing the pair $(\emptyset, f^{-1})$ with $(\emptyset, f)$ we conclude that $(\emptyset, f^{-1})$ is both a right and a left inverse for $(\emptyset, f)$. Conversely, assume that a left inverse (λ, ℓ) of (κ, f) exists. Then from (3.13) and (3.15) it follows that

$$\kappa \cup f^{-1}(\lambda) = \emptyset,$$

and this implies $\kappa = \emptyset$ and $f^{-1}(\lambda) = \emptyset$. On the other hand, because λ is a subset of $f(\mathcal{A})$, $f^{-1}(\lambda) = \emptyset$ implies $\lambda = \emptyset$. Because $\lambda = \kappa = \emptyset$, we obtain again from (3.13) and (3.15)

$$\ell \circ f = i_{\mathcal{A}},$$

i.e., $\ell = f^{-1}$. Following the same lines it can be proved that, if a right inverse (ρ, r) exists, then $\kappa = \rho = \emptyset$ and $r = f^{-1}$. ∎

With this proposition we have proved that right and left inverses exist only for simple deformations without cracking. If they exist, they are unique and coincide, and we can speak simply of the <u>inverse</u> of a deformation without cracking. It is natural to refer to simple deformations without cracking as <u>invertible simple deformations</u> and to write InvSid for the set of invertible simple deformations.

With reference to the decomposition (3.14), we see that only the factor $(\kappa, i_{\mathcal{A}\setminus\kappa})$ does not have, in general, an inverse. This reflects the irreversibility attributed to the process of formation of a crack. For simple deformations, the property (D3) of classical deformations is replaced by: <u>the inverse of a simple deformation, whenever exists, is a simple deformation</u>. This property is established in the following proposition.

<u>3.7 Proposition</u>: The inverse of an invertible simple deformation $(\emptyset, f)$ from $\mathcal{A}$ is a simple deformation from $f(\mathcal{A})$.

<u>Proof</u>: (Sid 1) and (Sid 2) are verified trivially for $(\emptyset, f^{-1})$. To verify (Sid 3), we take an admissible collection $\underline{A}$ for $(\emptyset, f)$ and consider the collection

$$\underline{A}' := \{f(\mathcal{A}_j) \mid j \in \{1,\ldots,J\}\}.$$

Each $f(\mathcal{A}_j)$ is a fit region. Moreover, because $f|_{\mathcal{A}_j}$ is a classical deformation with range $f(\mathcal{A}_j)$, its inverse exists and is a classical deformation from $f(\mathcal{A}_j)$. Thus, $\underline{A}'$ has the properties required by (Sid 3). ∎

The last result in this section is a more technical one: we show that the fundamental theorem of calculus applies to simple deformations.

<u>3.8 Theorem</u>: Let $\mathcal{A}$ be a piecewise fit region of $\mathcal{E}$, let (κ, f) be a simple deformation from

$\mathscr{A}$, and let a be a unit vector in $\mathscr{V}$. For almost every line L parallel to a there hold:

(i) to within a set of one-dimensional measure zero, $L \cap \mathscr{A}\backslash\kappa$ consists of finitely many pairwise disjoint closed intervals I_q, $q \in \{1, ..., Q\}$;

(ii) for every $q \in \{1, ..., Q\}$, $f|_{I_q \cap \mathscr{A}\backslash\kappa}$ extends to a piecewise continuously differentiable function f^e on I_q;

(iii) for every $q \in \{1, ..., Q\}$ and every $x, y \in I_q$ with $y = x + |y - x|a$, the fundamental formula of calculus

$$(3.16) \qquad f(y-) - f(x+) = \int_0^{|y-x|} \nabla f(x + ta)a \, dt + \sum_z (f(z+) - f(z-))$$

holds, where z runs through the points of discontinuity of f^e in (x,y) and $f(w+)$ and $f(w-)$ denote the right and left limits (with respect to a) of f at w, respectively.

Before proving this theorem, we introduce a subdivision $\underline{B}$ of $\mathscr{A} \backslash\kappa$ which will be useful in subsequent developments, and we prove some properties of $\underline{B}$. Let $\{\mathscr{A}_j \mid j \in \{1,...,J\}\}$ be an admissible collection for $(\kappa,f) \in \text{Sid}(\mathscr{A})$. Then $\underline{B}$ is the collection $\{\mathscr{B}_j \mid j \in \{1, ..., J\}\}$ defined recursively by

$$(3.17) \qquad \begin{cases} \mathscr{B}_1 := \mathscr{A}_1, \\ \mathscr{B}_j := \text{int}(\mathscr{A}_j \backslash \bigcup_{p=1}^{j-1} \mathscr{B}_p), \quad j \in \{2,...,J\}. \end{cases}$$

3.9 Lemma: The subdivision $\underline{B}$ has the following properties for each $j, k \in \{1,2,...,J\}$:

(i) $\mathscr{B}_j \subset \mathscr{A}_j$, and $\mathscr{B}_j \cap \mathscr{B}_k = \emptyset$ if $j \neq k$;

(ii) $\mathscr{B}_j$ is a fit region;

(iii) $\bigcup_{j=1}^{J} \mathscr{B}_j$ differs from $\mathscr{A} \backslash \kappa$ by a set of volume zero.

<u>Proof of the Lemma:</u> Property (i) follows directly from the definition of $\underline{B}$. To prove (ii) we proceed by induction, observing that $\mathcal{B}_1 = \mathcal{A}_1$ is fit. Let $j \in \{2, \ldots, J\}$ be given and assume that $\mathcal{B}_p$ is fit for all $p \in \{1,\ldots,j-1\}$. Then $\mathrm{int}(\mathcal{A}_j \setminus \mathcal{B}_p)$ is fit because the interior of the difference of fit regions is fit [17]. Moreover,

$$\mathcal{B}_j = \mathrm{int}\,(\mathcal{A}_j \setminus (\bigcup_{p=1}^{j-1} \mathcal{B}_p)) = \mathrm{int} \bigcap_{p=1}^{j-1} (\mathcal{A}_j \setminus \mathcal{B}_p) = \bigcap_{p=1}^{j-1} \mathrm{int}\,(\mathcal{A}_j \setminus \mathcal{B}_p).$$

Therefore, $\mathcal{B}_j$ is fit because it is the finite intersection of fit regions. To prove (iii), we first introduce the notation

$$\mathcal{A} \approx \mathcal{B} \tag{3.18}$$

to mean that two subsets $\mathcal{A}$, $\mathcal{B}$ of $\mathcal{E}$ differ by a set of volume zero, and we proceed again by induction. Clearly, $\mathcal{B}_1 \approx \mathcal{A}_1$ by (3.17). We let $j \in \{1,2,\ldots, J-1\}$ be given and assume that $\bigcup_{p=1}^{j} \mathcal{B}_p \approx \bigcup_{p=1}^{j} \mathcal{A}_p$. We now can write

$$\bigcup_{p=1}^{j+1} \mathcal{B}_p = \left[\bigcup_{p=1}^{j} \mathcal{B}_p\right] \cup \mathcal{B}_{j+1} = \left[\bigcup_{p=1}^{j} \mathcal{B}_p\right] \cup \mathrm{int}\left[\mathcal{A}_{j+1} \setminus \bigcup_{p=1}^{j} \mathcal{B}_p\right] \approx$$

$$\approx \left[\bigcup_{p=1}^{j} \mathcal{B}_p\right] \cup \left[\mathcal{A}_{j+1} \setminus \bigcup_{p=1}^{j} \mathcal{B}_p\right] = \left[\bigcup_{p=1}^{j} \mathcal{B}_p\right] \cup \mathcal{A}_{j+1} \approx \bigcup_{p=1}^{j+1} \mathcal{A}_p.$$

Here we have used the fact that $\mathrm{int}\,\mathcal{A} \approx \mathcal{A}$ whenever $\mathrm{vol}(\mathrm{bdy}\,\mathcal{A}) = 0$, as is the case for the sets $\mathcal{A}_{j+1} \setminus \left[\bigcup_{p=1}^{j} \mathcal{B}_p\right]$ and $\left[\bigcup_{p=1}^{j} \mathcal{B}_p\right] \cup \mathcal{A}_{j+1}$ whose boundaries are subsets of the set $\bigcup_{p=1}^{j+1} \mathrm{bdy}\,\mathcal{A}_p$ having volume zero, due to the fact that each region $\mathcal{A}_p$ is fit. ■

<u>Proof of Theorem 3.8</u>: Take an admissible collection $\underline{A}$ for (κ,f) and consider the subdivision $\underline{B}$ defined in (3.17) in terms of $\underline{A}$. Consider first the region $\mathcal{B}_1$ of $\underline{B}$; since $\mathcal{B}_1$ is fit, it

follows from (F3) that, for almost every line L parallel to a,

$$L \cap \mathscr{B}_1 \overset{1}{\approx} \bigcup_{q_1=1}^{Q_1} I_{q_1,1} \tag{3.19}$$

with $I_{q_1,1}$, $q_1 \in \{1, ..., Q_1\}$, closed pairwise disjoint intervals. Here $\overset{1}{\approx}$ is the counterpart in one dimension of the symbol $\approx$ defined in (3.18). Among all lines satisfying (3.19), we select those for which

$$L \cap \mathscr{B}_2 \overset{1}{\approx} \bigcup_{q_2=1}^{Q_2} I_{q_2,2} \tag{3.20}$$

with $I_{q_2,2}$, $q_2 \in \{1, ..., Q_2\}$, closed and pairwise disjoint. Since $\mathscr{B}_1$ and $\mathscr{B}_2$ are disjoint, the interior of each interval $I_{q_1,1}$ is disjoint from the interior of each interval $I_{q_2,2}$. Again by (F3), almost every line L parallel to a satisfies simultaneously (3.19) and (3.20). Proceeding recursively, we find that almost every L parallel to a satisfies

$$L \cap \mathscr{B}_j \overset{1}{\approx} \bigcup_{q_j=1}^{Q_j} I_{q_j,j} \quad \text{for all} \quad j \in \{1,...,J\}, \tag{3.21}$$

with $I_{q_j,j}$, $q_j \in \{1, ..., Q_j\}$, $j \in \{1, ..., J\}$, closed intervals with pairwise disjoint interiors. Denote by I_q, $q \in \{1,...,Q\}$ the connected components of the finite union $\bigcup_{j=1}^{J} \bigcup_{q_j=1}^{Q_j} I_{q_j,j}$. It follows from (3.21) that

$$\bigcup_{j=1}^{J} (L \cap \mathscr{B}_j) \overset{1}{\approx} \bigcup_{j=1}^{J} \bigcup_{q_j=1}^{Q_j} I_{q_j,j} = \bigcup_{q=1}^{Q} I_q. \tag{3.22}$$

In order to prove (i), it is sufficient to prove that

$$(3.23)\qquad \bigcup_{j=1}^{J} (L \cap \mathscr{B}_j) \overset{1}{\approx} L \cap \mathscr{A} \setminus \kappa$$

for almost every line L parallel to a. To do this, we observe that, by the assertion (iii) of the preceding lemma,

$$(3.24)\qquad \mathrm{vol}\left[(\mathscr{A} \setminus \kappa) \setminus (\bigcup_{j=1}^{J} \mathscr{B}_j)\right] = 0,$$

and that, by Fubini's Theorem, almost every line parallel to a given direction intersects each set of volume zero in a set of one–dimensional measure zero. Therefore, (3.23) holds for almost every line to which (3.22) applies. This proves the first assertion.

To prove (ii) it is sufficient to consider the restrictions of f to each $\mathscr{A}_j$. As we know from (Sid 3), they are classical deformations, and therefore they extend to C^1 – diffeomorphisms f_j of the whole space $\mathscr{E}$. It is sufficient to set

$$(3.25)\qquad f^e(x) := \begin{cases} f_j(x) & \text{for } x \in \ \mathrm{int}\ I_{q_j,j} \quad \text{for } q_j \in \{1,\dots,Q_j\},\ j \in \{1,\dots,J\} \\ f(x) & \text{for } x \in (I_q \cap \mathscr{A} \setminus \kappa) \setminus \bigcup_j \bigcup_{q_j} \ \mathrm{int}\ I_{q_j,j} \\ x & \text{for } x \in I_q \setminus (\bigcup_j \bigcup_{q_j} \mathrm{int}\ I_{q_j,j} \cup \mathscr{A} \setminus \kappa\ ) \end{cases}$$

to get a piecewise continuously differentiable extension of f to I_q for each $q \in \{1, \dots, Q\}$. Indeed, $f^e(x) = f_j(x)$ for each $j \in \{1, \dots, J\}$, $q_j \in \{1, \dots, Q_j\}$, and $x \in \mathrm{int}\ I_{q_j,j}$. Moreover, I_q is a finite union of intervals $I_{q_j,j}$, and each f_j has a continuously differentiable extension to $I_{q_j,j}$. At this point, (iii) follows from the fundamental formula of calculus applied to each $[w_i, w_{i+1}] := I_{q_j,j} \cap [x, y]$:

$$(3.26) \qquad f_j(w_{i+1}) - f_j(w_i) = \int_0^{|w_{i+1}-w_i|} \nabla f_j(w_i+t\alpha)\alpha \, dt,$$

for all $i \in \{1, ..., Q_j\}$, and from the observation that, for every $j \in \{1, ..., J\}$, f_j is defined in the whole space and that $f_j(x) = f(x)$ and $\nabla f_j(x) = \nabla f(x)$ almost everywhere in $I_{q_j,j}$. Therefore, ∇f_j can be replaced by ∇f in (3.26). Moreover,

$$f_j(w_i) = f(w_i+), \qquad f_j(w_{i+1}) = f(w_{i+1}-),$$

so that the addition of (3.26) over all intervals $I_{q_j,j}$ forming I_q leads to (3.16). ■

4. Limits of simple deformations

We have introduced simple deformations (κ, f) in order to describe the creation and opening of cracks. This corresponds to the idea we have of macroscopic fracture: we think of κ as the site of new macroscopic fractures that are revealed through the discontinuities of f across κ.

Our purpose in this section is more general: we wish to describe deformations for which fractures are allowed to diffuse throughout the body. This process of diffusion is obtained here from a limiting procedure on sequences $n \mapsto (\kappa_n, f_n)$ of simple deformations.

4.1 Definition: Let $\mathscr{A}$ be a piecewise fit region of $\mathscr{E}$. By LimSid($\mathscr{A}$) we mean the set of all triples (κ,g,G), with $\kappa \subset \mathscr{A}$, $g \in L^\infty(\mathscr{A},\mathscr{E})$, $G \in L^\infty(\mathscr{A},\mathrm{Lin}\,\mathscr{V})$, for which there is a sequence $n \mapsto (\kappa_n,f_n)$ in Sid($\mathscr{A}$) such that:

(i) $$\kappa = \liminf_{n\to\infty} \kappa_n,$$

(ii) $$\lim_{n\to\infty} \| g - f_n \|_{L^\infty(\mathscr{A},\mathscr{V})} = 0,$$

(iii) $$\lim_{n\to\infty} \| G - \nabla f_n \|_{L^\infty(\mathscr{A},\mathrm{Lin}\mathscr{V})} = 0.$$

We denote by LimSid the set

(4.1) $$\mathrm{LimSid} := \{(\kappa,g,G) \in \mathrm{LimSid}(\mathscr{A}) \mid \mathscr{A} \text{ is piecewise fit}\}.$$

We call each element of LimSid a limit of simple deformations. If $(\kappa,g,G) \in$ LimSid($\mathscr{A}$) and if $n \mapsto (\kappa_n,f_n)$ is a sequence in Sid($\mathscr{A}$) satisfying (i) – (iii) in the above definition, then we say that $n \mapsto (\kappa_n,f_n)$ determines (κ,g,G). In (i), by $\liminf_{n\to\infty} \kappa_n$ we mean the set

(4.2) $$\kappa := \bigcup_{p=1}^{\infty} \bigcap_{n=p}^{\infty} \kappa_n.$$

This is the set of all points x of $\mathscr{A}$ for which there exists a p such that $x \in \kappa_n$ for all

$n \geq p$. We recall that, by (Sid 3), each $\mathscr{A} \setminus \kappa_n$ is the union of finitely many regions $\mathscr{A}_j$ such that the restriction of f_n to each $\mathscr{A}_j$ extends to a C^1–diffeomorphism of $\mathscr{E}$. This implies that both f_n and ∇f_n are bounded. Since κ_n has volume zero by (Sid 1), it follows that f_n and ∇f_n can be identified with elements of the Lebesgue spaces $L^\infty(\mathscr{A},\mathscr{E})$ and $L^\infty(\mathscr{A},\mathrm{Lin}\,\mathscr{V})$, respectively.

The space Sid imbeds naturally into the space LimSid. Indeed, to each $(\kappa,f) \in$ Sid we can associate the element $(\kappa,f,\nabla f)$ of LimSid determined by the constant sequence $n \mapsto (\kappa,f)$. The fact, made evident by the following examples, that there are elements (κ,g,G) of LimSid with $\nabla g \neq G$ shows that the imbedding of Sid in LimSid is not surjective.

4.2 Example (the broken ramp sequence): Let $\mathscr{E} = \mathbb{R}$, $\mathscr{A} = (0,1)$ and, for each $n \in \mathbb{N}$, take (κ_n,f_n) to be the pair (σ_n,s_n) defined by (3.7) and (3.8). It is easy to see that the sequence $n \mapsto (\sigma_n,s_n)$ determines the triple (κ,g,G), where κ is the empty set and g and G are given by

$$g(x) = 2x, \quad G(x) = 1, \quad 0 < x < 1. \tag{4.3}$$

4.3 Example (the dyadic broken ramp sequence): Consider the subsequence $n \mapsto (\kappa_{2^n},f_{2^n})$ of the broken ramp sequence. Since each set κ_{2^n} consists of dyadic rationals in $(0,1)$ and since $\kappa_{2^n} \subset \kappa_{2^m}$ for every $n \leq m$, the set κ consists of all dyadic rationals in $(0,1)$. On the other hand, the L^∞ – limits of $n \mapsto f_{2^n}$ and $n \mapsto \nabla f_{2^n}$ are the functions defined in (4.3). We conclude that the dyadic broken ramp sequence determines the triple (κ,g,G), with κ the set of all dyadic rationals in $(0,1)$ and g, G given by (4.3).

Three–dimensional counterparts of the above examples are given by sequences of "decks of cards" constructed using the sets τ_n and the functions t_n defined in (3.9), (3.10). The last example shows that $\mathscr{A} \setminus \kappa$ is not, in general, a piecewise fit region; indeed, the complement of the dyadic rationals is not an open subset of (0,1). The same example also illustrates the following property of limits of simple deformations.

4.4 Proposition: If a sequence $n \mapsto (\kappa_n, f_n)$ determines a triple (κ, g, G) in Lim Sid, then each subsequence determines a triple (κ', g', G') in LimSid with $g' = g$, $G' = G$ and $\kappa' \supset \kappa$, the inclusion being strict, in general.

Of the next two examples, the first one shows that Lim Sid includes triples in which the second entry may be a function as nasty as the Cantor function. The second one shows that, if $n \mapsto (\kappa_n, f_n)$ determines (κ, g, G) and if all the f_n are of bounded variation, then g need not be of bounded variation.

4.5 Example (The Cantor fracture): Let $\mathscr{E} = \mathbb{R}$, $\mathscr{A} = (0,1)$, and let φ be any continuous non–decreasing map of (0,1) onto itself. For each $n \in \mathbb{N}$ consider the points x_h, $h \in \{0,1,...,n\}$, defined by

(4.4) $$x_0 := 0, \quad x_h := \min\{x \in (0,1) \mid \varphi(x) = \tfrac{h}{n}\}, \quad h \in \{1,...,n\}.$$

That there is at least one x such that $\varphi(x) = h / n$ is ensured by the fact that φ is continuous and surjective. Define $n \mapsto (\kappa_n, f_n)$ as follows:

$$\kappa_n := \{x_1, ..., x_{n-1}\},$$

$$f_n(x) := x + \tfrac{h}{n}, \quad x_h < x < x_{h+1}, \quad h \in \{0,1,..., n-1\}.$$

Each (κ_n, f_n) is a piecewise affine simple deformation. Moreover, $n \mapsto (\kappa_n, f_n)$ determines a triple (κ,g,G) in LimSid $(\mathscr{A})$. Indeed, $\nabla f_n(x) = 1$ for all $x \in \mathscr{A} \setminus \kappa_n$, so that the L^∞–limit of ∇f_n is the constant function $x \longmapsto G(x) = 1$, and $n \mapsto f_n$ has as L^∞–limit the function $f(x) = x + \varphi(x)$. To see this, note that by (4.4)

$$f_n(x) - x = \varphi(x_h)$$

for every $x \in (x_h, x_{h+1})$ and $h \in \{0,1, \ldots, n-1\}$, so that, in view of the monotonicity of φ,

$$0 = x + \varphi(x_h) - f_n(x) \leq x + \varphi(x) - f_n(x) \leq x + \varphi(x_{h+1}) - f_n(x) = \varphi(x_{h+1}) - \varphi(x_h) = \frac{1}{n}.$$

It is now sufficient to observe that the Cantor function has all the properties assumed for φ to conclude that (κ,g,G) may have as second entry the identity plus the Cantor function.

4.6 Example: Let $\mathscr{E} = \mathbb{R}$, $\mathscr{A} = (0,1)$ and let (κ_n, f_n) be the piecewise affine deformation:

$$\kappa_n := \{\tfrac{1}{2}, \tfrac{1}{3}, \ldots, \tfrac{1}{2n+1}\},$$

$$f_n(x) := \begin{cases} x - \frac{1}{h} & \text{if } \frac{1}{2h+1} < x < \frac{1}{2h}, \quad h \in \{1,\ldots,n\}, \\ x & \text{otherwise.} \end{cases}$$

It can be verified that each f_n is injective, and that $n \longmapsto f_n$ and $n \longmapsto \nabla f_n$ have L^∞–limits that we denote by g and G. The total variation of f_n is

$$V(f_n) = 1 + 2 \sum_{h=1}^{n} \frac{1}{h} < +\infty,$$

and that of g is

$$V(g) = \lim_{n \to \infty} V(f_n) = +\infty.$$

Therefore, each f_n is of bounded variation but g is not.

In Section 3 we showed that the function defined in (3.11) and representing cavitation is not a simple deformation. The following example shows that such a function may represent the second entry of an element of LimSid. For simplicity, we restrict ourselves to the special case $h(r) = r$.

4.7 Example: Let $\mathcal{A}$ be the unit disc of $\mathbb{R}^2$ without its center, and let $n \longmapsto (\kappa_n, f_n)$ be the sequence defined by

$$\kappa_n := \{x \mid r(x) \in (0,1)\,,\ \varphi(x) = \frac{2\pi h}{n},\ \ h \in \{0,1,\dots,n-1\}\},$$

$$f_n(x) := x + c\,(\cos\frac{2\pi h}{n},\ \sin\frac{2\pi h}{n}),\quad \frac{2\pi h}{n} < \varphi(x) < \frac{2\pi(h+1)}{n},\ h \in \{0,1,\dots,n-1\}.$$

Here c is a positive constant, and $r(x)$ and $\varphi(x)$ are the polar coordinates of x. Each $f_n(x)$ represents the piecewise rigid deformation in which each sector $2\pi h/n < \varphi(x) < 2\pi(h+1)/n$ experiences a translation of amount c in the radial direction $\varphi = 2\pi h/n$.

It is easy to verify that each (κ_n, f_n) is a simple deformation from $\mathcal{A}$ and that the L^∞–limit of $n \longmapsto \nabla f_n$ is the constant function $G(x) = I$. Moreover, the sequence $n \longmapsto f_n$ converges in L^∞ to the function g given by

$$g(x) := x + c\,(\cos\varphi(x),\ \sin\varphi(x)).$$

Indeed, by direct computation we find that

$$|f_n(x) - g(x)| = 2c\,|\sin\tfrac{1}{2}(\tfrac{2\pi h}{n} - \varphi(x))| \le c\,|\tfrac{2\pi h}{n} - \varphi(x)| \le \tfrac{2\pi c}{n}.$$

Our arguments prove that cavitation can indeed represent the second item of an element (κ, g, G) of LimSid. It is also interesting to remark that, in the present example, κ turns out to be the set

$$\kappa := \{x \in \mathcal{A} \mid r(x) \in (0,1),\ \varphi(x) = 0\}.$$

In general, it is not possible to compose limits of simple deformations. However, it is possible to compose a limit of simple deformations with a simple deformation, and the result is a limit of simple deformations.

4.8 Definition: Let $(\kappa,f) \in \mathrm{Sid}(\mathcal{A})$, and let $(\mu,h,H) \in \mathrm{LimSid}\,(f(\mathcal{A}\setminus\kappa))$. Then the composition of (μ,h,H) with (κ,f) is the triple

$$(4.5) \qquad (\mu,h,H) \circ (\kappa,f) := $$

$$= (\kappa \cup f^{-1}(\mu),\ h \circ f|_{\mathcal{A}\setminus(\kappa\cup f^{-1}(\mu))},\ (H \circ f|_{\mathcal{A}\setminus(\kappa\cup f^{-1}(\mu))})\ \nabla f|_{\mathcal{A}\setminus(\kappa\cup f^{-1}(\mu))}).$$

4.9 Proposition: Let $(\kappa,f) \in \mathrm{Sid}\,(\mathcal{A})$ and $(\mu,h,H) \in \mathrm{LimSid}\,(f(\mathcal{A}\setminus\kappa))$. It follows that $(\mu,h,H) \circ (\kappa,f) \in \mathrm{LimSid}(\mathcal{A})$. Specifically, for each sequence $n \mapsto (\mu_n,h_n) \in \mathrm{Sid}(f(\mathcal{A}\setminus\kappa))$ that determines (μ,h,H), the sequence

$$n \mapsto ((\mu_n,h_n) \circ (\kappa,f)) = n \mapsto (\kappa \cup f^{-1}(\mu_n),\ h_n \circ f|_{\mathcal{A}\setminus(\kappa\cup f^{-1}(\mu_n))})$$

determines $(\mu,h,H) \circ (\kappa,f)$.

Proof: We have

$$\liminf_{n\to\infty} (\kappa \cup f^{-1}(\mu_n)) = \bigcup_{p=1}^{\infty} \bigcap_{n=p}^{\infty} (\kappa \cup f^{-1}(\mu_n)) =$$

$$= \kappa \cup \Big(\bigcup_{p=1}^{\infty} \bigcap_{n=p}^{\infty} f^{-1}(\mu_n)\Big) = \kappa \cup f^{-1}\Big(\bigcup_{p=1}^{\infty} \bigcap_{n=p}^{\infty} \mu_n\Big) = \kappa \cup f^{-1}(\mu).$$

Moreover, there hold

$$\lim_{n\to\infty} \| h_n \circ f - h \circ f \|_{L^\infty(\mathcal{A},\mathcal{V})} = \lim_{n\to\infty} \| h_n - h \|_{L^\infty(f(\mathcal{A}\setminus\kappa),\mathcal{V})} = 0,$$

and

$$\lim_{n\to\infty} \| (\nabla h_n \circ f)\, \nabla f - (H \circ f)\, \nabla f \|_{L^\infty(\mathcal{A},\mathrm{Lin}\,\mathcal{V})} \leq$$

$$\leq \lim_{n\to\infty} \| \nabla h_n - H \|_{L^\infty(f(\mathcal{A}\setminus\kappa),\mathrm{Lin}\,\mathcal{V})} \| \nabla f \|_{L^\infty(\mathcal{A},\mathrm{Lin}\,\mathcal{V})} = 0. \blacksquare$$

In the remainder of this section, we establish a variety of properties of the elements of LimSid.

4.10 Theorem: Let $(\kappa,g,G) \in \mathrm{LimSid}\,(\mathcal{A})$. Then: (i) κ has volume zero; (ii) g and G have representatives g_o and G_o which are continuous on $\mathcal{A}\setminus\kappa$.

Before proving the theorem we state a lemma which shows that, in spite of the fact that the domains of the transplacements f_n depend upon n, a notion of uniform convergence can be established for $n \mapsto f_n$.

4.11 Lemma: Let $n \longmapsto (\kappa_n, f_n)$ be a sequence determining $(\kappa,g,G) \in \mathrm{LimSid}\,(\mathcal{A})$. Then g and G have representatives $g_o : \mathcal{A}\setminus\kappa \longrightarrow \mathcal{E}$, $G_o : \mathcal{A} \setminus \kappa \longrightarrow \mathrm{Lin}\,\mathcal{V}$ such that $n \mapsto f_n$ and $n \longmapsto \nabla f_n$ converge uniformly to g_o and G_o in the following sense: for every $\epsilon > 0$, there is an $N_\epsilon \in \mathbb{N}$ such that

(4.6a) $$\sup_{x\in\mathcal{A}\setminus(\kappa\cup\kappa_n)} | f_n(x) - g_o(x) | < \epsilon \qquad \text{for all } n > N_\epsilon ,$$

and

(4.6b) $$\sup_{x\in\mathcal{A}\setminus(\kappa\cup\kappa_n)} | \nabla f_n(x) - G_o(x) | < \epsilon \qquad \text{for all } n > N_\epsilon .$$

Proof: Let $m, n \in \mathbb{N}$. From the definition of the L^∞ norm,

$$\| f_n - f_m \|_{L^\infty(\mathcal{A},\mathcal{V})} = \inf \{ \sup_{\xi\in\mathcal{A}\setminus\mathcal{N}} | f_n(\xi) - f_m(\xi) | \mid \mathcal{N} \subset \mathcal{E},\ \mathrm{vol}\,\mathcal{N} = 0 \}.$$

From the continuity of f_n and f_m on $\mathcal{A}\setminus(\kappa_n \cup \kappa_m)$, it follows that

$$(4.7) \qquad \| f_n - f_m \|_{L^\infty(\mathcal{A},\mathcal{V})} \geq | f_n(x) - f_m(x) | \qquad \text{for all } x \in \mathcal{A}\setminus(\kappa_n \cup \kappa_m).$$

By the definition (4.2) of κ, for each $x \in \mathcal{A} \setminus \kappa$ there is a subsequence $n' \longmapsto (\kappa_{n'}, f_{n'})$, $n' \in \mathbb{N}' \subset \mathbb{N}$, such that $x \in \mathcal{A} \setminus \kappa_{n'}$ for all n'. Thus, by (4.7) and by the fact that $n' \longmapsto f_{n'}$ has an
L^∞ – limit g, we conclude that $n' \mapsto f_{n'}(x)$ has a limit, and we set

$$(4.8) \qquad g_0(x) := \lim_{n'\to\infty} f_{n'}(x) \qquad \text{for all } x \in \mathcal{A} \setminus \kappa.$$

Let x be in $\mathcal{A} \setminus (\kappa \cup \kappa_n)$. Then by (4.7), with m restricted to $\mathbb{N}'$, in the limit for $m \to \infty$ we have

$$(4.9) \qquad \| f_n - g \|_{L^\infty(\mathcal{A},\mathcal{V})} \geq | f_n(x) - g_0(x) |.$$

If we choose N_ϵ such that the L^∞ norm of $f_n - g$ is less than ϵ for all $n > N_\epsilon$, then (4.6a) follows from (4.9). Moreover,

$$\| f_n - g \|_{L^\infty(\mathcal{A},\mathcal{V})} \geq \| f_n - g_0 \|_{L^\infty(\mathcal{A},\mathcal{V})},$$

and, taking the limit as $n \to \infty$ in the last inequality we find that $\| g - g_0 \|_{L^\infty(\mathcal{A},\mathcal{V})} = 0$, i.e., that g_0 is a representative of g. A similar proof applies to G. ■

Proof of Theorem 4.10: By (4.2), κ is a countable union of sets each of which has volume zero by (Sid 1). Thus, κ has volume zero. We wish to prove that the representative
g_0 of g defined in Lemma 4.11 is continuous. Let $n \mapsto (\kappa_n, f_n)$ be a sequence in $\mathrm{Sid}(\mathcal{A})$ which determines (κ, g, G). By the lemma, for any fixed $\epsilon > 0$ we may choose $N_\epsilon \in \mathbb{N}$ such

that, for all $n > N_\epsilon$,

$$\sup_{\xi \in \mathscr{A} \setminus (\kappa \cup \kappa_n)} | f_n(\xi) - g_0(\xi) | < \frac{\epsilon}{3}. \tag{4.10}$$

Let $x \in \mathscr{A} \setminus \kappa$ be given. Choose $n' > N_\epsilon$ such that $x \in \mathscr{A} \setminus \kappa_{n'}$; because $f_{n'}$ is continuous with open domain $\mathscr{A} \setminus \kappa_{n'}$, we may choose $\delta > 0$ such that $\mathscr{B}(x,\delta) \subset \mathscr{A} \setminus \kappa_{n'}$ and, for all $y \in \mathscr{B}(x,\delta)$,

$$| f_{n'}(y) - f_{n'}(x) | < \frac{\epsilon}{3}. \tag{4.11}$$

Let $z \in \mathscr{B}(x,\delta) \cap (\mathscr{A} \setminus \kappa)$ be given. Because $\mathscr{B}(x,\delta) \subset \mathscr{A} \setminus \kappa_{n'}$, we have that both x and z are in $(\mathscr{A} \setminus (\kappa_{n'} \cup \kappa)) \cap \mathscr{B}(x,\delta)$ and, by (4.10) applied to both x and z and (4.11) with $y = z$ we obtain:

$$| g_0(x) - g_0(z) | \leq | g_0(x) - f_{n'}(x) | + | f_{n'}(x) - f_{n'}(z) | + | f_{n'}(z) - g_0(z) | < \epsilon. \tag{4.12}$$

A similar proof applies to G. ■

<u>4.12 Remark:</u> As shown by Example 4.3, $\mathscr{A} \setminus \kappa$ in general is not an open set. Theorem 4.10 establishes the continuity of g_0 on its domain $\mathscr{A} \setminus \kappa$ but does not guarantee that g_0 has a continuous extension to an open ball centered at a given point x in $\mathscr{A} \setminus \kappa$. Nevertheless, relation (4.12) does restrict the oscillation, and therefore the jumps, of g_0 in $\mathscr{B}(x,\delta) \cap (\mathscr{A} \setminus \kappa)$ to be no greater than 2ϵ.

<u>4.13 Remark:</u> It is not true in general that g_0 is C^1, or even differentiable, at the interior of $\mathscr{A} \setminus \kappa$. Indeed, referring to Example 4.5, choose the function φ to be

$$\varphi(x) = \begin{cases} 0 & \text{for } 0 < x < 1/2, \\ 2x-1 & \text{for } 1/2 \leq x < 1. \end{cases} \tag{4.13}$$

With this choice, κ is the empty set and yet the function $g_0 : x \mapsto x + \varphi(x)$ is not differentiable at $x = 1/2$.

Some of the examples in this section show that, although the permanent crack site κ for a limit of simple deformations (κ,g,G) may be empty, the crack sites κ_n for a determining sequence $n \mapsto (\kappa_n, f_n)$ can diffuse throughout the region $\mathcal{A}$. We now wish to identify precisely the region affected by such diffusion and to investigate the fact that, in the above examples, $\nabla g \neq G$ at the points where such diffusion of fracture occurs. Consider a sequence $n \mapsto \kappa_n$ in $\mathcal{A}$ and define the set

(4.14) $$\Lambda(n \mapsto \kappa_n) := \bigcap_{p=1}^{\infty} \operatorname{clo}\Big(\bigcup_{n=p}^{\infty} \kappa_n \Big),$$

where the closure is taken relative to $\mathcal{A}$. This set is closed in $\mathcal{A}$ because it is the intersection of sets that are closed in $\mathcal{A}$. It includes the set κ defined in (4.2) and the inclusion is in general strict. For example, for the sequence (3.7) κ is the empty set and Λ is the whole interval $(0,1)$. If two sequences in $\mathrm{Sid}(\mathcal{A})$ determine the same element of $\mathrm{LimSid}(\mathcal{A})$, the two sets Λ need not coincide. For example, the identity $(\emptyset, i_{\mathcal{A}}, I_{\mathcal{A}})$ of $\mathrm{LimSid}\,(\mathcal{A})$ is determined by the constant sequence $n \mapsto (\emptyset, i_{\mathcal{A}})$, as well as by the sequence $n \mapsto (\sigma_n, i_{\mathcal{A}\setminus\sigma_n})$, with σ_n as in (3.7). In the first case Λ is the empty set, and in the second case it coincides with $\mathcal{A}$. The intersection of the sets $\Lambda(n \mapsto \kappa_n)$ taken over all sequences which determine a given $(\kappa,g,G) \in \mathrm{LimSid}\,(\mathcal{A})$ will be denoted by $\Phi(\kappa,g,G)$. This set is closed in $\mathcal{A}$ and includes κ. The complement of $\Phi(\kappa,g,G)$ in $\mathcal{A}$ will be denoted by $\Psi(\kappa,g,G)$. It is the set of all points $x \in \mathcal{A}$ such that there is at least one sequence $n \mapsto (\kappa_n,f_n) \in \mathrm{Sid}\,(\mathcal{A})$ which determines (κ,g,G) and such that x belongs to the exterior of $\bigcup_{n=1}^{\infty} \kappa_n$. Ψ is an open set included in $\mathcal{A} \setminus \kappa$ and may be empty. $\Psi(\kappa,g,G)$ and $\Phi(\kappa, g, G)$ will be called the <u>unfractured zone</u> and the <u>fractured zone for</u> (κ,g,G), respectively.

In the above example with $(\kappa,g,G) = (\emptyset, i_{\mathscr{A}}, I_{\mathscr{A}})$, because there is a sequence determining (κ,g,G) for which $\Lambda = \emptyset$, we have $\Phi(\kappa,g,G) = \emptyset$ and therefore $\Psi(\kappa,g,G) = \mathscr{A}$.

4.14 Theorem: Let $(\kappa,g,G) \in \text{Lim Sid}(\mathscr{A})$. Then, at all points of the unfractured zone $\Psi(\kappa,g,G)$, g_0 is continuously differentiable and $\nabla g_0 = G_0$.
Proof: Let $x \in \Psi(\kappa,g,G)$. Then there is a sequence $n \mapsto (\kappa_n,f_n)$ in Sid $(\mathscr{A})$ which determines (κ,g,G) and there is a neighborhood $\mathscr{I}(x)$ of x such that $\mathscr{I}(x)$ and $\bigcup_{n=1}^{\infty} \kappa_n$ are disjoint. Consequently, each f_n is of class C^1 at x and the sequence of the derivatives ∇f_n converges uniformly to G_0 in $\mathscr{I}(x)$. Under these conditions, a corollary of the mean value theorem (see e.g. [2], Theorems 3.6.1, 3.6.2) ensures that g_0 is of class C^1 and that its derivative at x is $G_0(x)$. ∎

4.15 Theorem: Let $(\kappa,g,G) \in \text{Lim Sid}(\mathscr{A})$, and suppose that $\det G_0(x) > 0$ at each point x in the unfractured zone $\Psi(\kappa,g,G)$. Then the restriction of g_0 to $\Psi(\kappa,g,G)$ is a C^1 – diffeomorphism.
Proof: We first observe that the positivity of $\det G_0$ together with the continuity both of G_0 and the mapping of an invertible linear mapping into its inverse ([15], p. 250) imply that G_0^{-1} is defined and continuous on $\Psi(\kappa,g,G)$. Therefore, for each $x \in \Psi(\kappa,g,G)$ and for each $\delta > 0$ such that $\text{clo}\,\mathscr{B}(x,\delta) \subset \Psi(\kappa,g,G)$, there exists $M = M(x,\delta) > 0$ such that

$$0 < |\,G_0(\xi)^{-1}\,| \le M \qquad \text{for all } \xi \in \text{clo}\,\mathscr{B}(x,\delta). \tag{4.15}$$

Let us prove now that g_0 restricted to $\Psi(\kappa,g,G)$ is injective. Assume to the contrary that there are two distinct points x, y in $\Psi(\kappa,g,G)$ such that

$$g_0(y) = g_0(x). \tag{4.16}$$

Since $x \in \Psi(\kappa,g,G)$, there is a sequence $n \mapsto (\kappa_n,f_n)$ in Sid $(\mathscr{A})$ which determines (κ,g,G) and a number $\delta \in (0, |y-x|)$ such that clo $\mathscr{B}(x, \delta) \subset \Psi(\kappa, g, G)$ and clo $\mathscr{B}(x, \delta)$ and $\bigcup_{n=1}^{\infty} \kappa_n$ are disjoint. Because, for each $n \in \mathbb{N}$, f_n is a C^1 – diffeomorphism of clo $\mathscr{B}(x,\delta)$, $\mathscr{B}(x, \delta)$ is mapped by f_n onto a neighborhood of $f_n(x)$. Moreover, by (4.15) and by the uniform convergence of $n \mapsto \nabla f_n$ to G_0 in clo $\mathscr{B}(x, \delta)$, there exists $N \in \mathbb{N}$ such that $n > N$ implies

$$|\nabla f_n^{-1}(\xi)| \leq 2M \qquad \text{for all } \xi \in \text{clo } \mathscr{B}(x, \delta),$$

and therefore ([2], Prop. 3.3.1)

$$|f_n(\xi) - f_n(x)| \geq (2M)^{-1} |\xi - x| \qquad \text{for all } \xi \in \text{clo } \mathscr{B}(x, \delta).$$

Since a C^1 – diffeormorphism maps the interior of its domain onto the interior of its image and the boundary of its domain onto the boundary of its image, we conclude that f_n maps $\mathscr{B}(x, \delta)$ onto an open neighborhood of $f_n(x)$ which includes $\mathscr{B}(f_n(x), (2M)^{-1}\delta)$. Consider now the point y which, by assumption, belongs to $\Psi(\kappa,g,G)$ and therefore to $\mathscr{A} \setminus \kappa$. Thus, there is a subsequence $n' \mapsto (\kappa_{n'}, f_{n'})$ of $n \mapsto (\kappa_n,f_n)$ such that y belongs to all $\mathscr{A} \setminus \kappa_{n'}$. In view of (4.16) and of the property (4.6) of uniform convergence of $n \mapsto f_n$ to g_0, for sufficiently large n' we have

$$(4.17) \quad |f_{n'}(y) - f_{n'}(x)| \leq |f_{n'}(y) - g_0(y)| + |g_0(y) - g_0(x)| + |g_0(x) - f_{n'}(x)| \leq (2M)^{-1}\delta.$$

Thus, y does not belong to $\mathscr{B}(x,\delta)$, but, for sufficiently large n', its image under $f_{n'}$ belongs to $\mathscr{B}(f_{n'}(x), (2M)^{-1}\delta)$, and therefore to the image under $f_{n'}$ of $\mathscr{B}(x,\delta)$. This contradicts the injectivity of $f_{n'}$. We conclude that (4.16) holds only if $y = x$, and this proves that the restriction of g_0 to $\Psi(\kappa,g,G)$ is injective. By Theorem 4.14, g_0 is of class C^1 and $\nabla g_0 = G_0$ in $\Psi(\kappa,g,G)$; moreover, the determinant of G_0 is strictly positive by

These conditions ensure that g_o is a C^1-diffeomorphism of $\Psi(\kappa,g,G)$, (see [2], Corollary 4.2.2). ∎

4.16 Remark: That (4.15) does not imply that g_o is injective in $\mathscr{A} \setminus \kappa$ or even in $\mathscr{A} \setminus \operatorname{clo} \kappa$ is shown by the following example. Let $\mathscr{E} = \mathbb{R}$, $\mathscr{A} = (-1,1)$, and let $n \mapsto (\kappa_n, f_n)$ be given by

$$(4.18) \quad \begin{cases} \kappa_n := \{\frac{h}{n} \mid h \in \{1-n, 2-n, \ldots, -1, 0, 1, \ldots, n-1\}\} \\ f_n(x) := \begin{cases} x + \frac{k}{n} & \text{for } \frac{k}{n} < x < \frac{k+1}{n},\ k \in \{0,1,\ldots,n\text{-}1\} \\ x + 1 + \frac{k+n+1}{n} & \text{for } \frac{k}{n} < x < \frac{k+1}{n},\ k \in \{-n,-n+1,\ldots,-1\} \end{cases} \end{cases}$$

Each pair (κ_n, f_n) is a piecewise affine deformation; in particular, f_n is injective because the image of $(0,1)$ under f_n consists of the intervals $(0, 1/n)$, $(2/n, 3/n)$, $(4/n, 5/n)$,.. and that of $(-1,0)$ consists of the intervals $(1/n, 2/n)$, $(3/n, 4/n)$,...which are all pairwise disjoint. The sequence (4.18) determines the triple (κ,g,G) with $\kappa = \{0\}$, g_o given by

$$g_o(x) = \begin{cases} 2x & \text{in } (0,1), \\ 2x+2 & \text{in } (-1,0), \end{cases}$$

and $G_o(x) = 1$. Therefore, (4.15) is satisfied, but g_o is not injective.

4.17 Theorem: Let $(\kappa,g,G) \in \operatorname{Lim\,Sid}(\mathscr{A})$, and let $x \in \mathscr{A} \setminus \kappa$. Assume that there is an open neighborhood $\mathscr{T}(x)$ of x included in $\mathscr{A}$ such that g_o can be extended to an orientation preserving C^1-diffeomorphism g^e on $\mathscr{T}(x)$. Then

$$(4.19) \qquad \det G_o(x) \leq \det \nabla g^e(x).$$

Proof: Choose $\delta > 0$ such that the ball $\mathscr{B}(x,\delta)$ is included in $\mathscr{J}(x)$. For each $\epsilon > 0$, denote by $g^e(\mathscr{B}(x,\delta))^\epsilon$ the set

$$g^e(\mathscr{B}(x,\delta))^\epsilon := \bigcup_{\xi\in\mathscr{B}(x,\delta)} \mathscr{B}(g^e(\xi), \epsilon).$$

Since $\epsilon < \epsilon'$ implies $g^e(\mathscr{B}(x,\delta))^\epsilon \subset g^e(\mathscr{B}(x,\delta))^{\epsilon'}$ and since the intersection of all $g^e(\mathscr{B}(x,\delta))^\epsilon$ with $\epsilon > 0$ is the closure in $\mathscr{E}$ of $g^e(\mathscr{B}(x,\delta))$, we have that

$$\text{(4.20)} \qquad \operatorname{vol} g^e(\mathscr{B}(x,\delta))^\epsilon = \operatorname{vol} g^e(\mathscr{B}(x,\delta)) + \mathcal{O}(\epsilon).$$

Because g^e is an orientation preserving C^1 – diffeomorphism, (4.20) can be written as

$$\text{(4.21)} \qquad \operatorname{vol} g^e(\mathscr{B}(x,\delta))^\epsilon = \int_{\mathscr{B}(x,\delta)} \det \nabla g^e(\xi)\, d\xi + \mathcal{O}(\epsilon).$$

Choose a sequence $n \mapsto (\kappa_n, f_n)$ in Sid $(\mathscr{A})$ which determines (κ, g, G). Then there is an $N_\epsilon \in \mathbb{N}$ such that for all $n > N_\epsilon$

$$\text{(4.22)} \qquad \| f_n - g \|_{L^\infty(\mathscr{A},\mathscr{V})} + \| \nabla f_n - G \|_{L^\infty(\mathscr{A},\mathrm{Lin}\mathscr{V})} < \epsilon.$$

For each such n, by (4.22), by the definition of the set $g^e(\mathscr{B}(x,\delta))^\epsilon$, and by the fact that g^e is an extension of g,

$$f_n(\mathscr{B}(x,\delta)\setminus \kappa_n) \subset g^e(\mathscr{B}(x,\delta))^\epsilon,$$

so that, by (4.21),

$$\text{(4.23)} \qquad \operatorname{vol} f_n\, (\mathscr{B}(x,\delta)\setminus \kappa_n) \le \int_{\mathscr{B}(x,\delta)} \det \nabla g^e(\xi)\, d\xi + \mathcal{O}(\epsilon).$$

Since f_n is an orientation preserving C^1 – diffeomorphism of the open set $\mathscr{A} \setminus \kappa_n$ and since κ_n has volume zero,

$$\operatorname{vol} f_n(\mathscr{B}(x,\delta)\setminus \kappa_n) = \int_{\mathscr{B}(x,\delta)\setminus\kappa_n} \det \nabla f_n(\xi)\, d\xi = \int_{\mathscr{B}(x,\delta)} \det \nabla f_n(\xi)\, d\xi.$$

On the other hand, by (4.22), for the representative G_0 of G we have

$$\mathrm{vol}\, f_n\,(\mathscr{B}(x,\delta)\setminus \kappa_n) = \int_{\mathscr{B}(x,\delta)} \det G_0(\xi)\, d\xi + \mathscr{O}(\epsilon),$$

and, by (4.23),

$$\int_{\mathscr{B}(x,\delta)} \det G_0(\xi)\, d\xi \le \int_{\mathscr{B}(x,\delta)} \det \nabla g^e(\xi)\, d\xi + \mathscr{O}(\epsilon).$$

Because ϵ is arbitrary, we conclude that

$$\int_{\mathscr{B}(x,\delta)} \det G_0(\xi)\, d\xi \le \int_{\mathscr{B}(x,\delta)} \det \nabla g^e(\xi)\, d\xi.$$

It is clear that δ can be replaced by an arbitrary positive number δ' less than δ. Then, by letting δ' tend to zero and using the continuity of G_0 on $\mathscr{A}\setminus\kappa$, we obtain (4.19). ■

<u>4.18 Remark:</u> If x belongs to the interior of $\mathscr{A} \setminus \kappa$, Theorem 4.17 has the following consequence: assume that g_0 is of class C^1 in a neighborhood of x and that $\det \nabla g_0(x) > 0$. Then

$$\det G_0(x) \le \det \nabla g_0(x). \tag{4.24}$$

The last result in this section is an extension to LimSid of the fundamental formula of calculus for simple deformations proved in Theorem 3.8.

<u>4.19 Theorem:</u> Let a piecewise fit region $\mathscr{A}$, a triple $(\kappa,g,G) \in \mathrm{LimSid}\,(\mathscr{A})$, and a unit vector $\boldsymbol{a} \in \mathscr{V}$ be given. There hold:

(i) For almost every line L parallel to $\mathscr{a}$, $L \cap \mathscr{A} \setminus \kappa$ is, to within a one–dimensional set of measure zero, a finite union of pairwise disjoint closed intervals I_q, $q \in \{1, \ldots, Q\}$;

(ii) for each sequence $n \mapsto (\kappa_n, f_n)$ that determines (κ, g, G) and for almost every line L parallel to $\mathscr{a}$, not only does (i) hold, but also, for each $q \in \{1, \ldots, Q\}$ and $N \in \mathbb{N}$, $f_n|_{I_q \cap \mathscr{A} \setminus \kappa_n}$ extends to a piecewise continuously differentiable function f_n^e on I_q, and the fundamental formula (3.16) applies to f_n for every x, y in I_q with $y = x + |y-x| \, \mathscr{a}$;

(iii) in addition, for almost every $x, y \in I_q \setminus \kappa$ with $y = x + |y-x| \, \mathscr{a}$, the formula

$$g_0(y) - g_0(x) = \int_0^{|y-x|} G_0(x+t\mathscr{a})\mathscr{a}dt + \lim_{n\to\infty} \sum_{z_n} (f_n(z_n+) - f_n(z_n-)) \tag{4.25}$$

is valid, where g_0 and G_0 are the representatives of g and G introduced in Theorem 4.10 and z_n runs through the points of discontinuity of f_n^e in the interval (x,y).

<u>Proof:</u> Let $n \longmapsto (\kappa_n, f_n)$ be a sequence that determines (κ, g, G). By (Sid 1) in Definition 3.2, each set κ_n has volume zero, and by item (i) of Theorem 4.10 the set κ also has volume zero. Using the notation of (3.18) and (3.19), we have that $\mathscr{A}\setminus\kappa_n \approx \mathscr{A}\setminus\kappa$ and, therefore, for each $n \geq 1$ and for almost every line L parallel to $\mathscr{a}$, there holds

$$L \cap \mathscr{A}\setminus\kappa_n \overset{1}{\approx} L \cap \mathscr{A}\setminus\kappa. \tag{4.26}$$

By item (i) of Theorem 3.8, for each integer $n \geq 1$ and for almost every line L satisfying (4.26) there is a finite collection of pairwise disjoint closed intervals $I_{q_n,n}$, $q_n \in \{1, \ldots, Q_n\}$

such that

$$(4.27) \qquad L \cap \mathscr{A}\backslash\kappa_n \overset{1}{\approx} \bigcup_{q_n=1}^{Q_n} I_{q_n,n}.$$

For each $n \geq 1$, we define $\mathscr{L}_n$ to be the set of lines parallel to a for which not only (4.26) and (4.27) hold, but also (cf. items (ii), (iii) of Theorem 3.8)

(ii)$_n$ for each $I_{q_n,n}$, $q_n \in \{1, \ldots, Q_n\}$, $f_n|_{I_{q_n,n} \cap \mathscr{A}\backslash\kappa_n}$ extends to a piecewise continuously differentiable function f_n^e on $I_{q_n,n}$;

(iii)$_n$ for each $q_n \in \{1, \ldots, Q_n\}$ and every $x, y \in I_{q_n,n}$ with $y = x + |y-x|\,a$, there holds

$$(4.28) \qquad f_n(y-) - f_n(x+) = \int_0^{|y-x|} \nabla f_n(x + ta)a\,dt + \sum_{z_n} (f_n(z_n+) - f_n(z_n-)).$$

By items (ii) and (iii) of Theorem 3.8 and the above arguments, $\mathscr{L}_n$ has full measure, i.e., $\mathscr{L}_n$ differs from the collection of all lines parallel to a by a set of measure zero. It follows that $\bigcap_{n=1}^{\infty} \mathscr{L}_n$ has full measure and that both (4.26) and (4.27) hold for every $n \geq 1$ and for every $L \in \bigcap_{n=1}^{\infty} \mathscr{L}_n$. Relation (4.26) implies that for every such n and L

$$(4.29) \qquad L \cap \mathscr{A}\backslash(\bigcup_{m=1}^{\infty} \kappa_m) \overset{1}{\approx} L \cap \mathscr{A}\backslash\kappa_n \overset{1}{\approx} L \cap \mathscr{A}\backslash\kappa.$$

Let a line $L \in \bigcap_{n=1}^{\infty} \mathscr{L}_n$ be given. By (4.26) and (4.27), for each $n, m \geq 1$ there holds

$$\bigcup_{q_n=1}^{Q_n} I_{q_n,n} \overset{1}{\approx} \bigcup_{q_m=1}^{Q_m} I_{q_m,m}.$$

Since each member of this relation is the union of a collection of pairwise disjoint, closed intervals, this implies

$$\{I_{q_n,n} \mid q_n \in \{1, \dots, Q_n\}\} = \{I_{q_m,m} \mid q_m \in \{1, \dots, Q_m\}\}. \tag{4.30}$$

Hence, if we put $Q := Q_1$ and if, for each $q \in \{1, \dots, Q\}$, we put $I_q := I_{q,1}$, then (4.26) and (4.27) yield

$$L \cap \mathscr{A} \setminus \kappa \overset{1}{\approx} \bigcup_{q=1}^{Q} I_q$$

which proves item (i). Item (ii) is proved by observing that item $(ii)_n$ holds for the given line L and for every $n \geq 1$, and that, by (4.30), the intervals $I_{q_n,n}$ are now independent of n. Because item $(iii)_n$ holds for the given line L and for every $n \geq 1$, we can assert that (4.28) holds for each interval I_q and for every $x, y \in I_q$ with $y = x + |y - x| a$. In particular, if for a given I_q we take $x, y \in \mathscr{A} \cap I_q \setminus (\bigcup_{m=1}^{\infty} \kappa_m)$, then $x, y \in \mathscr{A} \cap I_q \setminus \kappa_n$ for every $n \geq 1$. Since f_n is continuous on $\mathscr{A} \setminus \kappa_n$, the limits $f_n(y-)$ and $f_n(x+)$ in (4.28) can be replaced by $f_n(y)$ and $f_n(x)$, respectively. If in that relation we let n tend to ∞, then by (4.6) $f_n(y)$ and $f_n(x)$ converge to $g_0(y)$ and $g_0(x)$, respectively. We wish now to show that the integral in (4.28) tends to the integral in (4.25) as n tends to ∞. We first observe that, for every $n \geq 1$, ∇f_n and G_0 are defined on $\mathscr{A} \setminus \kappa_n$ and $\mathscr{A} \setminus \kappa$, respectively, and, therefore, are defined almost everywhere on I_q. Moreover, (4.6b) tells us that for each $\epsilon > 0$ there exists an integer N_ϵ such that

$$\operatorname*{ess\,sup}_{\xi \in I_q} | \nabla f_n(\xi) - G_0(\xi) | < \epsilon$$

for all $n > N_\epsilon$. Consequently, for each $n > N_\epsilon$,

$$\left| \int_0^{|y-x|} (\nabla f_n(x + t\alpha) - G_0(x + t\alpha))\alpha dt \right| < \epsilon |y-x|$$

which yields the desired conclusion. The formula (4.25) then follows from $(iii)_n$ upon letting n tend to ∞. This establishes (4.25) for every x, y in the set $\mathcal{A} \cap I_q \setminus (\bigcup_{m=1}^{\infty} \kappa_m)$ which, by (4.27) and (4.29), differs from I_q by a set of one–dimensional measure zero. ■

5. Structured deformations

The examples in Section 4 show on the one hand that an element of LimSid may contain significant kinematical information not provided by any single term of a sequence defining it. Thus, in Example 3.2, the limit of the broken ramp sequence describes a mapping of the interval (0,1) onto the interval (0,2) in which the smaller interval can be viewed as being fractured into infinitely many infinitesimally small pieces that are scattered uniformly throughout the larger interval, a situation that cannot be described by any one term of the sequence. On the other hand, an element of LimSid may lose the injectivity or some of the regularity properties enjoyed by each term. In fact, Remark 4.16 provides an example of an element (κ, g, G) of LimSid $(\mathcal{A})$ in which the continuous representative g_0 of g is not injective, and in Remark 4.13 g_0 is not differentiable at an interior point of $\mathcal{A} \setminus \kappa$. Although lack of injectivity is useful in some situations, for example, to describe mixtures of two continua as we indicate in Section 7d, in many other situations it is natural to consider only limits of simple deformations (κ, g, G) in which g_0 is injective. Moreover, in order to be able to define compositions of triples (κ, g, G) it is natural to require further that g_0 and G_0 have smoothness properties stronger than the continuity guaranteed in Theorem 4.10, such as those enjoyed by f and ∇f, respectively, in the case where f is the second entry of a simple deformation (κ, f). In this section we define and study a collection Std of triples meeting these requirements.

5.1 Definition: Let a piecewise fit region $\mathcal{A}$ be given. A structured deformation from $\mathcal{A}$ is a triple (κ, g, G) for which there hold:

(Std 1) (κ, g) is a simple deformation from $\mathcal{A}$;

(Std 2) $G : \mathscr{A} \setminus \kappa \to \mathrm{Lin}\,\mathscr{V}$ is continuous and is piecewise continuous on $\mathrm{clo}\,\mathscr{A}$, i.e., there exists a finite collection of fit regions $\{\mathscr{G}_j \mid j \in \{1,\dots,J\}\}$ whose union is $\mathscr{A} \setminus \kappa$ such that, for each $j \in \{1,\dots, J\}$, $G|_{\mathscr{G}_j}$ has a continuous extension to $\mathrm{clo}\,\mathscr{G}_j$;

(Std 3) there exists $m > 0$ such that, for all $x \in \mathscr{A} \setminus \kappa$, $m < \det G(x) \leq \det \nabla g(x)$.

We emphasize that our definition of structured deformations makes no use of limits of simple deformations, even though both notions of deformation are described by triples (κ,g,G). Nevertheless, in the Approximation Theorem, Theorem 5.8, we will prove that every structured deformation is a limit of simple deformations. Of course, when we say that a given structured deformation (κ,g,G) is a limit of simple deformations, we mean that the set κ and the L^∞– mappings associated with the continuous functions g and G form a triple that satisfies Definition 4.1.

It is helpful to re–examine the examples of limits of simple deformations in Section 4 to determine those $(\kappa,g,G) \in \mathrm{LimSid}$ that also are structured deformations, in the sense that the triple (κ,g_0,G_0), with g_0 and G_0 the continuous mappings constructed in Theorem 4.10, satisfies Definition 5.1. The broken ramp sequence (Example 4.2) determines the limit of simple deformations $(\emptyset,g,G)$, with g and G given by (4.3), which also is a structured deformation. However, the dyadic broken ramp sequence (Example 4.3) determines the limit of simple deformations (κ,g,G), with g and G again given by (4.3), but with κ the set of dyadic rationals in $(0,1)$. Hence, $\mathscr{A} \setminus \kappa = (0,1) \setminus \kappa$ is not open and, therefore, not piecewise fit; consequently, (κ,g,G) is not a structured deformation from $(0,1)$. These two examples illustrate the role that the set κ can play in distinguishing structured deformations from general limits of simple deformations. The remaining examples in Section 4 include more pathological situations that may not lead to structured deformations. In Example 4.5, choosing

φ to be the Cantor function provides us with an element (κ,g,G) of LimSid $(0,1)$ in which g_0 is not differentiable, and, hence, (κ,g,G) is not a structured deformation. The limit of simple deformations (κ,g,G) in Example 4.6 is not a structured deformation, because the set $\mathcal{A} \setminus \kappa$, with $\kappa = \{\frac{1}{n} \mid n \in \mathbb{N} \setminus \{0,1\}\}$, is not a piecewise fit region. The two–dimensional cavitation $(\emptyset,g,G)$ in Example 4.7 is a limit of simple deformations but not a structured deformation, because our discussion in Section 3 showed that $(\emptyset,g)$ is not a simple deformation. Finally, the "one–dimensional kink" described in Remark 4.13 is not a structured deformation, because g_0 is not differentiable on $\mathcal{A} \setminus \kappa$.

We shall denote by $\mathrm{Std}(\mathcal{A})$ the set of structured deformations from a given piecewise fit region $\mathcal{A}$ and by Std the set of all structured deformations:

$$(5.1) \qquad \mathrm{Std} := \{ (\kappa,g,G) \in \mathrm{Std}(\mathcal{A}) \mid \mathcal{A} \text{ is piecewise fit}\}.$$

It is easy to see from (Sid 1) – (Sid 3) that for each $(\kappa,g) \in \mathrm{Sid}(\mathcal{A})$, the triple $(\kappa,g,\nabla g)$ satisfies (Std 1) – (Std 3); therefore, we may identify Sid with a subset of Std.

The appearance of a simple deformation (κ,g) in the triple (κ,g,G) denoting a structured deformation makes it easy to define the composition of structured deformations.

<u>5.2 Definition:</u> Let a piecewise fit region $\mathcal{A}$ and structured deformations $(\kappa,g,G) \in \mathrm{Std}(\mathcal{A})$, $(\mu,h,H) \in \mathrm{Std}(g(\mathcal{A} \setminus \kappa))$ be given. The <u>composition</u> $(\mu,h,H)\circ(\kappa,g,G)$ is defined to be the triple

$$(5.2) \qquad \left(\kappa \cup g^{-1}(\mu),\ h \circ g|_{\mathcal{A}\setminus(\kappa\cup g^{-1}(\mu))},\ ((H\circ g)G)|_{\mathcal{A}\setminus(\kappa\cup g^{-1}(\mu))} \right).$$

<u>5.3 Proposition:</u> A composition of structured deformations is a structured deformation.

<u>Proof:</u> The reader will notice that the first two components of the triple in (5.2) describe the simple deformation $(\mu,h) \circ (\kappa,g)$ defined in (3.13). By Proposition 3.4,

$(\mu,h) \circ (\kappa,g) \in \mathrm{Sid}(\mathscr{A})$, so that $(\mu,h,H) \circ (\kappa,g,G)$ obeys (Std 1). Moreover, we can write

$$\text{(5.3)} \qquad \det((H\circ g)G) = \det(H\circ g)\det G \leq \det(\nabla h\circ g)\det \nabla g =$$

$$= \det((\nabla h\circ g)\nabla g) = \det(\nabla(h\circ g)),$$

where we have dropped explicit indication of restrictions and have used (Std 3) for (μ,h,H) and for (κ,g,G) along with the chain rule. Relation (5.3) shows that the triple (5.2) also satisfies the second inequality in (Std 3). The remaining inequality in (Std 3) follows from the first equality in (5.3) and the fact that each factor in the second member in (5.3) is bounded below by a positive number. We wish finally to show that this triple also satisfies (Std 2). To this end, we choose finite collections of fit regions $\{\mathscr{G}_k \mid k \in \{1,...,K\}\}$ in $\mathscr{A} \setminus \kappa$ and $\{\mathscr{H}_\ell \mid \ell \in \{1,...,L\}\}$ in $g(\mathscr{A} \setminus \kappa)\setminus\mu$ satisfying (Std 2) for G and for H, respectively, as well as an admissible collection $\{\mathscr{A}_j \mid j \in \{1,...,J\}\}$ as in (Sid 3) for the simple deformation (κ,g) from $\mathscr{A}$. Because for each $j \in \{1, ..., J\}$ the restriction g_j of g to $\mathscr{A}_j$ is a classical deformation, its inverse g_j^{-1} is a classical deformation from the fit region $g_j(\mathscr{A}_j)$. Because for each $\ell \in \{1, .., L\}$ $\mathscr{H}_\ell$ is a fit region, the set $\mathscr{H}_\ell \cap g_j(\mathscr{A}_j)$ is a fit region and, therefore, so is $g_j^{-1}(\mathscr{H}_\ell \cap g_j(\mathscr{A}_j))$. Consequently, for every j, k, and ℓ as above, the set

$$\text{(5.4)} \qquad \mathscr{A}_{jk\ell} := g_j^{-1}(\mathscr{H}_\ell \cap g_j(\mathscr{A}_j)) \cap \mathscr{G}_k$$

is a fit region; we have

$$\text{(5.5)} \qquad g_j^{-1}(\mathscr{H}_\ell \cap g_j(\mathscr{A}_j)) = g^{-1}(\mathscr{H}_\ell \cap g(\mathscr{A}_j)) = g^{-1}(\mathscr{H}_\ell) \cap \mathscr{A}_j,$$

so that

$$\bigcup_{j,k,\ell} \mathscr{A}_{jk\ell} = \bigcup_{j,k,\ell}(g^{-1}(\mathscr{H}_\ell) \cap \mathscr{A}_j \cap \mathscr{G}_k) = (\bigcup_\ell g^{-1}(\mathscr{H}_\ell)) \cap (\bigcup_j \mathscr{A}_j) \cap (\bigcup_k \mathscr{G}_k) =$$

$$= g^{-1}(g(\mathscr{A}\setminus\kappa)\setminus\mu) \cap (\mathscr{A}\setminus\kappa) \cap (\mathscr{A}\setminus\kappa),$$

and, since the range of g^{-1} is included in $\mathscr{A}\setminus\kappa$,

$$\bigcup_{j,k,\ell} \mathscr{A}_{jk\ell} = g^{-1}(g(\mathscr{A}\setminus\kappa)\setminus\mu) = \mathscr{A}\setminus(\kappa \cup g^{-1}(\mu)).$$

Thus, the sets $\mathscr{A}_{jk\ell}$ form a collection of fit regions whose union is $\mathscr{A}\setminus(\kappa\cup g^{-1}(\mu))$. Because $\mathscr{A}_{jk\ell} \subset \mathscr{G}_k$, by (Std 2), G has a continuous extension to clo $\mathscr{A}_{jk\ell}$; because, by (5.4), (5.5) and the injectivity of g,

$$g(\mathscr{A}_{jk\ell}) = g(g^{-1}(\mathscr{H}_\ell)\cap\mathscr{A}_j\cap\mathscr{G}_k) = \mathscr{H}_\ell \cap g(\mathscr{A}_j) \cap g(\mathscr{G}_k) \subset \mathscr{H}_\ell,$$

(Std 2) implies that H has a continuous extension to clo $g(\mathscr{A}_{jk\ell})$. Moreover, g has a continuous extension to clo $\mathscr{A}_j \supset$ clo $\mathscr{A}_{jk\ell}$ and, thus, $H\circ g$ has a continuous extension to clo $\mathscr{A}_{jk\ell}$. We conclude that $(H\circ g)\,G$ has a continuous extension to clo $\mathscr{A}_{jk\ell}$ and that (Std 2) is satisfied by the triple in (5.2). ■

It is natural to consider the triple $(\emptyset, i_{\mathscr{A}}, I_{\mathscr{A}})$ as the <u>identity element</u> in Std $(\mathscr{A})$. Indeed, we observe that not only is $(\emptyset, i_{\mathscr{A}}, I_{\mathscr{A}}) \in$ Std $(\mathscr{A})$, but also that

(5.6) $$(\kappa,g,G)\circ(\emptyset, i_{\mathscr{A}}, I_{\mathscr{A}}) = (\emptyset, i_{g(\mathscr{A}\setminus\kappa)}, I_{g(\mathscr{A}\setminus\kappa)})\circ(\kappa,g,G) = (\kappa,g,G)$$

for all $(\kappa,g,G) \in$ Std $(\mathscr{A})$. The next proposition concerns the existence of a left inverse for a structured deformation and is a natural counterpart of Proposition 3.6 for simple deformations.

<u>5.4 Proposition:</u> Let $(\kappa,g,G) \in$ Std $(\mathscr{A})$ be given. There exists $(\lambda,\ell,L) \in$ Std $(g(\mathscr{A}\setminus\kappa))$ satisfying

(5.7) $$(\lambda,\ell,L)\circ(\kappa,g,G) = (\emptyset, i_{\mathscr{A}}, I_{\mathscr{A}})$$

if and only if

(5.8) $$\kappa = \emptyset \quad \text{and} \quad \det G = \det \nabla g.$$

In this case, we have the relations

$$(5.9) \qquad \lambda = \emptyset, \quad \ell = g^{-1}, \quad \text{and } L = G^{-1} \circ g^{-1},$$

as well as the relation

$$(5.10) \qquad (\kappa,g,G) \circ (\lambda,\ell,L) = (\emptyset, i_{g(\mathcal{B})}, I_{g(\mathcal{B})}).$$

Proof: Suppose that (5.7) holds for some structured deformation (λ,ℓ,L) from $g(\mathcal{B} \setminus \kappa)$. The definition of composition (5.2) then yields for the simple deformations (λ,ℓ) and (κ,g)

$$(\lambda,\ell) \circ (\kappa,g) = (\emptyset, i_{\mathcal{B}}),$$

and Proposition 3.6 implies

$$(5.11) \qquad \lambda = \kappa = \emptyset \quad \text{and} \quad \ell = g^{-1}.$$

Furthermore, (5.2) tells us that $(L \circ g)\, G = I_{\mathcal{B}}$, so that $L \circ g = G^{-1}$ and, thus,

$$(5.12) \qquad L = G^{-1} \circ g^{-1}.$$

Because $(\lambda,\ell,L) \in \mathrm{Std}(g\,(\mathcal{B} \setminus \kappa))$, the inequality (Std 3), (5.11), (5.12) and the Inverse Function Theorem tell us that

$$(\det G)^{-1} \circ g^{-1} = \det\,(G^{-1} \circ g^{-1}) = \det L \leq \det \nabla\ell = \det\,(\nabla(g^{-1})),$$

i.e.,

$$(5.13) \qquad (\det G)^{-1} \leq \det\,(\nabla(g^{-1}) \circ g) = (\det \nabla g)^{-1}.$$

It is immediate from (5.13) and (Std 3) that the relation $\det G = \det \nabla g$ in (5.8) holds.

Conversely, suppose that (5.8) holds and put $\lambda = \emptyset$, $\ell = g^{-1}$, and $L = G^{-1} \circ g^{-1}$. It is then clear that (5.7) and (5.10) hold, and it only remains to show that $(\emptyset, g^{-1}, G^{-1} \circ g^{-1}) \in \mathrm{Std}\,(g(\mathscr{A}))$. Because $(\emptyset, g) \in \mathrm{Sid}\,(\mathscr{A})$, Proposition 3.7 tells us that $(\emptyset, g^{-1}) \in \mathrm{Sid}\,(g(\mathscr{A}))$, so that (Std 1) is satisfied for $(\emptyset, g^{-1}, G^{-1} \circ g^{-1})$. To verify (Std 3) we note that

$$\text{(5.14)} \qquad \det(G^{-1} \circ g^{-1}) = (\det G^{-1}) \circ g^{-1} = (\det G)^{-1} \circ g^{-1}$$

and the second relation in (5.8) yields

$$\text{(5.15)} \qquad (\det G)^{-1} \circ g^{-1} = (\det \nabla g)^{-1} \circ g^{-1} = (\det (\nabla g)^{-1}) \circ g^{-1}$$

$$= \det ((\nabla g)^{-1} \circ g^{-1}) = \det(\nabla(g^{-1})),$$

which verifies with equality the second relation in (Std 3) for $(\emptyset, g^{-1}, G^{-1} \circ g^{-1})$. Moreover, this equality, relation (3.5), and the fact that $(\emptyset, g^{-1}) \in \mathrm{Sid}\,(g(\mathscr{A} \setminus \kappa))$ provide a positive lower bound for $\det (G^{-1} \circ g^{-1})$ on $g(\mathscr{A} \setminus \kappa)$. Thus, (Std 3) is satisfied. To verify (Std 2) for $(\emptyset, g^{-1}, G^{-1} \circ g^{-1})$, we choose an admissible collection $\{\mathscr{A}_k \mid k \in \{1,\ldots, K\}\}$ for $(\emptyset, g)$ and fit regions $\mathscr{G}_1, \ldots, \mathscr{G}_J$ satisfying (Std 2) for $(\emptyset, g, G)$ and note that, for each j, $G^{-1}|_{\mathscr{G}_j}$ has a continuous extension H_j to $\mathrm{clo}\,\mathscr{G}_j$. Put

$$\text{(5.16)} \qquad \mathscr{I}_{jk} := g_k(\mathscr{A}_k \cap \mathscr{G}_j) \subset g(\mathscr{A})$$

and observe that each set $\mathscr{I}_{jk}$ is a fit region, as it is the image of $\mathscr{A}_k \cap \mathscr{G}_j$ under the classical deformation g_k. Choosing an extension g_k^e to all of $\mathscr{E}$ that is a C^1 – diffeormorphism, we can write

$$\mathrm{clo}\,\mathscr{I}_{jk} = \mathrm{clo}\,(g_k^e(\mathscr{A}_k \cap \mathscr{G}_j)) = g_k^e(\mathrm{clo}\,(\mathscr{A}_k \cap \mathscr{G}_j)),$$

and we note that for each $y \in \text{int clo}\, \mathcal{J}_{jk} = \mathcal{J}_{jk}$,

$$H_j(g^{-1}(y)) = H_j(g_k^{-1}(y)) = (G^{-1} \circ g^{-1})(y).$$

Moreover, $H_j \circ g_k^{-1}$ has the extension $H_j \circ (g_k^e)^{-1}|_{\text{clo}\, \mathcal{J}_{jk}}$ that is continuous. Thus, $G^{-1} \circ g^{-1}$ is piecewise continuous on clo $\mathcal{A}$, and (Std 2) is satified. ■

Proposition 5.4 has the following immediate corollary.

<u>5.5 Proposition:</u> Let $(\kappa, g, G) \in \text{Std}$ be given. Then (κ, g, G) has an inverse $(\lambda, \ell, L) \in \text{Std}$ if and only if $\kappa = \emptyset$ and $\det G = \det \nabla g$, in which case

(5.17) $$(\lambda, \ell, L) = (\emptyset, g^{-1}, G^{-1} \circ g^{-1}).$$

Propositions 5.4 and 5.5 tell us that the existence of an inverse in Std is equivalent to the existence of a left inverse which, in turn, is equivalent to the relations (5.8). We use the notation

(5.18) $$(\emptyset, g, G)^{-1} := (\emptyset, g^{-1}, G^{-1} \circ g^{-1})$$

for the inverse of $(\emptyset, g, G)$, and we write InvStd for the set of invertible structured deformations:

(5.19) $$\text{InvStd} := \{(\kappa, g, G) \in \text{Std} \mid \kappa = \emptyset \text{ and } \det G = \det \nabla g\}.$$

We recall that, from Proposition 3.6, the condition $\kappa = \emptyset$ is both necessary and sufficient in order that the simple deformation (κ, g) be invertible. Thus, the conditions $\det G = \det \nabla g$ and $\kappa = \emptyset$ play the corresponding role in determining which structured deformations are invertible. From the imbedding $g \mapsto (\emptyset, g, \nabla g)$ of Cld into Std we conclude that each invertible simple deformation and, in particular, each classical deformation, when regarded as a structured deformation, is an invertible structured deformation.

A principal goal in the remainder of this section is to show that Std can be identified with a subset of LimSid. In order to do so, it is helpful to record a simple consequence of the definition of composition of structured deformations that gives a counterpart of the decomposition (3.14) for simple deformations.

<u>5.6 Proposition:</u> Each structured deformation (κ,g,G) is a composition of the simple deformation $(\kappa,g,\nabla g)$ and of a structured deformation of the form $(\emptyset,i,H)$:

$$(5.20)\qquad (\kappa,g,G) = \left[\emptyset,\ i_{g(\mathcal{A}\setminus\kappa,)},\ (G\circ g^{-1})((\nabla g)^{-1}\circ g^{-1})\right]\circ(\kappa,g,\nabla g).$$

The next result, the Approximation Lemma, is central to the Approximation Theorem which asserts that every structured deformation is a limit of simple deformations. The lemma shows that a structured deformation of the form $(\emptyset,i,H)$ can be approximated to any desired accuracy by a simple deformation (λ,h). Before stating the Approximation Lemma, we let a piecewise fit region $\mathcal{A}$ be given and choose a Cartesian coordinate system for $\mathcal{E}$ with origin O in such a way that clo $\mathcal{A}$ is included in the coordinate cube $(\frac{1}{3},\frac{2}{3})^3$:

$$(5.21)\qquad \text{clo}\,\mathcal{A} \subset (\tfrac{1}{3},\tfrac{2}{3})^3 .$$

(For definiteness and simplicity, we here consider the case where $\mathcal{E}$ is three–dimensional.) For each prime $p \in \mathbb{N}$ and subset Z of the integers $\mathbb{Z}$, we define a family $\Pi(p,Z)$ of coordinate planes:

$$(5.22)\qquad \Pi(p,Z) := \{\pi \subset \mathcal{E} \mid \pi \text{ is a coordinate plane whose distance from O is } m/p \text{ for some } m \in Z\}.$$

In particular, $\Pi(p,\mathbb{Z})$ is the set of all coordinate planes obtained from any one of the three coordinate planes through O by a translation of amount an integral multiple of p^{-1}.

In stating the Approximation Lemma for a given $(\emptyset,i,H) \in \mathrm{Std}(\mathcal{A})$, we also refer to sets occurring in (Std 2) of Definition 5.1 as well as sets constructed from them. Specifically, we choose fit regions $\mathcal{H}_1, \ldots, \mathcal{H}_J$ whose union is $\mathcal{A}$ and such that, for each $j \in \{1,\ldots,J\}$, $H|_{\mathcal{H}_j}$ has a continuous extension $H_j : \mathrm{clo}\,\mathcal{H}_j \to \mathrm{Lin}\,\mathcal{V}$. Consider the subdivision $\underline{B}$ of $\mathcal{A}$ into mutually disjoint fit regions $\mathcal{B}_j$, $j \in \{1,\ldots,J\}$, constructed using the procedure in (3.17) with $\mathcal{A}_j$ there replaced by $\mathcal{H}_j$. We now define

$$\Gamma(\underline{B}) := \bigcup_{j=1}^{J} ((\mathrm{bdy}\,\mathcal{B}_j) \cap \mathcal{A}), \tag{5.23}$$

i.e., $\Gamma(\underline{B})$ is the set of points in $\mathcal{A}$ that are in the boundary of at least one of the subdividing regions $\mathcal{B}_j$, and for each $\epsilon > 0$ we define

$$\Gamma(\underline{B})_\epsilon := \{x \in \Gamma(\underline{B}) \mid \mathrm{dist}\,(x, \mathrm{bdy}\,\mathcal{A}) < \epsilon\}. \tag{5.24}$$

<u>5.7 Approximation Lemma:</u> Let a piecewise fit region $\mathcal{A}$ be given and choose a Cartesian coordinate system for $\mathcal{E}$ satisfying (5.21). Let $(\emptyset,i,H) \in \mathrm{Std}(\mathcal{A})$ be given, choose sets $\{\mathcal{H}_j \mid j \in \{1,\ldots,J\}\}$ as in (Std 2), and consider the subdivision $\underline{B}$ as in (3.17), with $\mathcal{A}_j$ there replaced by $\mathcal{H}_j$. For each $\epsilon > 0$ and each prime $p \in \mathbb{N}$, there exist a piecewise affine simple deformation (λ,h) from $\mathcal{A}$ and primes p_1, p_2 greater than p such that

(i) λ is covered by the set $\Gamma(\underline{B})_\epsilon$ defined in (5.24) together with the planes $\pi \in \Pi(p_\ell, \{1, \ldots, p_\ell - 1\})$ with $\ell \in \{1, 2\}$;

(ii) $\| h - i \|_{L^\infty(\mathcal{A},\mathcal{V})} < \epsilon$;

(iii) $\| \nabla h - H \|_{L^\infty(\mathcal{A},\mathrm{Lin}\mathcal{V})} < \epsilon$.

Moreover, (λ, h), p_1, and p_2 can be chosen so that, if we put

$$\mathcal{G} := \operatorname{int}\{x \in \mathcal{A} \mid H(x) = I\}, \tag{5.25}$$

then $\lambda \cap \mathcal{G} = \emptyset$ and $h|_{\mathcal{G}} = i_{\mathcal{G}}$.

Proof: Let $\epsilon > 0$ and a prime p be given. Because for each $j \in \{1,\ldots,J\}$ the extension $H_j : \operatorname{clo} \mathcal{H}_j \to \operatorname{Lin} \mathcal{V}$ of H is continuous, H_j is uniformly continuous; moreover, by (Std 3), there exists $m > 0$ such that

$$m < \det H_j \leq 1, \tag{5.26}$$

and we may choose $\beta > 0$ satisfying

$$1 - \frac{\epsilon/2}{\sup_{x \in \mathcal{A}} |H(x)|} < \beta < 1. \tag{5.27}$$

The uniform continuity of the finitely many functions $H_1,\ldots,H_J$ tells us that we may choose $\delta > 0$ such that, for each $j \in \{1,\ldots,J\}$,

$$(x,y \in \operatorname{clo} \mathcal{H}_j \text{ and } |x-y| < \delta) \implies |H_j(x) - H_j(y)| < \frac{\epsilon}{2J}. \tag{5.28}$$

Choose a prime p_1 such that

$$p_1 > \max\left\{ p+1, \frac{\sqrt{3}}{\delta}, \frac{2\sqrt{3}\,M}{\epsilon} \right\}, \tag{5.29}$$

with

$$M := \max\left\{ \sup_{x \in \mathcal{A}} |\beta H(x) - I|,\ 1 \right\}, \tag{5.30}$$

and put

$$(5.31) \qquad \underline{C}(p_1) := \{\mathscr{C} \subset (0,1)^3 \mid \mathscr{C} = [\frac{k_1}{p_1}, \frac{k_1+1}{p_1}] \times [\frac{k_2}{p_1}, \frac{k_2+1}{p_1}] \times [\frac{k_3}{p_1}, \frac{k_3+1}{p_1}],$$

$$k_1, k_2, k_3 \in \{1, 2, \ldots, p_1-2\}\}$$

i.e., $\underline{C}(p_1)$ is the set of all closed cubes $\mathscr{C}$ in $(0,1)^3$ whose parallel faces are included in consecutive planes in $\Pi(p_1, \{1, 2, \ldots, p_1-1\})$. The condition (5.21) tells us that clo $\mathscr{A}$ is covered by $\underline{C}(p_1)$, and the relation (5.29) implies that the diagonal $\sqrt{3}/p_1$ of each cube is less than both $\epsilon/2M$ and δ. Consequently, we may write for each $\mathscr{C} \in \underline{C}(p_1)$

$$(5.32) \qquad x, y \in \mathscr{C} \implies |x-y| < \min\{\epsilon/2M, \delta\}$$

and for each $j \in \{1,\ldots,J\}$, by (5.28),

$$(5.33) \qquad x, y \in (\mathscr{C} \cap \text{clo}\,\mathscr{H}_j) \implies |H_j(x) - H_j(y)| < \frac{\epsilon}{2J}.$$

In order to define the set $\lambda \subset \mathscr{A}$ and the mapping $h : \mathscr{A} \setminus \lambda \longrightarrow \mathscr{E}$, we need to consider three cases for the cubes $\mathscr{C} \in \underline{C}(p_1)$, one of which, namely when $\mathscr{C}$ and $\mathscr{A}$ are disjoint, so that the domain of h does not intersect $\mathscr{C}$, is a trivial case and we need not mention it further. We now treat the remaining two cases: $\mathscr{C}$ is included in $\mathscr{A}$, or $\mathscr{C}$ is neither included in nor disjoint from $\mathscr{A}$.

It is convenient here to put

$$(5.34) \qquad \underline{D}_1 := \{\mathscr{C} \in \underline{C}(p_1) \mid \mathscr{C} \subset \mathscr{A}\}.$$

To treat the first case, let a cube $\mathscr{C} \in \underline{D}_1$ be given. Because $\mathscr{C} \subset \mathscr{A}$, H is continuous on $\mathscr{C}$, and, because $\mathscr{C}$ is closed, H is uniformly continuous on $\mathscr{C}$. Moreover, because the fit regions $\mathscr{H}_j$, $j \in \{1,\ldots,J\}$, cover $\mathscr{A}$, they cover $\mathscr{C}$. For each x, y in the convex set $\mathscr{C}$, we construct a list of points in the segment $[x,y] \subset \mathscr{C}$ as follows. Choose $j_1 \in \{1, ..., J\}$ such that $x \in \mathscr{H}_{j_1}$, and choose $w_1 \in \text{clo}\, \mathscr{H}_{j_1}$ such that

$$|w_1 - y| = \min \{ |z - y| \mid z \in \text{clo}\, \mathscr{H}_{j_1} \cap [x,y] \}.$$

Note that $[w_1,y] \cap \mathscr{H}_{j_1} = \emptyset$, so that $[w_1,y]$ is covered by the remaining $J-1$ sets $\mathscr{H}_j$. Choosing $j_2 \in \{1,...,J\} \setminus \{j_1\}$ such that $w_1 \in \mathscr{H}_{j_2}$, we obtain by the same procedure a list of points $w_0, w_1, w_2, ..., w_{J'}$ in $[x,y]$ with $J' \leq J$, $w_0 = x$, $w_{J'} = y$, and satisfying: for each $k \in \{1,...,J'\}$, w_{k-1} and w_k both are in the set $\text{clo}\, \mathscr{H}_{j_k}$. This list permits us to use (5.33) and the triangle inequality to write

$$|H(x) - H(y)| \leq \sum_{k=1}^{J'} |H(w_{k-1}) - H(w_k)| < \frac{J'\epsilon}{2J} \leq \frac{\epsilon}{2},$$

and to conclude that

$$x, y \in \mathscr{C} \implies |H(x) - H(y)| < \frac{\epsilon}{2}. \tag{5.35}$$

Now, we choose a point $c_{\mathscr{C}}$ in the cube $\mathscr{C}$, we define the affine map $a_{\mathscr{C}}: \mathscr{E} \longrightarrow \mathscr{E}$ by

$$a_{\mathscr{C}}(x) := c_{\mathscr{C}} + \beta\, H(c_{\mathscr{C}})\, [x - c_{\mathscr{C}}], \quad x \in \mathscr{E}, \tag{5.36}$$

and we note that $a_{\mathscr{C}}$ satisfies

$$\nabla a_{\mathscr{C}}(x) = \beta\, H(c_{\mathscr{C}}) \quad \text{for all } x \in \mathscr{C}. \tag{5.37}$$

By (5.27) and (Std 3) we also have for all $x \in \mathscr{C}$:

$$\det \nabla a_{\mathscr{C}}(x) \leq \beta^3 < 1, \tag{5.38}$$

and, by (5.27), (5.35), and (5.37),

$$(5.39) \qquad |\nabla a_{\mathscr{C}}(x) - H(x)| \leq \beta \mid H(c_{\mathscr{C}}) - H(x) \mid + (1-\beta) \mid H(x) \mid < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.$$

The last relation shows that, on $\mathscr{C}$, the requirement (iii) of the lemma is satisfied by $a_{\mathscr{C}}$.

The requirement (ii) is satisfied on $\mathscr{C}$ as well. In fact, for each $x \in \mathscr{C}$, by (5.36), (5.32) and (5.30), we have

$$(5.40) \qquad | a_{\mathscr{C}}(x) - i(x) | = | c_{\mathscr{C}} + \beta H(c_{\mathscr{C}}) [x - c_{\mathscr{C}}] - x | \leq$$

$$\leq | \beta H(c_{\mathscr{C}}) - I | \, |x - c_{\mathscr{C}}| < | \beta H(c_{\mathscr{C}}) - I | \tfrac{\epsilon}{M} \leq \epsilon.$$

However, if we define h restricted to the interior of each cube $\mathscr{C} \in \underline{D}_1$ to be $h|_{\text{int}\,\mathscr{C}} := a_{\mathscr{C}}|_{\text{int}\,\mathscr{C}}$, we find that h need not be injective, because the images $a_{\mathscr{C}}(\text{int}\,\mathscr{C})$ of the cubes $\mathscr{C}$ in $\underline{D}_1$ need not be disjoint. To remedy this situation we compose each affine mapping $a_{\mathscr{C}}$ with a piecewise rigid mapping $r_{\mathscr{C}}$ that fractures $a_{\mathscr{C}}(\mathscr{C})$ into smaller, mutually congruent parallelepipeds, and then moves the smaller parallelepipeds into $\mathscr{C}$ without interpenetration. To this end, we let a prime $p' > p$ be given and note that the set

$$(5.41) \qquad a_{\mathscr{C}}(\Pi(p', \mathbb{Z})) := \{ a_{\mathscr{C}}(\pi) \mid \pi \in \Pi(p', \mathbb{Z}) \}$$

is a collection of planes in space, each of which is parallel to one of the faces of the parallelepiped $a_{\mathscr{C}}(\mathscr{C})$. Two consecutive planes in $a_{\mathscr{C}}(\Pi(p', \mathbb{Z}))$ parallel to a given face of $a_{\mathscr{C}}(\mathscr{C})$ have distances γ / p' from one another, with γ a positive number depending only on $\nabla a_{\mathscr{C}}$ and the normal to the face. Therefore, the collection $a_{\mathscr{C}}(\Pi(p', \mathbb{Z}))$ subdivides $\mathscr{E}$ into infinitely many mutually congruent closed parallelepipeds, and we denote by $\underline{P}(p')$ this collection of parallelepipeds. We consider now two finite subsets of $\underline{P}(p')$:

$$(5.42) \qquad \underline{P}(p')_{a_{\mathscr{C}}} := \{ \mathscr{P} \in \underline{P}(p') \mid \mathscr{P} \cap a_{\mathscr{C}}(\mathscr{C}) \neq \emptyset \}$$

and

$$\underline{P}(p')_{\mathscr{C}} := \{\mathscr{P} \in \underline{P}(p') \ | \ \mathscr{P} \subset \text{int } \mathscr{C}\}; \tag{5.43}$$

thus, the elements of $\underline{P}(p')_{a_{\mathscr{C}}}$ form a cover of $a_{\mathscr{C}}(\mathscr{C})$ by mutually congruent non–overlapping parallelepipeds, and the elements of $\underline{P}(p')_{\mathscr{C}}$ are mutually congruent non–overlapping parallelepipeds in the interior of $\mathscr{C}$. We observe from (5.37), (5.38), and (5.26) that

$$\text{vol } a_{\mathscr{C}}(\mathscr{C}) = \beta^3 \det H(c_{\mathscr{C}}) \text{ vol } \mathscr{C} < \text{vol } \mathscr{C}. \tag{5.44}$$

Now, by (5.37), for each $\mathscr{P} \in \underline{P}(p')$ there holds

$$v(p') := \text{vol } \mathscr{P} = \beta^3 \det H(c_{\mathscr{C}}) \ (p')^{-3}, \tag{5.45}$$

because each $\mathscr{P}$ is the image of a cube of volume $(p')^{-3}$ under $a_{\mathscr{C}}$, and this tells us that

$$\lim_{p' \to \infty} v(p') = 0. \tag{5.46}$$

Using (5.42), (5.43) and the fact that vol bdy $\mathscr{C}$ = vol bdy $a_{\mathscr{C}}(\mathscr{C}) = 0$, we conclude that

$$\lim_{p' \to \infty} \text{vol} \cup \underline{P}(p')_{a_{\mathscr{C}}} = \text{vol } a_{\mathscr{C}}(\mathscr{C}) \tag{5.47}$$

and

$$\lim_{p' \to \infty} \text{vol} \cup \underline{P}(p')_{\mathscr{C}} = \text{vol } \mathscr{C}. \tag{5.48}$$

Relations (5.44), (5.47) and (5.48) imply that we can choose a prime $p_{\mathscr{C}} > p$ such that vol $\cup \underline{P}(p')_{a_{\mathscr{C}}} <$ vol $\cup \underline{P}(p')_{\mathscr{C}}$ for all $p' > p_{\mathscr{C}}$. Since all parallelepipeds in $\underline{P}(p')$ have the same volume $v(p')$, we conclude that the set $\underline{P}(p')_{a_{\mathscr{C}}}$ of non–overlapping parallelepipeds covering $a_{\mathscr{C}}(\mathscr{C})$ has fewer elements than the set $\underline{P}(p')_{\mathscr{C}}$ of non–overlapping parallelepipeds contained in int $\mathscr{C}$ whenever $p' > p_{\mathscr{C}}$. Consequently, since all the parallelepipeds in both collections are congruent, we can choose an injective, piecewise rigid mapping $r_{\mathscr{C}}: a_{\mathscr{C}}(\mathscr{C}) \setminus a_{\mathscr{C}}(\Pi(p', \mathbb{Z})) \longrightarrow \text{int}\,\mathscr{C}$; using (5.21) we may put

$$\lambda_{\mathscr{C}} := \{x \in \pi \cap \text{int}\,\mathscr{C} \mid \pi \in \Pi(p', \{1, \ldots, p'-1\})\}, \tag{5.49}$$

and define the mapping $h_{\mathscr{C}} : (\text{int}\,\mathscr{C}) \setminus \lambda_{\mathscr{C}} \longrightarrow \text{int}\,\mathscr{C}$ by

$$h_{\mathscr{C}}(x) := r_{\mathscr{C}}(a_{\mathscr{C}}(x)), \quad x \in (\text{int}\,\mathscr{C})\setminus\lambda_{\mathscr{C}}. \tag{5.50}$$

Because the range of $r_{\mathscr{C}}$ is included in $\mathscr{C}$ and $\mathscr{C}$ has diagonal $\sqrt{3}/p'$ less than ϵ, $h_{\mathscr{C}}$ satisfies item (ii) in the statement of the lemma on its domain, and, because $\nabla r_{\mathscr{C}} = I$, we have

$$\nabla h_{\mathscr{C}} = \nabla a_{\mathscr{C}}. \tag{5.51}$$

Relations (5.51) and (5.39) then tell us that $h_{\mathscr{C}}$ satisfies (iii) on its domain. Finally, relation (5.49) tells us that $\lambda_{\mathscr{C}}$ is consistent with (i). To summarize, we have shown that our construction yields a piecewise affine simple deformation $(\lambda_{\mathscr{C}}, h_{\mathscr{C}})$ from int $\mathscr{C}$ that satisfies (i) – (iii) and whose range is included in int $\mathscr{C}$ when $\mathscr{C}$ is in $\underline{D}_1$ and when p' is a prime greater than or equal to $p_{\mathscr{C}}$.

Let now a cube $\mathscr{C} \in \underline{C}(p_1)$ be given such that $\mathscr{C} \cap \mathscr{A} \neq \emptyset$ and $\mathscr{C} \setminus \mathscr{A} \neq \emptyset$, and note that

$$\mathscr{C} \cap \text{bdy}\,\mathscr{A} \neq \emptyset. \tag{5.52}$$

Therefore, $H|_{\mathscr{C}\cap\mathscr{A}}$ need not be uniformly continuous, and we need to subdivide $\mathscr{C} \cap \mathscr{A}$ into smaller regions before we can approximate H. We use the chosen subdivision $\underline{B} = \{\mathscr{B}_1,\dots,\mathscr{B}_J\}$ of $\mathscr{A}$ and put for each $j \in \{1,\dots,J\}$

$$(5.53) \qquad \mathscr{C}_j := (\text{clo } \mathscr{B}_j) \cap \mathscr{C};$$

we note that, by the construction of $\underline{B}$,

$$(5.54) \qquad \mathscr{C}_j \subset (\text{clo } \mathscr{H}_j) \cap \mathscr{C},$$

so that (5.33) yields

$$(5.55) \qquad x, y \in \mathscr{C}_j \quad \Rightarrow \quad |H(x) - H(y)| = |H_j(x) - H_j(y)| < \epsilon.$$

We also put

$$(5.56) \qquad \underline{D}_2 := \{\mathscr{C}_j = \text{clo } \mathscr{B}_j \cap \mathscr{C} \mid \mathscr{C}_j \neq \emptyset,\ \mathscr{C} \cap \mathscr{A} \neq \emptyset,\ \mathscr{C} \setminus \mathscr{A} \neq \emptyset,\ j \in \{1, \dots, J\},\ \mathscr{C} \in \underline{C}(p_1)\}.$$

For each $\mathscr{C}_j \in \underline{D}_2$, we choose $y_j \in \mathscr{C}_j$ and define $a_{\mathscr{C}_j} : \mathscr{E} \longrightarrow \mathscr{E}$ by

$$(5.57) \qquad a_{\mathscr{C}_j}(x) := y_j + \beta H(y_j)[x - y_j], \quad x \in \mathscr{E}.$$

Using the same procedure as described in the case $\mathscr{C} \in \underline{D}_1$, for each $\mathscr{C}_j \in \underline{D}_2$ we can choose a prime $p_{\mathscr{C}_j}$ such that, for all primes $p_j' \geq p_{\mathscr{C}_j}$, there exists an injective, piecewise rigid mapping $r_{\mathscr{C}_j}$ which maps $a_{\mathscr{C}_j}(\mathscr{C}_j) \setminus a_{\mathscr{C}_j}(\Pi(p_j', \mathbb{Z}))$ into int $\mathscr{C}_j$. Note that, in establishing the counterparts of (5.47) and (5.48), the property vol bdy $\mathscr{C} = 0$ is replaced by vol bdy $\mathscr{C}_j = 0$ which holds because $\mathscr{C}_j$, by (5.53), is the closure of a fit region. Let a prime $p_j' \geq p_{\mathscr{C}_j}$ be given. We define as in (5.49) and (5.50)

(5.58) $$\lambda_{\mathscr{C}_j} := \{x \in \pi \cap \operatorname{int}\mathscr{C}_j \mid \pi \in \Pi(p'_j, \{i, \dots, p'_j - 1\})\},$$

(5.59) $$h_{\mathscr{C}_j}(x) := r_{\mathscr{C}_j}(a_{\mathscr{C}_j}(x)), \quad x \in (\operatorname{int}\mathscr{C}_j) \setminus \lambda_{\mathscr{C}_j},$$

and note that $\lambda_{\mathscr{C}_j}$ is covered by the collection of planes $\Pi(p'_j, \{1, \dots, p'_j - 1\})$ and that $h_{\mathscr{C}_j}$ satisfies

$$\| h_{\mathscr{C}_j} - i \|_{L^\infty(\mathscr{C}_j, \mathscr{V})} < \epsilon$$

and

$$\| \nabla h_{\mathscr{C}_j} - H \|_{L^\infty(\mathscr{C}_j, \operatorname{Lin}\mathscr{V})} < \epsilon.$$

As in the preceding case, our goal is that of constructing a piecewise affine simple deformation, now from $(\operatorname{int}\mathscr{C}) \cap \mathscr{A}$, that satisfies (i) – (iii). To this end, for each j we put

(5.60) $$h_{\mathscr{C}}(x) := h_{\mathscr{C}_j}(x) \qquad \text{for all } x \in (\operatorname{int}\mathscr{C}_j) \setminus \lambda_{\mathscr{C}_j}$$

to obtain a mapping $h_{\mathscr{C}}: \bigcup_{\mathscr{C}_j \subset \mathscr{C}} ((\operatorname{int}\mathscr{C}_j) \setminus \lambda_{\mathscr{C}_j}) \longrightarrow \operatorname{int}\mathscr{C}$. For each $\mathscr{C}_j \subset \mathscr{C}$ we have

$$(\operatorname{int}\mathscr{C}_j) \setminus \lambda_{\mathscr{C}_j} = \operatorname{int}\mathscr{C}_j \cap ((\operatorname{int}\mathscr{C}) \setminus \lambda_{\mathscr{C}_j});$$

because $\operatorname{int}\mathscr{C}_j$ is a fit region and $(\operatorname{int}\mathscr{C}) \setminus \lambda_{\mathscr{C}_j}$ is a piecewise fit region, we may conclude that the domain of $h_{\mathscr{C}}$ is a piecewise fit region. Since the sets $\operatorname{int}\mathscr{C}_j$ are pairwise disjoint and $\lambda_{\mathscr{C}_j} \subset \operatorname{int}\mathscr{C}_j$, there holds

$$(5.61)\quad \bigcup_{\mathscr{C}_j \subset \mathscr{C}} ((\operatorname{int} \mathscr{C}_j) \setminus \lambda_{\mathscr{C}_j}) = (\bigcup_{\mathscr{C}_j \subset \mathscr{C}} \operatorname{int} \mathscr{C}_j) \setminus \bigcup_{\mathscr{C}_j \subset \mathscr{C}} \lambda_{\mathscr{C}_j} =$$

$$= (\bigcup_{\mathscr{C}_j \subset \mathscr{C}} (\mathscr{B}_j \cap \operatorname{int} \mathscr{C})) \setminus \bigcup_{\mathscr{C}_j \subset \mathscr{C}} \lambda_{\mathscr{C}_j} = ((\bigcup_{\mathscr{C}_j \subset \mathscr{C}} \mathscr{B}_j) \cap \operatorname{int} \mathscr{C}) \setminus \bigcup_{\mathscr{C}_j \subset \mathscr{C}} \lambda_{\mathscr{C}_j} =$$

$$= ((\mathscr{A} \setminus \Gamma(\underline{B})) \cap \operatorname{int} \mathscr{C}) \setminus \bigcup_{\mathscr{C}_j \subset \mathscr{C}} \lambda_{\mathscr{C}_j} = (\mathscr{A} \cap \operatorname{int} \mathscr{C}) \setminus ((\Gamma(\underline{B}) \cap \operatorname{int} \mathscr{C}) \cup (\bigcup_{\mathscr{C}_j \subset \mathscr{C}} \lambda_{\mathscr{C}_j})).$$

We now define

$$(5.62)\qquad \lambda_{\mathscr{C}} := (\Gamma(\underline{B}) \cap \operatorname{int} \mathscr{C}) \cup (\bigcup_{\mathscr{C}_j \subset \mathscr{C}} \lambda_{\mathscr{C}_j})$$

and observe that $(\lambda_{\mathscr{C}}, h_{\mathscr{C}})$ is a piecewise affine simple deformation from $(\operatorname{int} \mathscr{C}) \cap \mathscr{A}$. Indeed, $\operatorname{vol}(\bigcup_{\mathscr{C}_j \subset \mathscr{C}} \lambda_{\mathscr{C}_j}) = 0$ by (5.58) and $\operatorname{vol} \Gamma(\underline{B}) = 0$ by (5.23). Moreover, $h_{\mathscr{C}}$ is injective because each $h_{\mathscr{C}_j}$ is injective and $h_{\mathscr{C}_j}(\operatorname{int} \mathscr{C}_j \setminus \lambda_{\mathscr{C}_j}) \subset \operatorname{int} \mathscr{C}_j$. Finally, (5.58), (5.59), and (5.60) tell us that the domain of $h_{\mathscr{C}}$ is a finite union of fit regions, each of which is the intersection of $\operatorname{int} \mathscr{C}_j$ with an open cube of edge length $1/p_j'$, and the restriction of $h_{\mathscr{C}}$ to each of these fit regions extends to $\mathscr{E}$ as an affine deformation. Thus, $(\lambda_{\mathscr{C}}, h_{\mathscr{C}})$ is a piecewise affine simple deformation from $(\operatorname{int} \mathscr{C}) \cap \mathscr{A}$. To show that $(\lambda_{\mathscr{C}}, h_{\mathscr{C}})$ satisfies (i) we note that, by (5.32) and (5.52), the set $\Gamma(\underline{B}) \cap \mathscr{C}$ lies within a distance ϵ from $\operatorname{bdy} \mathscr{A}$, i.e., by (5.24),

$$(5.63)\qquad \Gamma(\underline{B}) \cap \mathscr{C} \subset \Gamma(\underline{B})_\epsilon,$$

so that, by (5.58) and (5.62), $\lambda_{\mathscr{C}}$ is covered by the collection

$$(5.64)\qquad \{\pi \cap \mathscr{C}_j \mid \pi \in \Pi(p_j', \{1, \ldots, p_j' - 1\})\} \cup \Gamma(\underline{B})_\epsilon.$$

That $h_{\mathscr{C}}$ satisfies (ii) and (iii) is a direct consequence of the fact that each $h_{\mathscr{C}_j}$ satisfies these conditions in its domain. We conclude that our procedure yields a piecewise affine simple deformation $(\lambda_{\mathscr{C}}, h_{\mathscr{C}})$ from $(\text{int } \mathscr{C}) \cap \mathscr{A}$ with the required properties (i) – (iii) when $\mathscr{C} \setminus \mathscr{A} \neq \emptyset$, $\mathscr{C} \cap \mathscr{A} \neq \emptyset$, and $p'_j > p_{\mathscr{C}_j}$ for all $\mathscr{C}_j \subset \mathscr{C}$.

We are now prepared to construct the simple deformation (λ, h) in the statement of the lemma. First of all, we define p_2 to be the maximum of the primes $p_{\mathscr{C}}$ and $p_{\mathscr{C}_j}$ as $\mathscr{C}$ varies over the cubes in $\underline{D}_1$ and $\mathscr{C}_j$ varies over the regions in $\underline{D}_2$, respectively, and we put $p' = p_2$ in (5.49) and $p'_j = p_2$ in (5.58). We define $h : \bigcup_{\mathscr{C}}(\mathscr{A} \cap \text{int } \mathscr{C} \setminus \lambda_{\mathscr{C}}) \longrightarrow \mathscr{E}$, with $\mathscr{C}$ running through those cubes in the set $\underline{C}(p_1)$ in (5.31) whose intersections with $\mathscr{A}$ are not empty, by setting:

$$h(x) := h_{\mathscr{C}}(x) \qquad \text{for all } x \in \mathscr{A} \cap \text{int } \mathscr{C} \setminus \lambda_{\mathscr{C}}, \tag{5.65}$$

with $h_{\mathscr{C}}$ given by (5.50) for $\mathscr{C}$ in $\underline{D}_1$ and by (5.60), (5.59) for $\mathscr{C}$ with $\mathscr{C} \cap \mathscr{A} \neq \emptyset$ and $\mathscr{C} \setminus \mathscr{A} \neq \emptyset$. We take λ to be the complement in $\mathscr{A}$ of the domain of h and note that, because the domain of h is

$$\bigcup_{\mathscr{C}} (\mathscr{A} \cap \text{int } \mathscr{C} \setminus \lambda_{\mathscr{C}}) = \mathscr{A} \setminus \bigcup_{\mathscr{C}} ((\mathscr{A} \cap \text{bdy } \mathscr{C}) \cup \lambda_{\mathscr{C}}), \tag{5.66}$$

the set λ obeys item (i); it follows from the above construction that (λ, h) is a piecewise affine simple deformation obeying (ii) and (iii).

We now wish to modify the definition of h on some of the regions in $\underline{D}_1 \cup \underline{D}_2$ in order to obtain a simple deformation (λ, h) that satisfies

$$\lambda \cap \mathscr{G} = \emptyset \quad \text{and} \quad h|_{\mathscr{G}} = i_{\mathscr{G}}, \tag{5.67}$$

with $\mathscr{G}$ given by (5.25). The key observation that permits us to do so is that if $\mathscr{C} \in \underline{D}_1$ and $\mathscr{C} \cap \mathscr{G} \neq \emptyset$, or if $\mathscr{C}_j \in \underline{D}_2$ and $\mathscr{C}_j \cap \mathscr{G} \neq \emptyset$, instead of the specifications (5.36) and (5.57) of $a_{\mathscr{C}}$ and $a_{\mathscr{C}_j}$, we may take

$$a_{\mathscr{C}} = a_{\mathscr{C}_j} = i. \tag{5.68}$$

We define

$$\mathscr{D} := \cup\{\mathscr{S} \in \mathbb{D}_1 \cup \mathbb{D}_2 \mid \mathscr{S} \cap \mathscr{G} \neq \emptyset\}, \tag{5.69}$$

and we note that each $x \in \mathscr{G}$ belongs to some $\mathscr{S} \in \mathbb{D}_1 \cup \mathbb{D}_2$ and, therefore, to $\mathscr{D}$. Thus $\mathscr{G} \subset \mathscr{D}$ and, because $\mathscr{G}$ is open, $\mathscr{G} \subset \operatorname{int} \mathscr{D}$; because $\mathscr{G} \subset \mathscr{A}$, we have

$$\mathscr{G} \subset \mathscr{A} \cap \operatorname{int} \mathscr{D}. \tag{5.70}$$

Moreover, $\mathscr{D}$ is a finite union of closures of fit regions, and, therefore, $\operatorname{int} \mathscr{D}$ is a fit region and $\mathscr{A} \cap \operatorname{int} \mathscr{D}$ is piecewise fit. We now modify the definition (5.65) of h by replacing all of the fit regions $\operatorname{int} \mathscr{S}$, with $\mathscr{S} \cap \mathscr{G} \neq \emptyset$, by the single piecewise fit region $\mathscr{A} \cap \operatorname{int} \mathscr{D}$ and by putting

$$h(x) := x \qquad \text{for all } x \in \mathscr{A} \cap \operatorname{int} \mathscr{D}. \tag{5.71}$$

We leave the definition of $h(x)$ for $x \in \mathscr{A} \setminus \operatorname{int} \mathscr{D}$ unchanged. This definition permits us to write the domain of h in the form:

$$\bigcup_{\substack{\mathscr{S} \in \mathbb{D}_1 \cup \mathbb{D}_2 \\ \mathscr{S} \cap \mathscr{G} = \emptyset}} (\mathscr{A} \cap \operatorname{int} \mathscr{S} \setminus \lambda_{\mathscr{S}}) \cup (\mathscr{A} \cap \operatorname{int} \mathscr{D}) = \mathscr{A} \cap \Big(\bigcup_{\substack{\mathscr{S} \in \mathbb{D}_1 \cup \mathbb{D}_2 \\ \mathscr{S} \cap \mathscr{G} = \emptyset}} (\operatorname{int} \mathscr{S} \setminus \lambda_{\mathscr{S}}) \cup \operatorname{int} \mathscr{D} \Big),$$

with $\lambda_{\mathscr{S}}$ as in (5.49) when $\mathscr{S} = \mathscr{C} \in \mathbb{D}_1$ and with $\lambda_{\mathscr{S}}$ as in (5.58) when $\mathscr{S} = \mathscr{C}_j \in \mathbb{D}_2$. If we define λ to be the complement in $\mathscr{A}$ of the domain of h, then the last relations yield

$$\begin{aligned} \lambda &= \bigcup_{\substack{\mathscr{S} \in \mathbb{D}_1 \cup \mathbb{D}_2 \\ \mathscr{S} \cap \mathscr{G} = \emptyset}} ((\mathscr{A} \cap \operatorname{bdy} \mathscr{S}) \cup \lambda_{\mathscr{S}}) \cup (\mathscr{A} \cap \operatorname{bdy} \mathscr{D}) = \\ &= \mathscr{A} \setminus \Big(\bigcup_{\substack{\mathscr{S} \in \mathbb{D}_1 \cup \mathbb{D}_2 \\ \mathscr{S} \cap \mathscr{G} = \emptyset}} (\operatorname{int} \mathscr{S} \setminus \lambda_{\mathscr{S}}) \cup \operatorname{int} \mathscr{D} \Big), \end{aligned} \tag{5.72}$$

and $(5.67)_1$ follows immediately from (5.70) and (5.72). Moreover, $(5.67)_2$ follows directly from (5.71) and (5.70). The observations that showed that the original pair (λ, h) is a piecewise affine simple deformation from $\mathcal{A}$ satisfying (i) – (iii) are easily adapted to show that the modified pair also is a piecewise affine simple deformation satisfying these conditions. ■

We now state and prove the Approximation Theorem.

5.8 Approximation Theorem: For each piecewise fit region $\mathcal{A}$ and each $(\kappa,g,G) \in \mathrm{Std}(\mathcal{A})$, there exists a sequence $n \mapsto (\kappa_n, f_n) \in \mathrm{Sid}(\mathcal{A})$ that determines (κ,g,G) in the sense of Definition 4.1.

Proof: Let $(\kappa,g,G) \in \mathrm{Std}(\mathcal{A})$ be given. Our first step is to reduce the problem of finding a sequence $n \mapsto (\kappa_n, f_n)$ that determines (κ,g,G) to that of finding a sequence that determines a structured deformation of the form $(\emptyset, i, H)$. Specifically, suppose that the sequence $n \mapsto (\lambda_n, h_n) \in \mathrm{Sid}(g(\mathcal{A}\backslash\kappa))$ determines the structured deformation $(\emptyset, i_{g(\mathcal{A}\backslash\kappa)}, H)$ from $g(\mathcal{A}\backslash\kappa)$, with

$$H := (G \circ g^{-1})((\nabla g)^{-1} \circ g^{-1}) \tag{5.73}$$

as given by (5.20). By (4.5) and Proposition 4.9, because $(\kappa,g) \in \mathrm{Sid}(\mathcal{A})$, the sequence $n \mapsto (\lambda_n,h_n) \circ (\kappa,g)$ determines the triple

$$\begin{aligned}(\emptyset, i_{g(\mathcal{A}\backslash\kappa)}, H) \circ (\kappa,g) &= (\kappa \cup g^{-1}(\emptyset),\ i_{g(\mathcal{A}\backslash\kappa)} \circ g, (H \circ g)\nabla g) = \\ &= (\kappa \cup \emptyset, g, G(\nabla g)^{-1}\nabla g) = (\kappa,g,G).\end{aligned} \tag{5.74}$$

Next, we let $(\emptyset,i,H) \in \mathrm{Std}(\mathcal{A})$ be given, and we define recursively a sequence $n \mapsto (\lambda_n, h_n)$ in $\mathrm{Sid}(\mathcal{A})$ that determines $(\emptyset,i,H)$ as follows. Put $p := 2$ and $\epsilon^1 := \frac{1}{2}$ in the Approximation Lemma to obtain primes p_1^1, p_2^1 greater than 3 and $(\lambda_1,h_1) \in \mathrm{Sid}(\mathcal{A})$

satisfying

(i)$_1$ λ_1 is covered by the collection of planes $\pi \in \Pi(p^1_\ell, \{1, ..., p^1_\ell - 1\})$ with $\ell \in \{1, 2\}$, together with the set $\Gamma(\underline{B})_{\epsilon^1} \subset \mathscr{B}$ defined as in (5.24),

(ii)$_1$ $\|h_1 - i\|_{L^\infty(\mathscr{B}, \mathscr{V})} < \epsilon^1$,

(iii)$_1$ $\|\nabla h_1 - H\|_{L^\infty(\mathscr{B}, \text{Lin } \mathscr{V})} < \epsilon^1$.

Let $n \in \mathbb{N} \setminus \{0\}$ be given. Suppose further that for each $k \in \{2, ..., n\}$ we have chosen primes p^k_1, p^k_2 and a simple deformation (λ_k, h_k), with p^k_1, p^k_2 both greater than $\max\{p^{k-1}_1, p^{k-1}_2\}$ and satisfying

(i)$_k$ λ_k is covered by $\bigcup_{\ell=1}^{2} \Pi(p^k_\ell, \{1, ..., p^k_\ell - 1\}) \cup \{\Gamma(\underline{B})_{\epsilon^k}\}$,

(ii)$_k$ $\| h_k - i \|_{L^\infty(\mathscr{B}, \mathscr{V})} < \epsilon^k$,

(iii)$_k$ $\| \nabla h_k - H \|_{L^\infty(\mathscr{B}, \text{Lin } \mathscr{V})} < \epsilon^k$,

with $\epsilon^k := 1/(k+1)$. To choose p^{n+1}_1, p^{n+1}_2 and $(\lambda_{n+1}, h_{n+1}) \in \text{Sid}(\mathscr{B})$, we again use the Approximation Lemma, with $\epsilon^{n+1} := 1/(n+2)$, and with $p := \max\{p^n_1, p^n_2\}$. The chosen primes p^{n+1}_1, p^{n+1}_2 and the simple deformation (λ_{n+1}, h_{n+1}) satisfy (i)$_{n+1}$, (ii)$_{n+1}$, and (iii)$_{n+1}$. This completes the recursive choice of a sequence of simple deformations $n \mapsto (\lambda_n, h_n) \in \text{Sid}(\mathscr{B})$. We note that properties (ii)$_n$ and (iii)$_n$ imply that conditions (ii) and (iii) in Definition 4.1 are satisfied, and it remains to verify that condition (i) holds, i.e., that $\liminf_{n\to\infty} \lambda_n = \emptyset$.

Let $x \in \mathscr{B}$ be given. Because the primes p^{n+1}_1, p^{n+1}_2 chosen at the $(n+1)^{st}$ stage are greater than all the primes chosen at preceding stages and because $\mathcal{A}$ obeys (5.21), the relation

$$x \in \bigcup_{\ell=1}^{2} \Pi(p_\ell^n, \{1,\dots,p_\ell^n - 1\})$$

can be satisfied at most for three values of n. Moreover, because dist (x, bdy $\mathcal{A}$) > 0 and $\lim_{n\to\infty} \epsilon^n = 0$, the relation $x \in \Gamma(\underline{B})_{\epsilon^n}$ can be satisfied at most for finitely many values of n. Hence, by $(i)_n$, the relation $x \in \lambda_n$ can be valid at most for finitely many values of n, and the definition of $\liminf_{n\to\infty} \lambda_n$ tells us that $x \notin \liminf_{n\to\infty} \lambda_n$. Because $x \in \mathcal{A}$ is arbitrary, we conclude that $\liminf_{n\to\infty} \lambda_n = \emptyset$. ■

5.9 Remark: It is convenient to restate the Approximation Theorem as follows: every structured deformation is a limit of simple deformations. Note that in order to consider a structured deformation (κ,g,G) as a limit of simple deformations, one must identify g and G with the L^∞ – functions that they represent. With this identification, the Approximation Theorem establishes the desired inclusion Std ⊂ LimSid.

An immediate consequence of the Approximation Theorem is that the fractured zone $\Phi(\kappa,g,G)$ and the unfractured zone $\Psi(\kappa,g,G)$, defined following (4.14) for each $(\kappa,g,G) \in$ LimSid ($\mathcal{A}$), also are defined for each structured deformation. This fact permits us to characterize the unfractured zone directly in terms of κ, ∇g and G, when (κ,g,G) is a structured deformation. We recall from Section 4 that for each $(\kappa,g,G) \in$ LimSid ($\mathcal{A}$), $\Psi(\kappa,g,G)$ is the set of all points $x \in \mathcal{A}$ such that, for at least one sequence $n \longmapsto (\kappa_n, f_n)$ that determines (κ,g,G), x belongs to ext $(\bigcup_{n=1}^{\infty} \kappa_n)$. We have already established that $\Psi(\kappa,g,G)$ is open and $\Psi(\kappa,g,G) \cap \kappa = \emptyset$. Moreover, by Theorem 4.14, $g_0|_{\Psi(\kappa,g,G)}$ is of class C^1 and $\nabla g_0 = G_0$ on $\Psi(\kappa, g, G)$, i.e.,

(5.75) $\Psi(\kappa,g,G) \subset \text{int}\,\{x \in \mathcal{B} \setminus \kappa \mid g_0 \text{ is differentiable at } x \text{ and } \nabla g_0(x) = G_0(x)\}$.

The next theorem establishes the opposite inclusion when (κ,g,G) is a structured deformation and tells us that $\Psi(\kappa,g,G)$ is the largest set on which (κ,g,G) is locally a classical deformation.

5.10 Theorem: For each structured deformation (κ,g,G) from $\mathcal{B}$ there holds

(5.76) $$\Psi(\kappa,g,G) = \text{int}\,\{x \in \mathcal{B} \setminus \kappa \mid \nabla g(x) = G(x)\}.$$

Proof: As in the proof of the Approximation Theorem, our first step is to show that it suffices to verify (5.76) for structured deformations of the form $(\emptyset, i, H)$. Let $(\kappa,g,G) \in \text{Std}(\mathcal{B})$ be given and consider $(\emptyset, i_{g(\mathcal{B}\setminus\kappa)}, H) \in \text{Std}(g(\mathcal{B}\setminus\kappa))$, with H as in (5.73). For each $x \in \mathcal{B} \setminus \kappa$, put $z := g(x)$ and note that

(5.77) $$\nabla g(x) = G(x) \Leftrightarrow I = G(x)(\nabla g(x))^{-1} \Leftrightarrow$$

$$\Leftrightarrow I = G(g^{-1}(z))(\nabla g(g^{-1}(z)))^{-1} \Leftrightarrow I = H(z).$$

Therefore, we may write

(5.78) $$g(\{x \in \mathcal{B}\setminus\kappa \mid \nabla g(x) = G(x)\}) = \{z \in g(\mathcal{B}\setminus\kappa) \mid H(z) = I\}.$$

Assume now that (5.76) holds for $(\emptyset, i_{g(\mathcal{B}\setminus\kappa)}, H)$, i.e.,

(5.79) $$\Psi(\emptyset, i_{g(\mathcal{B}\setminus\kappa)}, H) = \text{int}\,\{z \in g(\mathcal{B}\setminus\kappa) \mid H(z) = I\}.$$

Relations (5.78), (5.79) and the fact that g is a C^1 – diffeomorphism imply

(5.80) $$g^{-1}(\Psi(\emptyset, i_{g(\mathscr{A} \setminus \kappa)}, H)) = g^{-1}(\text{int}\, g(\{x \in \mathscr{A}\setminus\kappa \mid \nabla g(x) = G(x)\}) =$$
$$= \text{int}\, \{x \in \mathscr{A}\setminus\kappa \mid \nabla g(x) = G(x)\}.$$

Thus, the inclusion opposite to (5.75),

(5.81) $$\text{int}\, \{x \in \mathscr{A}\setminus\kappa \mid \nabla g(x) = G(x)\} \subset \Psi(\kappa,g,G),$$

will follow from (5.80) if we can prove the inclusion

(5.82) $$g^{-1}(\Psi(\emptyset, i_{g(\mathscr{A}\setminus\kappa)}, H)) \subset \Psi(\kappa,g,G).$$

To this end, let $z \in \Psi(\emptyset, i_{g(\mathscr{A}\setminus \kappa)}, H)$ be given. We may then choose $r > 0$ and a sequence $n \longmapsto (\mu_n, h_n) \in \text{Sid}(g(\mathscr{A} \setminus\kappa))$ that determines $(\emptyset, i_{g(\mathscr{A}\setminus\kappa)}, H)$ such that $\mathscr{B}(z, r) \subset g(\mathscr{A}\setminus\kappa)$ and

(5.83) $$\mathscr{B}(z, r) \cap (\bigcup_{n=1}^{\infty} \mu_n) = \emptyset.$$

Using the injectivity of g^{-1} and (5.83) we may write

(5.84) $$\emptyset = g^{-1}(\mathscr{B}(z, r)) \cap g^{-1}(\bigcup_{n=1}^{\infty} \mu_n) = g^{-1}(\mathscr{B}(z,r)) \cap (\bigcup_{n=1}^{\infty} g^{-1}(\mu_n)).$$

Because $g^{-1}(\mathscr{B}(z,r)) \subset \mathscr{A}\setminus \kappa$, we have

(5.85) $$g^{-1}(\mathscr{B}(z,r)) \cap \kappa = \emptyset,$$

and (5.84) then can be written as

(5.86) $$\emptyset = g^{-1}(\mathscr{B}(z,r)) \cap (\bigcup_{n=1}^{\infty} (g^{-1}(\mu_n) \cup \kappa)).$$

Moreover, g^{-1} is a C^1 – diffeomorphism, so there exists $\tilde{r} > 0$ such that $\mathscr{B}(g^{-1}(z), \tilde{r}) \subset g^{-1}(\mathscr{B}(z, r))$ and, by (5.86), we have

$$\mathscr{B}(g^{-1}(z), \tilde{r}) \cap \left(\bigcup_{n=1}^{\infty} (g^{-1}(\mu_n) \cup \kappa)\right) = \emptyset. \tag{5.87}$$

By Proposition 4.9, the sequence

$$n \longmapsto (\mu_n, h_n) \circ (\kappa, g) = (\kappa \cup g^{-1}(\mu_n), h_n \circ g \mid_{\mathscr{A}\setminus(\kappa \cup g^{-1}(\mu_n))}) \in \mathrm{Sid}(\mathscr{A})$$

determines (κ,g,G), and relation (5.87) then tells us that $g^{-1}(z) \in \Psi(\kappa,g,G)$, so that the inclusion (5.82) is established. Consequently, we have reduced the verification of (5.81) to that of (5.79) which, by Theorem 4.14 and the fact that $\Psi(\emptyset, i_{g(\mathscr{A} \setminus \kappa)}, H)$ is open, reduces finally to the verification of

$$\operatorname{int} \{z \in g(\mathscr{A}\setminus\kappa) \mid H(z) = I\} \subset \Psi(\emptyset, i_{g(\mathscr{A} \setminus \kappa)}, H). \tag{5.88}$$

Of course, it suffices to verify that, for each piecewise fit region $\mathscr{A}$ and for every $(\emptyset, i, H) \in \mathrm{Std}\,(\mathscr{A})$,

$$\mathscr{G} \subset \Psi(\emptyset, i_{\mathscr{A}}, H), \tag{5.89}$$

with $\mathscr{G}$ as defined in (5.25). By the Approximation Lemma, each term (λ_n, f_n) in the determining sequence for $(\emptyset, i_{\mathscr{A}}, H)$ in the Approximation Theorem can be chosen so that $\lambda_n \cap \mathscr{G} = \emptyset$. Because $\mathscr{G}$ is open, (5.89) follows from the definition of the unfractured zone $\Psi(\emptyset, i_{\mathscr{A}}, H)$. ■

Because each structured deformation (κ,g,G) from $\mathscr{A}$ satisfies both $(\kappa,g) \in \mathrm{Sid}\,(\mathscr{A})$ and $(\kappa,g,G) \in \mathrm{LimSid}\,(\mathscr{A})$, we may apply both Theorem 3.8 and Theorem 4.19 to obtain the following version of the Fundamental Theorem of Calculus for structured deformations.

<u>5.11 Theorem:</u> Let a piecewise fit region $\mathcal{A}$, a triple $(\kappa,g,G) \in \mathrm{Std}(\mathcal{A})$, and a unit vector $\boldsymbol{a}$ in $\mathcal{V}$ be given. There then hold:

(i) for almost every line L parallel to $\boldsymbol{a}$, $L \cap (\mathcal{A}\setminus\kappa)$ is, to within a one–dimensional set of measure zero, a finite union of pairwise disjoint intervals I_q, $q \in \{1, ..., Q\}$;

(ii) for each sequence $n \longmapsto (\kappa_n, f_n)$ which determines (κ,g,G) and for almost every line L parallel to $\boldsymbol{a}$, not only does (i) hold, but also, for each $q \in \{1, ..., Q\}$ and $n \in \mathbb{N}$, $f_n|_{I_q\setminus\kappa_n}$ and $g|_{I_q\setminus\kappa}$ extend to piecewise C^1 functions f_n^e and g^e on I_q, and the fundamental formula (3.16) applies to f_n and g for every x, y in I_q;

(iii) in addition, for every $x, y \in I_q$ with $y = x + |y-x|\boldsymbol{a}$, we have the formula

$$g(y-) - g(x+) = \int_0^{|x-y|} G(x+t\boldsymbol{a})\boldsymbol{a}dt + \lim_{n\to\infty} \sum_{z_n} (f_n(z_n+) - f_n(z_n-)), \tag{5.90}$$

where z_n are the discontinuity points of f_n^e in (x,y).

<u>Proof</u>: All of the assertions in this theorem are immediate consequences of Theorem 3.8 and Theorem 4.19, except for the assertion that formula (5.90) holds everywhere in I_q. Equation (4.25) tells us only that (5.90) holds for all x, y in the set $\mathcal{A}\cap I_q \setminus \bigcup_{m=1}^{\infty} \kappa_m$ which differs from I_q by a set of one–dimensional measure zero and, hence, is dense in I_q. To extend the validity of (5.90) to all x and y in I_q, it suffices to note that by (ii) of the present theorem, g extends to I_q as a piecewise continuous function and, therefore, has left and right limits at each point of I_q. Moreover, the proof of Theorem 4.19 shows that G is an integrable function on I_q, and it follows that the integral in (5.90) as a function of x and y extends to I_q as a continuous function. ∎

6. Interpretations and examples

The concepts and results in Sections 2 through 5 have been presented with only hints of possible interpretations in mechanics. In this section we make explicit such interpretations and provide justifications based on the mathematical results established in previous sections. In addition, we show how structured deformations can be used to describe the deformations of specific classes of continua with microstructure.

In our theory, a piecewise fit region $\mathscr{A}$ represents a region occupied by a continuous body, and the set $(\text{int clo } \mathscr{A}) \setminus \mathscr{A}$ represents the site of pre–existing, unopened cracks. In a simple deformation (κ, f) from $\mathscr{A}$, κ is the collection of all points of the body at which a new crack is created, and f specifies the position occupied by the remaining points of the body after the deformation. We call κ the new crack site, or briefly the crack site, and f the transplacement of the given simple deformation. We point out that new cracks can be created through a simple deformation, but pre–existing cracks cannot disappear; in other words, the process of forming a crack via simple deformations is irreversible. A theory in which all cracking is reversible recently has been formulated by NOLL [16].

A macroscopic fracture, or macrofracture, is here identified with the creation of a new crack. Besides macrofractures, our scheme allows for the presence of microfractures. The simultaneous formation of microfractures and macrofractures is described mathematically as the result of a limit procedure in which a sequence $n \longmapsto (\kappa_n, f_n)$ of simple deformations determines a limit of simple deformations (κ, g, G). In a limit of simple deformations, the site of all fractures, both macroscopic and microscopic, is the fractured zone $\Phi(\kappa, g, G)$ defined in Section 4, and the site of the macrofractures is the crack site κ. The broken ramp sequences in Examples 4.2, 4.3 illustrate a typical situation in which each individual term of the determining sequence involves macrofractures which, for growing values of n, spread all over the body with decreasing amplitudes of the associated jumps. In the limit, the jumps disappear, but κ turns out to be the empty set in the first example and the set of all dyadic rationals between

0 and 1 in the second example. Thus, there are no macrofractures in the first case, whereas in the second case the macrofractures are diffused throughout the body. In the examples under consideration, the fractured zone can easily be determined. Indeed, Theorem 4.14 tells us that the fractured zone includes all points x for which $\nabla g_0(x) \neq G_0(x)$, and in both examples this condition is satisfied at all points of the body.

Among the collection of limits of simple deformations, we have identified the subclass of structured deformations. This subclass has the property that, for each structured deformation (κ,g,G), the pair (κ,g) is a simple deformation. Another useful property is provided by Proposition 5.3: the composition of two structured deformations is a structured deformation. We now use these two properties to define local measures of deformations due to microfracture. First of all we observe that the gradient ∇f_n of each transplacement in the determining sequence $n \longmapsto (\kappa_n, f_n)$ can be regarded as a local measure of deformation at those points of $\mathcal{A}$ at which no fracture occurs. It is then natural to consider the limit element G of the sequence $n \longmapsto \nabla f_n$ as a local measure of deformation without fracture, in the sense that G is not affected by the presence of either microfractures of macrofractures. It is also natural to consider ∇g as a local measure of the macroscopic deformation determined by the macroscopically observed transplacements g. For these reasons, we call ∇g and G the <u>macroscopic deformation tensor</u> and the <u>tensor of deformation without fracture</u>, respectively. Theorem 5.10 characterizes the fractured zone for a structured deformation as the complement in $\mathcal{A}$ of the interior of the region in which these two deformation tensors agree.

For a structured deformation (κ,g,G) from $\mathcal{A}$, the fundamental formula of calculus for simple deformations (3.16) tells us that

$$g(y) - g(x) = \int_0^{|y-x|} \nabla g(x+t\mathbf{a})\mathbf{a}\,dt + \sum_z (g(z+) - g(z-)), \tag{6.1}$$

i.e., that the relative transplacement $g(y) - g(x)$ of y and x, with x, y in $\mathcal{B}\setminus\kappa$ and $y = x + |y-x|\,\boldsymbol{e}$, is a sum of an integral representing the relative transplacement due to macroscopic deformation and of a sum of jumps representing the relative transplacement due to macroscopic fracture. On the other hand, the fundamental formula of calculus for structured deformations (5.90)

$$g(y) - g(x) = \int_0^{|y-x|} G(x+t\boldsymbol{e})\boldsymbol{e}\,dt + \lim_{n\to\infty} \Sigma(f_n(z_n+) - (f_n(z_n-)) \tag{6.2}$$

shows that the relative transplacement of y and x also is the sum of an integral involving the relative transplacement without fracture, plus a term which accounts for all fractures occurring in the simple deformations (κ_n, f_n) of the determining sequence. Note that, in the limit, this term may result both from macroscopic and microscopic fractures. Subtracting (6.1) from (6.2) yields

$$\int_0^{|y-x|} (\nabla g(x+t\boldsymbol{e}) - G(x + t\boldsymbol{e}))\boldsymbol{e}\,dt = \tag{6.3}$$

$$= \lim_{n\to\infty} \sum_{z_n} (f_n(z_n+) - (f_n(z_n-)) - \sum_{z}(g(z+) - g(z-)),$$

where the right–hand side consists of the difference between the relative transplacement due to fracture and that due to macrofracture; therefore, the right–hand side of (6.3) represents the relative transplacement due to microfracture. This formula tells us that the relative transplacement due microfracture admits an integral representation in which the tensor field

$$M(x) := \nabla g(x) - G(x), \quad x \in \mathcal{B}\setminus\kappa, \tag{6.4}$$

provides a local measure of deformation due to microfracture. The additive decomposition

$$\nabla g = G + M \tag{6.5}$$

expresses the macroscopic deformation tensor as the sum of the tensor of deformation without fracture and of a local measure of deformation due to microfracture. For reasons that we give presently, we call M the Burgers microfracture tensor.

Multiplicative decompositions of ∇g involving G and appropriate local measures of deformation due to microfracture can be obtained from the global decomposition of structured deformations

$$(\kappa,g,G) = (\emptyset, i_{g(\mathcal{B}\setminus\kappa)}, (G\circ g^{-1})((\nabla g)^{-1}\circ g^{-1}))\circ(\kappa,g,\nabla g), \tag{6.6}$$

established in Proposition 5.6, and from its counterpart

$$(\kappa,g,G) = (\kappa, g, \nabla g)\circ(\emptyset, i_{\mathcal{B}\setminus\kappa}, (\nabla g)^{-1}G), \tag{6.7}$$

that also follows from the formula (5.2) for the composition of two structured deformations. In both cases, a structured deformation is decomposed into a simple deformation and a structured deformation of the form $(\emptyset,i,H)$. The latter can be interpreted as a purely microscopic deformation; indeed, because $\kappa = \emptyset$, no macrofracture occurs; because $g = i$, no point is macroscopically displaced. Thus, the factorization (6.6) represents a structured deformation as a simple deformation followed by a purely microscopic deformation, and (6.7) represents the same structured deformation as a purely microscopic deformation followed by a simple deformation. This suggests the local multiplicative decompositions

$$\nabla g = M_\ell G, \tag{6.8}$$

$$\nabla g = G M_r, \tag{6.9}$$

of the macroscopic deformation tensor ∇g, with $M_\ell := \nabla g\, G^{-1}$ and $M_r := G^{-1}\nabla g$; M_ℓ and M_r

will be called the <u>left microfracture tensor</u> and <u>the right microfracture tensor</u>, respectively.

In conclusion, both additive and multiplicative decompositions of ∇g involving G and a measure of deformation due to microfracture arise from our analysis. For each structured deformation (κ,g,G), the tensor fields M, M_ℓ and M_r are determined by g and G. Thus, there is no difference in principle between adopting one decomposition or the other, and, once G and one of the above tensor fields are known, the other two can be determined from their definitions. For a structured deformation (κ,g,G), the fact that (κ,g) is a simple deformation implies that g is differentiable in $\mathcal{B}\backslash\kappa$. Therefore, if we integrate relation (6.5) along a closed curve c in $\mathcal{B}\backslash\kappa$ we obtain

$$(6.10) \qquad 0 = \oint_{c} \nabla g \, d\mathbf{x} = \oint_{c} G \, d\mathbf{x} + \oint_{c} M \, d\mathbf{x};$$

this shows that the circulation of $-G$ along c equals that of M, and, therefore, measures the relative transplacement due to microfracture along the closed curve c. The circulation of M (or of $-G$) along c is called the <u>Burgers vector</u> in continuum theories of dislocations [8], [9]. This motivates our choice of the name <u>Burgers microfracture tensor</u> for M.

Of course, $\det \nabla g$ represents the macroscopic local volume change; analogously, $\det G$ represents the local volume change without fracture, so that the inequality (4.19), also occurring in (Std 3), Definition 5.1, asserts that <u>the local volume change without fracture cannot exceed the macroscopic local volume change</u>. In other words, microfracture can create voids but cannot consolidate the body. Accordingly, we interpret the scalar field

$$(6.11) \qquad \varphi := \frac{\det G}{\det \nabla g} = (\det M_r)^{-1} = (\det M_\ell)^{-1},$$

whose values lie in the interval (0,1], as the <u>volume fraction</u> and $1-\varphi$ as the <u>void fraction</u> for the given structured deformation (κ,g,G). When $\varphi = 1$, there is no volume change due to microfracture, i.e., deformation due to microfracture cannot entail the opening of cracks at the microscopic level. In particular, if we consider the set of invertable structured deformations as defined by (5.19), we see that, because for such deformations the crack site κ is empty, invertible structured deformations describe situations in which deformation due to microfracture occurs without the formation of new cracks; because for invertible structured deformations the volume fraction is one, by (6.11), they describe deformations in which no volume change occurs due to microfracture.

We are now in a position to describe the deformations of particular classes of continua. We recall the decompositions (6.6), (6.7), according to which a structured deformation can be thought of as the composition of a simple deformation and a purely microscopic deformation. For definiteness, we consider here the decomposition (6.6) which we rewrite in the form

$$(\kappa,g,G) = (\emptyset,i,H) \circ (\kappa,g,\nabla g). \tag{6.12}$$

Sets of deformations appropriate to particular classes of continua can be constructed by requiring that each factor in this decomposition be subject to further restrictions. For example, the requirement $\kappa = \emptyset$ yields the deformations of a <u>continuum without macrofractures</u>, and $H = I$ yields the deformations of a <u>continuum without microfractures</u>. Taking both $\kappa = \emptyset$ and $H = I$ and requiring further that all transplacements g be classical deformations yields the deformations of a <u>classical continuum</u>. Suitable restrictions on the transplacements g yield deformations of continua subject to <u>internal constraints</u>, such as rigidity, incompressibility, or inextensibility in a given material direction. Similarly, restrictions on H yield deformations of particular classes of continua with microfractures, some of which will be identified in the following.

For each purely microscopic deformation $(\emptyset,i,H)$ from $\mathcal{A}$ and for all $x \in \mathcal{A}$, set

$$U(x) := (\varphi(x))^{-1/3}\, H(x), \tag{6.13}$$

where the volume fraction φ, defined in (6.11), here reduces to $\det H$. Note also that the tensor field U is unimodular, i.e., $\det U(x) = 1$ for all $x \in \mathcal{A}$. Consider the decomposition

$$(6.14) \qquad (\emptyset,i,H) = (\emptyset,i,U) \circ (\emptyset,i,\varphi^{1/3} I)$$

in which both triples on the right–hand side are purely microscopic deformations. The deformation $(\emptyset,i,\varphi^{1/3} I)$ involves a creation of voids; the fact that the tensor of deformation without fracture is spherical at each point is expressed by saying that there is no distortion in the deformation without fracture. Accordingly, we call the two factors in the decomposition (6.14) a creation of voids without distortion and a purely microscopic distortion, respectively. Notice that, by (5.19), a purely microscopic distortion $(\emptyset,i,U)$ is an invertible structured deformation.

The term continuum with voids is commonly used to describe a continuum in which the only purely microscopic deformations that can occur are creations of voids without distortion. In a similar way, a continuum in which the only purely microscopic deformations are purely microscopic distortions may be called a continuum without voids.

Particular classes of continua without voids can be identified by imposing further restrictions on the field U. Examples taken from the list given in the book [1], Sect. 2, are shown in the following table.

Name of continuum	Range of U
continuum with spin	all proper orthogonal tensors with axes parallel to a fixed direction
Cosserat continuum	all proper orthogonal tensors
continuum without voids	all proper unimodular tensors

A continuum that can undergo arbitrary, purely microscopic deformations is called a micromorphic continuum. In another class of continua mentioned in reference [1], the continua with vector microstructure, the field H can take arbitrary values in the set of second–order tensors. Deformations of such continua need not be structured deformations because they can violate the condition (Std 3) on the positiveness of the determinant of H. (In fact, they need not even be limits of simple deformations.) Further classes of continua in the same list, namely

<u>liquid crystals</u> and <u>bodies with continuous distribution of dislocations</u>, will be considered in the next section.

7. Applications to specific continuum theories

In this section we examine several existing continuum theories and propose sets of non–classical deformations that clarify concepts in these theories.

7a Plasticity

The subject of plasticity treats large deformations of a continuous body that can occur under nearly constant stress and that cannot be reversed by reversal of the stress alone. The most widely used theories of plasticity are formulated in the kinematical context of classical deformations of a continuum and introduce notions of "elastic deformation" and "plastic deformation" not as purely kinematical quantities but in the context of internal variables that appear in decompositions of the deformation gradient and in other constitutive relations. This approach has the advantage of describing plastic behavior in the same classical kinematical framework that is used in elasticity and fluid mechanics. It has the disadvantage that, however well motivated on physical grounds the chosen notions of elastic and plastic deformation might be, certain additional choices must be made as to how the local deformation should decompose into elastic and plastic parts and how each part should transform under changes in observer and reference configuration. The multiplicity of such choices has led to lingering controversies [11].

Here we choose to describe elastic deformation as deformation without fracture and plastic deformation as deformation due to microfracture. Within the class of structured deformations (κ, g, G) we define the elastic deformation tensor to be G, the tensor of deformation without fracture. This definition agrees with descriptions of deformations at the microscopic level in metals, where deformations are considered to be elastic when no substantial activation of defects in the crystalline structure occurs. This definition also agrees with the caveat made in theories of plasticity: elastic deformation need not be the gradient of a displacement of the body, or even of a piece of the body. In fact, our results in Section 5 require only that G be a limit of gradients, and this limit need not be a gradient. Furthermore, the plastic behavior of many materials involves neither macroscopic fracture nor local volume changes due to microfracture. For such materials, an appropriate class of deformations is the class InvStd of invertible structured deformations for which, by definition, the crack site is empty and $\det \nabla g = \det G$.

An important consequence of our definition of the elastic deformation tensor, that usually enters as an assumption in existing theories of plasticity, is the fact that the tensor transforms in the same manner as the local macroscopic deformation ∇g under changes in observer and reference configuration (see [18]). Here, we consider changes in observer and reference configuration in the sense described in the book [7]. In particular, we consider changes in reference configuration that are classical deformations.

7.1 Proposition: Under changes in observer there holds

$$\nabla g \longrightarrow Q\,\nabla g \quad \text{and} \quad G \longrightarrow Q\,G, \tag{7.1}$$

and under changes in reference configuration there hold

$$\nabla g \longrightarrow \nabla g\,H \quad \text{and} \quad G \longrightarrow GH. \tag{7.2}$$

Proof: The laws of change in observer and change in reference configuration tell us that deformation gradients F transform according to $F \longrightarrow Q\,F$ and $F \longrightarrow F\,H$, which immediately yields the properties of ∇g in (7.1) and (7.2). Here, Q is the orthogonal tensor associated with the change in observer and H is the unimodular tensor associated with the change in reference configuration. Because G is the limit of the sequence $n \longmapsto \nabla f_n$ and, for every $n \in \mathbb{N}$, the deformation gradient ∇f_n transforms in the same way as ∇g, we conclude that G transforms as in (7.1) and (7.2). ∎

Our earlier description of plastic deformation as deformation due to microfracture leads naturally to the choice of the local measures of deformation due to microfracture

(7.3) $$M = \nabla g - G,$$

(7.4) $$M_\ell = \nabla g\, G^{-1},$$

(7.5) $$M_r = G^{-1} \nabla g,$$

as plastic deformation tensors for a given structured deformation (κ, g, G). Counterparts of M, M_ℓ and M_r are the tensors $F^p - I$ of NEMAT–NASSER [12], $\bar{F}^p$ of CLIFTON [3], and F^p of LEE & LIU [10], respectively (c.f. [18] for further discussion of this correspondence). The relations (7.3) – (7.5) give rise to corresponding decompositions of the macroscopic deformation tensor ∇g into elastic and plastic parts. When there is no local volume change due to microfracture the relation $\det \nabla g = \det G$ implies $\det M_\ell = \det M_r = 1$. In this case, the tensor field H in (6.12) satisfies $\det H = 1$, so that $(\emptyset, i, H)$ represents a purely microscopic distortion.

Just as we showed in Proposition 7.1 that the transformation laws for the elastic deformation tensor G are determined by those for deformation gradients, we obtain a corresponding result for each of the plastic deformation tensors M, M_ℓ, and M_r ([18]).

7.2 Proposition: The tensors M, M_ℓ, and M_r transform under changes in observer and changes in reference configuration according to the rules:

(7.6) $$M \longrightarrow Q M, \qquad M \longrightarrow M H$$

(7.7) $$M_\ell \longrightarrow Q M_\ell Q^T, \qquad M_\ell \longrightarrow M_\ell,$$

(7.8) $$M_r \longrightarrow M_r, \qquad M_r \longrightarrow H^{-1} M_r H.$$

We note that the plastic deformation tensor M transforms in the same way as do the elastic deformation tensor G and the macroscopic deformation tensor ∇g.

7b Liquid Crystals

Liquid crystals are substances that behave mechanically in many ways like fluids, but whose optical and electrical properties are more like those of anisotropic, crystalline solids. Based on the observed presence in many liquid crystals of nearly rigid, rod–like molecules that tend to align within small regions of the liquid crystal, theories of liquid crystals often postulate the existence of a director field for each global state of the liquid crystal, i.e., a field $\mathscr{n}$ whose value $\mathscr{n}(x)$ at a point x is a unit vector, interpreted as the average of the orientations of a collection of molecules. In addition, a scalar order–parameter S sometimes is introduced to represent deviations of molecular orientation from the director field $\mathscr{n}$.

We show here that a director field automatically arises within a specified class of structured deformations, and we also discuss the possibility of defining an order–parameter within that class. For simplicity, we discuss only "statical configurations" of liquid crystals, i.e., situations in which there is no movement of the continuum that represents the liquid crystal, but in which the director field $\mathscr{n}$ can change. To this end, we consider purely microscopic deformations $(\emptyset, i, H)$ in $\mathrm{Std}\,(\mathscr{A})$ for which $H(x)$ is an orthogonal tensor for every point x in the given region $\mathscr{A}$. By the Approximation Theorem, Theorem 5.8, we may choose a sequence $j \longmapsto (\kappa_j, f_j)$ of piecewise affine simple deformations such that, in the sense of L^∞ – convergence on $\mathscr{A}$,

(7.9) $$\lim_{j\to\infty} f_j = i,$$

(7.10) $$\lim_{j\to\infty} \nabla f_j = H,$$

and such that

(7.11) $$\liminf_{j\to\infty} \kappa_j = \emptyset.$$

Relations (7.9) and (7.10) tell us that f_j is approximately a piecewise rigid mapping and that the finitely many values of ∇f_j are approximately orthogonal. Therefore, it is natural to regard each of the affine restrictions of f_j as imparting an approximately rigid transplacement to one of the molecules of the liquid crystal, which we identify as one of the connected components of $\mathcal{A} \setminus \kappa_j$. If all the molecules in $\mathcal{A}$ have a common initial alignment defined by a unit vector ℓ, then the finitely many values of $\nabla f_j \ell$ give the corresponding alignments after the simple deformation (κ_j, f_j). Moreover, Lemma 4.11 tells us that, for each $x \in \mathcal{A}$,

$$H(x)\,\ell = \lim_{j'\to\infty} \nabla f_{j'}(x)\,\ell, \tag{7.12}$$

$$x = \lim_{j'\to\infty} f_{j'}(x), \tag{7.13}$$

for some subsequence $j' \longmapsto (\kappa_{j'}, f_{j'})$ of $j \longmapsto (\kappa_j, f_j)$ that may depend upon the point x. Therefore, (7.12) permits us to assert: <u>the unit vector</u> $H(x)\ell$ <u>is the limiting alignment of the orientations</u> $\nabla f_{j'}(x)\,\ell$ <u>of molecules at</u> x <u>as</u> j' <u>tends to infinity</u>. (Our identification of molecules with connected components of $\mathcal{A} \setminus \kappa_j$ suggests that the size of the molecules tends to zero as j and j' tend to ∞). Therefore, we define the <u>director field</u> n for the deformation $(\emptyset,i,H)$ and for the initial direction ℓ by the relation

$$n(x) = H(x)\,\ell \quad \text{for all } x \in \mathcal{A}. \tag{7.14}$$

The following proposition gives a precise sense in which the director field n is a limit of the average orientations of collections of molecules.

<u>7.3 Proposition</u>: Let a unit vector ℓ and $(\emptyset,i,H) \in \mathrm{Std}(\mathcal{A})$ be given with H orthogonal–valued. For each $x \in \mathcal{A}$ and each determining sequence $j \longmapsto (\kappa_j, f_j)$ for $(\emptyset,i,H)$, there holds

$$n(x) = \lim_{j\to\infty} \frac{1}{\mathrm{vol}(\mathcal{B}(x,(j+1)^{-1})\cap\mathcal{A})} \int_{\mathcal{B}(x,(j+1)^{-1})\cap\mathcal{A}} \nabla f_j(y)\,\ell\, dy. \tag{7.15}$$

Proof: For each $j \in \mathbb{N}$ we have

$$\int_{\mathscr{B}(x,(j+1)^{-1})\cap\mathscr{A}} \nabla f_j(y)\ell \, dy =$$

$$= \int_{\mathscr{B}(x,(j+1)^{-1})\cap\mathscr{A}} (\nabla f_j(y)\ell - H(y)\ell)\, dy \; +$$

$$+ \int_{\mathscr{B}(x,(j+1)^{-1})\cap\mathscr{A}} (H(y)\ell - H(x)\ell)\, dy + \int_{\mathscr{B}(x,(j+1)^{-1})\cap\mathscr{A}} H(x)\ell \, dy,$$

and (7.15) then follows from (7.10), (7.14), and the continuity of H on $\mathscr{A}$. ∎

We recall that in the relation (7.12) the choice of subsequence depends upon the given point x, as is also the case in the following formula for the director at x:

(7.16) $$n(x) = H(x)\,\ell = \lim_{j'\to\infty} \nabla f_{j'}(x)\,\ell.$$

In contrast to this fact, the relation (7.15) holds for every determining sequence $j \longmapsto (\kappa_j, f_j)$ and for every $x \in \mathscr{A}$. Thus, for a given point x in $\mathscr{A}$, no special determining sequence for $(\emptyset, i, H)$ need be chosen in order to guarantee the validity of (7.15).

It is interesting to try to use the present kinematical setting in order to identify the order–parameter S mentioned above. In descriptions of nematic liquid crystals [20] one finds a measure of deviation of molecular orientations from the director $n(x)$ that corresponds in our setting to the quantity

(7.17) $$S(x) := \lim_{j\to\infty} \frac{3}{2}\left[\frac{\int_{\mathscr{B}(x,(j+1)^{-1})\cap\mathscr{A}} (n(x)\cdot\nabla f_j(y)\ell)^2 dy}{\mathrm{vol}(\mathscr{B}(x,(j+1)^{-1})\cap\mathscr{A})} - \frac{1}{3}\right].$$

Here, $(n(x)\cdot\nabla f_j(y)\ell)^2$ measures the angular deviation from $n(x)$ of the axis of the molecule located at y. Using the same arguments given in the proof of Proposition 7.3, we conclude that $S(x) = 1$, a relation that describes the completely oriented nematic phase of the liquid crystal. Hence, the L^∞ – convergence of $j \longmapsto \nabla f_j$ to H leads to a class of liquid crystals in which the order–parameter takes on only the value 1. In further studies along these lines, it would be necessary to weaken the type of convergence of the gradients in order to obtain other values of the order parameter.

7c Crystals with Defects

Continuum theories of defective crystalline solids describe substances with microscopically regular atomic lattices that are weakened by the presence of defects, i.e., irregularities in the lattice structure that collectively can support large deformations of the crystal. In such theories, the discrete structure of the atomic lattice at the microscopic scale is replaced by a continuous distribution of basic lattice vectors that are specified mathematically as a triple $(\mathscr{e}_1, \mathscr{e}_2, \mathscr{e}_3)$ of vector fields such that the triple of vectors $(\mathscr{e}_1(x), \mathscr{e}_2(x), \mathscr{e}_3(x))$ is a basis of $\mathscr{V}$ at each point x of the body. The presence of defects is then inferred from topological properties associated with the fields $(\mathscr{e}_1, \mathscr{e}_2, \mathscr{e}_3)$. We show here that lattice vector fields automatically arise within a specified class of structured deformations in the same way as the director field for liquid crystals arose in the previous discussion, and this provides a mathematical description of the process of "continuizing" a crystalline structure [9]. We further indicate how continuous distributions of dislocations can be obtained as limits of simple deformations, and we identify the position of a class of "neutral changes in state" for defective crystals ([4], [5], [6]) within the class of structured deformations.

We restrict our attention to purely microscopic distortions of a defective crystal; other deformations can then be obtained via composition with arbitrary simple deformations. Let $(\emptyset, i, H)$ in $Std(\mathscr{A})$ be given such that $\det H = 1$, and let an orthonormal basis $(\mathscr{k}_1, \mathscr{k}_2, \mathscr{k}_3)$ of $\mathscr{V}$ be given. We choose a determining sequence $m \longmapsto (\kappa_m, f_m)$ for $(\emptyset, i, H)$ such that each simple deformation is piecewise affine. Note that, for each $m \in \mathbb{N}$ and $x \in \mathscr{A} \setminus \kappa_m$, $(\nabla f_m(x)\mathscr{k}_1, \nabla f_m(x)\mathscr{k}_2, \nabla f_m(x)\mathscr{k}_3)$ is a basis of $\mathscr{V}$, and that, as ∇f_m has only finitely many values, this basis takes on only finitely many values. Therefore, $\{(\nabla f_m(x)\ell_1, \nabla f_m(x)\ell_2, \nabla f_m(x)\ell_3) \mid x \in \mathcal{A} \setminus \kappa_m\}$ is a finite set of bases of $\mathcal{V}$ that we interpret as the set of discrete lattice bases for all the atomic sites of the crystal in the deformed state determined by (κ_m, f_m). The same arguments that led to (7.12) and (7.13) tell us that, for each $x \in \mathcal{A}$, there is a subsequence $m' \longmapsto (\kappa_{m'}, f_{m'})$ of $m \longmapsto (\kappa_m, f_m)$ such that

$$H(x)\ell_\ell = \lim_{m'\to\infty} \nabla f_{m'}(x)\ell_\ell, \qquad \ell \in \{1, 2, 3\}. \tag{7.18}$$

We call $(H(x)\ell_1, H(x)\ell_2, H(x)\ell_3)$ the <u>lattice basis</u> at x for $(\emptyset, i, H)$ and (ℓ_1, ℓ_2, ℓ_3). Moreover, the proof of Proposition 7.3 immediately shows that, for every $x \in \mathcal{A}$ and $\ell \in \{1, 2, 3\}$,

$$\lim_{m\to\infty} \frac{\displaystyle\int_{\mathcal{B}(x,(m+1)^{-1})\cap\mathcal{A}} \nabla f_m(y)\,\ell_\ell\,dy}{\mathrm{vol}(\mathcal{B}(x,(m+1)^{-1}) \cap \mathcal{A})} = H(x)\ell_\ell. \tag{7.19}$$

Thus, our use of structured deformations $(\emptyset, i, H)$ permits us to show that the <u>lattice basis field</u> $(H\ell_1, H\ell_2, H\ell_3)$ <u>is a limit of averages of discrete lattice bases</u>.

Two classes of structured deformations of interest in the study of defective crystals arise from the following factorization of a classical deformation f, regarded as a structured deformation $(\emptyset,f,\nabla f)$:

$$(\emptyset,f,\nabla f) = (\emptyset,f,I) \circ (\emptyset,i,\nabla f). \tag{7.20}$$

The factor $(\emptyset,f,I)$ has $G = I$, so there is no local deformation without fracture; because $\kappa = \emptyset$, there is no macrofracture. Therefore, we think of $(\emptyset,f,I)$ as a deformation due to microfracture. The limit of the deck of cards discussed following Example 4.3 is a deformation $(\emptyset,s,I)$ of this type with s a simple shear. The second factor $(\emptyset,i,\nabla f)$ in (7.20) involves some deformation due to microfracture whenever $\nabla i = I \neq \nabla f$. However, because $G = \nabla f$ is a gradient, the line integral $\oint_{\mathcal{C}} G \, d\boldsymbol{x}$ must vanish for every rectifiable closed curve $\mathcal{C}$ in the domain of f, i.e., the Burgers vector for $(\emptyset,i,\nabla f)$ vanishes. This situation is described by saying that the purely microscopic deformation $(\emptyset,i,\nabla f)$ introduces no new defects in a crystal. Extensive studies of classes of "changes of state" of defective crystals by DAVINI & PARRY ([4], [5]) and FONSECA & PARRY ([6]) have focused on a class of "neutral", or "defect – preserving", changes of state; when $\det \nabla f = 1$, our deformations $(\emptyset,i,\nabla f)$ correspond to a proper subset of the collection of neutral changes in state, and our deformations $(\emptyset,f,I)$ correspond to "rearrangements" in the above studies.

In closing this subsection, we indicate how limits of simple deformations permit one to describe continuous distributions of defects in a body. Indeed, the limit of the "deck of cards" $(\emptyset,s,I)$ with s a simple shear suffices to illustrate this point. Each simple deformation (τ_n, t_n) defined in (3.9) and (3.10) describes the effect of $n-1$ edge dislocations passing through the deck of cards. The glide plane of each dislocation is one of the interfaces between the cards, and one may view the $n-1$ edge dislocations as passing through the deck one after another, starting with the activation of the plane on the bottom card and proceeding to higher and higher cards in the deck. The structured deformation $(\emptyset,s,I)$ is determined by the sequence $n \longmapsto (\tau_n,t_n)$ and can be visualized as the effect of infinitely many parallel glide planes, each causing an infinitesimal displacement of the region above that plane parallel to that plane. Thus, $(\emptyset,s,I)$ describes the effect of a continuous distribution of edge dislocations passing through a body. Of course, our theory provides the possibility of generalizing this example in many directions, as our examples in Sections 4 and 5 suggest.

7d Mixtures

Continuum theories of mixtures describe a distinguished continuum, called the mixture, as well as auxiliary continua, called the constituents, that are permitted to interpenetrate. Such interpenetration cannot be described using a structured deformation, because the transplacement for such a deformation must be injective. Mixture theories often assign to each constituent its own classical deformation and then explore various methods for identifying an associated deformation of the mixture. The need to include in a mixture theory the possibility of diffusion among the constituents leads to many possible choices of an associated deformation of the mixture and, hence, to a diversity of mixture theories.

We here propose a description of deformations of a mixture and of its constituents in which we approximate a deformation of the mixture by a simple deformation that, by virtue of its injectivity, places all the constituents in space without interpenetration of matter. Each constituent undergoes in the approximation a simple deformation that separates the constituent into pieces, with spaces between the pieces left for other constituents to occupy. Thus, in the approximation, the constituents are dispersed in space without interpenetration of matter. A deformation of the mixture is defined to be a limit of approximating simple deformations. In the passage to the limit, the volume of the pieces of the constituents goes to zero and the number of pieces goes to infinity, so that, in the limit, different constituents are permitted to interpenetrate. For example, consider as an approximating simple deformation the cutting and shuffling of a deck of cards. The cut divides the deck in two parts: each part can be thought of as a constituent of the mixture and the shuffling as the dispersion of the two constituents without interpenetration. A sequence of cuts and shuffles of decks whose cards are taken to be thinner but more numerous at each stage in the sequence leads to a limit of simple deformations that describes each of the two constituents diffused throughout one and the same region in space. Thus, the occurrence of interpenetration of matter in the limit can be resolved by choosing arbitrarily accurate approximations of a given deformation by simple deformations in which interpenetration of matter does not occur.

A starting point toward a precise description of these ideas may be found in the following definition.

<u>7.4 Definition</u>: Let a piecewise fit region $\mathscr{A}$ and $(\kappa, g, G) \in \mathrm{LimSid}(\mathscr{A})$ be given. The triple (κ, g, G), is called a <u>mixing deformation from</u> $\mathscr{A}$ if there exists a finite collection $\{\mathscr{A}_j \mid j \in \{1, \ldots, J\}\}$ of pairwise disjoint, piecewise fit regions such that

$$\text{(Mix 1)} \quad \mathscr{A} \setminus \kappa = \bigcup_{j=1}^{J} \mathscr{A}_j ;$$

$$\text{(Mix 2)} \quad \text{for each } j \in \{1, \ldots, J\}, \quad (\emptyset, g_j, G_j) \in \mathrm{Std}(\mathscr{A}_j).$$

In (Mix 2), g_j and G_j denote the restrictions to $\mathscr{A}_j$ of g_0 and G_0, the continuous representatives of g and G on $\mathscr{A} \setminus \kappa$ defined in Lemma 4.11. We interpret each region $\mathscr{A}_j$ as a reference region for the j^{th} constituent of the mixture and $g_0(\mathscr{A}_j)$ as the region occupied by the j^{th} constituent of the mixture for the given mixing deformation. The transplacement g_j is injective, because $(\emptyset, g_j, G_j)$ is a structured deformation from $\mathscr{A}_j$. Because (κ, g, G) is assumed to be a limit of simple deformations and not a structured deformation, $g_0 : \mathscr{A} \setminus \kappa \longrightarrow \mathscr{E}$ need not be injective. We interpret $g_0(\mathscr{A})$ to be the region occupied by the mixture in the given mixing deformation. For example, the triple (κ, g, G) defined in Remark 4.16 is a mixing deformation that "shuffles" the two intervals $(-1,0)$ and $(0,1)$ to form the interval $(0,2)$.

Of course, condition (Mix 2) permits us to apply to the deformation $(\emptyset, g_j, G_j)$ of each constituent the concepts introduced in Section 6 for arbitrary structured deformations. For example, we define the <u>volume fraction</u> φ_j for the j^{th} constituent in the mixing deformation (κ, g, G) to be the scalar field $\varphi_j : g_0(\mathscr{A} \setminus \kappa) \longrightarrow \mathbb{R}$ given by

$$\varphi_j(x) = \begin{cases} 0 & \text{if } x \notin g_0(\mathscr{A}_j) \\ \dfrac{\det G_j}{\det \nabla g_j}(g_j^{-1}(x)) & \text{if } x \in g_0(\mathscr{A}_j). \end{cases}$$

If we decompose each structured deformation $(\emptyset, g_j, G_j)$ according to (6.12) and (6.14), the factor $(\emptyset, i, \varphi^{1/3} I)$ represents a creation of voids without distortion for the j^{th} constituent. This deformation gives a measure of the dispersion of the constituent in space, in the sense that it tells us what fraction of the volume in space is available for occupation by other constituents. It is then natural to think of the region $\text{int}\{x \in g_0(\mathscr{A} \setminus \kappa) \mid \varphi_j(x) = 1\}$ as the undispersed zone for the j^{th} constituent in the mixing deformation (κ, g, G).

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REFERENCES

1. Capriz, G., Continua with Microstructure, Springer Tracts in Natural Philosophy, Vol. 35, Springer–Verlag, 1989.

2. Cartan, H., Calcul Différentiel, Hermann, 1967.

3. Clifton, R., On the equivalence of F^eF^p and $\bar{F}^p\bar{F}^e$, *J. Appl. Mech.*, Transactions of the ASME, Series E, 39, 1, 287–289, 1972.

4. Davini, C., and G. P. Parry, On defect–preserving deformations in crystals, *Int. J. Plasticity* **5**, 337–369, 1989.

5. Davini, C., and G. P. Parry, A complete list of invariants for defective crystals, *Proc. R. Soc. Lond. A* **432**, 341–365, 1991.

6. Fonseca, I., and G. P. Parry, Equilibrium Configurations of defective crystals, *Arch. Rational Mech. Anal.*, forthcoming.

7. Gurtin, M. E., An Introduction to Continuum Mechanics, Academic Press, 1981.

8. Kröner, E., Kontinuumstheorie der Versetzungen und Eigenspanungen, Springer, 1958.

9. Kröner, E., The continuized crystal – a bridge between micro– and macromechanics?, *Z. Angew. Math. Mech.* **66**, T284–T292, 1986.

10. Lee, E. H., and D. T. Liu, Finite strain elastic–plastic theory with application to plane–wave analysis, *J. Appl. Physics*, **38**, 19–27, 1967.

11. Naghdi, P. M., A critical review of the state of finite plasticity, *Z. Angew. Math. Phuysik*, **41**, 316–387, 1990.

12. Nemat–Nasser, S., Decomposition of strain measures and their rates in finite deformation elastoplasticity, *Int. J. Solids Structures*, **17**, 859–872, 1979.

13. Noll, W., Lectures on the foundations of continuum mechanics and thermodynamics, *Arch. Rational Mech. Anal.* **52**, 62–92, 1973.

14. Noll, W., Continuum mechanics and geometric integration theory, 17–29, in: Categories in Continuum Physics, F. W. Lawvere & S. H. Schanuel, eds., Lecture Notes in Mathematics, n. 1174, Springer, 1980.

15. Noll, W., Finite–Dimensional Spaces, Vol. I, Nijhoff, 1987.

16. Noll, W., The geometry of contact, separation, and reformation of continuous bodies, *Arch. Rational Mech. Anal.*, forthcoming.

17. Noll, W., and E. Virga, Fit regions and functions of bounded variation, *Arch. Rational Mech. Anal* **102**, 1–22, 1988.

18. Owen, D. R., Deformations and stresses with and without microslip, Research Report No. 92–NA–019, Center for Nonlinear Analysis, Carnegie Mellon University, 1992; AMD – Vol. 148, Defects and Anelasticity in the Characterization of Crystalline Solids, edited by L. M. Brock, ASME, 1992.

19. Vol'pert, A. I., and S. J. Hudjaev, Analysis in Classes of Discontinuous Functions and the Equations of Mathematical Physics, Nijhoff, 1985.

20. Woijtowicz, P. J., Introduction to the molecular theory of nematic liquid crystals, in Introduction to Liquid Crystals, edited by E. B. Priestly, P. J. Woijtowicz and Ping Sheng, Plenum Press, 1975.

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