LOWER SEMICONTINUITY OF SURFACE ENERGIES

by

Irene Fonseca
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

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Department of Mathematics
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1. INTRODUCTION.

The energy functionals involved in variational problems associated to phase transitions correspond to nonconvex integrands. This feature induces failure of weak lower semicontinuity and typically these variational problems are characterized either by nonexistence or by nonuniqueness of solutions. Nonexistence is generally associated to highly oscillating minimizing sequences and nonuniqueness might be due also to the fact that, by neglecting small interfacial effects, interfaces can form without an increase of the energy.

When trying to either eliminate the oscillating behaviour or to find a selection criterion yielding the most likely observed solutions, we are lead naturally to the study of models involving bulk and surface energy terms.

Interfacial energies may be introduced either by direct penalization of sharp interfaces or by singular perturbations taking into account higher concentration gradients on a thin transition layer, as in the Van-der-Walls-Cahn-Hilliard theory of phase transitions for fluids (see FONSECA [18], [19], GURTIN [22], KINDERLEHRER & VERGARA - CAFFARELLI [27]). For solid crystals which have been subjected to thermal or mechanical treatments, HERRING [23] assumes that interfaces are sharp and he shows that the anisotropic surface energy may determine the surface structure and geometry of phase boundaries. Indeed, according to HERRING [23] if the dimensions of the crystal grains are sufficiently small, then the tendency of the crystal to lower its surface free energy is often the principal motivation for changes in the surface structure when approaching an equilibrium configuration of minimum free energy.

Therefore, the analysis involved in the study of such variational problems requires results concerning continuity and lower semicontinuity of surface energies of the type

$$J(E) := \int_{\partial E \cap \Omega} \Gamma(\nu_E(x))\, dH_{N-1}(x), \tag{1.1}$$

where E is a smooth subset of $\mathbb{R}^N$, ν_E is the outward unit normal to its boundary and Γ denotes the anisotropic surface energy density per unit area of the deformed configuration. In order to extend J(.) to sets of finite perimeter, (1.1) suggests the study of sequentially weak * lower semicontinuity properties of functionals of the type

$$I_f(\mu) := \int_{\Omega} f(x, \alpha(x))\, d\lambda(x), \tag{1.2}$$

where μ is a $\mathbb{R}^p$-valued measure with polar decomposition $d\mu = \alpha\, d\lambda$.

As we mentioned before, the nonexistence of minimizers is related to the fact that oscillations of minimizing sequences cannot be prevented due to the failure of lower semicontinuity of the energy functional . This situation is particularly interesting when the material has a crystalline structure, in which case the stored energy function has several potential wells (see ERICKSEN [12], [13], FONSECA [17], KINDERLEHRER [25]). Indeed, it may

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happen that the macroscopic weak limit of a minimizing sequence is not a minimizer of the total energy functional, although being stable enough to be observed. Therefore, the study of minimizing sequences becomes crucial for understanding the role in equilibria of the underlying oscillations at the microscopic level.

The tool generally used to control the oscillations of minimizing sequences of deformations relies on Young's probability measures (see TARTAR [31]). As an example, this is the treatment required by fine phase twinning of crystals (cf. BALL & JAMES [5], CHIPOT & KINDERLEHRER [8], JAMES & KINDERLEHRER [24], KINDERLEHRER & PEDREGAL [26]).

Similarly, indicator measures (see RESHETNYAK [30]) may be particularly useful to handle oscillating weakly convergent sequences of surfaces, and they may be relevant to treat those cases where the interfaces are not sharp but thin transition layers. Here, the limiting "surface" may be a generalized surface which macroscopic properties are described by the indicator measure of the sequence. These Radon measures turn out to be helpful for studying the weak lower semicontinuity properties of functionals of the type (1.2), and in particular, surface energy functionals as in (1.1).

In Section 2 we introduce the notation and we review briefly some concepts of the theory of functions of bounded variation.

In Section 3, using an approach similar to that of EVANS [14] and RESHETNYAK [30], we introduce the notion of slicing measures and indicator measure associated to a weakly * converging sequence of vector-valued measures (see Theorem 3.7). In Proposition 3.13 we establish the relations between the weak * limiting measure and the indicator measure. For the related notions of varifolds and generalized surfaces, see ALLARD [2] and ALMGREN [3] and L. C. YOUNG [32].

In Section 4, we prove that convexity of the integrand is a necessary and sufficient condition for sequential weak * lower semicontinuity of (1.2) (see Theorems 4.5 and 4.7).

In Corollary 5.3, we show that if the total variation of the weak * limit is equal to the limit of the total variations then continuity of functionals holds. Some of the results of Sections 4 and 5 can be found also in FEDERER [16], who relies heavily on concepts of geometric measure theory, in GOFFMAN & SERRIN [21] and in RESHETNYAK [30]. Here, we use the terminology of RESHETNYAK [30], and we provide a more detailed analysis of this problem in the context of functions of bounded variation and sets of finite perimeter.

As mentioned before, we are interested in the lower semicontinuity properties of surface energy densities of the type (1.1) associated to elastic solid materials that undergo a change of phase. These functionals are integrals of the type (1.2) where μ is the gradient of a function of bounded variation, namely $\mu = \nabla \chi_E$. As shown by DACOROGNA [9] (see also ACERBI &

FUSCO [1], MORREY [28]), in this case the $W^{1,1}$ - weak lower semicontinuous envelope of the functional is the integral of the quasiconvexification of the energy density, and so quasiconvexity becomes the natural constitutive assumption rather than convexity. We address this question on Section 6. In Theorem 6.6 we prove that lower semicontinuity holds when the density is quasiconvex and homogeneous of degree one, and when the sequence μ_ε and its weak * limit are absolutely continuous with respect to the Lebesgue measure. We conjecture that the result is still true even in the presence of a singular part with respect to the Lebesgue measure.

In Section 7 we use the notions of indicator measure and Young's measure associated to a weakly * converging sequence in order to obtain strong convergence. In particular, in Theorem 7.4 we show that if a sequence of twinned configurations of an elastic crystal converges in $W^{1,\infty}$ weakly * to a configuration and if the L^1 norms of the corresponding deformation gradients do not oscillate then the sequence itself has no oscillations and the limiting configuration is also twinned.

2. PRELIMINARIES.

Let $\Omega \subset \mathbb{R}^N$ be an open set and consider the canonical euclidean norm in $\mathbb{R}^N$. In what follows, $\|\,.\,\| : \mathbb{R}^p \to [0, +\infty)$ is a norm, $S^{p-1} := \{x \in \mathbb{R}^p \mid \|x\| = 1\}$ and L^N is the Lebesgue measure in $\mathbb{R}^N$.

Let μ be a $\mathbb{R}^p$ -valued measure in Ω. The *polar decomposition of* μ is represented by

$$d\mu = \alpha\, d\lambda,$$

where $\alpha : \Omega \to S^{p-1}$ is the *density of* μ and the positive and finite Radon measure λ is the *total variation of* μ (also denoted by $\|\mu\|$).

Definition 2.1.

We say that $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu_0$ (weakly *) in the sense of measures if

$$\int_\Omega \varphi(x)\, \alpha_\varepsilon(x)\, d\lambda_\varepsilon \to \int_\Omega \varphi(x)\, \alpha_0(x)\, d\lambda_0$$

for all $\varphi \in C_0(\Omega; \mathbb{R}) := \{\varphi \in C(\Omega; \mathbb{R}) : \text{support}\ \varphi \subset\subset \Omega\}$.

We recall briefly some results of the theory of functions of bounded variation (for details see De GIORGI [11], EVANS & GARIEPY [15], GIUSTI [20]).

Definition 2.2.

A function $u \in L^1(\Omega)$ is said to be a *function of bounded variation* ($u \in BV(\Omega)$) if

$$\int_\Omega |\nabla u(x)|\, dx := \sup\left\{ \int_\Omega u(x) \,.\, \mathrm{div}\, \varphi(x)\, dx \mid \varphi \in C_0^1(\Omega; \mathbb{R}^N), \|\varphi\|_\infty \le 1 \right\} < +\infty$$

Definition 2.3.

If A is a subset of $\mathbb{R}^N$ then the *perimeter of A in Ω* is defined by

$$\mathrm{Per}_\Omega(A) := \int_\Omega |\nabla \chi_A(x)|\, dx = \sup\left\{ \int_A \mathrm{div}\, \varphi(x)\, dx \mid \varphi \in C_0^1(\Omega; \mathbb{R}^N), \|\varphi\|_\infty \le 1 \right\},$$

where χ_A denotes the characteristic function of A.

Now we state the structure theorem for functions of bounded variation.

Theorem 2.4.

If $u \in BV(\Omega)$ then there exists a Radon measure $\|Du\|$ on Ω and a $\|Du\|$-measurable function $\alpha : \Omega \to \mathbb{R}^N$ such that

(i) $\|Du\|(\Omega) = \int_\Omega |\nabla u(x)|\, dx < +\infty$;

(ii) $\|\alpha\| = 1$ for $\|Du\|$ a.e. $x \in \Omega$;

(iii) $\int_\Omega u \dfrac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega \varphi\, \alpha_i\, d\|Du\|$

for $i = 1, ..., N$ and for every $\varphi \in C^1{}_0(\Omega)$.

We write $u \sim (\alpha, \|Du\|)$.

Remark 2.5.

(i) Note that by Theorem 2.4 (iii), $\alpha \|Du\| = \nabla u$ in the sense of distributions;

(ii) If $u \in W^{1,1}_{loc}$ then $\alpha(x) = \dfrac{\nabla u(x)}{\|\nabla u(x)\|}$ and $\|Du\| = L^N \llcorner \|\nabla u\|$, i. e. $d\|Du\| = \|\nabla u\| dx$;

(iii) Suppose that $u : \Omega \to \mathbb{R}^n$ is a function of bounded variation. Let $u = (u_1, ..., u_n)$, where $u_i \sim (\alpha_i, \lambda_i)$. Then $u \sim (F, \lambda)$, where $F \in S^* :=$ {nxN matrices with euclidean norm 1} and λ is a positive finite Radon measure, i, e.

$$\int_\Omega u_i(x)\ \mathrm{div}\, \varphi(x)\ dx = -\int_\Omega \left(\sum_{j=1}^{N} F_{ij}(x)\, \varphi_j(x) \right) d\lambda(x),$$

for all $i = 1, ..., n$ and for every $\varphi \in C_0^1(\Omega ; \mathbb{R}^N)$. It is easy to see that

$$F_{ij} = \frac{\alpha_{ij}\, f_i}{\|f\|} \text{ and } d\lambda = \|f\|\, d\zeta,$$

where $d\zeta := d\lambda_1 + \ldots + d\lambda_n$; as λ_i is absolutely continuous with respect to ζ ($\lambda_i << \zeta$), by the Radon-Nikodym Theorem there exists a nonnegative function $f_i \in L^1(\zeta)$ such that $d\lambda_i = f_i\, d\zeta$. As in (ii), if u is locally in $W^{1,1}$ then it turns out that

$$F_{ij} = \frac{\frac{\partial u_i}{\partial x_j}}{\|\nabla u\|}, \text{ where } \|\nabla u\| = \left(\sum_{i,j}\left(\frac{\partial u_i}{\partial x_j}\right)^2\right)^{1/2}.$$

(iv) Suppose that $E \subset \mathbb{R}^N$ is bounded and has finite perimeter in $\mathbb{R}^N$. Then

$$\|\partial E\| := \|D\chi_E\| = H_{N-1} \llcorner \partial^* E$$

where H_{N-1} is the N-1 Hausdorff measure and *∂^*E is the reduced boundary of E*, i. e.

$$\|\partial E\|\,(A) = H_{N-1}\,(A \cap \partial^* E).$$

Moreover, $\chi_E \sim (-\nu_E, \|\partial E\|)$ where ν_E is the outward unit normal to the reduced boundary of E, and, as in (i),

$$-\nu_E\,\|\partial E\| = \nabla \chi_E \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Theorem 2.6. (Generalized Green-Gauss theorem)

$$\int_E \operatorname{div} \varphi(x)\, dx = \int_{\mathbb{R}^N} \varphi(x) \,.\, \nu_E(x)\, d\,\|\partial E\|$$
$$= \int_{\partial^* E} \varphi(x) \,.\, \nu_E(x)\; dH_{N-1}$$

for all $\varphi \in C_0^1(\mathbb{R}^N; \mathbb{R}^N)$.

Let $\{h_\varepsilon\}$ be a sequence bounded in $BV(\Omega; \mathbb{R}^n)$, i. e.

$$\|h_\varepsilon\|_{BV} := \int_\Omega |h_\varepsilon(x)|\, dx + \int_\Omega |Dh_\varepsilon(x)|\, dx \leq \text{Const.} < +\infty,$$

and assume that $h_\varepsilon \to h_0$ strongly in $L^1(\Omega; \mathbb{R}^n)$. From Definition 2.2 it follows immediatly that

$$\int_\Omega |Dh_0(x)|\, dx \leq \liminf \int_\Omega |Dh_\varepsilon(x)|\, dx.$$

According to Theorem 2.4, let $h_\varepsilon \sim (\alpha_\varepsilon, \lambda_\varepsilon)$ and $h_0 \sim (\alpha_0, \lambda_0)$, where $\lambda_\varepsilon := \|Dh_\varepsilon\|$ and $\lambda_0 := \|Dh_0\|$, and define

$$d\mu_\varepsilon := \alpha_\varepsilon\, d\lambda_\varepsilon,\; d\mu_0 := \alpha_0\, d\lambda_0. \tag{2.7}$$

Lemma 2.8.

$\mu_\varepsilon \overset{*}{\rightharpoonup} \mu_0$ weakly *.

Proof. (a) Let $\varphi \in C_0^1(\Omega)$. By Theorem 2.4 and since $h_\varepsilon \to h_0$ in $L^1(\Omega)$,

$$\int_\Omega \varphi \, d\mu_\varepsilon = \int_\Omega \varphi \, \alpha_\varepsilon \, d\lambda_\varepsilon = - \int_\Omega h_\varepsilon \nabla\varphi \, dx \to - \int_\Omega h_0 \nabla\varphi \, dx = \int_\Omega \varphi \, \alpha_0 \, d\lambda_0 = \int_\Omega \varphi \, d\mu_0.$$

(b) If $\varphi \in C_0(\Omega)$, let $\varphi_n \in C_0^1(\Omega)$ be such that $\| \varphi - \varphi_n \|_\infty \to 0$. Given $\delta > 0$, and assuming

$$\|Dh_\varepsilon\| (\Omega) = \int_\Omega |\nabla h_\varepsilon(x)| \, dx \le M,$$

choose n_0 such that

$$\|\varphi - \varphi_{n_0}\|_\infty \le \frac{\delta}{3M}.$$

Then

$$\begin{aligned} \left|\int_\Omega \varphi \, \alpha_\varepsilon \, d\lambda_\varepsilon - \int_\Omega \varphi \, \alpha_0 \, d\lambda_0\right| &\le \left|\int_\Omega (\varphi - \varphi_{n_0}) \, \alpha_\varepsilon \, d\lambda_\varepsilon\right| + \left|\int_\Omega \varphi_{n_0} \, \alpha_\varepsilon \, d\lambda_\varepsilon - \int_\Omega \varphi_{n_0} \, \alpha_0 \, d\lambda_0\right| \\ &\quad + \left|\int_\Omega (\varphi - \varphi_{n_0}) \, \alpha_0 \, d\lambda_0\right| \\ &\le \frac{2\delta}{3} + \left|\int_\Omega \varphi_{n_0} \, \alpha_\varepsilon \, d\lambda_\varepsilon - \int_\Omega \varphi_{n_0} \, \alpha_0 \, d\lambda_0\right|. \end{aligned} \tag{2.9}$$

Finally, by (a) there exists an ε_0 such that for all $0 < \varepsilon < \varepsilon_0$

$$\left|\int_\Omega \varphi_{n_0} \, \alpha_\varepsilon \, d\lambda_\varepsilon - \int_\Omega \varphi_{n_0} \, \alpha_0 \, d\lambda_0\right| < \frac{\delta}{3}$$

which, together with (2.9) and the arbitrariness of δ, yields

$$\int_\Omega \varphi \, \alpha_\varepsilon \, d\lambda_\varepsilon \to \int_\Omega \varphi \, \alpha_0 \, d\lambda_0.$$

For the remaining of this section, we refer the reader to GOFFMAN & SERRIN [21]. Let μ be a $\mathbb{R}^p$-valued measure in Ω with *polar decomposition* $d\mu = \alpha \, d\lambda$, and let $f \in C_0(\Omega \times \mathbb{R}^p)$.

Definition 2.10.

$$\int_\Omega f(x, d\mu) := \int_\Omega f(x, \alpha(x)) \, d\lambda(x).$$

It is possible to show that for every Borel set $E \subset \Omega$ and for every $x_0 \in \Omega$ we have

$$\int_E f(x_0, d\mu) = \sup \sum_{i \in I} f(x_0, \mu(E_i)),$$

where the supremum is taken over all finite Borelian partitions $\{E_i \mid i \in I\}$ of E. Using the Lebesgue Decomposition Theorem (see EVANS & GARIEPY [15]), it follows immediatly from the definition that if $d\lambda(x) = a(x) \, dx + b(x) \, d\xi$, where L^N and ζ are mutually singular, then

$$\int_\Omega f(x, d\mu) := \int_\Omega f(x, \alpha(x)) \, a(x) \, dx + \int_\Omega f(x, \alpha(x)) \, b(x) \, d\xi(x).$$

3. SLICING MEASURES AND INDICATOR MEASURES.

We start by introducing the concept of *slicing measures* (see EVANS [14]). Let Λ be a finite, nonnegative Radon measure on $\Omega \times \mathbb{R}^p$ and consider its projection π onto Ω, i. e.

$$\pi(E) := \Lambda (E \times \mathbb{R}^p)$$

for every borel set $E \subset \Omega$. Clearly, $< \pi, \varphi > = < \Lambda, \varphi \otimes 1>$ for all $\varphi \in C_0(\Omega)$, i. e.

$$\int_\Omega \varphi(x) \ d\pi(x) = \int_{\Omega \times \mathbb{R}^p} \varphi(x) \ d\Lambda(x, y).$$

Given a borel set $B \subset \mathbb{R}^p$, define $\rho_B(A) := \Lambda(A \times B)$ for every borel set $A \subset \Omega$. As ρ_B is absolutely continuous with respect to π ($\rho_B << \pi$), by the Radon-Nikodym Theorem there exists $\lambda_B \in L^1(\pi)$ such that $d\rho_B = \lambda_B \, d\pi$, i. e.

$$\rho_B(A) = \int_A \lambda_B(x) \ d\pi(x).$$

Definition 3.1.

For π a. e. $x \in \Omega$, we define the *slicing measure* λ_x on $\mathbb{R}^p$ by $\lambda_x (B) := \lambda_B(x)$.

Proposition 3.2. ([14])

(i) $\Lambda(A \times B) = \int_A \lambda_x(B) \, d\pi(x)$

for all A and B borel sets of Ω and $\mathbb{R}^p$ respectively ;

(ii) λ_x is a nonnegative Radon probability measure, i. e. $\lambda_x (\mathbb{R}^p) = 1$ for π a. e. $x \in \Omega$;

(iii) (Fubini's decomposition) $\int_{\Omega \times \mathbb{R}^p} f(x, y) \ d\Lambda(x, y) = \int_\Omega \left(\int_{\mathbb{R}^p} f(x, y) \ d\lambda_x(y) \right) d\pi(x)$

for every $f \in C_0(\Omega \times \mathbb{R}^p)$.

Note that by (i), for all Borel subset A of Ω we have

$$\pi(A) = \Lambda(A \times \mathbb{R}^p) = \int_A \lambda_x(\mathbb{R}^p) \ d\pi(x),$$

and so, by the Lebesgue - Besicovitch Differentiation Theorem (see EVANS & GARIEPY [15]) we deduce (ii), i. e. $\lambda_x (\mathbb{R}^p) = 1$ for π a. e. $x \in \Omega$. Due to Proposition 3.2 (iii) , we write

$$\Lambda \equiv \lambda_x \otimes \pi.$$

Definition 3.3.

For π a. e. $x \in \Omega$, we define the *center of mass ν of Λ* by

$$\nu(x) := \int_{\mathbb{R}^p} y \ d\lambda_x(y).$$

Proposition 3.4.
$<\pi, \varphi\, \nu> = <\Lambda, \varphi \otimes y>$ for all $\varphi \in C_0(\Omega)$.

Proof. By Proposition 3.2 (iii) and by the definition of the center of mass,

$$\int_\Omega \varphi(x)\nu(x)\ d\pi(x) = \int_\Omega \varphi(x)\left(\int_{\mathbb{R}^p} y\ d\lambda_x(y)\right) d\pi(x)$$
$$= \int_{\Omega \times \mathbb{R}^p} \varphi(x)\, y\ d\Lambda(x, y).$$

Using the terminology of RESHETNYAK [30], we define indicator measure of a vector-valued measure.

Definition 3.5.
Let μ be a $\mathbb{R}^p$-valued measure on Ω with polar decomposition $d\mu = \alpha\, d\lambda$. The *indicator measure of* μ is the finite, nonnegative Radon measure Λ on $\Omega \times \mathbb{R}^p$ defined by

$$<\Lambda, f> := \int_\Omega f(x, \alpha(x))\ d\lambda(x)$$

for all $f \in C_0(\Omega \times \mathbb{R}^p)$.

Remark 3.6.
From the previous definition we deduce that
(i) $\Lambda(E) = \lambda(\{x \in \Omega \mid (x, \alpha(x)) \in E\})$ and $\Lambda(\Omega \times S^{p-1}) = \|\mu\|(\Omega)$;
(ii) support $\Lambda \subset$ (support λ) $\times S^{p-1}$;
(iii) Using the slicing measures (see Proposition 3.2 (iii)), $\Lambda \equiv \lambda_x \otimes \pi$ where
$\lambda_x = \delta_{y = \alpha(x)}$, $\pi = \lambda$ and $\nu = \alpha$.

Now we introduce the notion of *indicator measure of a weakly * converging sequence of measures.*

Theorem 3.7.
Let $\{\mu_\varepsilon\}$ be a sequence of $\mathbb{R}^p$-valued measures on Ω with polar decompositions $d\mu_\varepsilon = \alpha_\varepsilon d\lambda_\varepsilon$ and suppose that $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu_0$ weakly * in the sense of measures, with $d\mu_0 = \alpha_0 d\lambda_0$. Then there exists a subsequence $\{\mu_\eta\}$ and a nonnegative Radon measure $\Lambda_\infty \equiv \lambda_x^\infty \otimes \pi_\infty$ on $\Omega \times S^{p-1}$ such that

$$d\mu_0 = \nu_\infty d\pi_\infty \tag{3.8}$$

and for every $f \in C_0(\Omega \times \mathbb{R}^p)$ we have

$$\lim_{\eta \to 0} \int_{\Omega} f(x, \alpha_\eta(x))\, d\lambda_\eta(x) = \int_{\Omega x S^{p-1}} f(x, y)\, d\Lambda_\infty(x, y)$$
$$= \int_{\Omega} \left(\int_{S^{p-1}} f(x, y)\, d\lambda_x^\infty(y) \right) d\pi_\infty(x). \tag{3.9}$$

Proof. Let Λ_ε be the indicator measure of μ_ε (see Definition 3.5). As $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu_0$ weakly*, the sequence of total variations $\{\lambda_\varepsilon(\Omega)\}$ is bounded, and so, by Remark 3.6 (i), there is a subsequence $\{\mu_\eta\}$ such that $\Lambda_\eta \overset{*}{\rightharpoonup} \Lambda_\infty$ weakly *. Thus, by Definition 3.5 and Proposition 3.2 (iii), for every $f \in C_0(\Omega x \mathbb{R}^p)$ we have

$$\lim_{\eta \to 0} \langle \Lambda_\eta, f \rangle = \lim_{\eta \to 0} \int_{\Omega} f(x, \alpha_\eta(x))\, d\lambda_\eta(x) = \int_{\Omega x \mathbb{R}^p} f(x, y)\, d\Lambda_\infty(x, y)$$

$$= \int_{\Omega} \left(\int_{\mathbb{R}^p} f(x, y)\, d\lambda_x^\infty(y) \right) d\pi_\infty(x).$$

Setting $f(x, y) = \varphi(x)y$, where $\varphi \in C_0(\Omega)$, by Definition 3.3 and Proposition 3.4 we have

$$\lim_{\eta \to 0} \int_{\Omega} \varphi(x)\ \alpha_\eta(x)\ d\lambda_\eta(x) = \int_{\Omega x \mathbb{R}^p} \varphi(x)\, y\ \ d\Lambda_\infty(x, y) = \int_{\Omega} \varphi(x)\, \nu_\infty(x)\ \ d\pi_\infty(x).$$

On the other hand, as $\mu_\eta \overset{*}{\rightharpoonup} \mu_0$ weakly *,

$$\int_{\Omega} \varphi(x)\, \alpha_\eta(x) d\lambda_\eta(x) = \int_{\Omega} \varphi(x)\, d\mu_\eta(x) \to \int_{\Omega} \varphi(x)\, d\mu_0(x)$$

and so,

$$\int_{\Omega} \varphi(x)\ \ d\mu_0(x) = \int_{\Omega} \varphi(x)\, \nu_\infty(x)\ \ d\pi_\infty(x)$$

for all $\varphi \in C_0(\Omega)$, i. e. $d\mu_0 = \nu_\infty\, d\, \pi_\infty$.

Finally, as by Remark 3.6 (iii) support $\Lambda_\eta \subset \bar{\Omega} x S^{p-1}$, we conclude that support $\lambda_x^\infty \subset S^{p-1}$.

Corollary 3.10.

Let $\{h_\varepsilon\}$ be a sequence of $W^{1,1}$ functions bounded in $BV(\Omega\, ; \mathbb{R}^n)$ and assume that $h_\varepsilon \to h_0$ strongly in $L^1(\Omega; \mathbb{R}^n)$, with $h_0 \sim (\alpha_0, \lambda_0)$. Then there exists a subsequence $\{h_\eta\}$ and a nonnegative Radon measure $\Lambda_\infty \equiv \lambda_x^\infty \otimes \pi_\infty$ on Ω x S^*, with $S^* := \{F \in M^{nxN} \mid \|F\| = 1\}$, such that

$$\nabla h_0 = \alpha_0 d\lambda_0 = \nu_\infty d\pi_\infty \text{ in the sense of distributions,} \tag{3.11}$$

and for every $f \in C_0(\Omega \text{ x } M^{nxN})$ we have

$$\lim_{\eta \to 0} \int_{\Omega} f\left(x, \frac{\nabla h_\eta(x)}{\|\nabla h_\eta(x)\|} \right) \|\nabla h_\eta(x)\|\, d(x) = \int_{\Omega x S^*} f(x, F)\, d\Lambda_\infty(x, F)$$

$$= \int_{\Omega} \left(\int_{S^*} f(x, F)\, d\lambda_x^\infty(F) \right) d\pi_\infty(x). \tag{3.12}$$

Note : On (3.12) it is understood that

$$f\left(x, \frac{\nabla h_\eta(x)}{\|\nabla h_\eta(x)\|}\right)\|\nabla h_\eta(x)\| = \begin{cases} f\left(x, \frac{\nabla h_\eta(x)}{\|\nabla h_\eta(x)\|}\right)\|\nabla h_\eta(x)\| & \text{if } \|\nabla h_\eta(x)\| \neq 0 \\ 0 & \text{if } \|\nabla h_\eta(x)\| = 0. \end{cases}$$

Proof. (3.12) follows immediatly from (3.9) and Remark 2.5 (ii), (iii). By Remark 2.5 (i), (ii), (iii), and by Remark 3.6 (iii)

$\nabla h_\varepsilon = \alpha_\varepsilon\, d\lambda_\varepsilon = \nu_\varepsilon\, d\pi_\varepsilon \to \nu_\infty d\pi_\infty$ in the sense of distributions.

Indeed,

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi(x)\,\nu_\varepsilon(x)\ d\pi_\varepsilon(x) = \lim_{\varepsilon\to 0}\int_{\Omega x S^{N-1}} \varphi(x)\, y\, d\Lambda_\varepsilon(x, y)$$

$$= \int_{\Omega x S^{N-1}} \varphi(x)\, y\, d\Lambda_\infty(x, y)$$

$$= \int_\Omega \varphi(x)\,\nu_\infty(x)\ d\pi_\infty(x)$$

for all $\varphi \in \mathcal{D}(\Omega)$. On the other hand, as $h_\varepsilon \to h_0$ in $L^1(\Omega;\mathbb{R}^n)$ it follows that $\nabla h_\varepsilon \to \nabla h_0$ in $\mathcal{D}'(\Omega)$, and so $\nu_\infty d\pi_\infty = \nabla h_0$ in $\mathcal{D}'(\Omega)$.

Next, we search for relations between $\Lambda_\infty \equiv \lambda_x^\infty \otimes \pi_\infty$ and the indicator measure of μ_0 (see Remark 3.6 (iii)),

$\Lambda_0 = \delta_{y=\alpha_0(x)} \otimes \lambda_0.$

Proposition 3.13.

Under the hypotheses of Theorem 3.7, we have

(i) $d\lambda_0 = \|\nu_\infty\|\, d\pi_\infty$;

(ii) $\nu_\infty = \|\nu_\infty\|\,\alpha_0$ for λ_0 a. e. $x \in \Omega$ and $\|\nu_\infty\| \leq 1$ for π_∞ a. e. $x \in \Omega$;

(iii) support ν_∞ = support λ_0.

Proof. By (3.8) we have

$$d\mu_0 = \frac{\nu_\infty}{\|\nu_\infty\|}\ \|\nu_\infty\|\ d\pi_\infty$$

which yields $d\lambda_0 = \|\nu_\infty\|\ d\pi_\infty$ and $\nu_\infty = \|\nu_\infty\|\,\alpha_0$ for λ_0 a. e.$x \in \Omega$. On the other hand, since λ_x^∞ is a probability measure (see Proposition 3.2 (ii)), we have that $\nu(x) \in$ closed convex hull of S^{p-1} i. e. $\|\nu(x)\| \leq 1$. This proves (i) and (ii).

(iii) Let $U \subset \mathbb{R}^N$ be an open set such that $\lambda_0(U) = 0$. Let $x_0 \in U$ and consider $B(x_0, \varepsilon)$ with $0<\varepsilon<\varepsilon_0$ where $B(x_0, \varepsilon_0) \subset U$. Then, by (i),

$$0 = \int_{B(x_0, \varepsilon)} d\lambda_0 = \int_{B(x_0, \varepsilon)} \|v_\infty(x)\| \, d\pi_\infty(x)$$

and so $v_\infty(x) = 0$ π_∞ a. e. $x \in U$, i. e. support $v_\infty \subset \mathbb{R}^N \setminus U$. Thus

support $v_\infty \subset$ support λ_0.

Conversely, if $v_\infty(x) = 0$ π_∞ a. e. $x \in U$, then by (i) we have

$$\lambda_0(U) = \int_U \|v_\infty(x)\| \, d\pi_\infty(x) = 0$$

and so,

support $\lambda_0 \subset$ support v_∞.

Corollary 3.14.

Under the hypotheses of Corollary 3.10, the *density of π_∞ with respect to $\|Dh_0\|$*

$$\xi(x) := \lim_{r \to 0} \frac{\pi_\infty(B(x, r))}{\|Dh_0\|(B(x, r))}$$

exists, is finite and $\xi(x) \geq 1$ for $\|Dh_0\|$ a. e. $x \in \Omega$.

Proof. By (3.11) and since $\| v_\infty\| \leq 1$ (see Proposition 3.4 (i)),

$$\|Dh_0\| (E) \leq \pi_\infty(E) \tag{3.15}$$

for every Borel set $E \subset \Omega$. Finally, by Besicovitch Differentiation Theorem (see BESICOVITCH [7], EVANS & GARIEPY [15] Theorem 1.6.1) $\xi(x)$ exists and is finite for $\|Dh_0\|$ a. e. $x \in \Omega$, which, together with (3.15) concludes the proof.

Remark 3.16.

If h_0 turns out to be a $W^{1,1}$ function, then by Remark 2.5 (ii), (iii), (3.11) and (3.15)

$$\|\nabla h_0\| dx = d\lambda_0 = \|v_\infty\| \, d\pi_\infty \leq d\pi_\infty$$

and so,

$$\|\nabla h_0\| \llcorner L^N \leq \pi_\infty.$$

Proposition 3.17.

Let $\{E_\varepsilon\}$ be a sequence of bounded sets of finite perimeter in $\mathbb{R}^N$ such that meas $(E_\varepsilon) \to k$ and $\{\text{Per}\,(E_\varepsilon)\}$ is bounded. Suppose that $\Lambda_\varepsilon \overset{*}{\rightharpoonup} \Lambda_\infty$ weakly *, where Λ_ε is the indicator measure of $(\alpha_\varepsilon, \lambda_\varepsilon) \sim \chi_{E_\varepsilon}$, and let $E_\varepsilon \subset\subset B(0, R)$ for some $R > 0$. Then

(i) $\int_\Omega x . v_\infty(x) \; d\pi_\infty(x) = -N k$;

(ii) $\int_\Omega \nu_\infty(x)\ d\pi_\infty(x) = 0.$

Proof. Let $\varphi \in C_0(\mathbb{R}^N)$ be such that $\varphi = 1$ in $B(0, 2R)$.

(i) By Theorem 2.6 and as support $\pi_\infty \subset \bar{B}(0, R)$, we have

$$k = \lim_{\varepsilon \to 0} \text{meas}(E_\varepsilon) = \lim_{\varepsilon \to 0} \int_{E_\varepsilon} \frac{\text{div } x}{N}\ dx = -\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \frac{x}{N} \cdot \alpha_\varepsilon(x)\ d\lambda_\varepsilon$$

$$= -\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N x S^{N-1}} \varphi(x) \frac{x}{N} \cdot y\ \ d\Lambda_\varepsilon(x, y) = -\int_{\mathbb{R}^N x S^{N-1}} \varphi(x) \frac{x}{N} \cdot y\ \ d\Lambda_\infty(x, y)$$

$$= -\int_{\mathbb{R}^N} \frac{x}{N} \cdot \nu_\infty(x)\ \ d\pi_\infty(x).$$

(ii) Once again by Theorem 2.6 and for all $\varepsilon > 0$,

$$\int_{\mathbb{R}^N} \alpha_\varepsilon(x)\ d\lambda_\varepsilon = 0$$

and so

$$0 = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \alpha_\varepsilon(x)\ d\lambda_\varepsilon = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N x S^{N-1}} \varphi(x) y\ \ d\Lambda_\varepsilon(x, y)$$

$$= \int_{\mathbb{R}^N x S^{N-1}} \varphi(x) y\ \ d\Lambda_\infty(x, y) = \int_{\mathbb{R}^N} \nu_\infty(x)\ \ d\pi_\infty(x).$$

4. LOWER SEMICONTINUITY OF SURFACE ENERGIES : THE CONVEX CASE.

In this section we search for necessary and sufficient conditions ensuring the lower semicontinuity of a surface energy functional of the type

$$J(E) := \int_{\partial E \cap \Omega} \Gamma(\nu_E(x))\ dH_{N-1}(x),$$

where E is a smooth subset of $\mathbb{R}^N$, ν_E is the outward unit normal to its boundary and Γ denotes the anisotropic surface energy density per unit area of the deformed configuration. In order to extend J(.) to sets of finite perimeter, and according to Remark 2.5 (iv), we rewrite the energy functional as

$$J(E) = \int_\Omega \Gamma(-\alpha_E(x))\ d\lambda_E(x), \tag{4.1}$$

where $\chi_E \sim (\alpha_E, \lambda_E) = (-\nu_E, H_{N-1}\lfloor \partial^* E)$, $\partial^* E$ is the reduced boundary of E and ν_E is the normal to $\partial^* E$. Using the notation introduced in Section 2, the formulation (4.1) suggests the study of sequentially weak * lower semicontinuity properties of functionals of the type

$$I_f(\mu) := \int_\Omega f(x, \alpha(x))\ d\lambda(x),$$

where μ is a $\mathbb{R}^p$-valued measure with polar decomposition $d\mu = \alpha\, d\lambda$. Note that if E is a set of finite perimeter, then $J(E) = I_f(\mu)$ where $\mu = -\nabla\chi_E$ and $f(x, y) = \Gamma(y)$.

In what follows, let Ω be an open subset of $\mathbb{R}^N$, and let μ_ε and μ_0 be $\mathbb{R}^p$- valued measures with polar decomposition $d\mu_\varepsilon = \alpha_\varepsilon\, d\lambda_\varepsilon$ and $d\mu_0 = \alpha_0\, d\lambda_0$.

Lemma 4.2.

If $\Lambda \equiv \lambda_x \otimes \pi$ is the indicator measure of the $\mathbb{R}^p$- valued measure μ with polar decomposition $d\mu = \alpha d\lambda$ then

$$\int_\Omega f(x, \alpha(x))\; d\lambda(x) = \int_\Omega f(x, v(x))\; d\pi(x)$$

for all $f \in C_0(\Omega x \mathbb{R}^p)$ such that $f(x, .)$ is homogeneous of degree one, for every $x \in \Omega$.

Proof. Define

$$\gamma(x) := \begin{cases} \dfrac{v(x)}{\|v(x)\|} & \text{if } \|v(x)\| \neq 0 \\ \\ 0 & \text{if } \|v(x)\| = 0. \end{cases}$$

As $f(x, .)$ is homogeneous of degree one, by Proposition 3.13 (i) we have

$$\begin{aligned} \int_\Omega f(x, v(x))\, d\pi(x) &= \int_\Omega f(x, \gamma(x))\, \|v(x)\|\, d\pi(x) \\ &= \int_\Omega f(x, \gamma(x))\; d\lambda(x). \end{aligned} \tag{4.3}$$

By Proposition 3.13 (ii), (iii), (iv), $v = \|v_\infty\|\,\alpha$ and $v \neq 0$ for λ a. e. $x \in \Omega$, thus $\gamma = \alpha$ for λ a. e. $x \in \Omega$ which, together with (4.3), concludes the proof.

Definition 4.4.

Given $f \in C_0(\Omega x S^{p-1})$, we define the homogeneous of degree one extension of f, H_f, by

$$H_f(x, y) := \begin{cases} \|y\|\, f\left(x, \dfrac{y}{\|y\|}\right) & \text{if } y \neq 0 \\ \\ 0 & \text{if } y = 0. \end{cases}$$

The following lower semicontinuity result was proved independently by GOFFMAN & SERRIN [21] and by RESHETNYAK [30].

Theorem 4.5 (sufficient condition) ([30]).

If $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu_0$ and if $H_f(x, .)$ is convex for all $x \in \Omega$, then

$$\int_{\Omega} f(x, \alpha_0(x)) \ d\lambda_0(x) \leq \liminf_{\varepsilon \to 0} \int_{\Omega} f(x, \alpha_\varepsilon(x)) \ d\lambda_\varepsilon(x).$$

Proof. Given a subsequence $\{\mu_{\varepsilon'}\}$ of $\{\mu_\varepsilon\}$, by Theorem 3.7 there exists a subsequence $\{\mu_\eta\}$ and a Radon measure $\Lambda_\infty \equiv \lambda_x^\infty \otimes \pi_\infty$ on $\Omega x S^{p-1}$ such that

$$\lim_{\eta \to 0} \int_{\Omega} f(x, \alpha_\eta(x)) \ d\lambda_\eta(x) = \lim_{\eta \to 0} \int_{\Omega} H_f(x, \alpha_\eta(x)) \ d\lambda_\eta(x)$$

$$= \int_{\Omega} \left(\int_{S^{p-1}} H_f(x, y) \ d\lambda_x^\infty(y) \right) d\pi_\infty(x).$$

Due to the convexity of $H_f(x, .)$ and since λ_x^∞ is a probability measure (see Proposition 3.2 (ii)), by Jensen's inequality we deduce that

$$\lim_{\eta \to 0} \int_{\Omega} f(x, \alpha_\eta(x)) \ d\lambda_\eta(x) \geq \int_{\Omega} H_f(x, v_\infty(x)) \ d\pi_\infty(x).$$

Finally, by Lemma 4.2 we conclude that

$$\liminf_{\varepsilon \to 0} \int_{\Omega} f(x, \alpha_\varepsilon(x)) \ d\lambda_\varepsilon(x) \geq \int_{\Omega} H_f(x, \alpha_0(x)) \ d\lambda_0(x) = \int_{\Omega} f(x, \alpha_0(x)) \ d\lambda_0(x).$$

Corollary 4.6.

(i) Let $h_\varepsilon \in W^{1,1}(\Omega)$ be such that $h_\varepsilon \to h_0$ strongly in $L^1(\Omega)$ and $\{\| h_\varepsilon \|_{1,1}\}$ is bounded. If $h_0 \sim (\alpha_0, \lambda_0)$, then

$$\int_{\Omega} f(x, \alpha_0(x)) \, d\lambda_0(x) \leq \liminf_{\varepsilon \to 0} \int_{\Omega} f\left(x, \frac{\nabla h_\varepsilon(x)}{\|\nabla h_\varepsilon(x)\|} \right) \|\nabla h_\varepsilon(x)\| \, dx$$

for all $f \in C_0(\Omega x \mathbb{R}^p)$ such that $H_f(x, .)$ is convex for all $x \in \Omega$.

(ii) Let $E_\varepsilon \subset \mathbb{R}^N$ be bounded with finite perimeter in $\mathbb{R}^N$. If the sequence $\{\text{meas}(E_\varepsilon) + \text{Per}(E_\varepsilon)\}$ is bounded and if $\chi_{E_\varepsilon} \to \chi_{E_0}$ strongly in $L^1(\mathbb{R}^N)$, then

$$\int_{\partial^* E_0 \cap \Omega} f(x, v_{E_0}(x)) \, dH_{N-1}(x) \leq \liminf_{\varepsilon \to 0} \int_{\partial^* E_\varepsilon \cap \Omega} f(x, v_{E_\varepsilon}(x)) \, dH_{N-1}(x)$$

for all $f \in C_0(\Omega x \mathbb{R}^p)$ such that $H_f(x, .)$ is convex for all $x \in \Omega$. Moreover, if $g \in C(\mathbb{R}^p)$ is a nonnegative, convex, homogeneous of degree one function then

$$\int_{\partial^* E_0} g(v_{E_0}(x)) \, dH_{N-1}(x) \leq \liminf_{\varepsilon \to 0} \int_{\partial^* E_\varepsilon} g(v_{E_\varepsilon}(x)) \, dH_{N-1}(x).$$

Proof. (i) Let $d\mu_\varepsilon = \alpha_\varepsilon \, d\lambda_\varepsilon$, where $h_\varepsilon \sim (\alpha_\varepsilon, \lambda_\varepsilon)$ (see Theorem 2.4). Since by Remark 2.5 (ii)

$$\alpha_\varepsilon = \frac{\nabla h_\varepsilon}{\|\nabla h_\varepsilon\|} \quad \text{and} \quad d\lambda_\varepsilon = \|\nabla h_\varepsilon\| \ dx$$

the result follows immediatly from Theorem 4.5 and Lemma 2.8.

(ii) As in part (i), let $d\,\mu_\varepsilon = \alpha_\varepsilon d\lambda_\varepsilon$, where $\chi_{E\varepsilon} \sim (\alpha_\varepsilon, \lambda_\varepsilon) = (-\,\nu_E, H_{N-1}\lfloor\partial^*E)$. As in Lemma 2.8, it is easy to check that $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu_0$ which, together with Theorem 4.5 concludes the proof. Suppose now that $g \in C(\mathbb{R}^p)$ is a nonnegative, convex, homogeneous of degree one function, and consider an increasing sequence of cut-off functions $\varphi_n \in C_0(\Omega)$ such that $0 \le \varphi_n \le 1$ and for all $x \in \Omega$, $\lim \varphi_n(x) = 1$. Then, for all $n \in \mathbb{N}$ we have

$$\int_\Omega \varphi_n(x)\, g(\nu_{E_0}(x))\, dH_{N-1}(x) \le \liminf_{\varepsilon \to 0} \int_{\partial^* E_\varepsilon} \varphi_n(x)\, g(\nu_{E_\varepsilon}(x))\, dH_{N-1}(x)$$

$$\le \liminf_{\varepsilon \to 0} \int_{\partial^* E_\varepsilon} g(\nu_{E_\varepsilon}(x))\, dH_{N-1}(x).$$

Finally, by Lebesgue's Monotone Convergence Theorem,

$$\int_\Omega g(\nu_{E_0}(x))\, dH_{N-1}(x) = \lim_{n \to \infty} \int_\Omega \varphi_n(x)\, g(\nu_{E_0}(x))\, dH_{N-1}(x)$$

and so we conclude that

$$\int_{\partial^* E_0} g(\nu_{E_0}(x))\, dH_{N-1}(x) \le \liminf_{\varepsilon \to 0} \int_{\partial^* E_\varepsilon} g(\nu_{E_\varepsilon}(x))\, dH_{N-1}(x).$$

Now we prove the converse of Theorem 4.5.

Theorem 4.7 (necessary condition).
If $f \in C_0(\Omega \times \mathbb{R}^p)$ is such that

$$\int_\Omega f(x, \alpha_0(x))\ \ d\lambda_0(x) \le \liminf_{\varepsilon \to 0} \int_\Omega f(x, \alpha_\varepsilon(x))\ \ d\lambda_\varepsilon(x)$$

whenever $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu_0$, then $H_f(x, .)$ is convex for all $x \in \Omega$.

Proof. Let $x_0 \in \Omega$, let $\theta \in (0, 1)$, let $a, b \in \mathbb{R}^p$, $a \neq b$, and let $\xi = (1, 0, \dots, 0)$ be a unit vector in $\mathbb{R}^N$. Let χ be the characteristic function of the interval $(0, \theta)$ extended periodically to $\mathbb{R}$ with period 1. Clearly, the function

$$x \to \chi(x.\ \xi)$$

is periodic with period $Y := [0, 1]^N$. Define the sequence of functions

$$u_\varepsilon(x) := \begin{cases} \dfrac{1}{\varepsilon^N}\left[b + \chi\left(\dfrac{x - x_0}{\varepsilon^2} .\ \xi \right) (a - b) \right] & \text{if } x \in x_0 + \varepsilon Y \\ \\ 0 & \text{otherwise.} \end{cases}$$

Setting

$$d\mu_\varepsilon := u_\varepsilon\, dx,$$

we have

$$\mu_\varepsilon \to (\theta a + (1-\theta) b)\ \delta_{x = x_0} \quad \text{weakly * in the sense of measures.} \tag{4.8}$$

Indeed, if $\varphi \in \mathcal{D}(\Omega)$ then

$$\int_\Omega \varphi(x)\, u_\varepsilon(x)\ dx = \int_{x_0+\varepsilon Y} \frac{1}{\varepsilon^N}\, \varphi(x)\left[b + \chi\left(\frac{x-x_0}{\varepsilon^2}\cdot\xi\right)(a-b)\right] dx$$

$$= \int_Y \varphi(x_0+\varepsilon y)\left[b + \chi\left(\frac{y}{\varepsilon}\cdot\xi\right)(a-b)\right] dy$$

and so, as

$$\chi\left(\frac{y}{\varepsilon}\cdot\xi\right) \to \theta \quad \text{in } L^\infty \text{ weak } *,$$

we deduce that

$$\int_\Omega \varphi(x)\, u_\varepsilon(x)\ dx \ \to \varphi(x_0)\,(\theta a + (1-\theta)b) = \langle\, (\theta a + (1-\theta)b)\ \delta_{x=x_0}, \varphi \,\rangle$$

which proves (4.8). Since

$$\alpha_\varepsilon = \frac{u_\varepsilon}{\|u_\varepsilon\|},\ \lambda_\varepsilon = \|u_\varepsilon\|\ dx,\ \alpha_0 = \frac{\theta a + (1-\theta)b}{\|\theta a + (1-\theta)b\|} \quad \text{and } \lambda_0 = \|\theta a + (1-\theta)b\|\ \delta_{x=x_0},$$

by (4.8) we have

$$\int_\Omega f(x, \alpha_0(x))\ \ d\lambda_0(x)\ \le \liminf_{\varepsilon\to 0} \int_\Omega f(x, \alpha_\varepsilon(x))\ \ d\lambda_\varepsilon(x)$$

i. e.

$$H_f(x_0, \theta a + (1-\theta)b) \le \liminf_{\varepsilon\to 0} \int_\Omega H_f(x, u_\varepsilon)\, dx$$

$$= \liminf_{\varepsilon\to 0} \int_{x_0+\varepsilon Y} H_f\left(x, \frac{1}{\varepsilon^N}\left[b + \chi\left(\frac{x-x_0}{\varepsilon^2}\cdot\xi\right)(a-b)\right]\right) dx$$

$$= \liminf_{\varepsilon\to 0} \int_Y \varepsilon^N \left\{ \chi\left(\frac{y}{\varepsilon}\cdot\xi\right) H_f\left(x_0+\varepsilon y, \frac{a}{\varepsilon^N}\right) + \left(1-\chi\left(\frac{y}{\varepsilon}\cdot\xi\right)\right) H_f\left(x_0+\varepsilon y, \frac{b}{\varepsilon^N}\right)\right\} dy$$

$$= \theta H_f(x_0, a) + (1-\theta)\, H_f(x_0, b).$$

5. CONTINUITY OF SURFACE ENERGY DENSITIES.

Here we provide necessary and sufficient conditions for the sequential weakly * continuity of the surface energy functionals. As in Section 4, in what follows Ω is an open subset of $\mathbb{R}^N$, μ_ε and μ_0 are vector valued measures with values in $\mathbb{R}^p$ and with polar decomposition, respectively, $d\mu_\varepsilon = \alpha_\varepsilon\, d\lambda_\varepsilon$ and $d\mu_0 = \alpha_0\, d\lambda_0$. Let $\Lambda_\varepsilon \equiv \lambda_\varepsilon \otimes \pi_\varepsilon$ be the indicator measure of μ_ε (see Definition 3.5).

Theorem 5.1.

Let $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu_0$ in the sense of measures and assume that $\Lambda_\varepsilon \overset{*}{\rightharpoonup} \Lambda_\infty \equiv \lambda_x^\infty \otimes \pi_\infty$. Then λ_x^∞ is a Dirac mass if and only if

$$\int_\Omega f(x, \alpha_0(x))\ d\lambda_0(x) = \lim_{\varepsilon \to 0} \int_\Omega f(x, \alpha_\varepsilon(x))\ d\lambda_\varepsilon(x)$$

for all $f \in C_0(\Omega x \mathbb{R}^p)$.

Proof. Suppose that λ_x^∞ is a Dirac mass, $\lambda_x^\infty = \delta_{y\,=\,\xi(x)}$. By Proposition 3.2(ii), support $\lambda_x^\infty \subset S^{p-1}$ and $\lambda_x^\infty(S^{p-1}) = 1$, therefore $||\xi(x)|| = 1$ for π_∞ a. e. $x \in \Omega$ and

$\nu_\infty(x) = \xi(x)$, with $||\nu_\infty(x)|| = 1$ π_∞ a. e. $x \in \Omega$.

By Proposition 3.13 (i), (ii), we conclude that

$$\nu_\infty = \alpha_0 \ \text{ and } \pi_\infty = \lambda_0. \tag{5.2}$$

Therefore, and according to Definition 3.3,

$$\lambda_x^\infty = \delta_{y\,=\,\alpha_0(x)}$$

and by (5.2) and Proposition 3.2 (iii),

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_\Omega f(x, \alpha_\varepsilon(x))\ d\lambda_\varepsilon(x) &= \lim_{\varepsilon \to 0} \int_{\Omega x S^{p-1}} f(x, y)\ d\Lambda_\varepsilon(x, y)\\ &= \int_{\Omega x S^{p-1}} f(x, y)\ d\Lambda_\infty(x, y)\\ &= \int_\Omega \left(\int_{S^{p-1}} f(x, y)\ d\lambda_x^\infty(y)\right) d\pi_\infty(x)\\ &= \int_\Omega f(x, \alpha_0(x))\ d\lambda_0(x).\end{aligned}$$

Conversely, if for all $f \in C_0(\Omega x \mathbb{R}^p)$

$$\int_\Omega f(x, \alpha_0(x))\ d\lambda_0(x) = \lim_{\varepsilon \to 0} \int_\Omega f(x, \alpha_\varepsilon(x))\ d\lambda_\varepsilon(x)$$

then $\Lambda_0 = \Lambda_\infty$, and by Remark 3.6 (iii) we conclude that

$$\lambda_x^\infty = \lambda_{0x} = \delta_{y\,=\,\alpha_0(x)}.$$

Corollary 5.3.

Assume that $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu_0$ in the sense of measures, $\lambda_\varepsilon(\Omega) \to \lambda_0(\Omega)$ and $\Lambda_\varepsilon \overset{*}{\rightharpoonup} \Lambda_\infty$. In addition, suppose that the norm $||.||$ in $\mathbb{R}^p$ is an euclidean norm. Then

$$\int_\Omega f(x, \alpha_0(x))\ d\lambda_0(x) = \lim_{\varepsilon \to 0} \int_\Omega f(x, \alpha_\varepsilon(x))\ d\lambda_\varepsilon(x)$$

for all $f \in C_0(\Omega x \mathbb{R}^p)$.

The proof of this result is based on Theorem 5.1 and on the following lemma.

Lemma 5.4.

If $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu_0$ in the sense of measures, if $\lambda_\varepsilon(\Omega) \to \lambda_0(\Omega)$ and if $\Lambda_\varepsilon \overset{*}{\rightharpoonup} \Lambda_\infty \equiv \lambda_x^\infty \otimes \pi_\infty$ in the sense of measures, then $\nu_\infty = \alpha_0$ and $\pi_\infty = \lambda_0$.

Proof. Let $\varphi \in C_0(\Omega)$. Then

$$\int_\Omega \varphi(x)\ d\lambda_\varepsilon(x) = \int_\Omega \varphi(x)\ \|\alpha_\varepsilon(x)\|\ d\lambda_\varepsilon(x)$$

$$= \int_{\Omega x S^{p-1}} \varphi(x)\ \|y\|\ d\Lambda_\varepsilon(x, y)$$

and so,

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi(x)\ d\lambda_\varepsilon(x) = \int_{\Omega x S^{p-1}} \varphi(x)\ \|y\|\ d\Lambda_\infty(x, y)$$

$$= \int_\Omega \varphi(x)\ d\pi_\infty(x). \tag{5.5}$$

On the other hand, as $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu_0$ and as $\lambda_\varepsilon = \|\ \mu_\varepsilon\ \|$, $\lambda_0 = \|\ \mu_0\ \|$, we have that $\lambda_0 \leq \liminf \lambda_\varepsilon$. Thus, since by hypothesis $\lambda_\varepsilon(\Omega) \to \lambda_0(\Omega)$, we deduce that

$$\lambda_\varepsilon \rightharpoonup \lambda_0 \text{ weakly * in the sense of measures.} \tag{5.6}$$

Therefore

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi(x)\ d\lambda_\varepsilon(x) = \int_\Omega \varphi(x)\ d\lambda_0(x)$$

which, together with (5.5) implies that

$$\pi_\infty = \lambda_0. \tag{5.7}$$

By Proposition 3.13 (i), (ii), we have $d\lambda_0 = \|\nu_\infty\|\ d\pi_\infty$ and $\nu_\infty = \|\nu_\infty\|\alpha_0$, and so, by (5.7) we conclude that $\|\nu_\infty\| = 1$ and $\nu_\infty = \alpha_0$.

Proof of Corollary 5.3. Let $\varphi \in C_0(\Omega\ ;\mathbb{R}^p)$. By (5.6) we have

$$\lim_{\varepsilon \to 0} \int_\Omega \|\alpha_\varepsilon(x) - \varphi(x)\|^2\ d\lambda_\varepsilon(x) = \lim_{\varepsilon \to 0} \int_\Omega [1 - 2\alpha_\varepsilon(x).\varphi(x) + \|\varphi(x)\|^2\ ]\ d\lambda_\varepsilon(x)$$

$$= \lambda_0(\Omega) - 2\int_\Omega \alpha_0(x).\varphi(x)\ d\lambda_0 + \int_\Omega \|\varphi(x)\|^2\ d\lambda_0(x). \tag{5.8}$$

On the other hand,

$$\lim_{\varepsilon \to 0} \int_\Omega \|\alpha_\varepsilon(x) - \varphi(x)\|^2\ d\lambda_\varepsilon(x) = \lim_{\varepsilon \to 0} \int_{\Omega x S^{p-1}} \|y - \varphi(x)\|^2\ d\Lambda_\varepsilon(x, y)$$

$$= \int_{\Omega x S^{p-1}} \|y - \varphi(x)\|^2\ d\Lambda_\infty(x, y)$$

$$= \int_\Omega \left(\int_{S^{p-1}} \|y - \varphi(x)\|^2\ d\lambda_x^\infty(y) \right) d\pi_\infty(x)$$

and so, by (5.8) we deduce that

$$\int_\Omega\left(\int_{S^{p-1}} \|y-\varphi(x)\|^2 \, d\lambda_x^\infty(y)\right)d\pi_\infty(x) = \lambda_0(\Omega) - 2\int_\Omega \alpha_0(x).\varphi(x)\, d\lambda_0 + \int_\Omega \|\varphi(x)\|^2 \, d\lambda_0(x).$$

By Lemma 5.4, $\nu_\infty = \alpha_0$ and $\pi_\infty = \lambda_0$, and so taking $\varphi_n \in C_0(\Omega; \mathbb{R}^p)$ such that $\|\varphi_n\| \leq 1$ and $\varphi_n \to \alpha_0$ in $L^1(\lambda_0)$, we obtain

$$\int_\Omega\left(\int_{S^{p-1}} \|y-\alpha_0(x)\|^2 \, d\lambda_x^\infty(y)\right)d\pi_\infty(x) = 0.$$

Therefore, as $\|\alpha_0(x)\| = 1$, we conclude that $y = \alpha_0(x)$ for λ_x^∞ a. e. $y \in S^{p-1}$ and for π_∞ a. e. $x \in \Omega$, i. e.

$$\lambda_x^\infty = \delta_{y=\alpha_0(x)}.$$

The conclusion follows from Theorem 5.1.

Corollary 5.9.

(i) Let $h_\varepsilon, h_0 \in W^{1,1}(\Omega ; \mathbb{R}^n)$ be such that $h_\varepsilon \to h_0$ strongly in $L^1(\Omega; \mathbb{R}^n)$ and

$$\int_\Omega \|\nabla h_\varepsilon(x)\|\, dx \to \int_\Omega \|\nabla h_0(x)\|\, dx.$$

Then

$$\int_\Omega f\left(x, \frac{\nabla h_0(x)}{\|\nabla h_0(x)\|}\right) \|\nabla h_0(x)\|\, dx = \lim_{\varepsilon\to 0} \int_\Omega f\left(x, \frac{\nabla h_\varepsilon(x)}{\|\nabla h_\varepsilon(x)\|}\right) \|\nabla h_\varepsilon(x)\|\, dx$$

for all $f \in C_0(\Omega x M^{nxN})$.

(ii) Let $E_\varepsilon \subset \mathbb{R}^N$ be bounded with finite perimeter in $\mathbb{R}^N$. If $\chi_{E_\varepsilon} \to \chi_{E_0}$ strongly in $L^1(\mathbb{R}^N)$ and if $Per(E_\varepsilon) \to Per(E_0)$ then

$$\int_{(\partial^* E_0)\cap\Omega} f(x, \nu_{E_0}(x))\, dH_{N-1} = \lim_{\varepsilon\to 0} \int_{(\partial^* E_\varepsilon)\cap\Omega} f(x, \nu_{E_\varepsilon}(x))\, dH_{N-1}$$

for all $f \in C_0(\Omega x \mathbb{R}^p)$.

Proof. (i) As in the proof of Corollary 4.6, let $d\,\mu_\varepsilon = \alpha_\varepsilon d\lambda_\varepsilon$, where $h_\varepsilon \sim (\alpha_\varepsilon, \lambda_\varepsilon)$ (see Theorem 2.4) and, by Remark 2.5 (ii),

$$\alpha_\varepsilon(x) = \frac{\nabla h_\varepsilon(x)}{\|\nabla h_\varepsilon(x)\|} \quad \text{and } d\lambda_\varepsilon = \|\nabla h_\varepsilon(x)\|\, dx.$$

By Lemma 2.8 and since $\|\nabla h_\varepsilon\|_1 \to \|\nabla h_0\|_1$, we have that $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu_0$ and $\lambda_\varepsilon(\Omega) \to \lambda_0(\Omega)$. Now the result follows from Corollary 5.3.

(ii) Here $\chi_{E_\varepsilon} \sim (\alpha_\varepsilon, \lambda_\varepsilon) = (-\nu_{E_\varepsilon}(x), \; H_{N-1} \llcorner \partial^* E_\varepsilon)$. Since μ_ε converges weakly * in measure to μ_0 and as $\lambda_\varepsilon(\Omega) = Per(E_\varepsilon) \to Per(E_0) = \lambda_0(\Omega)$, we can apply Corollary 5.3.

Example 5.10.

In $\mathbb{R}^2$ consider the canonical euclidean norm. Consider the sets E_0 and E_k as in Figures 1 and 2, respectively.

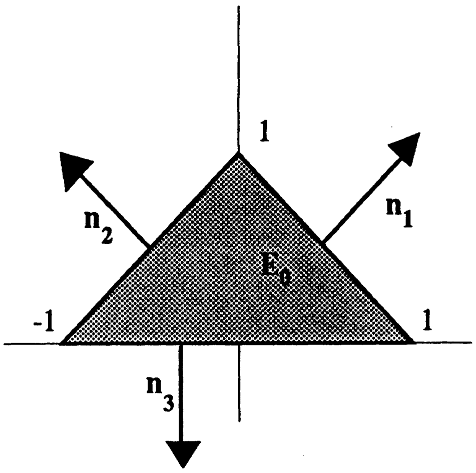

Fig. 1

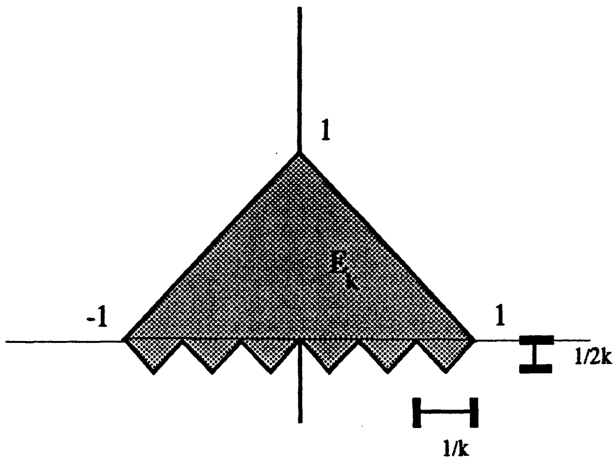

Fig. 2

Clearly

$$\chi_{E_k} \to \chi_{E_0} \quad \text{strongly in } L^1$$

and

$$2 + 2\sqrt{2} = \mathrm{Per}(E_0) < \lim_{k \to +\infty} \mathrm{Per}(E_k) = 4\sqrt{2}.$$

Setting

$$n_1 = \frac{(1, 1)}{\sqrt{2}}, \quad n_2 = \frac{(-1, 1)}{\sqrt{2}}, \quad n_3 = (0, -1)$$

and

$L_1 :=$ segment joining the points (1, 0) and (0, 1),
$L_2 :=$ segment joining the points (0, 1) and (-1, 0),
$L_3 :=$ segment joining the points (-1, 0) and (1, 0),

it is easy to verify that

1. support ν_∞ = support π_∞ = support α_0 = support $\lambda_0 = \partial E_0$, with

$$-\alpha_0 = \begin{cases} n_1 & \text{on } L_1 \\ n_2 & \text{on } L_2 \\ n_3 & \text{on } L_3 \end{cases}$$

and

$$\lambda_0 = H_{N-1} \llcorner \partial E_0.$$

2. $\pi_\infty \llcorner L_i = \lambda_0 \llcorner L_i$ for $i = 1, 2$ and $\pi_\infty \llcorner L_3 = \sqrt{2}\, \lambda_0 \llcorner L_3$.

3.
$$v_\infty = \begin{cases} \alpha_0 & \text{on } L_1 \cup L_2 \\ \frac{\sqrt{2}}{2}\alpha_0 & \text{on } L_3. \end{cases}$$

4.
$$\lambda_x^\infty = \begin{cases} \delta_{y=n_i} & \text{if } x \in L_i, \text{ for } i = 1, 2 \\ \frac{1}{2}(\delta_{y=-n_1} + \delta_{y=-n_2}) & \text{if } x \in L_3. \end{cases}$$

6. LOWER SEMICONTINUITY OF SURFACE ENERGIES : THE QUASICONVEX CASE.

We are interested in the lower semicontinuity properties of surface energy densities of the type (1.1) associated to elastic solid materials that undergo a change of phase. According to Remark 2.5 (iv) and Definition 2.10, these functionals are integrals of the type

$$\int_\Omega f(\nabla \chi_E)$$

where E is a set of finite perimeter, or equivalently, χ_E is a function of bounded variation. Consider the class of functionals

$$I_f(\nabla u) := \int_\Omega f(\nabla u)$$

defined for $u \in BV(\Omega;\mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^N$ and $N, n \geq 1$. It was shown on Theorem 4.5 that if $\{u_\varepsilon\}$ is a sequence bounded in BV, if $u_\varepsilon \to u$ strongly in L^1 and if f(x, .) is convex and homogeneous of degree one then

$$I_f(\nabla u) \leq \liminf I_f(\nabla u_\varepsilon).$$

However, as shown by DACOROGNA [9] (see also ACERBI & FUSCO [1], MORREY [28]), the $W^{1,1}$ weak lower semicontinuous envelope of the functional I_f is the integral of the quasiconvexification of the energy density, and so quasiconvexity becomes the natural constitutive assumption rather than convexity. Precisely

Definition 6.1([6]).

Let $1 \leq p \leq +\infty$. A function $f : M^{n \times N} \to \mathbb{R}$ is said to be *$W^{1,p}$-quasiconvex* if

$$f(F) \leq \frac{1}{\text{meas}(D)} \int_D f(F + \nabla\varphi(x))\, dx,$$

for all $F \in M^{n \times N}$ and for all $\varphi \in W_0^{1,p}(D; \mathbb{R}^n)$.

Proposition 6.2.

Let $f : M^{nxN} \to \mathbb{R}$ be a nonnegative continuous function such that

$$f(F) \leq C(1 + ||F||) \tag{6.3}$$

for some positive constant C and for all $F \in M^{nxN}$. Then f is $W^{1,1}$-quasiconvex if and only if f is $W^{1,\infty}$-quasiconvex.

ACERBI & FUSCO [1] and DACOROGNA [9] showed that if f satisfies (6.3), then the sequential $W^{1,1}$ weak lower semicontinuous envelope of

$$J(u) := \int_\Omega f(\nabla u(x))\, dx$$

is its quasiconvexification QJ(.), namely

$$QJ(u) := \int_\Omega Qf(\nabla u)\, dx,$$

where Qf is the biggest $W^{1,1}$-quasiconvex function smaller than or equal to f.

It turns out that if f is homogeneous of degree one then f verifies (6.3) and there exists a positive constant C' such that

$$|f(F) - f(G)| \leq C' \, ||F - G|| \tag{6.4}$$

for all $F, G \in M^{nxN}$ (see DACOROGNA [10], EVANS [14]). Therefore, if in addition f is quasiconvex then

$$\int_\Omega f(\nabla u(x))\, dx \leq \liminf_{\varepsilon \to 0} \int_\Omega f(\nabla u_\varepsilon(x))\, dx \tag{6.5}$$

whenever $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,1}$.

Conjecture : If f is quasiconvex and homogeneous of degree one and if $\{u_\varepsilon\}$ is a sequence bounded in BV and $u_\varepsilon \to u$ strongly in L^1, then

$$I_f(\nabla u) \leq \liminf I_f(\nabla u_\varepsilon).$$

In Theorem 6.6 we prove the conjecture in the case where ∇u_ε and ∇u are absolutely continuous with respect to the Lebesgue measure. Let $M^{nxN} :=$ {nxm real matrices} and set $S^* := \{F \in M^{nxN} \mid ||F|| = 1\}$, where

$$||F|| := \left(\sum_{i,\, j} F_{ij}^2 \right)^{1/2} .$$

Theorem 6.6.

Let $f : \Omega x M^{nxN} \to \mathbb{R}$ be a nonnegative continuous function such that f(x,.) is homogeneous of degree one and $W^{1,\infty}$-quasiconvex. Suppose further that there exists a continuous function g with $g(0) = 0$ such that $|f(x, A) - f(y, A)| \leq g(\|x - y\|)(1 + \|A\|)$ for all $x, y \in \Omega$ and for all $A \in M^{nxN}$. Let $\{u_\varepsilon\}$ be a bounded sequence in $W^{1,1}(\Omega; \mathbb{R}^n)$ and assume that $u_\varepsilon \to u$ strongly in $L^1(\Omega; \mathbb{R}^n)$. If $u \in W^{1,1}(\Omega; \mathbb{R}^n)$ then

$$\int_\Omega f(x, \nabla u(x))\, dx \leq \liminf_{\varepsilon \to 0} \int_\Omega f(x, \nabla u_\varepsilon(x))\, dx. \tag{6.7}$$

Proof. (a) Suppose that f does not depend on the variable x. By Corollary 3.10, given a subsequence $\{u_{\varepsilon'}\}$ of $\{u_\varepsilon\}$ there exists a subsequence $\{u_\eta\}$ and a nonnegative Radon indicator measure $\Lambda_\infty \equiv \lambda_x^\infty \otimes \pi_\infty$ on $\Omega \times S^*$, such that

$$\lim_{\eta \to 0} \int_\Omega G\left(x, \frac{\nabla u_\eta}{\|\nabla u_\eta\|}\right) \|\nabla u_\eta\|\, dx = \int_\Omega \left(\int_{S^*} G(x, F)\, d\lambda_x^\infty(F)\right) d\pi_\infty(x) \tag{6.8}$$

for all $G \in C_0(\Omega x M^{nxN})$. By the Lebesgue-Besicovitch Differentiation Theorem (see EVANS & GARIEPY [15], Theorem 1.7.1), by Corollary 3.14 and by (3.15), there exists a set $E \subset \Omega$ such that $\|Du\|(E) = 0$ and for all $x_0 \notin E$ the following hold :

$$\nabla u(x_0) \neq 0, \quad \lim_{r \to 0} \frac{1}{\text{meas}(B(x_0,r))} \int_{B(x_0,r)} \|\nabla u(x) - \nabla u(x_0)\|\, dx = 0, \tag{6.9}$$

$$1 \leq \xi(x_0) < +\infty, \text{ where } \xi(x_0) = \lim_{r \to 0} \frac{\pi_\infty(B(x_0, r))}{\|Du\|(B(x_0, r))} \tag{6.10}$$

and

$$\lim_{r \to 0} \frac{1}{\pi_\infty(B(x_0,r))} \int_{B(x_0,r)} \left(\int_{S^*} f(F)\, d\lambda_x^\infty(F)\right) d\pi_\infty(x) = \int_{S^*} f(F)\, d\lambda_{x_0}^\infty(F). \tag{6.11}$$

Fix $x_0 \notin E$ and let $B_k := B(x_0, 1/k)$ with $k \in \mathbb{N}$. Consider a family of cut-off functions $\varphi_k \in \mathcal{D}(\Omega)$ such that $0 \leq \varphi_k \leq 1$, $\varphi_k = 1$ on B_k and $\varphi_k = 0$ outside B_{k-1} (set $B_0 := \Omega$). By Proposition 6.2 we have

$$\begin{aligned} f(\nabla u(x_0))\, \text{meas}(\Omega) &\leq \int_\Omega f(\nabla u(x_0) + \nabla[\varphi_k(x)(u_\eta(x) - u(x))])\, dx \\ &= f(\nabla u(x_0))\, \text{meas}(\Omega \backslash B_{k-1}) + \int_{B_{k-1} \backslash B_k} f(\nabla u(x_0) + \nabla[\varphi_k(x)(u_\eta(x) - u(x))])\, dx \\ &\quad + \int_{B_k} f(\nabla u(x_0) + \nabla u_\eta(x) - \nabla u(x))\, dx. \end{aligned}$$

Therefore, by (6.4) we deduce that

$$\begin{aligned} f(\nabla u(x_0)) &\leq C\, \frac{\text{meas}(B_{k-1}) - \text{meas}(B_k)}{\text{meas}(B_{k-1})} + \frac{C}{\text{meas}(B_{k-1})} \|\nabla \varphi_k\|_\infty \int_{B_{k-1} \backslash B_k} \|u_\eta(x) - u(x)\|\, dx \\ &\quad + \frac{C}{\text{meas}(B_{k-1})} \int_{B_{k-1} \backslash B_k} \|\nabla u_\eta(x)\| + \frac{C}{\text{meas}(B_{k-1})} \int_{B_{k-1} \backslash B_k} \|\nabla u(x)\|\, dx \\ &\quad + \frac{C}{\text{meas}(B_{k-1})} \int_{B_k} \|\nabla u(x) - \nabla u(x_0)\|\, dx \end{aligned}$$

$$+\frac{1}{\mathrm{meas}(B_{k-1})}\int_{B_k} f(\nabla u_\eta(x))\,dx. \tag{6.12}$$

By Remark 2.5 (ii) and by (6.8), we have

$$\begin{aligned}\limsup_{\eta\to 0}\int_{B_{k-1}\setminus B_k}\|\nabla u_\eta(x)\|\,dx &\le \limsup_{\eta\to 0}\int_\Omega(\varphi_{k-1}(x)-\varphi_{k+1}(x))\,\|\nabla u_\eta(x)\|\,dx\\ &=\int_\Omega(\varphi_{k-1}(x)-\varphi_{k+1}(x))\,d\pi_\infty(x)\\ &\le \pi_\infty(B_{k-2})-\pi_\infty(B_{k+1}).\end{aligned} \tag{6.13}$$

Thus, as by (6.8)

$$\begin{aligned}\limsup_{\eta\to 0}\int_{B_k} f(\nabla u_\eta(x))\,dx &\le \limsup_{\eta\to 0}\int_{B_k}\varphi_k(x)\, f(\nabla u_\eta(x))\,dx\\ &=\int_\Omega \varphi_k(x)\left(\int_{S^*} f(F)\,d\lambda_x^\infty(F)\right)d\pi_\infty(x),\end{aligned}$$

and as $u_\eta \to u$ strongly in $L^1(\Omega;\mathbb{R}^n)$, (6.12) yields

$$\begin{aligned}f(\nabla u(x_0)) \le\; & C\,\frac{\mathrm{meas}(B_{k-1})-\mathrm{meas}(B_k)}{\mathrm{meas}(B_{k-1})}+C\,\frac{\pi_\infty(B_{k-2})-\pi_\infty(B_{k+1})}{\mathrm{meas}(B_{k-1})}\\ &+\frac{C}{\mathrm{meas}B_{k-1}}\int_{B_{k-1}\setminus B_k}\|\nabla u(x)\|\,dx+\frac{C}{\mathrm{meas}B_{k-1}}\int_{B_k}\|\nabla u(x)-\nabla u(x_0)\|\,dx\\ &+\frac{1}{\mathrm{meas}(B_{k-1})}\int_\Omega\varphi_k(x)\left(\int_{S^*} f(F)\,d\lambda_x^\infty(F)\right)d\pi_\infty(x).\end{aligned} \tag{6.14}$$

As

$$\frac{\mathrm{meas}(B_k)}{\mathrm{meas}(B_{k-1})}\to 1 \quad \text{as } k\to+\infty,$$

by (6.9) and (6.10) we have

$$\frac{\pi_\infty(B_{k-2})}{\mathrm{meas}(B_{k-1})}=\frac{\pi_\infty(B_{k-2})}{\|Du\|(B_{k-2})}\,\frac{\|Du\|(B_{k-2})}{\mathrm{meas}(B_{k-2})}\,\frac{\mathrm{meas}(B_{k-2})}{\mathrm{meas}(B_{k-1})}\to\xi(x_0)\,\|\nabla u(x_0)\|,$$

which, together with (6.11) implies that

$$\begin{aligned}&\limsup_{k\to+\infty}\frac{1}{\mathrm{meas}(B_{k-1})}\int_\Omega\varphi_k(x)\left(\int_{S^*} f(F)\,d\lambda_x^\infty(F)\right)d\pi_\infty(x)\le\\ &\le\limsup_{k\to+\infty}\frac{1}{\mathrm{meas}(B_{k-1})}\int_{B_{k-1}}\left(\int_{S^*} f(F)\,d\lambda_x^\infty(F)\right)d\pi_\infty(x)\\ &=\limsup_{k\to+\infty}\frac{\pi_\infty(B_{k-1})}{\mathrm{meas}(B_{k-1})}\,\frac{1}{\pi_\infty(B_{k-1})}\int_{B_{k-1}}\left(\int_{S^*} f(F)\,d\lambda_x^\infty(F)\right)d\pi_\infty(x)\\ &=\xi(x_0)\,\|\nabla u(x_0)\|\int_{S^*} f(F)\,d\lambda_{x_0}^\infty(F).\end{aligned}$$

Therefore, (6.14) reduces to

$$f(\nabla u(x_0))\le\xi(x_0)\,\|\nabla u(x_0)\|\int_{S^*} f(F)\,d\lambda_{x_0}^\infty(F)$$

for all $x_0 \notin E$ and so, as $\|Du\|(E)=0$ and $f(x_0,.)$ is homogeneous of degree one,

$$\int_{\Omega} f(\nabla u(x))\,dx = \int_{\Omega\setminus E} f(\nabla u(x))\,dx \le \int_{\Omega}\left(\int_{S^*} f(F)\,d\lambda_x^{\infty}(F)\right)\xi(x)\,\|\nabla u(x)\|\,dx.$$

Finally, by the Lebesgue Decomposition Theorem (see EVANS & GARIEPY [15]),

$$d\pi_\infty = \xi(x)\,d\,\|Du\| + \beta(x)\,d\mu = \xi(x)\,\|\nabla u(x)\|\,dx + \beta(x)\,d\mu$$

with $\beta \ge 0$ and where $\|Du\|$ and μ are nonnegative mutually singular Radon measures, thus by (6.8) we conclude that

$$\begin{aligned}\int_{\Omega} f(\nabla u(x))\,dx &\le \int_{\Omega}\left(\int_{S^*} f(F)\,d\lambda_x^{\infty}(F)\right)d\pi_\infty(x)\\ &= \lim_{\eta\to 0}\int_{\Omega} f(\nabla u_\eta(x))\,dx.\end{aligned}$$

(b) Fix $\delta > 0$ and let $\{Q_i \mid i = 1, ..., q\}$ be a disjoint collection of subcubes of Ω such that

$$\text{meas}(\Omega\setminus\cup Q_i)<\delta,\ g(\|x-y\|)<\delta \text{ if } x, y \in Q_i,\ \pi_\infty(\Omega\setminus\cup Q_i)<\delta \text{ and } \int_{\Omega\setminus\cup Q_i}\|\nabla u(x)\|dx<\delta. \tag{6.15}$$

In each cube Q_i select a point x_i. Then, by (6.3) and as f is nonnegative we have

$$\begin{aligned}&\liminf_{\varepsilon\to 0}\left[\int_{\Omega} f(x, \nabla u_\varepsilon(x))\,dx - \int_{\Omega} f(x, \nabla u(x))\,dx\right] =\\ &= \liminf_{\varepsilon\to 0}\Bigg[\int_{\Omega\setminus\cup Q_i} f(x,\nabla u_\varepsilon(x))\,dx - \int_{\Omega\setminus\cup Q_i} f(x,\nabla u(x))\,dx +\\ &\qquad + \sum_i \int_{Q_i} [\,f(x,\nabla u_\varepsilon(x)) - f(x,\nabla u(x))]\,dx\Bigg]\\ &\ge \liminf_{\varepsilon\to 0}\Bigg[-\int_{\Omega\setminus\cup Q_i} C(1+\|\nabla u(x)\|)\,dx + \sum_i\int_{Q_i}[\,f(x, \nabla u_\varepsilon(x)) - f(x_i, \nabla u_\varepsilon(x)]\,dx +\\ &\quad + \sum_i\int_{Q_i}[\,f(x_i, \nabla u_\varepsilon(x)) - f(x_i, \nabla u(x))]\,dx + \sum_i\int_{Q_i}[\,f(x_i, \nabla u(x)) - f(x, \nabla u(x))]\,dx\Bigg].\end{aligned}$$

By (a)

$$\liminf_{\varepsilon\to 0}\int_{Q_i}[\,f(x_i, \nabla u_\varepsilon(x)) - f(x_i, \nabla u(x))]\,dx \ge 0$$

and by (6.15)

$$\begin{aligned}\sum_i\int_{Q_i} |\,f(x,\nabla u(x)) - f(x_i,\nabla u(x))|\,dx &\le \sum_i\int_{Q_i} g(\|x-x_i\|)\,(1+\|\nabla u(x)\|)\,dx\\ &\le \delta\,(\text{meas}(\Omega) + \|\nabla u\|_{L^1}),\end{aligned}$$

$$\begin{aligned}\limsup_{\varepsilon\to 0}\sum_i\int_{Q_i} |\,f(x, \nabla u_\varepsilon(x)) - f(x_i, \nabla u_\varepsilon(x))|\,dx &\le \limsup_{\varepsilon\to 0}\sum_i\int_{Q_i} g(\|x-x_i\|)\,(1+\|\nabla u_\varepsilon(x)\|)\,dx\\ &\le \delta\,(\text{meas}(\Omega) + \pi_\infty(\Omega)).\end{aligned}$$

We conclude that

$$\liminf_{\varepsilon \to 0} \left[\int_\Omega f(x, \nabla u_\varepsilon(x))\, dx - \int_\Omega f(x, \nabla u(x))\, dx \right] \geq -C\delta$$

and so, as δ is arbitrarily small, we obtain (6.7).

Remark 6.15.

Under the hypotheses of Theorem 6.6, (6.5) holds when f satisfies the growth condition (6.3) but is not necessarily homogeneous of degree one in F. The proof is a replica of that for the case where $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,1}$ (see DACOROGNA [10]), using (6.13) instead of De La Vallée-Poussin Theorem to estimate

$$\int \|\nabla u_\varepsilon(x)\|\, dx.$$

However, the proof of Theorem 6.6 presented above seems to be more adapted to deal with the case where ∇u has a singular part with respect to the Lebesgue measure.

7. OSCILLATIONS OF TWINNED CONFIGURATIONS OF ELASTIC CRYSTALS.

In this section we are going to use indicator measures and Young probability measures to show that deformations supoported on two potential wells with non-oscillating L^1 norms cannot oscillate.

We start by describing briefly the notion of twinned configuration of an elastic crystal. The foundations of this theory are due to ERICKSEN (see ERICKSEN [12], [13]). Assuming isothermal conditions, in what follows $\Omega \subset \mathbb{R}^N$ represents the reference configuration, $W: M^{N\times N} \to [0, +\infty]$ is the stored energy density and $u : \Omega \to \mathbb{R}^N$ is the deformation, where $N>1$. In order to prevent changes in orientation and interpenetration of matter, we prescribe that

$W(F) < +\infty$ if and only if $\det(F) > 0$ and $W(F_n) \to +\infty$ if $\det(F_n) \to 0^+$.

Moreover, due to frame indifference,

$$W(F) = W(RF) \tag{7.1}$$

for all $F \in M^{N\times N}_+ := \{F \in M^{N\times N}_+ \mid \det(F) > 0\}$ and for all rotation $R \in \mathcal{O}^+(N) := \{F \in M^{N\times N} \mid F^T F = \mathbb{1}$ and $\det(F) = 1\}$. Moreover, and according to ERICKSEN [12], if the solid has crystalline structure then W should be independent of the choice of lattice basis. This implies that W is invariant by the action of an infinite discrete group $\mathcal{G}^*$ conjugate to $\mathcal{G} := \{F \in M^{N\times N} \mid F_{ij} \in \mathbb{Z}$ for all $i, j = 1, \ldots, N$ and $\det(F) = 1\}$, precisely

$$W(F) = W(FH) \tag{7.2}$$

for all $F \in M^{N\times N}_+$ and for all $H \in \mathcal{G}^*$, where $\mathcal{G}^* = L\mathcal{G}L^{-1}$ for some $L \in M^{N\times N}$ (for details, see FONSECA [17], JAMES & KINDERLEHRER [24], KINDERLEHRER [25]). Due to (7.1) and (7.2) W is periodic in many directions and the total energy

$$E(u) := \int_\Omega W(\nabla u(x))\, dx$$

is not sequentially weakly * lower semicontinuous. Thus, oscillations may develop and the study of oscillating twinned configurations is particularly relevant to the understanding of stable and metastable configurations of ordered materials (see BALL & JAMES [5], CHIPOT & KINDERLEHRER [8], FONSECA [17], [18], JAMES & KINDERLEHRER [24], KINDERLEHRER & PEDREGAL [26], KINDERLEHRER & VERGARA-CAFFARELLI [27]). Precisely, a configuration is said to be *twinned* if it corresponds to a deformation u such that

$$\nabla u \in \{A, B\} \quad \text{for a. e. } x \in \Omega,$$

where A and B are *symmetry related*, i. e.

$$B = RAH \tag{7.3}$$

for some $R \in \mathcal{O}^+(N)$ and some $H \in \mathcal{G}^*$. Then

$$\det(A) = \det(B)$$

and by (7.1) and (7.2), if W has a minimum at A then $\mathcal{O}^+(N)A$ and $\mathcal{O}^+(N)B$ are two orbits of mimima for W.

Here, we will study the oscillations of a sequence of twinned configurations : if $\nabla u_\varepsilon \in \mathcal{O}^+(N)A \cup \mathcal{O}^+(N)B$ for a. e. $x \in \Omega$ and if $u_\varepsilon \overset{*}{\rightharpoonup} u$ weakly * in $W^{1,\infty}$, what can we say about the structure of u ?

In what follows, we assume that A and B lie on two distinct wells, precisely

(H) $\det(A) > 0$, $\det(B) > 0$ and $\mathcal{O}^+(N)A \cap \mathcal{O}^+(N)B = \emptyset$.

Theorem 7.4.

Let $u_\varepsilon \in W^{1,\infty}(\Omega\,;\,\mathbb{R}^N)$ be such that $\nabla u_\varepsilon(x) \in \{RA, RB \mid R \in \mathcal{O}^+(N)\}$ for a. e. $x \in \Omega$ and let $u_\varepsilon \overset{*}{\rightharpoonup} u$ weakly * in $W^{1,\infty}$. If $\det(A) = \det(B)$ and if

$$\lim_{\varepsilon \to 0} \int_\Omega \|\nabla u_\varepsilon(x)\|\, dx = \int_\Omega \|\nabla u(x)\|\, dx,$$

then $u_\varepsilon \to u$ strongly in $W^{1,p}$ for all $1 \le p < +\infty$, and $\nabla u(x) \in \{RA, RB \mid R \in \mathcal{O}^+(N)\}$ for a.e. $x \in \Omega$.

We will prove this result using indicator measures (see Section 3) and some properties of Young measures summarized in the following theorem (for details, see EVANS [14], TARTAR [31]).

Theorem 7.5.

Let $\{h_\varepsilon\}$ be a sequence bounded in $L^\infty(\Omega; C)$, where C is a closed subset of $\mathbb{R}^m$, and let $h_\varepsilon \overset{*}{\rightharpoonup} h$ weakly * in L^∞. Then there exists a subsequence $\{h_\eta\}$ and for a. e. $x \in \Omega$ a Borel probability measure μ_x on $\mathbb{R}^m$ such that $\text{spt}\, \mu_x \subset C$,

$$h(x) = \int_{\mathbb{R}^m} y \, d\mu_x(y) \qquad (\text{a. e. } x \in \Omega)$$

and for every $G \in C(\mathbb{R}^m)$ we have

$$G(h_\eta) \overset{*}{\rightharpoonup} G^* \text{ weakly * in } L^\infty$$

where

$$G^*(x) := \int_{\mathbb{R}^m} G(y) \, d\mu_x(y) \qquad (\text{a. e. } x \in \Omega).$$

Moreover, μ_x is a Dirac mass for a. e. $x \in \Omega$ if and only if

$$\mu_x = \delta_{h(x)} \text{ and } h_\eta \to h \text{ strongly in } L^p(\Omega ; \mathbb{R}^m) \text{ for all } 1 \leq p < +\infty.$$

We will use also the weak continuity property of the minors of $\{\nabla u_\varepsilon\}$ (see BALL [4], MÜLLER [29]). In what follows, adj(F) is the matrix of cofactors of F, i. e. $F^{-1} = (\det F)^{-1} \text{adj}(F)^T$.

Theorem 7.6.

Assume that $u_\varepsilon \in W^{1,\infty}(\Omega ; \mathbb{R}^N)$ is such that $u_\varepsilon \overset{*}{\rightharpoonup} u$ weakly * in $W^{1,\infty}$. Then $\det(\nabla u_\varepsilon) \rightharpoonup \det(\nabla u)$ and $\text{adj}(\nabla u_\varepsilon) \rightharpoonup \text{adj}(\nabla u)$ weakly in L^p, for all $1 \leq p \leq +\infty$.

We divide the proof of Theorem 7.4 into a series of lemmas and propositions, the first of which is well known.

Lemma 7.7.

Let $\{h_\varepsilon\}$ be a sequence of characteristic functions, i. e. $h_\varepsilon \in \{0, 1\}$ for a. e. $x \in \Omega$, such that $h_\varepsilon \overset{*}{\rightharpoonup} h$ weakly * in L^∞, with $h \in \{0, 1\}$ for a. e. $x \in \Omega$. Then $h_\varepsilon \to h$ strongly in L^p, for all $1 \leq p < +\infty$.

Proof. Consider a subsequence $\{h_{\varepsilon'}\}$ and let $\{\mu_x\}$be the Young probability measures corresponding to a subsequence $\{h_\eta\}$ of $\{h_{\varepsilon'}\}$. By Theorem 7.5 we have that $\text{spt}\, \mu_x \subset \{0, 1\}$, and so

$$\mu_x = \theta(x)\, \delta_{y=0} + (1 - \theta(x))\, \delta_{y=1}$$

for some $\theta(x) \in [0, 1]$. Thus, setting $a(x) := 1 - \theta(x)$, we obtain

$$\mu_x = \delta_{a(x)}$$

which, by Theorem 7.5, implies that $a(x) = h(x)$ a. e. $x \in \Omega$ and $h_\eta \to h$ strongly in L^p, for all $1 \le p < +\infty$.

Lemma 7.8.

Let $u_\varepsilon \in L^\infty(\Omega ; \mathbb{R}^n)$ be such that $u_\varepsilon \overset{*}{\rightharpoonup} u$ weakly * in L^∞, $\|u_\varepsilon\| \to g$ strongly in L^1 and $\int_\Omega \|u(x)\| \, dx = \int_\Omega g(x) \, dx$. Then $u_\varepsilon \to u$ strongly in L^p, for all $1 \le p < +\infty$.

Proof. Consider a subsequence $\{u_{\varepsilon'}\}$ and let $\{\mu_x\}$ be the Young probability measures corresponding to a subsequence $\{u_\eta\}$ of $\{u_{\varepsilon'}\}$. As $\|u_\varepsilon\| \overset{*}{\rightharpoonup} k^*$ weakly * in L^∞, with

$$k^*(x) = \int_{\mathbb{R}^n} \|y\| \, d\mu_x(y) \quad (\text{a. e. } x \in \Omega),$$

and since $\|u_\eta\| \to g$ strongly in L^1, we conclude that

$$g(x) = k^*(x) = \int_{\mathbb{R}^n} \|y\| \, d\mu_x(y) \quad (\text{a. e. } x \in \Omega).$$

On the other hand, for a. e. $x \in \Omega$

$$u(x) = \int_{\mathbb{R}^n} y \, d\mu_x(y), \tag{7.9}$$

and so, since μ_x is a probability measure,

$$\|u(x)\| \le g(x) \quad (\text{a. e. } x \in \Omega).$$

Therefore, as by hypothesis

$$\int_\Omega \|u(x)\| \, dx = \int_\Omega g(x) \, dx,$$

we deduce that $\|u(x)\| = g(x)$, and so, by the Dominated Convergence Theorem and by Theorem 7.5, given $G \in C(\mathbb{R})$ we have

$$G(\|u_\eta\|) \to G(\|u\|) \text{ strongly in } L^1$$

and

$$G(\|u_\eta\|) \overset{*}{\rightharpoonup} G^* \text{ weakly * in } L^\infty$$

where

$$G^*(x) := \int_{\mathbb{R}^n} G(\|y\|) \, d\mu_x(y) \quad (\text{a. e. } x \in \Omega).$$

Thus,

$$G(\|u(x)\|) = G^*(x) := \int_{\mathbb{R}^n} G(\|y\|) \, d\mu_x(y) \quad (\text{a. e. } x \in \Omega)$$

and so, given the arbitrariness of G, we deduce that

$$\operatorname{spt} \mu_x \subset \{y \in \mathbb{R}^n \mid \|y\| = \|u(x)\|\}$$

which, together with (7.9) and since μ_x is a probability measure, implies that

$$\mu_x = \delta_{y = u(x)}.$$

The conclusion of the proposition follows from Theorem 7.5.

Proposition 7.10.

Let $u_\varepsilon \in W^{1,\infty}(\Omega ; \mathbb{R}^N)$ be such that $\nabla u_\varepsilon(x) = R_\varepsilon(x)\ \chi_\varepsilon(x)A + R_\varepsilon(x)(1 - \chi_\varepsilon(x))B$, where $R_\varepsilon(x) \in \mathcal{O}^+(N)$ for a.e. $x \in \Omega$ and $\chi_\varepsilon(.)$ is a characteristic function. Let $u_\varepsilon \stackrel{*}{\rightharpoonup} u$ weakly * in $W^{1,\infty}$ and assume that

$$\lim_{\varepsilon \to 0} \int_\Omega \|\nabla u_\varepsilon(x)\|\, dx = \int_\Omega \|\nabla u(x)\|\, dx.$$

(i) If $\chi_\varepsilon \to \chi$ strongly in L^1, then $u_\varepsilon \to u$ strongly in $W^{1,p}$ for all $1 \le p < +\infty$, and

$\nabla u(x) = R(x)\ \chi(x)A + R(x)(1 - \chi(x))B$, where $R(x) \in \mathcal{O}^+(N)$ for a.e. $x \in \Omega$.

(ii) If $u_\varepsilon \to u$ strongly in $W^{1,p}$ for some $1 \le p < +\infty$, then $\chi_\varepsilon \to \chi$ strongly in L^1 and

$\nabla u(x) = R(x)\ \chi(x)A + R(x)(1 - \chi(x))B$, where $R(x) \in \mathcal{O}^+(N)$ for a.e. $x \in \Omega$.

Proof. (i) Without loss of generality, we can assume that $R_\varepsilon \stackrel{*}{\rightharpoonup} R_\infty$ weakly * in L^∞. Clearly,

$$\det(\nabla u_\varepsilon(x)) = \chi_\varepsilon(x)\det(A) + (1 - \chi_\varepsilon(x))\det(B),$$

and so, by Theorem 7.6

$$\det(\nabla u(x)) = \chi(x)\det(A) + (1 - \chi(x))\det(B) \quad \text{for a. e. } x \in \Omega. \tag{7.11}$$

On the other hand, since $\chi_\varepsilon \to \chi$ strongly in L^1, for a. e. $x \in \Omega$

$$\chi(x) \in \{0, 1\},\ \nabla u(x) = R_\infty(x)\ \chi(x)A + R_\infty(x)(1 - \chi(x))B \tag{7.12}$$

and

$$\det(\nabla u(x)) = \det(R_\infty(x))\ [\chi(x)\det(A) + (1 - \chi(x))\det(B)]$$

which, together with (7.13) implies that

$$\det(R_\infty(x)) = 1 \quad \text{a. e. in } \Omega. \tag{7.13}$$

Also, $\mathrm{adj}(\nabla u_\varepsilon) = R_\varepsilon(x)\ \chi_\varepsilon(x)\ \mathrm{adj}(A) + R_\varepsilon(x)(1 - \chi_\varepsilon(x))\ \mathrm{adj}(B)$ converges weakly * to

$$R_\infty(x)\ \chi(x)\ \mathrm{adj}(A) + R_\infty(x)(1 - \chi(x)),$$

thus, by Theorem 7.6, (7.12) and (7.13)

$$R_\infty(x)\ [\chi(x)\ \mathrm{adj}(A) + (1 - \chi(x))\mathrm{adj}(B)] = R_\infty^{-T}(x)\ \det(R_\infty(x))[\chi(x)\ \mathrm{adj}(A) + (1-\chi(x))\mathrm{adj}(B)]. \tag{7.14}$$

As $\det(A), \det(B) > 0$, we have $\det[\chi(x)\ \mathrm{adj}(A) + (1 - \chi(x))\mathrm{adj}(B)] > 0$ and so (7.12), (7.13) and (7.14) imply that $R_\infty(x) \in \mathcal{O}^+(N)$ and

$\nabla u(x) = R(x)\ \chi(x)A + R(x)(1 - \chi(x))B$, where $R(x) = R_\infty(x) \in \mathcal{O}^+(N)$ for a.e. $x \in \Omega$.

Therefore

$$\|\nabla u_\varepsilon\| = \chi_\varepsilon \|A\| + (1 - \chi_\varepsilon)\|B\| \to \chi \|A\| + (1 - \chi)\|B\| = \|\nabla u\| \qquad \text{strongly in } L^1$$

and by Lemma 7.8 we conclude that $u_\varepsilon \to u$ strongly in $W^{1,p}$ for all $1 \le p < +\infty$.

(ii) Assume that $u_\varepsilon \to u$ strongly in $W^{1,p}$ for some $1 \le p < +\infty$. As $\{\nabla u_\varepsilon\}$ is bounded in L^∞ we have $(\nabla u_\varepsilon)^T(\nabla u_\varepsilon) = \chi_\varepsilon A^TA + (1 - \chi_\varepsilon) B^TB \to (\nabla u)^T(\nabla u)$ strongly in L^1 and, i. e.

$\chi_\varepsilon (A^TA - B^TB)$ converges strongly in L^1.

Due to the hypothesis (H), $A^TA - B^TB \neq 0$ and so

$$\chi_\varepsilon \to \chi \text{ strongly in } L^1.$$

By (i) we conclude that

$$\nabla u(x) = R(x)\ \chi(x)A + R(x)(1 - \chi(x))B, \text{ where } R(x) \in \mathcal{O}^+(N) \text{ for a.e. } x \in \Omega.$$

Corollary 7.15.

Let $u_\varepsilon \in W^{1,\infty}(\Omega\ ; \mathbb{R}^N)$ be such that $\nabla u_\varepsilon(x) \in \{RA, RB \mid R \in \mathcal{O}^+(N)\}$ for a. e. $x \in \Omega$ and let $u_\varepsilon \overset{*}{\rightharpoonup} u$ weakly * in $W^{1,\infty}$. If

$$\|A\| = \|B\| = \frac{1}{\text{meas}(\Omega)} \int_\Omega \|\nabla u(x)\|\, dx$$

then $u_\varepsilon \to u$ strongly in $W^{1,p}$ for all $1 \le p < +\infty$ and $\nabla u(x) \in \{RA, RB \mid R \in \mathcal{O}^+(N)\}$ for a.e. $x \in \Omega$.

Proof. As $\|A\| = \|B\|$, it is clear that $\|\nabla u_\varepsilon(x)\| = \|A\| =: g(x)$ for a. e. $x \in \Omega$ and, by hypothesis

$$\int_\Omega \|\nabla u(x)\|\, dx = \int_\Omega g(x)\, dx.$$

Thus, by Lemma 7.8 we have that $u_\varepsilon \to u$ strongly in $W^{1,p}$ for all $1 \le p < +\infty$, and so, by Proposition 7.10 (i) we deduce that $\nabla u(x) \in \{RA, RB \mid R \in \mathcal{O}^+(N)\}$ for a.e. $x \in \Omega$.

Proof of Theorem 7.4. As in Proposition 7.10, let

$$\nabla u_\varepsilon(x) = R_\varepsilon(x)\ \chi_\varepsilon(x)A + R_\varepsilon(x)(1 - \chi_\varepsilon(x))B,$$

where $R_\varepsilon(x) \in \mathcal{O}^+(N)$ for a.e. $x \in \Omega$ and $\chi_\varepsilon(.)$ is a characteristic function. By Proposition 7.10 (i), it suffices to show that

$$\chi_\varepsilon \to \chi \text{ strongly in } L^1. \tag{7.16}$$

As

$$\lim_{\varepsilon \to 0} \int_\Omega \|\nabla u_\varepsilon(x)\|\, dx = \int_\Omega \|\nabla u(x)\|\, dx, \tag{7.17}$$

by the Sobolev Embedding Theorem and by Corollary 5.3 we have

$$\lim_{\varepsilon \to 0} \int_\Omega f\left(x, \frac{\nabla u_\varepsilon(x)}{\|\nabla u_\varepsilon(x)\|}\right) \|\nabla u_\varepsilon(x)\|\, d(x) = \int_\Omega f\left(x, \frac{\nabla u(x)}{\|\nabla u(x)\|}\right) \|\nabla u(x)\|\, d(x) \tag{7.18}$$

for all $f \in C_0(\Omega x M^{N x N})$. On the other hand, as $\det(A) = \det(B)$ by Theorem 7.6 we obtain

$$\det(\nabla u_\varepsilon) = \det(B) \rightharpoonup \det(\nabla u), \text{ i. e. } \det(\nabla u(x)) = \det(B) \text{ for a. e. } x \in \Omega. \tag{7.19}$$

Since for all Borel subsets E of Ω

$$\int_E \|\nabla u(x)\|\, dx \le \liminf_{\varepsilon \to 0} \int_E \|\nabla u_\varepsilon(x)\|\, dx = \int_E [\chi(x)\, \|A\| + (1 - \chi(x))\, \|B\|]\, dx,$$

(7.17) yields

$$\|\nabla u(x)\| = \chi(x)\,\|A\| + (1 - \chi(x))\,\|B\| \quad \text{for a. e. } x \in \Omega. \tag{7.20}$$

Finally, setting in (7.18) $f(x, F) = \varphi(x) \det(F)$ with $\varphi \in C_0(\Omega)$, by (7.19) and (7.20) we have

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi(x) \det\left(\frac{\nabla u_\varepsilon(x)}{\|\nabla u_\varepsilon(x)\|}\right) \|\nabla u_\varepsilon(x)\|\, d(x) = \int_\Omega \varphi(x) \det\left(\frac{\nabla u(x)}{\|\nabla u(x)\|}\right) \|\nabla u(x)\|\, d(x),$$

i. e.

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi(x) \frac{\det(B)}{\|\nabla u_\varepsilon(x)\|^{N-1}}\, d(x) = \int_\Omega \varphi(x) \frac{\det(B)}{\|\nabla u(x)\|^{N-1}}\, dx$$

and so

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi(x) \frac{1}{\chi_\varepsilon(x)\|A\|^{N-1} + (1-\chi_\varepsilon(x))\|B\|^{N-1}}\, d(x) =$$

$$= \lim_{\varepsilon \to 0} \int_\Omega \varphi(x) \,[\chi_\varepsilon(x)\, \|A\|^{1-N} + (1 - \chi_\varepsilon(x))\, \|B\|^{1-N}]\, dx$$

$$= \int_\Omega \varphi(x) \frac{1}{[\chi(x)\, \|A\| + (1 - \chi(x))\, \|B\|]^{N-1}}\, dx.$$

Therefore, for a. e. $x \in \Omega$

$$\frac{1}{[\chi(x)\, \|A\| + (1 - \chi(x))\, \|B\|]^{N-1}} = \frac{\chi(x)}{\|A\|^{N-1}} + \frac{1 - \chi(x)}{\|B\|^{N-1}}.$$

Setting

$$\xi := \frac{\|B\|}{\|A\|},$$

we obtain

$$1 = \frac{\chi(x)}{\|A\|^{N-1}} [\chi(x)\, \|A\| + (1 - \chi(x))\, \|B\|]^{N-1} + \frac{1 - \chi(x)}{\|B\|^{N-1}} [\chi(x)\, \|A\| + (1 - \chi(x))\, \|B\|]^{N-1}$$

$$= \chi(x)\, [\chi(x) + (1 - \chi(x))\xi]^{N-1} + (1 - \chi(x)) \left[\frac{\chi(x)}{\xi} + (1 - \chi(x))\right]^{N-1}. \tag{7.21}$$

If $\xi = 1$ then by Corollary 7.15 we conclude that $u_\varepsilon \to u$ strongly in $W^{1,p}$ for all $1 \le p < +\infty$, and the result follows from Proposition 7.10 (ii).

Now suppose that $\xi \neq 1$. Then (7.21) is equivalent to saying that

$$h(\xi) = 1 \tag{7.22}$$

where

$$\theta = \chi(x) \text{ and } h(t) := [\theta + (1 - \theta)t]^{N-1} \left[\theta + \frac{1 - \theta}{t^{N-1}}\right].$$

Clearly

$$\min h = h(1) = 1 \text{ and } h'(t) = (N - 1)(1 - \theta)\,[\theta + (1 - \theta)t]^{N-2}\, t^{-N}\, (\theta t^N - \theta).$$

On the other hand, if $0 < \theta < 1$ then $h'(t) = 0$ if and only if $\theta t^N - \theta = 0$, i. e. $t = 1$. As $\xi \neq 1$ we deduce that $h(\xi) > 1$, which contradicts (7.22). Therefore, $\theta \in \{0, 1\}$ and by Lemma 7.7 we have (7.16), which concludes the proof.

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