NAMT
93-001

On Distribution Functions of the Derivatives of Weakly Differentiable Mappings

Victor J. Mizel
Carnegie Mellon University

Research Report No. 93-NA-001

January 1993

Sponsors

U.S. Army Research Office
Research Triangle Park
NC 27709

National Science Foundation
1800 G Street, N.W.
Washington, DC 20550

On Distribution Functions of the Derivatives of Weakly Differentiable Mappings

Victor J. Mizel,
Carnegie Mellon University, Pgh.Pa 15213 USA

A recent investigation into the behavior under homogenization of second order materials with negative capillarity [CMM] led to the formulation of certain questions concerning distribution functions of absolutely continuous mappings on a real interval,none of which appears to have been examined previously. The present article describes and resolves one of these questions: *For a function on the unit interval possessing an absolutely continuous first derivative, does the joint distribution of function and first derivative already determine the distribution of the second derivative?*

Given any measurable mapping [for brevity, we will also apply this term to L_N equivalence classes] $\mathbf{w}: \Omega \to \mathbb{R}^m$, with Ω a bounded measurable subset of $\mathbb{R}^N$, the (mass) distribution $\mu = \mu^{\mathbf{w}}$ denotes that Borel measure on $\mathbb{R}^m$ defined by

(0) $\quad \mu(B) = L_N \{x \in \Omega: \mathbf{w}(x) \in B\}, \qquad B \in \mathbb{B}(\mathbb{R}^m),$

where L_N denotes N-dimensional Lebesgue measure and $\mathbb{B}(\mathbb{R}^m)$ denotes the Borel σ-algebra on $\mathbb{R}^m$.

Given a function $u \in W^{2,1}(0,1)$, where $W^{2,1}(0,1)$ denotes the Sobolev space of real functions on (0,1) possessing two summable generalized derivatives, there are several associated (mass) distributions to be considered: one for each of the functions u, u', u" as well as for the mappings $\mathbf{f} = (u, u')$, $\mathbf{g} = (u, u', u")$, etc. We are here interested in the linkage between certain of these distributions, in particular between the distribution $\mu := \mu^{u"}$ and the distribution $\pi := \mu^{\mathbf{f}}$; π is hereafter called a *joint* distribution for emphasis. Our main result shows that μ is characterized by π; in particular, if u, v are such that

1991 Mathematics subject classification. Primary 26A46,28A75,49Q15.

$\mu^{(u,u')} = \mu^{(v,v')}$ then it necessarily follows that $\mu^{u''} = \mu^{v''}$.

Our arguments will utilize the following result [F, Theorem 2.10.10].

Proposition Suppose X and Y are separable metric spaces, ν is a metric outer measure on Y, f maps X into Y, and f(A) is ν-measurable whenever A is a Borel subset of X. If

$$\zeta(S) = \nu[f(S)] \quad \text{for} \quad S \subset X$$

and ψ is the metric outer measure over X resulting from ζ by Caratheodory's construction on the family of all Borel subsets of X, then

$$\psi(A) = \int N(f|A, y)\, d\nu(y) \quad \text{for every Borel set } A \subset X,$$

where for any mapping g: $Z \to Y$

$N(g, y) = \#\{x \in Z \mid g(x) = y\} \in \{0, 1, \ldots, n, \ldots, \infty\}$.

Theorem A For each u in $W^{2,1}(0, 1)$, the joint distribution $\pi = \mu^{(u,u')} =: \mu^{\mathbf{f}}$ determines the distribution $\mu = \mu^{u''}$. More precisely, for every A in $\mathbb{B}(\mathbb{R})$ one has the line integral formula [with $y = (y_0, y_1) \in \mathbb{R}^2$] :

$$(*) \quad \mu(A) = \int_{E_A} [N(\mathbf{f}, y)/y_1] dy_0 + L_1(Z)\, \delta_{\{0\}}(A),$$

where

$E_A = \{y \in \mathbf{f}([0, 1]) : y_1 dy_1/dy_0 \in A\}$, $Z = \{x \in [0,1] : u'(x) = 0\}$ and $\delta_{\{0\}}$ is the unit mass at {0}. Here π determines the integrand in (*) through the relation

$$(1)\, \pi(B) = \int_{F_B} [N(\mathbf{f}, y)/y_1]\, dy_0 + \pi^s(B \cap (\mathbb{R} \times \{0\})), \quad \text{all } B \in \mathbb{B}(\mathbb{R}^2),$$

where $F_B = \mathbf{f}([0, 1]) \cap B$, and π^s is supported on an L^1 null set of $\mathbb{R} \times \{0\}$ with $\pi^s(\mathbb{R} \times \{0\}) = L_1(Z)$.

Remarks: The line integrals in (*) and (1) are to be interpreted as integrals with respect to arclength, i.e. Hausdorff 1-measure. The symbol dy_1/dy_0 appearing in the definition of E_A refers to the slope of the (bilateral) tangent line to the set $\mathbf{f}([0, 1])$ at the point $y \in \mathbf{f}([0, 1])$ (so that formally, $y_1 dy_1/dy_0 = u$"), provided that y is a point where the tangent ["contingent"] cone $T_S(y)$ to $S = \mathbf{f}([0, 1])$ consists of a single line. Here we are utilizing terminology stemming from the following definition of Bouligand [S,p263][F,p233] [MM1,p304], see also [AE].

<u>Definition.</u> An element $\mathbf{v} \in \mathbb{R}^m$ is a **bilateral unit tangent** to the closed set S of $\mathbb{R}^m$ at the point $y \in S$ provided that there exist sequences $\{y_i\},\{y_i'\}$ in S converging to y and satisfying

$$(y_i - y)/|y_i - y| \to \mathbf{v}, \qquad (y_i' - y)/|y_i' - y| \to -\mathbf{v}$$

The collection of all such $\mathbf{v}$ is denoted by $U_S(y)$, and the cone consisting of all lines $\mathbb{R}\mathbf{v}$ with $\mathbf{v} \in U_S(y)$ is denoted by $T_S(y)$ and called the <u>contingent</u> cone to S at y.

A key feature utilized in our development is the fact that since $(u, u') = \mathbf{f}: [0, 1] \to \mathbb{R}^2$ is an absolutely continuous curve in $\mathbb{R}^2$, the following result applies [MM1,§3] [in what follows H_1 denotes one-dimensional Hausdorff measure on $\mathbb{R}^m$]:

<u>Lemma</u> If $S \subset \mathbb{R}^m$ is the track of an absolutely continuous curve $\mathbf{f}$ then for H_1-a.e. $y \in S$

(a) $U_S(y)$ consists of a single pair of opposing unit vectors

(b) $\mathbf{f}^{-1}(y)$ is a finite set

(c) $\mathbf{f}'(t) \neq \mathbf{0}$ for all $t \in \mathbf{f}^{-1}(y)$

(d) $U_S(y) = \{\pm \mathbf{f}'(t)/|\mathbf{f}'(t)|: t \in \mathbf{f}^{-1}(y)\}$.

Furthermore, $\mathbf{f}$ has the following property

(N) $L_1(A) = 0 \Rightarrow H_1(\mathbf{f}(A)) = 0, \qquad A \subset \mathrm{Dom}(\mathbf{f})$.

We proceed to present the proof of Theorem A.

Proof: Note that whenever Y, Z are disjoint measurable subsets of [0, 1] it follows from (0) that the distribution function $\mu_3 = \mu^{\mathbf{f}|Y\cup Z}$ is simply the sum of $\mu_1 = \mu^{\mathbf{f}|Y}$ and $\mu_2 = \mu^{\mathbf{f}|Z}$. In particular this holds for $Y = [0, 1]\setminus Z$, with Z as in the statement of the theorem. Now set for each $\varepsilon > 0$, $Y_\varepsilon = \{x\in[0,1]: |u'(x)| > \varepsilon\}$, $\pi_\varepsilon := \mu^{\mathbf{f}|Y_\varepsilon}$, and $\mu_\varepsilon := \mu^{u''|Y_\varepsilon}$. Then it follows from (0) and the Vitali-Hahn-Saks theorem that

$$\lim_{\varepsilon\to 0}\pi_\varepsilon = \mu^{\mathbf{f}|Y} =: \pi_0 ,$$

in the sense that var $(\pi_0 - \pi_\varepsilon)\to 0$, and similarly,

$$\lim_{\varepsilon\to 0}\mu_\varepsilon = \mu^{u''|Y} =: \mu_0.$$

Now the behavior of μ restricted to $\mathbb{B}(\mathbb{R}\setminus[-\varepsilon,\varepsilon])$ for all positive ε determines μ since $\mu(\mathbb{R}) = 1$. Thus it will suffice for our purposes to prove that for each $\varepsilon > 0$ μ_ε is given by the line integral formula

(2) $\mu_\varepsilon(A) = \int_{E_{\varepsilon,A}} [N(\mathbf{f},y)/y_1]\, dy_0$, all $A\in\mathbb{B}(\mathbb{R})$,

with $E_{\varepsilon,A} = \{y\in\mathbf{f}(Y_\varepsilon): y_1 dy_1/dy_0 \in A\}$, while π_ε determines the integrand in (2) via

(3) $\pi_\varepsilon(B) = \int_{F_{\varepsilon,B}} [N(\mathbf{f},y)/y_1]\, dy_0$, all $B\in\mathbb{B}(\mathbb{R}^2)$,

with $F_{\varepsilon,B} = \mathbf{f}(Y_\varepsilon)\cap B$.

In (2) and (3) we have utilized the fact that, by the definition of Y_ε,

$\mathbf{f}(Y_\varepsilon) = \mathbf{f}([0, 1]) \cap \{y\in\mathbb{R}^2: |y_1|>\varepsilon\}$, so that

(4) $N(\mathbf{f}|Y_\varepsilon, y) = N(\mathbf{f}, y)$, for each $y\in\mathbf{f}(Y_\varepsilon)$.

We have also denoted by $y_1 dy_1/dy_0$ the product of y_1 with the slope of the tangent line to the set $\mathbf{f}([0, 1])$ at the point $y = (y_0, y_1)$ [by the Lemma there is a unique tangent line at H_1-a.e. $y \in \mathbf{f}([0,1])$]. The existence of the integrals in (2) and (3) as integrals with respect to H_1 [taking $dy_0 = \cos \theta(y)\, dH_1$ where $\theta(y)$ denotes the angle with the horizontal made by the tangent line to $\mathbf{f}([0,1])$ at y] follows by the Proposition with $X = [0,1]$ and $\nu = H_1$ on $Y = \mathbb{R}^2$. Namely, by the Lemma the absolute continuity of $\mathbf{f}$ ensures that this mapping has the (N) property of Lusin [S] from which it easily follows that $\mathbf{f}(S)$ is H_1-measurable for each L_1-measurable subset S of [0 ,1]. Thus on defining the set function Π by

$$\Pi(A) = H_1(\mathbf{f}(A)) \text{ on subsets } A \subset [0, 1],$$

and letting β denote the metric (outer) measure generated by $\Pi | \mathbb{B}([0,1])$, it follows that

$$\text{(5)} \quad \beta(A) = \int_{\mathbf{f}(A)} N(\mathbf{f}|A,y)\, dH_1(y) \qquad \text{for all } A \in \mathbb{B}([0,1]).$$

It follows from (5) that the line integral in (3) is also well defined, for each $\varepsilon>0$. The validity of (3) is now an easy consequence of the formal relation $dx = du/u'$, since the equality is evident for $B = \mathbf{f}(J)$ on each component J of the open sets

$C_+(\varepsilon) = \{x|\ u'(x) > \varepsilon\}, \qquad C_-(\varepsilon) = \{x|\ u'(x) < -\varepsilon\}.$

The validity of formula (2) utilizes the fact that the function u" can be factored as $h \circ \mathbf{f}$ where $h: \mathbf{f}[0, 1] \to \mathbb{R}$ is the measurable mapping defined H_1 -a.e. by $h(y) = y_1 dy_1/dy_0$. More precisely, if C,D,E denote the subsets of [0, 1] consisting, respectively, of points x where u"(x) is not defined as a real number, of points x in Z at which Z doesn't have unit density, and of points x such that $\mathbf{f}([0,1])$ fails to have a unique (nonvertical) tangent line,then

$$L_1(C) = L_1(D) = L_1(E) = 0$$

[the validity of the last relation follows from the proof of the Lemma, cf.e.g. [S,Ch IX Lemma 3.1]]. Hence by setting $X = [0, 1]\backslash(C \cup D \cup E)$ one finds that $\mu = \mu^{u''|X}$. Moreover u'': $X \rightarrow \mathbb{R}$ can be factored as $u'' = h \circ \mathbf{f}$, where h: $\mathbf{f}(X) \rightarrow \mathbb{R}$ is given by $h(y) = y_1 dy_1/dy_0$. It follows from this that (2) is valid for each set $A \in \mathbb{B}(\mathbb{R})$ such that $A \subset \mathbb{R}\backslash\{0\}$. The validity of (2) for all A follows immediately.

Remarks: We observe that the arguments used in the proof of Theorem A demonstrate that π also characterizes $\mu^{\mathbf{g}}$, where $\mathbf{g} = (u,u',u'')$. So π characterizes $\mu^{(u',u'')}$, and hence if $u \in W^{3,1}(0, 1)$ it also characterizes $\mu^{u'''}$, and so on.

Although Theorem A leads fairly straightforwardly (using [MM2]) to the conclusion that for $\Omega \subset \mathbb{R}^N$ and $u \in W^{2,p}(\Omega)$, $p > N$, the joint distribution $\pi = \mu^{\mathbf{f}}$ where $\mathbf{f} = (u, \text{grad } u)$, characterizes $\mu_i = \mu^{u,x^ix^i}$, $i = 1, \ldots, N$, as well as $\mu^* = \mu^{\Delta u}$, one has to utilize the additional fact that π characterizes the distributions $\mu_{\mathbf{a}} = \mu^{(\partial_{\mathbf{a}})^2 u}$ for all second directional derivatives $(\partial_{\mathbf{a}})^2$ to obtain a multidimensional analogue of Theorem A involving nonsymmetric differential operators. Of course the use of rectifiable currents would be more suitable for this multidimensional context ([F,ch 4] or [M, ch 4]). In any event Theorem A supplies a key result for the relaxation analysis of the second order model in [CMM], as will be shown in a subsequent publication.

Acknowledgements. The work during this project was partially supported by the National Science Foundation under Grants DMS 90 02562 and DMS 92 01221.

The author wishes to express thanks to Bernard Coleman, Dick James and Moshe Marcus for very enlightening conversations on the subject of this manuscript.

Bibliography

[AE] J.-P. Aubin & I. Ekeland, Applied Nonlinear Analysis Wiley-Interscience New York 1984

[CMM] B. D. Coleman,M. Marcus & V. J. Mizel, On the thermodynamics of periodic phases, ARMA **117** (1992), 321-347.

[F] H. Federer, Geometric Measure Theory Springer Verlag New York 1969.

[M] F. Morgan, Geometric Measure Theory a beginner's guide Academic Press Boston 1988.

[MM1] M. Marcus & V.J. Mizel, Absolute continuity on tracks and mappings of Sobolev spaces, ARMA **45**(1972), 294-320

[MM2] ____________________, Transformation by functions in Sobolev spaces and lower semicontinuity for parametric variational problems, Bull. AMS **79** (1973), 790-795.

[S] S. Saks, Theory of the Integral, Dover reprint 1964