# A DUALITY METHOD FOR OPTIMAL CONSUMPTION AND INVESTMENT UNDER SHORT-SELLING PROHIBITION

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**Gan-Lin Xu**
**and**
**Steven E. Shreve**

Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

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**Abstract.** A continuous–time, consumption/investment problem on a finite horizon is considered for an agent seeking to maximize expected utility from consumption plus expected utility from terminal wealth. The agent is prohibited from selling stocks short, so the usual martingale methods for solving this problem do not directly apply. A dual problem is posed and solved, and the solution to the dual problem provides information about the existence and nature of the solution to the original problem. When the market coefficients are constant, the value functions for both problems are provided in terms of solutions to *linear*, second–order, partial differential equations. If, furthermore, the utility functions are of the power form, the solutions to these equations take a particularly simple form, as do the formulas for the optimal consumption and investment processes.

**Key words.** portfolio and consumption processes, utility functions, stochastic control, martingale representation theorems, duality

**AMS (MOS) subject classifications.** Primary 93E20; secondary 60G44, 90A16, 49B60

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## 1. INTRODUCTION

This paper treats a consumption/investment decision problem for a single agent, endowed with some initial wealth, who can consume the wealth at some rate $C(t)$ and invest it in any of $d+1$ available assets. The agent is attempting to maximize a linear combination of two quantities: namely:

(i) $E\int_0^T U_1(t,C(t))dt$, the total expected discounted *utility from consumption* over the time interval $[0,T]$, and

(ii) $EU_2(X(T))$, the expected *utility from terminal wealth.*

The $d+1$ *assets* or *securities* available to the agent are very general. One of them is a *bond,* a security whose instantaneous rate of return may fluctuate (possibly randomly), but which is otherwise riskless. The other assets are *stocks,* risky securities whose prices have randomly fluctuating mean rates of return $b_i(t)$ and dispersion coefficients $\sigma_{ij}(t)$. Section 2 provides a careful exposition of these matters. The stock prices are driven by independent Wiener processes; these represent the sources of uncertainty in the market model, which we assume to be *complete* in the sense of Harrison & Pliska (1981, 1983) and Bensoussan (1984).

In our context, completeness amounts to nondegeneracy of the "diffusion" matrix $a(t) = \sigma(t)\sigma^T(t)$, as imposed in condition (2.3). This condition guarantees, roughly speaking, that there are exactly as many stocks as there are sources of uncertainty in the market model. It also enables us to construct a new probability measure under which the stock prices, discounted at the rate $r(t)$ of the bond, become a local martingale; this fact is of great importance in the modern theory of financial economics, and we refer the reader to Harrison & Pliska (1981, 1983) for a fuller account of its ramifications.

The processes $r(t)$, $b_i(t)$, $\sigma_{ij}(t)$, $1 \le i, j \le d$, will be collectively referred to as the *coefficients of the market model.* We assume that our agent is a "small investor," in that his decisions do not influence the asset prices, which are treated as exogenous.

Single agent consumption/portfolio problems have been investigated by a number of authors. A significant plateau was reached by Merton (1969, 1971), who found closed–form solutions to the Hamilton–Jacobi–Bellman equation for a constant–coefficient model with power utility functions. Karatzas, Lehoczky, Sethi & Shreve (1986) generalized this work to allow general utility functions. More recently, Cox & Huang (1987), Pliska (1986), and Karatzas, Lehoczky & Shreve (1987) used martingale methods to study the problem with non–constant market coefficients. Using the Girsanov Theorem to change to a probability measure under which all the stock prices discounted by the bond rate become martingales, these authors found a simple expression for the optimal consumption process. The fact that every martingale relative to a Brownian filtration can be represented as a stochastic integral with respect to the underlying Brownian motion played a key role in the proof that this consumption process can be *financed*, i.e., that there is a corresponding portfolio process which, together with the consumption process, results in a nonnegative wealth process. However, the portfolio process which is obtained by this method may require short–selling of the stocks. This paper examines the model in which such short–selling is prohibited.

The approach of this paper is to define a *dual problem* for the original consumption/portfolio problem, hereafter referred to as the *primal problem*. Rockafellar & Wets (1976) have developed such a duality theory for discrete–time stochastic control, and Bismut (1973) has studied the continuous–time case. This approach has been used to get necessary conditions for optimal control processes, e.g., Rockafellar & Wets (1978), Frank (1984). For the problem at hand, one can establish existence of an optimal dual control process under fairly general conditions, and then use complementary slackness to obtain existence in the primal problem and to characterize the optimal consumption and portfolio processes in that problem. The dual problem is defined in Section 3, and the relations between the two problems are developed in Section 4. Section 5 proves the existence of the optimal dual and primal control processes. Section 6 specializes the earlier sections to the constant coefficient market.

**When the market coefficients are constant, the value functions for the primal and dual problems can be characterized by respective Hamilton--Jacobi—Bellman equations. The Hamilton—Jacobi—Bellman equation for the dual problem turns out to be linear, and its solution can be transformed into a solution for the nonlinear HJB equation corresponding to the primal problem. The end result is that the value function for the primal problem is obtained in terms of the solutions to a pair of *linear*, second—order, partial differential equations. For general utility functions, the solutions to the latter are available in Karatzas, Lehoczky & Shreve (1987). In Section 6, we provide all the relevant formulas when the market coefficients are constant and the utility functions are of the power form.**

**This paper is derived from the first author's PhD dissertation. The duality method presented in this paper is also useful in the study of optimal consumption and investment in incomplete markets; we refer the reader to Karatzas, Lehoczky, Shreve & Xu (1989) for an analysis of the maximization of the utility of terminal wealth in an incomplete market. He & Pearson (1989) have developed a closely related approach for both the incomplete market problem and the complete market problem with short-selling prohibition.**

## 2. FORMULATION OF THE PRIMAL PROBLEM

In this section we formulate the problem of optimal consumption and investment when short—selling of the stocks is prohibited.

**2.1 Assets.** To model uncertainty, we will consider our problem on a probability space (fi^jP). We assume that the a—field & is rich enough to support a d—dimensional Brownian motion {w(t), ^(t); 0 < t < T}, where T is a fixed finite horizon and {^(t)} is the augmentation by null sets of the filtration generated by w. There are d + 1 assets being traded continuously on the finite horizon [0,T]. One of them is a *bond,* whose price $p_o(t)$ at time t evolves according to the differential equation

(2.1) $$dpo(t) = r(t)p_o(t)dt, \quad 0 \le t \le T.$$

The remaining d assets are *stocks,* and their prices are modelled by the stochastic differential equations

(2.2) $$dpi(t) = {}_{Pi}(t)[bi(t)dt + \overset{d}{E}\ aij(t)dw< J) (t)l, \quad 0 \le t \le T,$$

for i = l,...,d.

*The interest rate process* r(-) as well as the vector process b(-) = (bi(•),...,bd(*))$^T$ °* *mean rates of return* and the d * d matrix *volatility process &(•)* = (θij(*)) are assumed to be {^(t)}—progressively measurable and uniformly bounded. We introduce the covariance process a(-) ≜ cr(*)a$^T$(•) and assume the strong nondegeneracy condition

(2.3) $$£^Ta(t)\ £ \ge *_0||£||2 \ _v \ £z «d_f \ _V t € [0,T], \text{ a.s.,}$$

for some $\kappa_0 > 0$. This implies that there is a constant $\kappa_1$ such that (see, e.g., Karatzas & Shreve (1987), Problem 5.8.1 with solution on page 393)

$$(2.4) \qquad \max\{\|(\sigma^T(t))^{-1}\xi\|,\ \|(\sigma(t))^{-1}\xi\|\} \le \kappa_1 \|\xi\| \quad \forall\, \xi \in \mathbb{R}^d, \forall\, t \in [0,T], \quad \text{a.s.,}$$

$$(2.5) \qquad \min\{\|(\sigma^T(t))^{-1}\xi\|,\ \|(\sigma(t))^{-1}\xi\|\} \ge \frac{1}{\kappa_1} \|\xi\| \quad \forall\, \xi \in \mathbb{R}^d, \forall\, t \in [0,T], \quad \text{a.s.}$$

For specificity, we assume that $p_i(0) = 1,\ i = 0,\dots,d$. The solutions to (2.1) with this initial condition is $p_0(t) = \exp(\int_0^t r(s)ds)$. We define for future reference a *discount process*

$$(2.6) \qquad \beta(t) \triangleq \frac{1}{p_0(t)} = \exp(-\int_0^t r(s)ds), \quad 0 \le t \le T.$$

**2.2 Portfolio and consumption processes.**

**Definition 2.1.** A *portfolio process* $\pi(\cdot) = (\pi_1(\cdot),\dots,\pi_d(\cdot))^T$ is a measurable, $\{\mathcal{F}(t)\}$–adapted, $\mathbb{R}^d$–valued process satisfying $\int_0^T \|\pi(t)\|^2 dt < \infty$ a.s. A *consumption process* $C(\cdot)$ is a measurable, $\{\mathcal{F}(t)\}$–adapted, $dt \times dP$–almost everywhere nonnegative process satisfying $\int_0^T C(t)dt < \infty$ a.s.

In Definition 2.1 we regard $\pi_i(t)$ as the amount of money invested by an agent in stock $i$ at time $t$ and we regard $C(t)$ as the rate of the agent's consumption at time $t$. If $X(t)$ denotes the wealth of the agent at time $t$, then the amount of money invested in the bond is $X(t) - \underline{1}^T\pi(t)$, where $\underline{1}$ denotes the d–dimensional vector of ones. In view of (2.1), (2.2), the agent's wealth must evolve according to the equation

$$\text{(2.7)}\quad dX(t) = (r(t)X(t) - C(t))dt + \pi^T(t)(b(t) - r(t)\underline{1})dt + \pi^T(t)\sigma(t)dw(t), \quad 0 \leq t \leq T,$$

whose solution is given by

$$\text{(2.8)}\qquad \beta(t)X(t) = x + \int_0^t \beta(s)[-C(s) + \pi^T(s)(b(s) - r(s)\underline{1})]ds + \int_0^t \beta(s)\pi^T(s)\sigma(s)dw(s),$$

where $x \geq 0$ denotes the agent's initial wealth.

DEFINITION 2.2. Let an initial wealth $x \geq 0$ and a consumption/portfolio process pair $(C,\pi)$ be given. We say that $(C,\pi)$ is *admissible for* $x$ if for $i = 1,\ldots,d$, we have

$$\text{(2.9)}\qquad \pi_i(t) \geq 0, \quad dt \times dP - \text{a.e.},$$

and the wealth process $X(\cdot)$ defined by (2.8) satisfies

$$\text{(2.10)}\qquad X(t) \geq 0, \quad 0 \leq t \leq T, \quad \text{a.s.}$$

The set of all consumption/portfolio process pairs which are admissible for $x$ will be denoted by $A(x)$.

Condition (2.9) rules out short–selling of stocks. However, $X(t) - \underline{1}^T\pi(t)$ is allowed to become negative, i.e., borrowing from the bond is permitted.

Following the notation of Karatzas & Shreve (1987), Section 5.8, we define the *relative risk process*

(2.11) $$\theta(t) \triangleq (\sigma(t))^{-1}[b(t) - r(t)\underset{\sim}{1}], \quad 0 \le t \le T.$$

We then introduce the martingale

(2.12) $$Z(t) \triangleq \exp\{-\int_0^t \theta^T(s)dw(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\}, \quad 0 \le t \le T,$$

and the new probability measure $\tilde{P}$ defined by

(2.13) $$\tilde{P}(A) \triangleq E[Z(T)\, 1_A] \quad \forall \ A \in \mathscr{F},$$

and the drifted Brownian motion

(2.14) $$\tilde{w}(t) \triangleq w(t) + \int_0^t \theta(s)ds, \quad 0 \le t \le T.$$

According to Girsanov's theorem, $\tilde{w}$ is a standard Brownian motion under $\tilde{P}$. In terms of $\tilde{w}$, we may rewrite (2.8) as

(2.15) $$\beta(t)X(t) + \int_0^t \beta(s)C(s)ds = x + \int_0^T \beta(s)\pi^T(s)\sigma(s)d\tilde{w}(s).$$

The left–hand side of (2.15) is nonnegative and the right–hand side is a local $\{\mathscr{F}(t)\}$–martingale under $\tilde{P}$. But Fatou's lemma shows that any nonnegative local martingale is a supermartingale, and the supermartingale property in (2.15) yields

(2.16) $$E[\beta(T)Z(T)X(T) + \int_0^T \beta(t)Z(t)C(t)dt] = \tilde{E}[\beta(T)X(T) + \int_0^T \beta(t)C(t)dt] \le x.$$

We have obtained the following necessary condition for admissibility.

2.3 P*OPOSITIOI. If (C,TT) € A(x) and X(-) is the corresponding wealth process, then (2.16) is satisfied.

**2.3 Utility functions.**

**2.4 DEFIIITIOI. A** utility function U is a strictly increasing, strictly concave, twice continuously differentiable, real—valued function defined on [O,OD) which satisfies

(2.17) $$U(0) = 0,$$

(2.18) $$U'(0) \triangleq \lim_{xJ0} U'(x) = {}_o, \quad U'(a>) \triangleq \lim_{x\to\infty} U'(X) = 0,$$

(2.19) $$0 \le U(X) \le m\{l + X^p<>) \quad V\ x \ge 0,$$

for some constants $K\backslash > 0$, $0 < po < 1$.

Condition (2.17) can be replaced by the assumption that U(0) > -©; we assume (2.17) only for notational convenience. However, our model does not include utility functions such as log for which U(0) = *-m.* Condition (2.18) ensures that the strictly decreasing, $C^1$ function IT maps (O,OD) onto (O,OD), and hence has a strictly decreasing, $C^1$ inverse I: (O,GD) -* (O,CD), i.e.,

(2.20) $$U'(I(y)) = y \quad V\ y > 0, \quad I(U'(x)) = x \quad V\ x > 0.$$

We define

(2.21) $$I(0) \triangleq \lim_{y \downarrow 0} I(y) = \infty,$$

the last equality resulting from $U'(\infty) = 0$.

Throughout the remainder of the paper, we will have a *terminal wealth utility function* $U_2 : [0,\infty) \to \mathbb{R}$ satisfying the properties in Definition 2.4 and with $I_2$ denoting the inverse of $U_2'$, and we will have a *consumption utility function* $U_1$: $[0,T] \times [0,\infty) \to \mathbb{R}$ which is (jointly) Borel measurable. For every $t \in [0,T]$, $U_1(t,\cdot)$ is assumed to satisfy Definition 2.4 with $\kappa_1$ and $\rho_0$ independent of $t$. We denote by $U_1'(t,x)$ the derivative of $U_1$ with respect to its second variable, and we denote by $I_1(t,\cdot)$ the inverse of $U_1'(t,\cdot)$.

**2.4 The value function.**

For $x \geq 0$ and $(C,\pi) \in A(x)$, we define the *expected utility* of $(C,\pi)$ as

(2.22) $$J(x,C,\pi) \triangleq E\int_0^T U_1(t,C(t))dt + EU_2(X(T)),$$

where $X(\cdot)$ is given by (2.8) (or equivalently, (2.15)). The *primal value function* is

(2.23) $$V(x) \triangleq \sup\{J(x,C,\pi) \mid (C,\pi) \in A(x)\}, \quad \forall\, x \geq 0.$$

An *optimal consumption/portfolio process pair* is one which attains the supremum in (2.23). Because of the strict concavity of $U_1(t,\cdot)$ and $U_2$, if such a pair exists, the consumption process component $C(\cdot)$ and the corresponding terminal wealth $X(T)$ are uniquely determined (see Xu (1990), Theorem 2.4.5).

Our goal is to characterize $V$, to obtain conditions under which an optimal consumption/portfolio process pair exists, and to characterize this pair. We begin with the following description of $V$.

**2.5 PEOPOSITIOH. The primal value function V : [O,OD) ->[0,a>) is a continuous, nondecreasing concave function.**

**PROOF: We first obtain an upper bound on V(x). With $p_0$ as in (2.19), choose qi G [1, $\frac{1}{\rho_0}$) and define $q_2$ = 1 - qi$p_0$ E (0,1). Given (C,TT) 6 A(x) and the corresponding wealth process X(-) of (2.15), we use (2.19), the inequality (a + *bf*$^{1}$ <_2$^{Ql}$(a$^{qi}$ + b$^{qi}$) V a,b >_ 0, and the boundedness of r( •) to write**

$$E\ f_{\smile}^{rT}\ [U\wedge.CWjNt\ <\ (2«0^{Ql}E\ f_0^{rT}[1 + (C(t))^{Qie_o}]dt$$

$$\leq (2\kappa_1)^{q_1}[T + e^{Tq_1\rho_0 \max|^r COI}\ E\int_0^T [\beta(t)C(t)]^{q_1 \varrho_0}dt].$$

**Hölder's inequality and Proposition 2.3 imply**

$$E\ f_{J\,0}^{T}\ [\#t)C(t)]^{qiQ_o}dt = E\ f_{J\,0}^{T}\ z^{"qiQ_o}(t)[/?(t)Z(t)C(t)]^{qiQ_o}dt$$

$$\leq (E\ f_{J\,0}^{T}\ \wedge^{"qie_o/q2}(t)dt)^{<l2}(E\ f_{J\,0}^{T})9(t)Z(t)C(t)dt)^{qie_o}$$

$$\leq (\int_0^T E\ Z^{-q_1\varrho_0/q_2}(t)dt)^{q_2}\ x^{q_1\varrho_0}.$$

**Because $0$ appearing in (2.12) is bounded, E Z$^{\wedge lQ_o/q2}$(t) is bounded uniformly in t 6 [0,T]. Therefore, for some constant /c(qi) independent of x, C and TT, we have**

(2-24) $$E\ f_{J\,0}^{T}\ [U,(t,C(t))]^{qi}dt \leq \kappa(q_1)(1 + x^{q_1\varrho_0}).$$

A similar estimation applied to $E([U_2(X(T))]^{q_1})$ results in the inequality

(2.25) $$E([U_2(X(T))]^{q_1}) \le \kappa(q_1)\,(1 + x^{q_1\varrho_0}).$$

Setting $q_1 = 1$ in (2.24), (2.25), we obtain an upper bound on $J(x,C,\pi)$ which is independent of $C$ and $\pi$; the finiteness of $V(x)$ follows.

Since the sets $A(x)$ increase with $x$, $V$ must be nondecreasing. To prove concavity, note that for $x_1, x_2 \ge 0$, $\lambda \in (0,1)$, $(C_1,\pi_1) \in A(x_1)$, and $(C_2,\pi_2) \in A(x_2)$, the linearity of the wealth equation (2.7) implies that $(\lambda C_1 + (1-\lambda)C_2, \lambda\pi_1 + (1-\lambda)\pi_2) \in A(\lambda x_1 + (1-\lambda)x_2)$. The concavity of $U_1(t,\cdot)$ and $U_2$ allows us to conclude that

$$\begin{aligned}
&\lambda\, J(x_1,C_1,\pi_1) + (1-\lambda)\, J(x_2,C_2,\pi_2) \\
&\qquad \le J(\lambda x_1 + (1-\lambda)x_2,\ \lambda C_1 + (1-\lambda)C_2,\ \lambda\pi_1 + (1-\lambda)\pi_2) \\
&\qquad \le V(\lambda x_1 + (1-\lambda)x_2).
\end{aligned}$$

Maximize the left–hand side of this inequality over $(C_1,\pi_1) \in A(x_1)$ and $(C_2,\pi_2) \in A(x_2)$ to obtain the concavity of $V$.

The concavity of $V$ implies its continuity on $(0,\infty)$. Now $V(0) = 0$ (recall (2.17)), so to establish the continuity of $V$ at zero, it suffices to show

(2.26) $$\lim_{x \downarrow 0} V(x) \le 0.$$

For every $\epsilon \in (0,1)$, choose $(C_\epsilon,\pi_\epsilon) \in A(\epsilon)$ such that

(2.27) $$V(\epsilon) \le J(\epsilon, c_\epsilon, \pi_\epsilon) + \epsilon.$$

Let $X_\epsilon(\cdot)$ be the associated wealth process. Inequality (2.16) implies

$$E\,\beta(T)Z(T)X_\epsilon(T) \le \epsilon, \quad E\int_0^T \beta(t)Z(t)c_\epsilon(t)\,dt \le \epsilon.$$

Because $L^1$ convergence implies convergence almost everywhere along a subsequence, we can choose $\{\epsilon_n\}_{n=1}^\infty$ such that $\epsilon_n \downarrow 0$, $\beta(T)Z(T)X_{\epsilon_n}(T) \to 0$ P-a.e., and $\beta(\cdot)Z(\cdot)c_{\epsilon_n}(\cdot) \to 0$ $dt\times dP$ - a.e. But (2.25) with $q_1 > 1$ implies that $\{\beta(T)Z(T)X_{\epsilon_n}(T)\}_{n=1}^\infty$ is uniformly P-integrable, and (2.24) with $q_1 > 1$ implies that $\{\beta(\cdot)Z(\cdot)c_{\epsilon_n}(\cdot)\}_{n=1}^\infty$ is uniformly $dt\times dP$ - integrable. Therefore, $\lim_{n\to\infty} J(\epsilon_n, c_{\epsilon_n}, \pi_{\epsilon_n}) = 0$, and (2.26) follows from (2.27). □

## 3. FORMULATION OF THE DUAL PROBLEM

In this section we introduce a stochastic control problem which is dual to the problem of Section 2. We define the dual value function and establish its basic properties. The relationship between the dual problem of this section and the primal problem of Section 2 will be explored in Sections **4** and 5.

### 3.1 Concave/convex conjugate function pairs.

**3.1 DEFINITION.** Let $U$ be a utility function (Definition 2.4). The *convex conjugate* of $U$ is defined by

(3.1) $$\tilde U(y) \triangleq \sup_{x\ge 0}\{U(x) - xy\} \quad \forall\, y > 0.$$

It is an easy exercise to verify that

(3.2) $$\tilde U(0) \triangleq \lim_{y \downarrow 0} \tilde U(y) = U(\infty), \quad \tilde U(\infty) \triangleq \lim_{y \to \infty} \tilde U(y) = U(0) = 0.$$

From (2.20) we have

(3.3) $$\tilde U(y) = U(I(y)) - yI(y), \quad \forall\, y > 0,$$

so

(3.4) $$\tilde U'(y) = -I(y), \quad \tilde U''(y) = -I'(y) > 0, \quad \forall\, y > 0.$$

In particular, $\tilde U$ is a strictly decreasing, strictly concave, $C^2$ function. Equation (3.1) implies

(3.5) $$U(x) \le \tilde U(y) + xy, \qquad \forall\, x \ge 0, \forall\, y > 0,$$

and equality holds if and only if $x = I(y)$, or equivalently, $y = U'(x)$. It follows that

(3.6) $$U(x) = \inf_{y>0} \{\tilde U(y) + xy\} = \tilde U(U'(x)) + xU'(x), \quad \forall\, x > 0.$$

Finally, (3.1) and (2.19) imply

(3.7) $$0 \le \tilde U(y) \le \sup_{x>0} \{\kappa_1(1 + x^{\alpha_0}) - xy\} \le K_2 (1 + y^{-\alpha}) \quad \forall\, y > 0,$$

where $K_2$ is a positive constant and $\alpha = \frac{\alpha_0}{1-\alpha_0}$.

Associated with the utility functions $U_1$ and $U_2$ introduced in Section 2.3, we have the convex conjugate functions $\tilde U_1, \tilde U_2$ defined by

$$\text{(3.8)} \qquad \tilde{U}_1(t,y) = \sup_{x>0}\{U_1(t,x) - xy\} \quad \forall\, t \in [0,T],\ \forall\ y > 0,$$

$$\text{(3.9)} \qquad \tilde{U}_2(y) = \sup_{x>0}\{U_2(x) - xy\} \quad \forall\, y > 0.$$

We define these functions at $y = 0$ and $y = \infty$ as in (3.2). We denote by $\tilde{U}_1'(t,y)$ the derivative of $\tilde{U}_1$ with respect to its second argument.

**3.2 Dual control processes.**

**3.2 Definition.** A *dual control process* is a measurable, $\{\mathcal{F}(t)\}$–adapted, $\mathbb{R}^d$–valued process $\tilde{\pi}(\cdot) = (\tilde{\pi}_1(\cdot),\ldots,\tilde{\pi}_d(\cdot))^T$ which satisfies $E\int_0^T \|\tilde{\pi}(t)\|^2 dt < \infty$ and

$$\text{(3.10)} \qquad \tilde{\pi}_i(t) \geq 0, \qquad dt \times dP - \text{a.e.}$$

The set of all dual control processes will be denoted by $\tilde{A}$.

For $\tilde{\pi} \in \tilde{A}$, we define the nonnegative local martingale (hence supermartingale)

$$\text{(3.11)}\ Z_{\tilde{\pi}}(t) \triangleq \exp\{-\int_0^t [\theta(s) + \sigma^{-1}(s)\tilde{\pi}(s)]^T dw(s) - \frac{1}{2}\int_0^t \|\theta(s) + \sigma^{-1}(s)\tilde{\pi}(s)\|^2 ds\}, \quad 0 \leq t \leq T.$$

**3.3 Lemma.** The set of processes $\mathcal{Z} \triangleq \{Z_{\tilde{\pi}}(\cdot) \mid \tilde{\pi} \in \tilde{A}\}$ is convex.

**Proof:** For every $\lambda > 0$, $\mu > 0$ with $\lambda + \mu = 1$, and for every $\tilde{\pi}_1, \tilde{\pi}_2 \in \tilde{A}$, define $\xi = \lambda Z_{\tilde{\pi}_1} + \mu Z_{\tilde{\pi}_2}$, $\tilde{\pi} = \frac{1}{\xi}(\lambda\, \tilde{\pi}_1 Z_{\tilde{\pi}_1} + \mu\, \tilde{\pi}_2 Z_{\tilde{\pi}_2})$. Then $\tilde{\pi} \in \tilde{A}$, $\xi(0) = 1$, and

$$d\xi(t) = \lambda\, dZ_{\tilde{\pi}_1}(t) + \mu\, dZ_{\tilde{\pi}_2}(t)$$

$$= -\lambda\, Z_{\tilde{\pi}_1}(t)[\theta(t) + \sigma^{-1}(t)\tilde{\pi}_1(t)]^T dw(t) - \mu\, Z_{\tilde{\pi}_2}(t)[\theta(t) + \sigma^{-1}(t)\tilde{\pi}_2(t)]^T dw(t)$$

$$= -\,\xi(t)\,[\theta(t) + \sigma^{-1}(t)\tilde{\pi}(t)]dw(t).$$

Therefore, $\xi = Z_{\tilde{\pi}} \in \mathscr{Z}$.

**3.3 The dual control problem.**

Let $\tilde{U}_1$ and $\tilde{U}_2$ be defined by (3.8), (3.9). For $y \geq 0$ and $\tilde{\pi} \in \tilde{A}$, define the *dual objective function*

(3.12) $$\tilde{J}(y,\tilde{\pi}) \triangleq E\int_0^T \tilde{U}_1(t,y\beta(t)Z_{\tilde{\pi}}(t))dt + E\tilde{U}_2(y\beta(T)Z_{\tilde{\pi}}(T)).$$

The dual problem is to minimize $\tilde{J}(y,\tilde{\pi})$ over $\tilde{A}$ for fixed $y$. The *dual value function* $\tilde{V}$ is defined by

(3.13) $$\tilde{V}(y) \triangleq \inf\{\tilde{J}(y,\tilde{\pi}) \mid \tilde{\pi} \in \tilde{A}\} \quad \forall\, y \geq 0.$$

An *optimal process for the dual problem with initial condition* $y$ is a process $\tilde{\pi}_y \in \tilde{A}$ which attains the infimum in (3.13). Because of the strict convexity of $\tilde{U}_1(t,\cdot)$ and $\tilde{U}_2$, if such a process exists, it must be unique (see Xu (1990), Theorem 3.3.1).

**3.4 Theorem.** Restricted to $(0,\infty)$, the dual value function $\tilde{V}$ is finite, nonnegative, continuous, nonincreasing, and convex. Moreover,

$$\tilde{V}(0) \triangleq \int_0^T \tilde{U}_1(t,0)dt + \tilde{U}_2(0) = \lim_{y \downarrow 0} \tilde{V}(y), \tag{3.14}$$

but $\tilde{V}(0)$ may be infinite. If $\tilde{V}(0)$ is finite, then

$$\tilde{V}'(0) \triangleq \lim_{y \downarrow 0} \frac{\tilde{V}(y)-\tilde{V}(0)}{y} = -\infty. \tag{3.15}$$

**Proof**: Because $\tilde{U}_1(t,\cdot)$ and $\tilde{U}_2$ are nonincreasing and nonnegative, $\tilde{V}$ is also. Let $\tilde{0}$ denote the identically zero dual control process, and note that $Z_{\tilde{0}}$ is the martingale $Z$ defined by (2.12). Inequality (3.7) implies that for every $y > 0$,

$$\tilde{V}(y) \le \tilde{J}(y,\tilde{0}) \le E \int_0^T \kappa_2[1 + (y\beta(t)Z(t))^{-\alpha}]dt$$

$$+ E\, \kappa_2[1 + (y\beta(T)Z(T))^{-\alpha}].$$

Because $\beta(\cdot)$ and $\theta(\cdot)$ are uniformly bounded, the above expectations are finite, so $0 \le \tilde{V}(y) < \infty \;\forall\, y > 0$.

We now prove convexity of $\tilde{V}$. For $y_1,y_2 > 0$, $\lambda,\mu > 0$ such that $\lambda + \mu = 1$, and $\tilde{\pi}_1,\tilde{\pi}_2 \in \tilde{A}$, by Lemma 3.3 there exists $\tilde{\pi} \in \tilde{A}$ such that $Z_{\tilde{\pi}} = \frac{1}{\lambda y_1+\mu y_2}(\lambda y_1 Z_{\tilde{\pi}_1} + \mu y_2 Z_{\tilde{\pi}_2})$. Therefore,

$$\tilde{V}(\lambda y_1 + \mu y_2) \le \tilde{J}(\lambda y_1 + \mu y_2, \tilde{\pi})$$

$$= E \int_0^T \tilde{U}_1(t,\beta(t)(\lambda y_1 Z_{\tilde{\pi}_1}(t) + \mu y_2 Z_{\tilde{\pi}_2}(t)))dt$$

$$+ E\tilde{U}_2(\beta(T)(\lambda y_1 Z_{\tilde{\pi}_1}(T) + \mu y_2 Z_{\tilde{\pi}_2}(T)))$$

$$\leq E\int_0^T [\lambda\tilde{U}_1(t,y_1\beta(t)Z_{\tilde{\pi}_1}(t)) + \mu\tilde{U}_1(t,y_2\beta(t)Z_{\tilde{\pi}_2}(t))]dt$$

$$+ E[\lambda\tilde{U}_2(y_1\beta(T)Z_{\tilde{\pi}_1}(T)) + \mu\tilde{U}_2(y_2\beta(T)\tilde{Z}_{\tilde{\pi}_2}(T))]$$

$$= \lambda\,\tilde{J}(y_1,\tilde{\pi}_1) + \mu\,\tilde{J}(y_2,\tilde{\pi}_2).$$

Minimization of the right–hand side of this inequality over $\tilde{\pi}_1,\tilde{\pi}_2 \in \tilde{A}$ yields the convexity of $\tilde{V}$. The continuity of $\tilde{V}$ on $(0,\infty)$ follows from its convexity.

The monotonicity of $\tilde{V}$ implies $\tilde{V}(0) \geq \lim_{y\downarrow 0} \tilde{V}(y)$. For the reverse inequality, let $\kappa$ be an upper bound on $\beta(\cdot)$. The monotonicity of $\tilde{U}_1(t,\cdot)$ and $\tilde{U}_2$, Jensen's inequality, and the supermartingale property imply that for $y > 0$, $\tilde{\pi} \in \tilde{A}$,

$$\tilde{J}(y,\tilde{\pi}) \geq E\int_0^T \tilde{U}_1(t,y\kappa Z_{\tilde{\pi}}(t))dt + E\tilde{U}_2(y\kappa Z_{\tilde{\pi}}(T))$$

$$\geq \int_0^T \tilde{U}_2(t,\, y\kappa\, EZ_{\tilde{\pi}}(t))dt + \tilde{U}_2(y\kappa EZ_{\tilde{\pi}}(T))$$

$$\geq \int_0^T \tilde{U}_1(t,y\kappa)dt + \tilde{U}_2(y\kappa).$$

Therefore, $\tilde{V}(y) \geq \int_0^T \tilde{U}_1(t,y\kappa)dt + \tilde{U}_2(y\kappa)$, and the monotone convergence theorem implies

$$\lim_{y\downarrow 0} V(y) \geq \int_0^T \tilde{U}_1(t,0)dt + \tilde{U}_2(0) = V(0).$$

If $\tilde V(0) < \infty$, then

$$-\tilde V'(0) = \lim_{y\downarrow 0} \frac{\tilde V(0)-\tilde V(y)}{y} \ge \lim_{y\downarrow 0} \frac{\tilde V(0)-\tilde J(y,\tilde 0)}{y}$$

$$= \lim_{y\downarrow 0}\Big\{\frac{1}{y}E\Big[\int_0^T [\tilde U_1(t,0) - \tilde U_1(t,y\beta(t)Z(t))]dt$$

$$+ \frac{1}{y}E[\tilde U_2(0)-\tilde U_2(y\beta(T)Z(T))]\Big\}$$

$$\ge \lim_{y\downarrow 0}\frac{1}{y}E[\tilde U_2(0)-\tilde U_2(y\beta(T)Z(T))].$$

The convexity of $\tilde U_2$, equations (3.4), (2.21), and the monotone convergence theorem imply

$$\lim_{y\downarrow 0}\frac{1}{y}E[\tilde U_2(0) - \tilde U_2(y\beta(T)Z(T))] \ge \lim_{y\downarrow 0} \beta(T)Z(T)I_2(y\beta(T)Z(T)) = \infty.$$

□

**3.5 COROLLARY.** For every $x > 0$, there exists $y_x > 0$ such that

(3.16) $$\tilde V(y_x) + xy_x = \inf_{y>0}\{\tilde V(y) + xy\}.$$

PROOF: Define $f: (0,\infty) \to \mathbb{R}$ by $f(y) = \tilde V(y) + xy$. Note that $f$ is continuous and $\lim_{y\to\infty} f(y) = \infty$. If $\tilde V(0) = \infty$, then $\lim_{y\downarrow 0} f(y) = \infty$, and $f$ attains its minimum on $(0,\infty)$. If $\tilde V(0) < \infty$, then $f$ has a continuous extension to $[0,\infty)$ and must attain its minimum at some $y_x \in [0,\infty)$. According to (3.15), $f'(0) = -\infty$, so $y_x$ must be positive. □

## 4. RELATIONS BETWEEN THE PRIMAL AND DUAL PROBLEMS

**In this section we show that, for any dual control process 5r, the objective function J(- ,5r) in the dual problem provides a bound on the value function V for the primal problem. Moreover, the existence of an optimal dual control process 5r implies the existence of an optimal consumption/portfolio process pair (C,ir), and *ir* is related to 5r by the complementarity condition (4.4) below.**

### 4.1 Weak duality.

**4.1 WEAK DUALITY TMEOIEM. For every x > 0, y > 0, (C,ir) e A(x) and 5r6A, the inequality**

**(4.1)** $$J(x,C,x) < \tilde{J}(y,5r) + xy$$

**holds. Equality holds in (4.1) if and only if**

**(4.2)** $$C(t) = Ii(t,y/?(t)Z_{\pi}\text{-}(t)), \quad dtxdP\text{ - a.e.}$$

**(4.3)** $$X(T) = I_2(y/?(T)\ Z_{\pi}\sim(T)), \quad \text{a.s.,}$$

**(4.4)** $$T^T(t)5r(t) = 0, \quad dt \ll dP\text{ - a.e.}$$

**(4.5)** $$E\int_0^T Z_{\pi}\sim(t)/?(t)C(t)dt + E\ Z_{\pi}\sim(T)/?(T)X(T) = x,$$

**where X( •) is the wealth process associated with x, C(•) and *ir*(•) (see (2.8)), and $Z_{\tilde\pi}$(•) is given by (3.11).**

**PlOOF: From (2.8), (3.11), (2.11) and Itô's rule, we have**

$$d(Z_{\pi}(t)\beta(t)X(t)) = -Z_{\pi}(t)\beta(t)C(t)dt - Z_{\pi}(t)\beta(t)\pi^T(t)\delta(t)dt$$

$$+ Z_{\pi}(t)\beta(t)[\pi^T(t)\sigma(t) - X(t)(\theta(t) + \sigma^{-1}(t)\delta(t))^T]dw(t),$$

so

$$Z_{\pi}(t)\beta(t)X(t) + \int_0^t Z_{\pi}(s)\beta(s)C(s)ds + \int_0^t Z_{\pi}(s)\beta(s)\pi^T(s)\delta(s)\, ds, \quad 0 \le t \le T,$$

is a nonnegative local martingale, hence a supermartingale. This supermartingale has initial condition $x$, so

$$\text{(4.6)} \qquad E\, Z_{\pi}(T)\beta(T)X(T) + E\int_0^T Z_{\pi}(s)\beta(s)C(s)ds + E\int_0^T Z_{\pi}(s)\beta(s)\pi^T(s)\delta(s)ds \le x.$$

From (3.5) we have

$$U_1(t,C(t)) \le \tilde{U}_1(t,y\beta(t)Z_{\pi}(t)) + y\beta(t)Z_{\pi}(t)C(t), \quad dt\times dP\text{-a.e.}$$

$$U_2(X(T)) \le \tilde{U}_2(y\beta(T)Z_{\pi}(T)) + y\beta(T)Z_{\pi}(T)X(T), \quad \text{a.s.},$$

and equality holds if and only if (4.2), (4.3) hold. Therefore,

$$\text{(4.7)} \qquad J(x,C,\pi) \le \tilde{J}(y,\pi) + y\{E\int_0^T \beta(t)Z_{\pi}(t)C(t)dt + E\beta(T)Z_{\pi}(T)X(T)\}$$

$$\le \tilde{J}(y,\pi) + yx$$

because of (4.6) and the fact that $\pi^T(t)\tilde{\pi}(t) \geq 0$, $0 \leq t \leq T$, a.s. Equality holds in (4.7) if and only if (4.2)–(4.5) hold. □

4.2 COROLLARY. For every $x \geq 0$ and $y > 0$,

$$V(x) \leq \tilde{V}(y) + xy. \tag{4.8}$$

If $(C,\pi_y) \in A(x)$ and $\tilde{\pi}_y \in \tilde{A}$ satisfy (4.2) – (4.5), then they are optimal in their respective problems, i.e.,

$$V(x) = J(x,C,\pi_y), \quad \tilde{V}(y) = \tilde{J}(y,\tilde{\pi}_y). \tag{4.9}$$

4.3 REMARK. Corollary 4.2 implies that

$$\tilde{V}(y) \geq \sup_{x \geq 0} \{V(x) - xy\} \quad \forall\, y > 0, \tag{4.11}$$

i.e., $\tilde{V}$ dominates the convex conjugate of $V$. We provide conditions in Corollary 4.9 and Remark 5.7 and under which the reverse inequality holds.

4.2 Strong duality.

In order to construct pairs $(C,\pi) \in A(x)$ and $\tilde{\pi} \in \tilde{A}$ which are related by the duality conditions (4.2) – (4.5), we begin with $y > 0$ and $\tilde{\pi} \in \tilde{A}$. We can define $C(\cdot)$ by (4.2) and $x$ by (4.5) (with $X(T)$ given by (4.3)), and we must then ask whether there is a portfolio process $\pi \in A(x)$ satisfying (4.4) such that the wealth process $X(\cdot)$ associated with $x$, $C(\cdot)$ and $\pi(\cdot)$ satisfies (4.3). We first construct a portfolio process $\pi$ such that (4.3) is satisfied, but $\pi$ may take negative values and so may fail to be admissible. We subsequently show that if $\tilde{\pi}$ is optimal, then $\pi$ is indeed admissible and (4.4) holds.

**Lemma 4.4.** Let $y > 0$ and $\tilde{\pi} \in \tilde{A}$ be given. Define $C(\cdot)$ by (4.2) and assume the finiteness of

$$x \triangleq E\int_0^T Z_{\tilde{\pi}}(t)\beta(t)C(t)dt + E[Z_{\tilde{\pi}}(T)\beta(T)I_2(y\beta(T)Z_{\tilde{\pi}}(T))]. \tag{4.11}$$

Then there exists a portfolio process $\pi(\cdot)$, which may take negative values, and there exists a continuous, nonnegative process $\tilde{X}(\cdot)$, such that

$$\tilde{X}(0) = x, \quad \tilde{X}(T) = I_2(y\beta(T)Z_{\tilde{\pi}}(T)), \tag{4.12}$$

$$d\tilde{X}(t) = (r(t)\tilde{X}(t) - C(t))dt + \pi^T(t)(b(t) - r(t)\underset{\sim}{1} + \tilde{\pi}(t))dt + \pi^T(t)\sigma(t)dw(t), \quad 0 \le t \le T. \tag{4.13}$$

**Proof:** Define

$$D \triangleq \int_0^T Z_{\tilde{\pi}}(t)\beta(t)C(t)dt + Z_{\tilde{\pi}}(T)\beta(T)I_2(y\beta(T)Z_{\tilde{\pi}}(T)),$$

so $x = ED$. We may assume that $P-$a.e. path of the martingale $B(t) \triangleq E(D\,|\,\mathcal{F}(t))$ is right–continuous (Karatzas & Shreve (1987), Theorem 1.3.13), and so $B$ has a representation as

$$B(t) = x + \int_0^t Y^T(s)dw(s), \quad 0 \le t \le T,$$

where $Y(\cdot)$ is an $\mathbb{R}^d$–valued, $\{\mathcal{F}(t)\}$–progressively measurable process satisfying $\int_0^T \|Y(t)\|^2 dt < \infty$ a.s. (use Karatzas & Shreve (1987), Theorem 3.4.15 and a localization argument). In particular, B is actually continuous. Define

$$\xi(t) = B(t) - \int_0^t \beta(s)C(s)Z_{\tilde{\pi}}(s)ds,$$

$$\tilde{X}(t) = \xi(t)\,[\beta(t)Z_{\tilde{\pi}}(t)]^{-1},$$

$$\pi(t) = \tilde{X}(t)(\sigma^T(t))^{-1}[\theta(t) + \sigma^{-1}(t)\tilde{\pi}(t) + \frac{1}{\xi(t)}\,Y(t)]. \tag{4.14}$$

Then $\tilde{X}(0) = x$, $\tilde{X}(T) = I_2(y\beta(T)Z_{\tilde{\pi}}(T))$. To verify (4.13), we observe that

$$d\xi(t) = Y^T(t)dw(t) - \beta(t)C(t)Z_{\tilde{\pi}}(t)dt,$$

$$d[\beta(t)Z_{\tilde{\pi}}(t)]^{-1} = [\beta(t)Z_{\tilde{\pi}}(t)]^{-1}[r(t) + \|\theta(t) + \sigma^{-1}(t)\tilde{\pi}(t)\|^2]dt$$

$$+ [\beta(t)Z_{\tilde{\pi}}(t)]^{-1}[\theta(t) + \sigma^{-1}(t)\tilde{\pi}(t)]^T dw(t).$$

Therefore

$$d\tilde{X}(t) = \tilde{X}(t)[r(t) + \|\theta(t) + \sigma^{-1}(t)\tilde{\pi}(t)\|^2]dt + \tilde{X}(t)[\theta(t) + \sigma^{-1}(t)\tilde{\pi}(t)]^T dw(t)$$

$$+ \frac{\tilde{X}(t)}{\xi(t)}\,Y^T(t)dw(t) - C(t)dt + \frac{\tilde{X}(t)}{\xi(t)}\,Y^T(t)[\theta(t) + \sigma^{-1}(t)\tilde{\pi}(t)]^T dt$$

$$= (r\tilde{X}(t) - C(t))dt + \pi^T(t)\sigma(t)[\theta(t) + \sigma^{-1}(t)\tilde{\pi}(t)]dt + \pi^T(t)\sigma(t)dw(t),$$

which agrees with (4.13). □

4.5 REMARK. Note that (4.13) differs from the wealth equation (2.8) because of the term $\pi^T(t)\nu(t)dt$ in (4.13); when the complementary slackness condition (4.4) holds, the two equations agree. The solution to (4.12), (4.13) satisfies

$$\text{(4.15)} \qquad \beta(t)\tilde{X}(t) = M(t) - \int_0^t \beta(s)C(s)ds, \quad 0 \le t \le T,$$

where

$$\text{(4.16)} \qquad M(t) \triangleq x + \int_0^t \beta(s)\pi^T(s)\sigma(s)dw_{\nu}(s), \quad 0 \le t \le T,$$

$$\text{(4.17)} \qquad w_{\nu}(t) \triangleq w(t) + \int_0^t (\theta(s) + \sigma^{-1}(s)\nu(s))ds, \quad 0 \le t \le T.$$

From the definitions in the proof of Lemma 4.4, we also have the useful formula

$$\text{(4.18)} \qquad Z_{\nu}(\tau)\beta(\tau)\tilde{X}(\tau) = E\left[\int_\tau^T Z_{\nu}(s)\beta(s)C(s)ds \;\middle|\; \mathcal{F}(\tau)\right] + E[Z_{\nu}(T)\beta(T)\tilde{X}(T) \mid \mathcal{F}(\tau)],$$

for any $\{\mathcal{F}(t)\}$-stopping time $\tau$ taking values in $[0,T]$. □

Let $y > 0$ be given, and assume the dual problem with initial condition $y$ has an optimal solution $\nu_y$, i.e.,

$$(4.19) \qquad \tilde{J}(y,\tilde{\pi}_y) = \tilde{V}(y).$$

In the remainder of this section, we show that the corresponding portfolio $\pi$ given by Lemma 4.4 is optimal in the primal problem with initial wealth $x$ given by (4.11) when $\tilde{\pi}_y$ is substituted for $\tilde{\pi}$. In order to obtain this result, we define

$$(4.20) \quad g_y(\lambda) \triangleq \tilde{J}(\lambda y,\tilde{\pi}_y) = E\int_0^T \tilde{U}_1(t,\lambda y\beta(t)Z_{\tilde{\pi}_y}(t))dt + E\tilde{U}_2(\lambda y\beta(T)Z_{\tilde{\pi}_y}(T)), \ \forall\, \lambda > 0,$$

and we need to assume

$$(4.21) \qquad \exists\, \delta_y \in (0,1) \text{ such that } g_y(\lambda) < \infty \ \ \forall\ \lambda \in (1-\delta_y, 1+\delta_y).$$

A sufficient condition for (4.21) is that for some $\alpha \in (0,1)$, $\gamma \in (1,\infty)$,

$$(4.22) \qquad \alpha U_1'(t,x) \geq U_1'(t,\gamma x), \ \ \alpha U_2'(x) \geq U_2'(\gamma x) \quad \forall\, t \in [0,T], \ x > 0;$$

see Karatzas, Lehoczky, Shreve & Xu (1989), Lemma 11.5.

**4.6 Lemma.** Let $y > 0$ be given, assume $\tilde{\pi}_y \in \tilde{A}$ satisfies (4.19), and assume (4.21). Then $g_y$ is differentiable at 1, and

$$(4.23) \quad g_y'(1) = -E\int_0^T y\beta(t)Z_{\tilde{\pi}_y}(t)I_1(t,y\beta(t)Z_{\tilde{\pi}_y}(t))dt - E[y\beta(T)Z_{\tilde{\pi}_y}(T)I_2(y\beta(T)Z_{\tilde{\pi}_y}(T))].$$

**Proof:** Because of the convexity of $\tilde{U}_2$, we have for $\lambda \in (1-\frac{\delta_y}{2}, \infty)$,

$$\frac{1}{|\lambda-1|}\,|\tilde{U}_2(\lambda\, y\beta(T)Z_{\tilde{\pi}_y}(T)) - \tilde{U}_2(y\,\beta(T)Z_{\tilde{\pi}_y}(T))|$$

$$\leq \frac{2}{\delta_y} \left| \tilde{U}_2((1-\tfrac{1}{2}\delta_y)y\beta(T)Z_{\tilde{\pi}_y}(T)) - \tilde{U}_2(y\beta(T)Z_{\tilde{\pi}_y}(T)) \right|.$$

The right–hand side is integrable, so the dominated convergence theorem and (3.4) imply

$$\frac{\partial}{\partial\lambda} E\tilde{U}_2(\lambda y\beta(T)Z_{\tilde{\pi}_y}(T))\big|_{\lambda=1} = -E[y\beta(T)Z_{\tilde{\pi}_y}(T)I_2(y\beta(T)Z_{\tilde{\pi}_y}(T))].$$

A similar analysis applies to $\tilde{U}_1$, and we thereby obtain (4.23). □

Let $\tilde{\pi}_y \in \tilde{A}$ satisfy (4.19), and let $\tilde{\pi}$ be another process in $\tilde{A}$. For any $\epsilon \in [0,1]$, the "perturbed" process $\tilde{\pi}_\epsilon \triangleq \tilde{\pi}_y + \epsilon(\tilde{\pi} - \tilde{\pi}_y)$ is also in $\tilde{A}$, so we can study the sensitivity of $J(y,\tilde{\pi}_\epsilon)$ to variations in $\epsilon$. In order to carry out this program, we introduce some notation. Define

(4.24) $$N(t) \triangleq \int_0^t [\sigma^{-1}(s)(\tilde{\pi}(s) - \tilde{\pi}_y(s))]^T dw_{\tilde{\pi}_y}(s), \quad 0 \leq t \leq T,$$

where $w_{\tilde{\pi}_y}$ is defined by (4.17). Corresponding to $\tilde{\pi}_y$, let $\pi_y$ be the portfolio process constructed in Lemma 4.4 and let $C(\cdot)$ and $\tilde{X}(\cdot)$ be given by (4.2), (4.12) and (4.13) when $\pi$ and $\tilde{\pi}$ are replaced by $\pi_y$ and $\tilde{\pi}_y$, respectively. For each positive integer n, define the stopping time

(4.25) $$\tau_n \triangleq T \wedge \inf\{t \in [0,T] \mid |N(t)| + |\tilde{X}(t)| + |Z_{\tilde{\pi}_y}(t)| + \int_0^t \|\theta(s) + \sigma^{-1}(s)\tilde{\pi}_y(s)\|^2 ds$$

$$+ \int_0^t \beta(s)C(s)ds + \int_0^t \|\sigma^{-1}(s)(\tilde{\pi}(s) - \tilde{\pi}_y(s))\|^2 ds + \int_0^t \|\sigma^T(s)\pi_y(s)\|^2 ds \geq n\},$$

and note that

(4.26) $r_n \mid T$ as n -»».

Set <(t) 4 *£(t)l{t<Tn} , 0 ≤ t ≤ T.

4.7 LEMMA. Assume (4.21). Then

(4.27) y E | / Z-y(t)/?(t)7rJ(t)(*(t) - *y(t))dt = lim | j (y , ^ ) - J(y,5ry)] ≥ 0.

PEOOF: Because 5ry satisfies (4.19), the inequality

$$\lim_{\epsilon\downarrow 0} \frac{1}{\epsilon} [\tilde{J}(y,\tilde{\pi}^n_\epsilon) - \tilde{J}(y,\tilde{\pi}_y)] \geq 0$$

holds. As for the equality in (4.27), direct computation reveals

(4.28) Z-n(t) = Z~ (t) exp{-e N(t A rn) - i e² ∫ Jo ||cr-¹(s)(5r(s) - Sy(s))||²ds}.

From the definition of $r_n$, we have

e-2ne Z^(t) ≤ Z-jn(t) ≤ ene z ^ t ) V n ≥ 1, e € [0,1], t 6 [0,T].

Choose Co 6 (0,1] such that l-e-^^≤^6y for all e e (0,e0). If c e (0,c0) and Z^(t) ≠ Z- (t), then the convexity of Ũ2 implies

$$\frac{1}{\epsilon}| \tilde{U}_2(y\beta(T)Z_{\tilde{\pi}^n_\epsilon}(T)) - \tilde{U}_2(y\beta(T)Z_{\tilde{\pi}_y}(T))|$$

$$\le \frac{|Z_{\tilde\pi^n_\epsilon}(T)Z^{-1}_{\tilde\pi_y}(T) - 1|}{\epsilon} \cdot \frac{|\tilde U_2(y\beta(T)Z_{\tilde\pi^n_\epsilon}(T)) - \tilde U_2(y\beta(T)Z_{\tilde\pi_y}(T))|}{|Z_{\tilde\pi^n_\epsilon}(T)Z^{-1}_{\tilde\pi_y}(T) - 1|}$$

$$\le \frac{1}{\epsilon}\max\{1 - e^{-2n\epsilon}, e^{n\epsilon} - 1\} \frac{|\tilde U_2(y\beta(T)(1 - \frac{1}{2}\delta_y)Z_{\tilde\pi_y}(T)) - \tilde U_2(y\beta(T)Z_{\tilde\pi_y}(T))|}{\frac{1}{2}\delta_y}.$$

If $Z_{\tilde\pi^n_\epsilon}(t) = Z_{\tilde\pi_y}(t)$, the first expression in the above string of inequalities is still dominated by the last expression. The last expression is the product of a bounded function of $\epsilon \in (0,\epsilon_0]$ and an integrable random variable, because of assumption (4.21). By the dominated convergence theorem, (4.28), and (3.4), we have

(4.29)
$$\lim_{\epsilon\downarrow 0} \frac{1}{\epsilon} E[\tilde U_2(y\beta(T)Z_{\tilde\pi^n_\epsilon}(T)) - \tilde U_2(y\beta(T)Z_{\tilde\pi_y}(T))]$$

$$= E\frac{\partial}{\partial\epsilon} \tilde U_2[y\beta(T)Z_{\tilde\pi_y}(T)\exp\{-\epsilon N(\tau_n) - \frac{1}{2}\epsilon^2 \int_0^{\tau_n} \|\sigma^{-1}(s)(\delta(s) - \delta_y(s))\|^2 ds\}]\,\Big|_{\epsilon=0}$$

$$= E[y\beta(T)Z_{\tilde\pi_y}(T)I_2(y\beta(T)Z_{\tilde\pi_y}(T))N(\tau_n)]$$

$$= E[y\beta(T)Z_{\tilde\pi_y}(T)\tilde X(T)N(\tau_n)].$$

A similar analysis for $U_1$ results in the formula

(4.30)
$$\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon} E\Big[\int_0^T \tilde U_1(t,y\beta(t)Z_{\tilde\pi^n_\epsilon}(t))dt - \int_0^T \tilde U_1(t,y\beta(t)Z_{\tilde\pi_y}(t))dt\Big]$$

$$= E\Big[\int_0^T y\beta(t)Z_{\tilde\pi_y}(t)C(t)N(t\wedge\tau_n)dt\Big].$$

Summing (4.29) and (4.30), we obtain

$$\lim_{\epsilon\downarrow 0} \frac{1}{\epsilon}[\tilde{J}(y,\tilde{\pi}^n_\epsilon) - \tilde{J}(y,\tilde{\pi}_y)] \tag{4.31}$$

$$= y\ E[\int_0^T Z_{\tilde{\pi}_y}(s)\beta(s)C(s)N(s\wedge\tau_n)ds + Z_{\tilde{\pi}_y}(T)\beta(T)\tilde{X}(T)N(\tau_n)].$$

It remains to show that the right–hand side of (4.31) agrees with the left–hand side of (4.27). Note first that (4.18) with $\tilde{\pi}_y$ replacing $\tilde{\pi}$ implies

$$E[\int_0^T Z_{\tilde{\pi}_y}(s)\beta(s)C(s)N(s\wedge\tau_n)ds + Z_{\tilde{\pi}_y}(T)\beta(T)\tilde{X}(T)N(\tau_n)] \tag{4.32}$$

$$= E\int_0^{\tau_n} Z_{\tilde{\pi}_y}(s)\beta(s)C(s)N(s)ds$$

$$+ E\{N(\tau_n)\ E[\int_{\tau_n}^T Z_{\tilde{\pi}_y}(s)\beta(s)C(s)ds + Z_{\tilde{\pi}_y}(T)\beta(T)\tilde{X}(T)\,|\,\mathcal{F}(\tau_n)]\}$$

$$= E[\int_0^{\tau_n} Z_{\tilde{\pi}_y}(t)\beta(t)C(t)N(t)dt + Z_{\tilde{\pi}_y}(\tau_n)\beta(\tau_n)\tilde{X}(\tau_n)N(\tau_n)],$$

so it suffices to prove that this last expression equals

$$E\int_0^{\tau_n} Z_{\tilde{\pi}_y}(t)\beta(t)\pi_y^T(t)(\tilde{\pi}(t) - \tilde{\pi}_y(t))dt.$$

Since $\int_0^{\tau_n}\|\theta(s) + \sigma^{-1}(s)\tilde{\pi}_y(s)\|^2 ds \le n$ a.s., the Novikov condition (see, e.g., Karatzas & Shreve, Corollary 3.5.13) implies that $Z_{\tilde{\pi}_y}(t\wedge\tau_n)$ is an $\{\mathcal{F}(t)\}$–martingale. Define a new probability measure $P_n$ on $\mathcal{F}$ by $P_n(A) \triangleq E[1_A Z_{\tilde{\pi}_y}(\tau_n)]$ $\forall A \in \mathcal{F}$. Girsanov's Theorem implies that, under $P_n$, the process $w_{\tilde{\pi}_y}(t \wedge \tau_n)$ is a standard Brownian motion stopped at time $\tau_n$. According to Remark 4.5,

$$d(\beta(t)\tilde{X}(t)) = dM(t) - \beta(t)C(t)dt,$$

where $dM(t) = \beta(t)\pi_y^T(t)\sigma(t)dw_{\tilde{\pi}_y}(t)$. Therefore,

$$\text{(4.33)} \qquad d(\beta(t)\tilde{X}(t)N(t)) = \beta(t)\tilde{X}(t)dN(t) + N(t)dM(t) - N(t)\beta(t)C(t)dt$$

$$+ \beta(t)\pi_y^T(t)(\tilde{\pi}(t) - \tilde{\pi}_y(t))dt.$$

Integrating (4.33) and taking expectation under $P_n$, with respect to which $N(t \wedge \tau_n)$ and $M(t \wedge \tau_n)$ are martingales, we obtain

$$\text{(4.34)} \qquad E[Z_{\tilde{\pi}_y}(\tau_n)\beta(\tau_n)\tilde{X}(\tau_n)N(\tau_n) + \int_0^{\tau_n} Z_{\tilde{\pi}_y}(t)\beta(t)C(t)N(t)dt]$$

$$= E\int_0^{\tau_n} Z_{\tilde{\pi}_y}(t)\beta(t)\pi_y^T(t)(\tilde{\pi}(t) - \tilde{\pi}_y(t))dt.$$

Equation (4.27) follows from (4.31), (4.32) and (4.34). □

**4.8 Strong Duality Theorem.** Let $y > 0$ be given and let $\tilde{\pi}_y \in \tilde{A}$ be optimal for the dual problem with initial condition $y$. Assume that (4.21) holds. With $\tilde{\pi}_y$ replacing $\tilde{\pi}$, let $C(\cdot)$ be given by (4.2), $x$ by (4.11), and let $\pi_y$ be the portfolio process whose existence is guaranteed by Lemma 4.4. Then $\pi_y \in A(x)$ and

(4.35) $\overset{T}{T}y(t)5r(t) = 0$, $dt * dP$ - a.e.

**In particular, the pair (C,$7r_y$) is optimal in the primal problem with initial wealth x, i.e., (4.9) holds.**

**PROOF:** **According to Corollary 4.2 and Lemma 4.4, we need only to verify that**

(4.36) $7r_y(t) \geq 0$, $dt \times dP$ - a.e.,

**and that (4.35) holds. Define 5́r = (7ri,...,5r<i) € Ã by**

$\tilde{\pi}_j(t) \triangleq (\tilde{\pi}_y)_j(t) -$ 1 7 1 5 * ^W < «}• 6 <ı T> j=1 ,d

**Lemma 4.7 implies**

$$- E \int_0^{\tau_n} Z_{\tilde{\pi}_y}(t)\beta(t)\frac{1}{1 + \|\pi_y(t)\|^2} \sum_{j=1}^{d} (\pi_y)_j^2(t)\, 1_{\{(\pi_y)_j(t)<0\}} dt \geq 0,$$

**from which we conclude that $(7r_y)j \geq 0$ dt x dP — a.e. on the set {t,a;) | $0 \leq t \leq r_n((J)\}$. Because of (4.26), we have (4.36).**

**Now take $\tilde{ic} = \lambda$ $5r_y$ and apply Lemma 4.7 again to conclude**

(4.37) $- E\, J_0^T \backslash_y(t)/?(t)$ Tr^tJ^tJdt $\geq 0$, n = 1,2,....

**Since $7r_y(t) \geq 0$, $5r_y(t) \geq 0$, dt x dP - almost everywhere, (4.37) implies $\wedge\overset{T}{r}_y(t)7r_y(t) = 0$, first on {(t,o;) | $0 \leq t \leq r_n((J)\}$ and then on [0,T] x n, dt x dP - almost everywhere, D**

**4.9 Corollary.** Under the assumptions of Theorem 4.8,

$$\tilde{V}(y) = \sup_{\xi \geq 0} \{V(\xi) - \xi y\}.$$

**Proof:** With $\tilde{\pi}_y$, $\pi_y$ and $x$ as in Theorem 4.8, we have from the Weak Duality Theorem 4.1

$$\tilde{V}(y) = \tilde{J}(y, \tilde{\pi}_y) = J(x, C, \pi_y) - xy \leq V(x) - xy \leq \sup_{\xi \geq 0}\{V(\xi) - \xi y\}.$$

The reverse inequality follows from Remark 4.3. □

The Strong Duality Theorem 4.8 begins with a dual variable $y > 0$ and an optimal dual process $\tilde{\pi}_y$, and then constructs an optimal consumption/portfolio process pair $(C, \pi_y)$ for the primal problem with initial wealth $x$, where $x$ is defined in terms of $y$ and $\tilde{\pi}_y$ by (4.11) with $\tilde{\pi}$ replaced by $\tilde{\pi}_y$. We now show how, beginning with $x$, to find the corresponding dual variable $y$ which permits this construction.

**4.10 Theorem.** Assume that for every $y > 0$, there exists an optimal control process $\tilde{\pi}_y \in \tilde{A}$ for the dual problem with initial condition $y$. Assume further that (4.21) holds for every $y > 0$. For every $x > 0$, let $y_x > 0$ be a minimizer of $\tilde{V}(y) + xy$ (the existence of $y_x$ is guaranteed by Corollary 3.5). Then (4.11) holds with $\tilde{\pi}$ replaced by $\tilde{\pi}_{y_x}$. In particular, the consumption/portfolio process $(C, \pi_{y_x})$ constructed in Theorem 4.8 is optimal for the primal problem with initial wealth $x$.

**Proof:** We are given that $y_x$ satisfies (3.16), and must prove that

$$(4.38)\qquad x = E\int_0^T Z_{\pi}(t)\gamma(t)I_1(t,y\gamma(t)Z_{\pi}(t))dt + E[Z_{\pi}(T)\gamma(T)I_2(y\gamma(T)Z_{\pi}(T))]$$

the last equality being a restatement of (4.23). We have

$$\inf_{\lambda>0}\{\tilde{J}(\lambda y_x, \tilde{\pi}_{y_x}) + \lambda x\, y_x\} = \inf_{y>0}\{\tilde{J}(y,\tilde{\pi}_{y_x}) + xy\} \geq \inf_{y>0}\{\tilde{V}(y) + xy\}$$

$$= \tilde{V}(y_x) + xy_x = \tilde{J}(y_x, \tilde{\pi}_{y_x}) + xy_x.$$

Therefore the function $\lambda \mapsto g_{y_x}(\lambda) + \lambda x y_x$ is minimized by $\lambda = 1$, and consequently, $g'_{y_x}(1) + xy_x = 0$. ∎

**4.11 COROLLARY.** Under the hypotheses of Theorem 4.8, we have

$$(4.39)\qquad V(x) = \min_{y>0}\{\tilde{V}(y) + xy\} \quad \forall\, x > 0.$$

PROOF: Given $x > 0$, let $y_x$, $\tilde{\pi}_{y_x}$ and $(C, \pi_{y_x})$ be as in theorem 4.8. These processes were constructed to satisfy (4.2) — (4.5), so (4.1) holds with equality. From (4.6) we have

$$V(x) \leq \min_{y>0}\{\tilde{V}(y) + xy\} = \tilde{V}(y_x) + xy_x = \tilde{J}(y_x, \tilde{\pi}_{y_x}) + xy_x = J(x,C,\pi) \leq V(x). \quad ∎$$

## 5. EXISTENCE OF OPTIMAL DUAL PROCESSES

A key assumption in the Strong Duality Theorem of the previous section was the existence of an optimal dual process. In this section, we show that if

$$(5.1)\qquad -\frac{xU_2''(x)}{U_2'(x)} \le 1, \quad -\frac{xU_1''(t,x)}{U_1'(t,x)} \le 1 \;\; \forall\, t \in [0,T],\; x > 0,$$

then for every $y > 0$, the dual problem with initial condition $y$ has an optimal solution. The ratios appearing on the left–hand side of the inequalities in (5.1) are called the *Arrow–Pratt indices of relative risk aversion.*

5.1 **Lemma.** Let $U : [0,\infty) \to [0,\infty)$ be a utility function (Definition 2.4). Then

$$(5.2)\qquad -\frac{xU''(x)}{U'(x)} \le 1 \quad \forall\, x > 0$$

if and only if the mapping from $\mathbb{R}$ to $[0,\infty)$ given by $s \mapsto \tilde{U}(e^s)$ is convex. In this case,

$$(5.3)\qquad U(\infty) = \tilde{U}(0) = \infty.$$

**Proof:** From (2.20) and (3.3), we have

$$\frac{d}{ds}\tilde{U}(e^s) = U'(I(e^s))I'(e^s)e^s - e^sI(e^s) - e^{2s}I'(e^s) = -e^sI(e^s),$$

$$\frac{d^2}{ds^2}\tilde{U}(e^s) = -e^{2s}I'(e^s) - e^sI(e^s) = \frac{-e^s}{U''(I(e^s))}\frac{d}{dx}(xU'(x))\Big|_{x=I(e^s)}.$$

Therefore, $\tilde{U}(e^s)$ is a convex function of $s$ if and only if

$$(5.4)\qquad \frac{d}{dx}(xU'(x)) \ge 0 \quad \forall\, x > 0.$$

But (5.4) is equivalent to (5.2). Moreover, (5.4) implies $U'(x) \ge \frac{U'(1)}{x}\ \forall\, x \ge 1$, and integration of this inequality yields $U(\infty) - U(1) = \infty$. The remainder of (5.3) is a restatement of (3.2). □

Let H denote the set of all measurable, {^(t)}—adapted, $R^d$—valued processes 5r satisfying E $\int_0^T ||5r(t)||^2 dt$ < ©. We impose on H the inner product

(5.5) $$<\tilde{x}i,5r_2> \triangleq E \int_0^T 5rT(t)\ 5r_2(t)dt, \quad V\ S\text{^}Sfe \in H,$$

and we denote the associated norm by | [5r] | $\triangleq$ \ <5r,5r> V 5r e H. The set of dual control process $\tilde{A}$ of Definition 3.2 is a closed convex set in the Hilbert space H. For every fixed y > 0, $\tilde{J}$(y,*) given by (3.12) is a possibly ar-valued nonlinear functional on $\tilde{A}$. The finiteness of $\tilde{J}$(y,x) for at least some Sr e $\tilde{A}$ follows from Theorem 3.4. For 5r 6 H\\$\tilde{A}$, we define $\tilde{J}$(y,5r) = CD.

5.2 LEMMA. For every y > 0, the extended real—valued functional $\tilde{J}$(y,0 is lower semicontinuous on H.

PEOOF: It suffices to show that if $\{5rn\}_{n=1}^{\infty}$ is a sequence in A which converges in norm to 5r e A, then

(5.6) $$\tilde{J}(y,5r) \le \liminf_{n\to\infty} \%,5r_n).$$

Define $y_n \triangleq 0 + a^{"1}?_{,,,}$ y 4 $0 + {}_a\text{-}\%$ and note that $\lim E \int^T ||y_n(t) - y(t)||^2 dt$

$= \lim_{n\to\infty} |[y_n - y]| = 0,$

$$\lim_{n\to\infty} E\int_0^T \big|\, \|y_n(t)\|^2 - \|y(t)\|^2 \big| dt = \lim_{n\to\infty} E\int_0^T |(y_n(t)-y(t))^T(y_n(t)+y(t))|\, dt$$

$$\leq \lim_{n\to\infty} |[y_n - y]| \cdot |[y_n + y]| = 0.$$

It follows that

$$(5.7)\ \lim_{n\to\infty} E\int_0^T |[\int_0^t y_n(s)dw(s) + \tfrac{1}{2}\int_0^t \|y_n(s)\|^2 ds] - [\int_0^t y(s)dw(s) + \tfrac{1}{2}\int_0^t \|y(s)\|^2 ds]|\, dt = 0,$$

$$(5.8)\quad \lim_{n\to\infty} E|[\int_0^T y_n(t)dw(t) + \tfrac{1}{2}\int_0^T \|y_n(t)\|^2 dt] - [\int_0^T y(t)dw(t) + \tfrac{1}{2}\int_0^T \|y(t)\|^2 dt]| = 0.$$

Because $L^1$ convergence implies convergence almost surely along a subsequence, there exists a subsequence, also denoted by $\{y_n\}_{n=1}^{\infty}$, along which the convergences in (5.7), (5.8) are almost sure. Consequently, $\lim_{n\to\infty} Z_{\tilde{\pi}_n}(t) = Z_{\tilde{\pi}}(t)$, $dt \times P$ almost everywhere on $[0,T] \times \Omega$, and $\lim_{n\to\infty} Z_{\tilde{\pi}_n}(T) = Z_{\tilde{\pi}}(T)$, almost surely on $\Omega$. Inequality (5.6) follows from Fatou's lemma and the nonnegativity of $\tilde{U}_1$ and $\tilde{U}_2$. □

5.3 **Lemma**. If $U_1$ and $U_2$ satisfy (5.1), then for every $y > 0$, $\tilde{J}(y,\cdot)$ is a convex, extended real–valued functional on $H$.

**Proof**: It suffices to prove convexity of $\tilde{J}(y,\cdot)$ on the convex set $\tilde{A}$. Let $\tilde{\pi}_1,\tilde{\pi}_2 \in \tilde{A}$ and $\lambda_1 > 0$, $\lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$ be given. The convexity of the Euclidean norm implies

$$Z_{\lambda_1\tilde{\pi}_1+\lambda_2\tilde{\pi}_2}(t) \geq (Z_{\tilde{\pi}_1}(t))^{\lambda_1}(Z_{\tilde{\pi}_2}))^{\lambda_2}, \quad 0 \leq t \leq T, \text{ a.s.}$$

The monotonicity of $\tilde{U}_2$ and Lemma 5.1 imply

$$\tilde{U}_2(y\beta(T)Z_{\lambda_1\tilde{\pi}_1+\lambda_2\tilde{\pi}_2}(T)) \le \tilde{U}_2(y\beta(T)(Z_{\tilde{\pi}_1}(T))^{\lambda_1}(Z_{\tilde{\pi}_2}(T))^{\lambda_2})$$

$$\le \lambda_1\tilde{U}_2(y\beta(T)Z_{\tilde{\pi}_1}(T)) + \lambda_2\tilde{U}_2(y\beta(T)Z_{\tilde{\pi}_2}(T)), \text{ a.s.}$$

A similar inequality holds for $\tilde{U}_1$, and the convexity of $\tilde{J}(y,\cdot)$ follows. □

**5.4 Lemma.** If $U_2$ satisfies (5.1), then for every $y > 0$, we have

$$\lim_{\|[\tilde{\pi}]\| \to \infty} \tilde{J}(y,\tilde{\pi}) = \infty. \tag{5.9}$$

**Proof:** Let $\kappa$ be a constant such that $\beta(t) \le \kappa$, $0 \le t \le T$, a.s. From the monotonicity of $\tilde{U}_2$, Lemma 5.1, and Jensen's inequality, we have for all $\tilde{\pi} \in \tilde{A}$,

$$\tilde{J}(\tilde{y},\tilde{\pi}) \ge E\tilde{U}_2(y\kappa Z_{\tilde{\pi}}(T)) \ge \tilde{U}_2(y\kappa \exp(-\tfrac{1}{2}\|[\theta + \sigma^{-1}\tilde{\pi}]\|)).$$

The result follows from (2.5) and (5.3). □

**5.5 Dual Existence Theorem.** Assume that the utility functions $U_1$ and $U_2$ satisfy (5.1). Then, for each $y > 0$, there exists an optimal solution $\tilde{\pi}_y \in \tilde{A}$ to the dual problem (3.13) with initial condition $y$.

**Proof:** This follows immediately from Lemmas 5.2, 5.3 and 5.4. See, e.g., Ekeland & Temam (1976), Corollary 1.2.2. □

**We now state the principal result of this work. In Section 6, we will use Corollary 5.6 to compute optimal solutions.**

**5.6 CO*OLLII.Y. Assume that Ui and $U_2$ satisfy (5.1), and (4.21) (or (4.22)) is satisfied as well. Then, for every $x > 0$, the optimal consumption/investment problem has an optimal solution (C,x). Moreover, let $y > 0$ solve the equation**

**(5.10)** $$xy + gj(l) = 0,$$

**and let 5ry 6 Ã be the optimal solution for the dual problem with initial condition y. Then optimal consumption and wealth processes are given by**

**(5.H)** $$C(t) = I,(t,y/?(t)Z_{\tilde{\pi}_y}(t)), \quad 0 \le t \le T, \text{ a.s.},$$

**(5.12)** $$X(T) = I_2(y/?(T)Z_{\tilde{\pi}_y}(T)), \text{ a.s.}$$

**(5-13)** $$X(t) = £(t)\ Wt)Z_{\tilde{\pi}_y}(t)]-», \quad 0 \le t \le T, \text{ a.s.},$$

**where**

**(5-14)** $$£(t) = B(t) - \int_0^t 0(s)C(s)Z_{\tilde{\pi}_y}(s)ds, \quad 0 \le t \le T, \text{ a.s.},$$

**and B is a continuous version of**

**(5.15)** $$B(t) = E[\int_0^T Z_{\tilde{\pi}_y}(t)/?(t)C(t)dt + Z_{\tilde{\pi}_y}(T)\wedge(T)X(T)|«?(t)], \quad 0 \le t \le T, \text{ a.s.}$$

The process $B(\cdot)$ has a representation as

$$B(t) = x + \int_0^t Y^T(s)dw(s), \quad 0 \le t \le T, \tag{5.16}$$

for some $\mathbb{R}^d$–valued, $\{\mathcal{F}(t)\}$–progressively measurable process satisfying $\int_0^T \|Y(t)\|^2 dt < \infty$ a.s., and in terms of $Y$, the optimal portfolio process is

$$\pi(t) = X(t)(\sigma^T(t))^{-1}[\theta(t) + \sigma^{-1}(t)\tilde{\pi}_y(t) + \frac{1}{\xi(t)} Y(t)], \quad 0 \le t \le T, \text{ a.s.} \tag{5.17}$$

**Proof:** This corollary is a restatement of Theorem 4.10 which takes advantage of the Dual Existence Theorem 5.5 and the characterization (4.38) of the minimizer $y_x$ of $\tilde{V}(y) + xy$. □

5.7 **Remark.** Under the assumptions of the Dual Existence Theorem 5.5, Corollary 4.9 implies that the dual value function $\tilde{V}$ is the convex conjugate of the primal value function $V$. When there is no utility for terminal wealth, i.e., $U_2 \equiv 0$, the proof of the Dual Existence Theorem breaks down and we do not know if the conclusion of that theorem holds. However, the conclusion of Corollary 4.9 still holds, as can be proved by introducing an artificial utility for terminal wealth $U_2(x) = \epsilon\sqrt{x}$, and then letting $\epsilon \downarrow 0$. See Xu (1990), Theorem 5.2.1 for details.

## 6. THE MODEL WITH CONSTANT COEFFICIENTS

The martingale and duality methods of the previous sections are powerful tools for proving the existence of optimal solutions. However, they do not provide much information about the properties of the optimal solutions. To amend this drawback, the present section considers the case of constant market coefficients, and obtains in feedback form an optimal consumption/portfolio pair for the primal problem. We also show in this case that even when the hypotheses of the Dual Existence Theorem 5.5 are not satisfied, there exists a dual optimal process, and this process is constant and independent of the initial condition and the utility functions.

We assume throughout this section that

(6.1) b(t) = b, r(t) = r, a{t) = σ, $0 \le t \le T$, a.s.,

where r e R, b 6 $R^d$, and a is a nonsingular, d*d matrix. We define the vector

**(6.2)** ***O^a^h-il).***

**6.1 The mean comparison theorem.**

**In** this subsection we prove a comparison theorem which will be instrumental in solving the dual problem. It is an easy consequence of Jensen's inequality for conditional expectations that if M(-) is a martingale and *(p* : R -> IR is convex and satisfies E| y>(M(t))| < © for all $0 \le t \le T$, then y>(M(*)) is a submartingale. If M is only a local martingale, then *<p(M(-))* can fail to be a submartingale. To see this, let M( •) be a positive local martingale which is not a martingale (and is therefore a supermartingale), and let *<p* be the identity function. However, we have the following result for a convex function of a local martingale.

**6.1 LEMMA.** Let {M(t), ^(t); $0 \le t \le T$} be a continuous, positive, local martingale, and let *<p:* (0,a>) -• R be a nonincreasing, lower-bounded, convex function satisfying

(6.3) E *<p(M(t))* < GO V t € [0,T].

Then {y<M(t)), ^(t); $0 \le t \le T$} is a submartingale.

PEOOF: Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of stopping times converging up to T almost surely such that for each n, {M(tAr$_n$), ^(t); 0 ≤ t ≤ T} is a martingale. For e > 0, define the bounded, nonincreasing, convex function y>$_e$ by

$$\varphi_\epsilon(x) = \begin{cases} V(c) + (x-e)\wedge'(e) & \text{if } 0 < x < e, \\ <p(x) & \text{if } x \geq e. \end{cases}$$

Then {^$_c$(M(tAr$_n$)), ^(t); 0 ≤ t ≤ T} is a bounded submartingale, so for every s ≤ t and A 6 ^(s), we have

$$\int_A p_£(M(sAr_n))dP \leq \int_A y>_£(M(tAr_n))dP.$$

Now let n -• OD, using the bounded convergence theorem, and then let e j 0, using the monotone convergence theorem, to obtain

$$\int_A v<M(s))dP \leq \int_A \varphi(M(t))dP. \qquad \square$$

Hajek (1985) and Borkar (1987) have proved mean comparison theorems for solutions to stochastic differential equations. In the cited references, the dominating process is a Markov process and a martingale; in the following theorem, the situation is reversed.

6.2 MEAI COUPI&ISOI TKEOBJBM. Let $(p : (0,©) \to R$ be a nonincreasing, lower—bounded, convex function, let $\tilde{p}$ be an {<5$^r$(t)}-adapted, R$^d$—valued processes satisfying $\int_0^T ||p(t)||^2 dt < OD$ almost surely, and let $\hat{p}$ e R$^d$ be a vector such that

(6.4) $$\|\tilde{\rho}(t)\| \geq \|\hat{\rho}\|\ ,\quad dt \times dP \ \text{a.e. on } [0,T] \times \Omega.$$

Define

$$\tilde{Z}(t) = \exp\{-\int_0^t \tilde{\rho}^T(s)dw(s) - \frac{1}{2}\int_0^t \|\tilde{\rho}(s)\|^2 ds\},\quad 0 \leq t \leq T,$$

$$\hat{Z}(t) = \exp\{-\hat{\rho}w(t) - \frac{1}{2}\|\hat{\rho}\|^2 t\},\quad 0 \leq t \leq T.$$

Then

(6.5) $$E\,\varphi(\tilde{Z}(t)) \geq E\varphi(\hat{Z}(t)),\quad 0 \leq t \leq T.$$

**Proof:** The process $\tilde{Z}$ is a local martingale. According to Lemma 6.1, $E\varphi(\tilde{Z}(t)) \geq \varphi(\tilde{Z}(0)) = \varphi(1)$ for all $t \in [0,T]$. If $\hat{\rho} = 0$, then $\hat{Z} \equiv 1$ and (6.5) follows.

We now assume that $\hat{\rho} \neq 0$. Consequently, $\|\tilde{\rho}(t)\| > 0$ for all $t$, and we can find an $\{\mathcal{F}(t)\}$–adapted, $d \times d$ orthonormal matrix–valued process $O(\cdot)$such that

$$\frac{\tilde{\rho}(t)}{\|\tilde{\rho}(t)\|} = O(t)\frac{\hat{\rho}}{\|\hat{\rho}\|},\quad dt \times dP - \text{a.e. on } [0,T] \times \Omega.$$

Define

$$\lambda(t) \triangleq \int_0^t \frac{\|\tilde{\rho}(s)\|^2}{\|\hat{\rho}\|^2}ds,\quad 0 \leq t \leq T,$$

so $\lambda'(t) \geq 1$ for all $t$. The inverse function $\lambda^{-1}$ is defined on $[0,\lambda(T)] \supset [0,T]$, and for each

$\tau \in [0,T]$, $\lambda^{-1}(\tau)$ is an $\{\mathcal{F}(t)\}$–stopping time. Set

$$\tilde{W}(t) \triangleq \int_0^t \frac{\|\tilde{\rho}(s)\|}{\|\hat{\rho}\|} O^T(s) dw(s), \quad 0 \le t \le T,$$

$$\hat{W}(\tau) \triangleq \tilde{W}(\lambda^{-1}(\tau)), \quad 0 \le \tau \le T.$$

Relative to the filtration $\{\mathcal{F}(\lambda^{-1}(\tau))\}$, the process $\hat{W}$ is a martingale and

$$<\hat{W}_i, \hat{W}_j>(\tau) = \delta_{ij} \int_0^{\lambda^{-1}(\tau)} \frac{\|\tilde{\rho}(s)\|^2}{\|\hat{\rho}\|^2} ds = \delta_{ij}\tau, \; 0 \le \tau \le T.$$

By Lévy's Theorem (Karatzas & Shreve (1987), Theorem 3.3.16), $\{\hat{W}(\tau), \mathcal{F}(\lambda^{-1}(\tau)); 0 \le \tau \le T\}$ is a standard, d–dimensional Brownian motion. According to Proposition 3.4.8 of Karatzas & Shreve (1987),

(6.6)
$$1 - \int_0^\tau \tilde{Z}(\lambda^{-1}(\nu))\hat{\rho}^T d\hat{W}(\nu) = 1 - \int_0^{\lambda^{-1}(\tau)} \tilde{Z}(s)\hat{\rho}^T d\tilde{W}(s)$$

$$= 1 - \int_0^{\lambda^{-1}(\tau)} \tilde{Z}(s) \frac{\|\tilde{\rho}(s)\|}{\|\hat{\rho}\|} (O(s)\hat{\rho})^T dw(s)$$

$$= 1 - \int_0^{\lambda^{-1}(\tau)} \tilde{Z}(s)\, \tilde{\rho}^T(s) dw(s)$$

$$= \tilde{Z}(\lambda^{-1}(\tau)), \quad 0 \le \tau \le T.$$

But $d\hat{Z}(\tau) = -\hat{Z}(\tau)\hat{\rho}^T \, dw(\tau)$, $0 \le \tau \le T$, so

$$\hat{Z}(\tau) = 1 - \int_0^{\tau} \hat{Z}(\tau)\hat{\rho}^T dw(\tau),\ 0 \le \tau \le T, \tag{6.7}$$

and weak uniqueness of the solution to (6.7) implies that $\tilde{Z}(\lambda^{-1}(\cdot))$ appearing in (6.6) has the same distribution as $\hat{Z}(\cdot)$. Therefore, for any $t \in [0,T]$,

$$E\varphi(\hat{Z}(t)) = E\varphi(\tilde{Z}(\lambda^{-1}(t))).$$

But $\lambda^{-1}(t) \le t$ and Lemma 6.1 implies that $\varphi(\tilde{Z}(\cdot))$ is a submartingale, so $E\varphi(\tilde{Z}(\lambda^{-1}(t)) \le E\varphi(\tilde{Z}(t))$. □

**6.2 The optimal dual control.**

In this section we show that the optimal dual control process is identically equal to the constant vector which is the unique minimizer of

$$f(\tilde{\pi}) \triangleq \frac{1}{2} \|\theta + \sigma^{-1}\tilde{\pi}\|^2 \tag{6.8}$$

over $\tilde{\pi} \in [0,\infty)^d$. Clearly $f$ is a continuous, strictly convex function satisfying $\lim_{\|\tilde{\pi}\| \to \infty} f(\tilde{\pi}) = \infty$. Therefore, $f$ has a unique minimizer $\hat{\pi} \in [0,\infty)^d$, i.e.,

$$\|\theta + \sigma^{-1}\hat{\pi}\| \le \|\theta + \sigma^{-1}\tilde{\pi}\| \quad \forall\ \tilde{\pi} \in [0,\infty). \tag{6.9}$$

We define

$$\hat{\theta} \triangleq \theta + \sigma^{-1}\hat{\pi}. \tag{6.10}$$

**6.3 Theorem.** Under assumption (6.1), for every $y > 0$, the dual control process identically equal to $\hat{\pi}$ is optimal for the dual control problem with initial condition y.

**PROOF:** Let $\tilde{\pi} \in \tilde{\mathcal{A}}$ be given. From (6.9) we have

$$\|\hat{\theta}\| = \|\theta + \sigma^{-1}\hat{\pi}\| \le \|\theta + \sigma^{-1}\tilde{\pi}(t)\|, \quad 0 \le t \le T.$$

Applying the Mean Comparison Theorem 6.2 to the nonincreasing, lower—bounded, convex functions $z \mapsto \tilde{U}_1(t, ye^{-\beta t}z)$ and $z \mapsto \tilde{U}_2(ye^{-\beta T}z)$ with $\tilde{p}(t) = \theta + \sigma^{-1}\tilde{\pi}(t)$ and $\hat{p} = \hat{\theta}$, we obtain $\tilde{J}(y,\tilde{\pi}) \ge \tilde{J}(y,\hat{\pi})$. □

**6.4 REMARK** Theorem 6.3 states that under condition (6.1), $J(\lambda y,\hat{\pi}) = V(\lambda y)$ for every $y > 0$, $\lambda > 0$. According to Theorem 3.4, $V(\lambda y)$ is finite, so condition (4.21) is satisfied. Theorem 4.10 now implies that for any $x > 0$, there is an optimal consumption/portfolio process pair for the primal problem with initial wealth x. The complementary slackness condition (4.35) shows that the optimal portfolio process thus obtained does not invest in any stock i for which $\hat{\pi}_i > 0$. We show in the next subsection that this optimal portfolio is a scalar process times $(\sigma^T)^{-1}\hat{\theta}$, so $\hat{\theta} = 0$ corresponds to never investing in stocks at all.

**6.3 The Hamilton—Jacobi—Bellman equation.**

The assumption of constant coefficients allows us to employ the Hamilton—Jacobi—Bellman (HJB) equation from dynamic programming. This nonlinear equation is often intractable, but using duality theory as a guide, we will be able to decompose it into two linear Cauchy problems much as has been done by Cox & Huang (1987) and Karatzas, Lehoczky & Shreve (1987) for the problem without a prohibition on short-selling. In terms of the solutions to these Cauchy problems, we will obtain formulas for optimal consumption and portfolio processes. When the utility functions are power functions, these formulas become very explicit.

It is assumed through this subsection that the utility function $U_1(t,x)$ has the special form $e^{-\delta t}U_1(x)$, where $\delta$ is a real constant and $U_1(x)$ is a utility function. In addition to the conditions of Definition 2.4, we assume that there exist constants $\kappa$ and $\alpha$ such that

$$(6.11)\qquad U_j(I_j(y)) + yI_j(y) \le \kappa(1 + y^\alpha + y^{-\alpha})\ \forall\, y > 0,\ j = 1,2.$$

A sufficient condition for (6.11) is that for some $\alpha > 2$,

$$(6.12)\qquad \lim_{x\downarrow 0} \frac{(U_j'(x))^2}{U_j''(x)} \text{ exists and } \lim_{x\to\infty} \frac{(U_j(x))^\alpha}{U_j''(x)} = 0,\ \ j = 1,2$$

(see the appendix of Karatzas, Lehoczky & Shreve (1987)).

We will need to consider the optimal consumption/portfolio problem of Section 2 for initial times other than zero. For $(t,x) \in [0,T] \times (0,\infty)$, we consider such a problem with consumption utility function $(s,x) \mapsto e^{-\delta(s-t)}U_1(x)$. The value function for this problem is

$$(6.13)\qquad V(t,x) \triangleq \sup_{(C,\pi)\in A(t,x)} E\{\int_t^T e^{-\delta(s-t)}U_1(C(s))ds + e^{-\delta(T-t)}U_2(X^{(t,x)}(T))\},$$

where $A(t,x)$ consists of those consumption/portfolio process pairs $(C,\pi)$ for which $\pi$ is nonnegative and the wealth process determined by

$$(6.14)\quad X^{(t,x)}(s) = x + \int_t^s (rX^{(t,x)}(u) - C(u))du + \int_t^s \pi^T(u)(b - r\underset{\sim}{1})du + \int_t^s \pi^T(u)\sigma dw(u)$$

remains nonnegative for all $s \in [t,T]$, almost surely.

The convex conjugate of the function $x \mapsto e^{-\delta t}U_j(x)$ is the function $y \mapsto e^{-\delta t}\tilde{U}_j(e^{\delta t}y)$, so the dual problem associated with (6.13) is defined for $0 \le t \le T,\ y \ge 0$ by

(6.15) $\tilde{V}(t,y)$ 4 inf E{ f e*< «t) XJiCy e< «*> <s-t) Z-(t,s))ds
tEA J t T *

$$- e^{-\delta(T-t)}\tilde{U}_2(ye^{(\delta-r)(T-t)}Z_{\tilde{\pi}}(t,T))\},$$

where

(6.16) $Z_{\tilde{\pi}}(t,s)$ 4 exp{-J V+ ^5r(u))$^T$dw(u) - J j $\|\theta + \sigma^{-1}\tilde{\pi}(u)\|^2 du\}$, $t \le s \le T$, $y > 0$.

According to Theorem 6.3, the optimal dual control process is identically equal to the vector TT satisfying (6.9). With $\hat{0}$ given by (6.10), we have

(6.17) Z^(t,s) = exp{- &$^T$(w(s) - w(t)) - \ ||^||$^2$(s-t)}, $t \le s \le T$,

and we define

(6.18) C(M) $\stackrel{\Delta}{=}$ e<6-TH*-t) $z_{\pi}(t,s)$, $t \le s \le T$.

Then

(6.19) $\tilde{V}(t,y)$ = E{ [ e*c s-t) $\tilde{U}_1$(y^(t,s))ds + e-^$^{T-t}$)u$_2$($\tilde{y}$C(t,T))}, t <_8 ≤ T, $y \ge 0$.

The Markov process C(*>) *has* differential generator 1 *- ^ p||$^2$ ^ + fl2 (^-r)y -^y fj, and we may use this fact to derive a linear partial differential equation satisfied by $\tilde{V}$. It is convenient to apply this Feynman—Kac analysis to two functions whose difference is $\tilde{V}$, rather than to $\tilde{V}$ directly. For (t,y) E [0,T] x (o,a>), define

$$(6.20) \qquad G(t,y) \triangleq E\left\{\int_t^T e^{-\delta(s-t)} U_1(I_1(y\zeta(t,s)))ds + e^{-\delta(T-t)} U_2(I_2(y\zeta(t,T)))\right\},$$

$$(6.21) \qquad S(t,y) \triangleq E\left\{\int_t^T e^{-\delta(s-t)} y\zeta(t,s) I_1(y\zeta(t,s))ds + e^{-\delta(T-t)} y\zeta(t,T) I_2(y\zeta(t,T))\right\}.$$

According to Lemma 7.1 of Karatzas, Lehoczky & Shreve (1987), under condition (6.11), G and S are finite and continuous on $[0,T] \times (0,\infty)$, of class $C^{1,2}$ on $[0,T) \times (0,\infty)$, and are the unique solutions to the respective linear Cauchy problems

$$(6.22) \qquad \left(\frac{\partial}{\partial t} + L\right) G(t,y) + U_1(I_1(y)) = 0, \quad 0 \le t < T, \; y > 0,$$

$$(6.23) \qquad G(T,y) = U_2(I_2(y)), \quad y > 0,$$

and

$$(6.24) \qquad \left(\frac{\partial}{\partial t} + L\right) S(t,y) + yI_1(y) = 0, \quad 0 \le t < T, \; y > 0,$$

$$(6.25) \qquad S(T,y) = yI_2(y), \quad y > 0$$

where L is the second-order differential operator defined by

$$(6.26) \qquad L\varphi = \frac{1}{2}\|\theta\|^2 y^2 \varphi_{yy} + (\delta - r)y\varphi_y - \delta\varphi.$$

From (3.3) and (6.19), we have

$$(6.27) \qquad \tilde{V}(t,y) = G(t,y) - S(t,y), \quad 0 \le t \le T, \; y > 0,$$

so $\tilde{V}$ is continuous on $[0,T] \times (0,\infty)$, of class $C^{1,2}$ on $[0,T) \times (0,\infty)$, and solves the Cauchy problem

$$(6.28) \qquad \left(\frac{\partial}{\partial t} + L\right) \tilde{V}(t,y) + \tilde{U}_1(y) = 0, \quad 0 \le t < T, \quad y > 0,$$

$$(6.29) \qquad \tilde{V}(T,y) = \tilde{U}_2(y), \quad y > 0.$$

**6.5 Theorem.** Under assumptions (6.1) and (6.11), $\tilde{V}$ is given by (6.27) and satisfies the Hamilton–Jacobi–Bellman equation

$$(6.30) \qquad \tilde{V}_t(t,y) + (\delta - r) y \tilde{V}_y(t,y) - \delta \tilde{V}(t,y)$$

$$+ \inf_{\tilde{\pi} \in [0,\infty)^d} \left\{\frac{1}{2} \|\theta + \sigma^{-1}\tilde{\pi}\|^2 y^2 \tilde{V}_{yy}(t,y)\right\} + \tilde{U}_1(y) = 0, \quad 0 \le t \le T, \quad y > 0.$$

**Proof:** Recall from Theorem 3.4 that $\tilde{V}(t,\cdot)$ is convex, so

$$(6.31) \qquad \inf_{\tilde{\pi} \in [0,\infty)^d} \left\{\frac{1}{2} \|\theta + \sigma^{-1} \tilde{\pi}\|^2 y^2 \tilde{V}_{yy}(t,y)\right\} = \frac{1}{2} \|\hat{\theta}\|^2 y^2 \tilde{V}_{yy}(t,y).$$

Therefore, (6.30) reduces to (6.28). □

The HJB equation for $\tilde{V}$ turned out to be linear. The HJB equation for $V$ is considerably more complicated, but it is possible to obtain it as a transformation of the equation for $\tilde{V}$. We begin by differentiating in (6.19), using the argument in the proof of Lemma 4.6, to obtain

$$(6.32)\qquad -\tilde{V}_y(t,y) = E\{\int_t^T e^{-\delta(s-t)}\zeta(t,s)I_1(y\zeta(t,s))ds + e^{-\delta(T-t)}\zeta(t,T)I_2(y\zeta(t,T))\}$$

$$= \frac{1}{y}S(t,y), \quad 0 \le t \le T, \ y > 0.$$

Because $I_1$ and $I_2$ are strictly decreasing mappings from $(0,\infty)$ onto $(0,\infty)$, $-\tilde{V}_y(t,\cdot)$ is also a strictly decreasing mapping from $(0,\infty)$ onto $(0,\infty)$. Indeed, the Mean Value Theorem implies that for $(t,y) \in [0,T] \times (0,\infty)$ and $0 < \epsilon < 1$,

$$\frac{1}{\epsilon}[\tilde{V}_y(t,y+\epsilon) - \tilde{V}_y(t,y)] \ge E\{\int_t^T e^{-\delta(s-t)}\zeta(t,s) \min_{z\in[y,y+1]} |I_1'(z\zeta(t,s))|ds$$

$$+ e^{-\delta(T-t)}\zeta(t,T) \min_{z\in[y,y+1]} |I_2'(z\zeta(t,T))|\} > 0,$$

so

$$(6.33)\qquad \tilde{V}_{yy}(t,y) > 0, \quad 0 \le t \le T, \quad y > 0.$$

Define $Y(t,\cdot) : (0,\infty) \to (0,\infty)$ to be the inverse of $-\tilde{V}_y(t,\cdot)$, i.e.,

$$(6.34)\qquad \tilde{V}_y(t,Y(t,x)) = x, \quad Y(t,-\tilde{V}_y(t,y)) = y, \quad 0 \le t \le T, \ x > 0, \ y > 0.$$

Then $Y$ is of class $C^1$ and

$$(6.35)\qquad Y_x(t,x) = -\frac{1}{\tilde{V}_{yy}(t,Y(t,x))} < 0, \quad 0 \le t \le T, \ x > 0.$$

The expression $\tilde{V}(t,y) + xy$ is minimized over $y > 0$ by $Y(t,x)$, so Corollary 4.11 implies

(6.36) $$V(t,x) = \tilde{V}(t,Y(t,x)) + xY(t,x), \quad 0 \le t \le T, \quad x > 0.$$

Differentiation in (6.36), coupled with (6.34) and (6.35), yields

(6.37y$_t$(t,x) = $\tilde{V}_t$(t,Y(t,x)), $V_x(t,x) = Y(t,x)$, V^tpc) = -^$\frac{}{V_{yy}(t,x)}$, 0 < t <_T, x > 0.

Substitution of (6.34), (6.36) and (6.37) into the HJB equation (6.30), where (6.31) is taken into account, results in the equation

(6.38) $V_t(t,x)$ - *6V(t,x)* + $rxV_x(t,x)$ $\tilde{+}$ U,($V_x(t,x)$) -\ *M** $\frac{V_x^2(t,x)}{V_{xx}(t,x)}$ = °'

0 ≤ t ≤ T, x > 0.

To see that (6.38) is the HJB equation for the primal stochastic control problem, we need the following lemma.

6.6 LEMMA. For every nonnegative number a, the unique minimizer of

(6.39) g(T)4^$^T{}_a$cr$^T$7r-a/(b-rl)

over ir € [0,a>)$^d$ is a(a$^T$)"$^1$tf. Furthermore, x̂$^T$(tr$^T$)~$^1$& = 0 and

(6.40) $$g(a(\sigma^T)^{-1}\hat{\theta}) = -\frac{1}{2}a^2\|\hat{\theta}\|^2.$$

PIOOF: The Kuhn-Tucker conditions for the minimization of f in (6.8) over [0,a>)$^d$ imply the existence of a vector A e [0,a>)$^d$ such that

$$\lambda = \nabla f(\hat{\pi}) = (\sigma^T)^{-1}\hat{\theta}, \ \lambda^T\hat{\pi} = 0.$$

For $\pi \in [0,\infty)^d$, we have

$$g(\pi) = g(a\lambda) + \frac{1}{2}(\pi - a\lambda)^T\sigma\sigma^T(\pi - a\lambda) + (\pi - a\lambda)^T[\sigma\sigma^T a\lambda - a(b - r\underset{\sim}{1})]$$

$$= g(a\lambda) + \frac{1}{2}\|\sigma^T(\pi - a\lambda)\|^2 + a(\pi - a\lambda)^T\hat{\pi}.$$

Because $a(\pi - a\lambda)^T\hat{\pi} = a\pi\hat{\pi} \geq 0$, we see that $g$ attains its minimum at $\pi = a\lambda = a(\sigma^T)^{-1}\hat{\theta}$. Furthermore

$$g(a(\sigma^T)^{-1}\hat{\theta}) = \frac{1}{2}a^2\|\hat{\theta}\|^2 - a^2\hat{\theta}\sigma^{-1}(b - r\underset{\sim}{1} + \hat{\pi}) + a\lambda^T\hat{\pi} = -\frac{1}{2}a^2\|\hat{\theta}\|. \qquad \square$$

**6.7 Theorem.** Under assumptions (6.1) and (6.11), the primal value function $V$ is given by

$$V(t,x) = G(t,Y(t,x)), \quad 0 \leq t \leq T, \ x > 0, \tag{6.41}$$

and satisfies the Hamilton–Jacobi–Bellman equation

$$V_t(t,x) - \delta V(t,x) + \sup_{\substack{c \geq 0 \\ \pi \geq 0}}\{[(rx - c) + \pi^T(b - r\underset{\sim}{1})]V_x(t,x) + \frac{1}{2}\|\sigma^T\pi\|^2 V_{xx}(t,x) + U_1(c)\} = 0, \ 0 \leq t \leq T, \ x > 0. \tag{6.42}$$

**Proof:** Equation (6.41) follows from (6.36), (6.27), (6.34) and (6.32). From Lemma 6.6 with $a = -\frac{V_x(t,x)}{V_{xx}(t,x)}$, we have

$$(6.43)\qquad \sup_{\pi\geq 0}\{\pi^T(b-r\underset{\sim}{1})V_x(t,x)+\tfrac{1}{2}\|\sigma^T\pi\|^2V_{xx}(t,x)\}$$

$$= V_{xx}(t,x)\inf_{\pi\geq 0}\{\tfrac{1}{2}\pi^T\sigma\sigma^T\pi+\frac{V_x(t,x)}{V_{xx}(t,x)}\pi^T(b-r\underset{\sim}{1})\}$$

$$= -\frac{1}{2}\frac{V_x^2(t,x)}{V_{xx}(t,x)}\|\hat{\theta}\|^2.$$

Therefore, equation (6.42) is equivalent to (6.38). □

The supremum in the HJB equation (6.42) is attained by

$$(6.44)\qquad c = I_1(V_x(t,x)) = I_1(Y(t,x))$$

$$(6.45)\qquad \pi = -\frac{V_x(t,x)}{V_{xx}(t,x)}(\sigma^T)^{-1}\hat{\theta} = -\frac{Y(t,x)}{Y_x(t,x)}(\sigma^T)^{-1}\hat{\theta}.$$

We have thus obtained optimal consumption and portfolio processes in feedback form, a fact we now state precisely and verify properly.

**6.8 Theorem.** Let $(t,x)\in[0,T]\times(0,\infty)$ be given. Under assumptions (6.1) and (6.11), the optimal wealth process for the consumption/portfolio problem with initial time $t$ and initial wealth $x$ is

$$(6.46)\qquad X^{(t,x)}(s) \triangleq \frac{S(s,Y(t,x)\zeta(t,s))}{Y(t,x)\zeta(t,s)}\ ,\quad t\leq s\leq T.$$

This process satisfies (6.14) with $C(\cdot)$ and $\pi(\cdot)$ replaced by

(6.47) $C^*(s) \triangleq I_1(Y(s) x^{(t,x)}(s)))$, ,*(.) 4 - y y , g (*V>. $t \le s \le T$.

**PIOOF:** To simplify notation, we define the function $H : [0,T] \times (0,\infty) \to (0,\infty)$ by $H = -\tilde{V}_y$, i.e., (see (6.32))

(6.48) $$H(t,y) = \frac{\partial}{\partial y} S(t,y), \quad 0 \le t \le T, \; y > 0.$$

Because $S$ is continuous on $[0,T] \times (0,\infty)$ and of class $C^{1,2}$ on $[0,T) \times (0,\infty)$, $H$ has these properties as well. From (6.24), (6.25), we derive the formulas

(6.49) $$H_t(t,y) + £ \|\hat{\theta}\| flHV'Hyy^y) + (6 - r + \|\hat{\theta}\|^2)yH_y(t,y)$$

$$-rH(t,y) + I_1(y) = 0, \quad 0 \le t < T, \quad y > 0,$$

(6-50) $$H(T,y) = I_2(y), \quad y > 0.$$

In terms of $H$, (6.46) becomes

(6.51) $$x^{(t,x)}(s) = H(s,Y(t,x)C(t,s)), \quad t \le s \le T.$$

Because $d\,C(t,s) = (£-r)C(t,s)ds - C(t,s)Fdw(s)$, Itô's lemma and (6.49) imply

(6.52) $$dx^{(t,x)}(s) = [H_8 + \backslash\, U\&\|^2Y^2(t,x)C^2(t,s)H_{yy} + (\wedge-r)Y(t,x)C(t,s)H_y]ds$$

$$- Y(t,x)\zeta(t,s)H_y\hat{\theta}^T dw(s)$$

$$= [rH(s,Y(t,x)\zeta(t,s)) - I_1(Y(t,x)\zeta(t,s))]ds$$

$$- \|\hat{\theta}\|^2 Y(t,x)\zeta(t,s)H_y(s,Y(t,x)\zeta(t,s))ds$$

$$- Y(t,x)\zeta(t,s)H_y(s,Y(t,x)\zeta(t,s))\hat{\theta}^T dw(s), \quad t \le s \le T.$$

We now examine the three terms on the right–hand side of (6.52). The functions $H(s,\cdot)$ and $Y(s,\cdot)$ are inverses (see (6.34)), so (6.51) can be rewritten as

(6.53) $$Y(s,X^{(t,x)}(s)) = Y(t,x)\zeta(t,s), \qquad t \le s \le T.$$

Therefore,

(6.54) $$C^*(s) = I_1(Y(t,x)\zeta(t,s))$$

and

(6.55) $$r\,H(s,Y(t,x))\zeta(t,s) - I_1(Y(t,x)\zeta(t,s)) = rX^{(t,x)}(s) - C^*(s).$$

Because $H(t,\cdot)$ and $Y(t,\cdot)$ are inverses, we also have

(6.56) $$H_y(s,y) = \frac{1}{Y_x(s,H(s,y))}\cdot$$

From the equality $\hat{\pi}^T(\sigma^T)^{-1}\hat{\theta}$ obtained in Lemma 6.6 we see that

(6.57) $$\| \$\|^{2} = 0^{T}a'\backslash b - rl + TT) = F\ a'\backslash b - rl).$$

Therefore,

(6.58) $$-|\hat{M}I^{2}\ Y(t,x)C(t,s)H_{y}(s,Y(t,x)C(t,s)) = (/)^{T}(s)(b-rl).$$

Finally,

(6.59) $$-\ Y(t,x)C(t,s)H_{y}(s,Y(t,x)C(t,s))\wedge^{T} = (ir*f(s)a.$$

Substituting (6.55), (6.58), and (6.59) into (6.52), we verify the last sentence in the theorem.

To verify optimality for the problem with initial time t and initial wealth x, we first note that

$$x(*'^{x})(t) = H(t,Y(t,x)) = x.$$

According to (6.50),

(6.60) $$x(*>^{x})(T) = H(T,\ Y(t,x)C(t,T)) = I_{2}(Y(t,x)C(t,T)).$$

By (6.54), (6.60), (6.20) and (6.41), the utility associated with $(C^{*},ic^{*})$ is

$$E\{\int_{t}^{T} e*»-»Ui(I,(Y(t,x)<(s,t)))ds + e-\wedge^{T}-^{t})u_{2}(I_{2}(Y(t,x)C(t,T)))\}$$

$$= G(t,\ Y(t,x)) = V(t,x).$$ □

**6.9 Remark.** Karatzas, Lehoczky & Shreve (1987) obtain the analogues of Theorems 6.7 and 6.8 for the problem with no prohibition on short–selling. Under the additional assumption that the utility functions are of class $C^3$, Proposition 7.3 of Karatzas et al provides integral formulas for G and S; these formulas can be adapted to our model by replacing $\theta$ in them by $\hat{\theta}$.

### 6.4 Power utility functions.

In this subsection, we specialize the formulas of the previous subsection to the case of power utility functions

$$U_1(x) = \frac{1}{p_1} x^{p_1}, \quad U_2(x) = \frac{1}{p_2} x^{p_2}, \quad x > 0, \tag{6.61}$$

where $p_1, p_2 \in (0,1)$. We have then

$$\tilde{U}_1(y) = \frac{1}{q_1} y^{-q_1}, \quad \tilde{U}_2(y) = \frac{1}{q_2} y^{-q_2}, \quad y > 0, \tag{6.62}$$

where $q_j = \frac{p_j}{1-p_j}$, $j = 1,2$. Direct evaluation of (6.19) yields

$$\tilde{V}(t,y) = \frac{a_1(t)}{q_1} y^{-q_1} + \frac{a_2(t)}{q_2} y^{-q_2}, \quad 0 \le t \le T, \; y > 0, \tag{6.63}$$

where

$$a_1(t) \triangleq \begin{cases} \frac{1}{k(p_1)} [1 - e^{-k(p_1)(T-t)}], & \text{if } k(p_1) \neq 0, \\ T - t, & \text{if } k(p_1) = 0, \end{cases} \tag{6.64}$$

$$a_2(t) \triangleq e^{-k(p_2)(T-t)}, \tag{6.65}$$

and

(6-66) $$k(p) \triangleq \frac{1}{1-p}\left[\delta - rp - \frac{p\|\hat{\theta}\|^2}{2(1-p)}\right].$$

It is straight—forward to verify that $\tilde{V}$ solves the Cauchy problem (6.28), (6.29).

For each $t \in [0,T]$, the function

$$-\tilde{V}_y(t,y) = a_1(t)\, y^{\frac{1}{p_1-1}} + a_2(t)\, y^{\frac{1}{p_2-1}}$$

is strictly decreasing with $\lim_{y \downarrow 0}(-\tilde{V}_y(t,y)) = \infty$ and $\lim_{y \to \infty}(-\tilde{V}_y(t,y)) = 0$. Thus, there is an inverse function $Y(t,\cdot)$ as in (6.34), and the value function for the primal problem and the optimal consumption and portfolio policies in feedback form are given by (6.36) and (6.47). In the special case $p_1 = p_2 = p$, these formulas become

(6.67) $$Y(t,x) = \left(\frac{x}{a_1(t) + a_2(t)}\right)^{p-1}, \quad 0 \le t \le T, \ x > 0,$$

(6.68) $$V(t,x) = \left(\frac{1}{p}\right)(a_1(t) + a_2(t))^{1-p} x^p, \quad 0 \le t \le T, \ x > 0,$$

(6.69) $$C(t,x) = \frac{1}{a_1(t) + a_2(t)}\, x, \quad \pi(t,x) = \frac{1}{1-p}\, x\, (\sigma^T)^{-1}\hat{\theta}, \quad t \le s \le T$$

Note that the optimal consumption and portfolio policies are linear in wealth.

It is also possible to obtain explicit formulas when either $U_1 \equiv 0$ and $U_2(x) = \frac{1}{p}x^p$ or else $U_1(x) = \frac{1}{p}x^p$ and $U_2 \equiv 0$. In the former case, one simply omits $a_1(t)$ from (6.67) — (6.69); in the latter case, $a_2(t)$ should be omitted.

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