# Multiplicity Results For An Inhomogeneous Neumann Problem With Critical Exponent

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# MULTIPLICITY RESULTS FOR AN INHOMOGENEOUS NEUMANN PROBLEM WITH CRITICAL EXPONENT

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Introduction

In this note we shall discuss some multiplicity results for a class of inhomogeneous Neumann problem involving the critical Sobolev exponent.

We will place particular emphasis on the existence of changing sign solutions which for constant data, will yield non constant solutions. More precisely, let $X > 0$ and let ft C $IR^N$, $N \geq 3$ be a bounded domain with smooth boundary 5ft. For a given function f we seek solutions for the following problem:

$$(1)_f \begin{cases} -AU + Jlu = |u|^{2^*-2} u + f & \text{in ft} \\ Is = 0 & \text{in } dQ \end{cases}$$

with n the outward pointing normal on $dQ$ and $2^* = jq^{2N}$ ^y *^e ^est exponent in the Sobolev embedding.

Even though our discussion extends to include f 6 $H^{-1}$ (the dual of $H^1$(ft)), we prefer to simplify the techniqualities and assume f € $L^{\frac{2N}{N+2}}$(ft).

The homogeneous case, i.e. f = 0, has been treated by several authors (cf [A—M], [G-K], {W]). They have established the existence of a positive solution for $(1)_{f=0}$ for all $X > 0$. It must be noticed however that when f = 0, problem $(1)_{f=Q}$ always admits the constant positive solution $u = X^{\frac{1}{2^*-2}}$. The above mentioned results can guarantee a non—constant positive solution only when $X$ is large.

The problem of finding non constant solutions for $(1)_{f=Q}$ has been examined in [C—T]. There the oddness of the problem has allowed to obtain changing sign solutions for all $A > 0$, provided $N \geq 5$.

Here, we extend these results to include the case where f ^ 0. It should be noticed that for f $f$ 0 problem (1)j is no longer odd and the techniques used in [C—T] become unsuitable. .However, extending an approach introduced in [Ta] for the corresponding Dirichlet problem, we are able to construct "ad hoc" minimization problems which yield the desired solutions.

More precisely, following [Ta] let $c_N = \frac{4}{N-2}\left(\frac{N-2}{N+2}\right)^{\frac{N+2}{4}}$ and define:

$$\mu_f = \inf_{\|u\|_{2^*}=1} \left\{ c_N \left(\| \nabla u \|_2^2 + \lambda \| u \|_2^2\right)^{\frac{N+2}{4}} - \int_\Omega f u \right\}. \tag{1.1}$$

The role of $\mu_f$ will become clear from the discussion below. Our main result states the following:

<u>Theorem 1:</u>

Let $N \geq 5$ and $f \not\equiv 0$. If $\mu_f > 0$, then $(1)_f$ admits at least <u>three</u> (weak) solutions one of which necessarily changes sign. Furthermore, if $f \geq 0$ then the other two solutions $u_0$ and $u_1$ satisfy: $0 \leq u_0 \leq u_1$. ■

Obviously regular data $f$ will yield classical solutions for $(1)_f$. Also we have, $0 < u_0 < u_1$ in case $f \geq 0$ and $f \not\equiv 0$. Furthermore, putting together the results of [A–M] (see also [W] and [C–K]) and [C–T] we see that the given Theorem continues to hold for $f = 0$; only that, in this case, the "smallest" solution $u_0$ reduces to the trivial one, i.e., $u_0 = 0$. So, Theorem 1 can be viewed as a bifurcation type result. In fact, the condition $\mu_f > 0$ (to be compared with (*) in [Ta]), is essentially a "smallness" condition on $f$, since it certainly holds when $f$ satisfies:

$$\| f \|_{\frac{2N}{N+2}} < c_N \left(S_N(\lambda)\right)^{\frac{N+2}{4}}$$

with

$$S_N(\lambda) = \inf_{\|u\|_{2^*} = 1} \left\{ \| \nabla u \|_2^2 + \lambda \| u \|_2^2 \right\} \tag{1.2}$$

Incidentally, let us also mention that the minimization problem (1.2) attains its infinum at a positive function in $H^1(\Omega)$ (cf [A–M], [C–K] and [W]).

When $f$ = constant $> 0$ (not too large), the claimed two positive solutions could correspond to (suitable) constants. While it follows from our construction that this is not the case for $\lambda$ large, our result asserts that, in any case, problem $(1)_f$ admits nonconstant solutions for all $\lambda > 0$.

We also point out that our result holds in the subcritical case (where one replaces the power $2^*$ in $(1)_f$ with $p \in (2, 2^*)$) under both Neumann or Dirichlet boundary condition. The proof is simpler in this situation and therefore left to the reader.

Finally, let us mention that our approach can be applied to handle the case $\lambda = 0$ and $\int_\Omega f = 0$. This is done via a dual variational principle as introduced by Clarke [Cl] and discussed in [C–K] in this context.

In this situation one finds a "dual" correspondent for the value $\mu_f$ as given by:

$$\mu_f^* = \inf \{c_N \left(\int_\Omega wKw\right)^{\frac{2-N}{4}} + \int_\Omega wKw;\ w \in E, \| w \|_{\frac{2N}{N+2}} = 1 \} \tag{1.3}$$

where

$$E = \{ w \in L^{\frac{2N}{N+2}}(\Omega) : \int_\Omega w = 0\} \tag{1.4}$$

and $K : E \longrightarrow E$ is the inverse of $-\Delta$ in $E$, that is:

$$Kf = g \Leftrightarrow -\Delta g = f \text{ in } H^1(\Omega) \text{ and } \int_\Omega g = 0. \tag{1.5}$$

We have:

<u>**Theorem 2:**</u> Let $N \geq 5$. If $f \neq 0$ satisfies $\int_\Omega f = 0$ and $\mu_f^* > 0$, then the problem:

$$(2)_f \begin{cases} -\Delta u = |u|^{2^*-2} u + f & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{in } \Omega \end{cases}$$

admits at least <u>two</u> (weak) solutions. ∎

Notice that, since $\int_\Omega f = 0$, all solutions of $(2)_f$ must change sign.

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## THE EXISTENCE OF THE FIRST TWO SOLUTIONS:

This section will be devoted to prove the following:

**Theorem 2.1:** Let $N \geq 5$ and $f \not\equiv 0$ satisfy $\mu_f > 0$. Problem $(1)_f$ admits at least two solutions $u_0$ and $u_1$. Furthermore, if $f \geq 0$ then $0 \leq u_0 \leq u_1$. ■

Such a result should be compared to the analogous one obtained in [Ta] for the corresponding Dirichlet problem. In fact, the proof is essentially the same and we shall refer to [Ta] for several of the details.

To start, let us observe that (weak) solutions for $(1)_f$ are the critical points for the functional,

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \lambda |u|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*} - \int_\Omega f u; \quad u \in H^1(\Omega).$$

Denote by $(\cdot, \cdot)$ the scalar product in $H^1(\Omega)$ corresponding to the norm:

$$\| u \|^2 = \| \nabla u \|_2^2 + \lambda \| u \|_2^2, \quad u \in H^1(\Omega).$$

Easy computations show that $I$ is bounded from below in the set,

$$\Lambda = \{ u \in H^1(\Omega) : (I'(u), u) = 0 \}.$$

So, in the search for solution of $(1)_f$, a first candidate would be the minimizer for the following problem:

$$c_0 = \inf_{\Lambda} I \tag{2.1}$$

On the other hand, to insure that $c_0$ is indeed critical for $I$, we require that $\Lambda$ admits no boundary. This is guaranteed if the function:

$$\varphi(t) \equiv I(tu), \quad t \geq 0$$

admits a nonzero critical point for each direction $u \in H^1(\Omega)$, $u \neq 0$

Following [Ta], this corresponds to require that:

$$I'(t_0(u)\, u) > 0 \quad \forall\, u \neq 0$$

with

$$t_0(u) = \left[ \frac{\|\nabla u\|_2^2 + \lambda \|u\|_2^2}{(2^*-1)\ \|u\|_{2^*}^{2^*}} \right]^{\frac{1}{2^*-2}} \tag{2.2}$$

Equivalently,

$$(2^*-2) \left[ \frac{1}{2^*-1} \right]^{\frac{2^*-1}{2^*-2}} \left[ \frac{(\|\nabla u\|_2^2 + \lambda \|u\|_2^2)^{2^*-1}}{\|u\|_{2^*}^{2^*}} \right]^{\frac{1}{2^*-2}} - \int_\Omega f u > 0, \ \forall\, u \neq 0$$

that is,

$$\|u\|_{2^*} \left[ c_N \left[ \frac{\|\nabla u\|_2^2 + \lambda \|u\|_2^2}{\|u\|_{2^*}^2} \right]^{\frac{2^*-1}{2^*-2}} - \int_\Omega f \frac{u}{\|u\|_{2^*}} \right] > 0 \quad \forall\, u \neq 0$$

which is exactly the condition $\mu_f > 0$.

A straightforward consequence of this observation is the following,

<u>Lemma 2.1:</u> Assume $\mu_f > 0$.

For every $u \neq 0$ there exists unique $t^-(u) < t^+(u)$ such that,

(i) $0 \leq t^-(u) < t_0(u) < t^+(u)$ ($t_0(u)$ given in (2.2))

(ii) $t^{\pm}(u)u \in \Lambda$

(iii) $I(t^-(u)u) = \min_{t \in [0, t^+(u)]} I(tu)$; $I(t^+(u)u) = \max_{t \geq 0} I(tu)$.

Furthermore, $t^-(u) > 0$ when $\int_\Omega fu > 0$. ■

The proof of Lemma 2.1 follows exactly as in Lemma 2.1 of [Ta].

Next, we derive some other useful consequences from the condition $\mu_f > 0$.

<u>Lemma 2.2:</u> Set,

$$\Lambda_0 = \{ u \in \Lambda : \| \nabla u \|_2^2 + \lambda \| u \|_2^2 - (2^* - 1) \| u \|_{2^*}^{2^*} = 0 \}.$$

If $\mu_f > 0$ then,

$$\Lambda_0 = \{ 0 \}. \tag{2.3}$$

Furthermore, for every $u \in \Lambda - \{ 0 \}$ there exist $\epsilon > 0$ and a $C^1$ – map:

$$t : B_\epsilon \longrightarrow \mathbb{R}^+$$

such that,

(i) $t(w)(u+w) \in \Lambda, \quad \forall\, w \in B_\varepsilon = \{ w \in H^1_0(\Omega) : \|w\| < \varepsilon \};$

(ii) $t(0) = 1$ and
$$\langle t'(0), \varphi\rangle = \frac{2\int_\Omega \nabla u \nabla \varphi + 2\int_\Omega \lambda u \varphi - 2^*\int_\Omega |u|^{2^*-2} u \varphi}{\|\nabla u\|_2^2 + \lambda\|u\|_2^2 - (2^*-1)\|u\|_{2^*}^{2^*}}$$

$\forall\, \varphi \in H^1_0(\Omega)$.

**Proof:**

To obtain (2.3) let us argue by contradiction and assume that there exist $u \in \Lambda - \{0\}$:
$$\|\nabla u\|_2^2 + \lambda\|u\|_2^2 = (2^*-1)\|u\|_{2^*}^{2^*}.$$

This implies that,
$$\|u\| \geq \gamma \quad \text{for suitable } \gamma > 0 \text{ and } \int_\Omega fu = (2^*-2)\|u\|_{2^*}^{2^*}.$$

But this is impossible since,
$$0 < \gamma \Lambda_f \le \|u\|_{2^*}\mu_f \le (2^*-2)\left(\frac{1}{2^*-1}\right)^{\frac{2^*-1}{2^*-2}}\left[\frac{\left(\|\nabla u\|_2^2 + \lambda\|u\|_2^2\right)^{\frac{2^*}{2}}}{\|u\|_{2^*}^{2^*}}\right]^{\frac{1}{2^*-2}} -$$
$$-\int_\Omega fu = (2^*-2)\|u\|_{2^*}^{2^*} - \int_\Omega fu = 0.$$

At this point we obtain the second part of our claim as a straightforward application of the Implicit Function Theorem applied to the function:
$$F(t,w) = t\left(\|\nabla(u+w)\|_2^2\right) - t^{2^*-1}\|u+w\|_{2^*}^{2^*} - \int_\Omega f(u+w)$$

at the point $(1,0) \in \mathbb{R} \times H^1_0(\Omega)$. ∎

Remark 2.1:

Notice that necessarily,

$$\| \nabla t^+(u)u \|_2^2 + \lambda \| t^+(u)u \|_2^2 - (2^*-1) \| t^+(u)u \|_{2^*}^{2^*} < 0,$$

while for $\int_\Omega f u > 0$ we have:

$$\| \nabla t^-(u)u \|_2^2 + \lambda \| t^-(u)u \|_2^2 - (2^*-1) \| t^-(u)u \|_{2^*}^{2^*} > 0.$$

Assertion (2.3) can be strengthened as follows:

Lemma 2.3: Assume $\mu_f > 0$ and let $\{ u_n \} \subset \Lambda$ such that,

$$\lim_{n \to +\infty} \| \nabla u_n \|_2^2 + \lambda \| u_n \|_2^2 - (2^*-1) \| u_n \|_{2^*}^{2^*} = 0;$$

then, $\liminf_{n \to +\infty} \| u_n \| = 0.$

Proof: Argue by contradiction and assume that $\| u_n \| \geq \gamma > 0, \forall n.$

Then,

$$\int_\Omega f u_n = (2^*-2) \| u_n \|_{2^*}^{2^*} + o(1)$$

and

$$\frac{\|\nabla u_n\|_2^2 + \lambda \|u_n\|_2^2}{(2^*-1) \; \|u_n\|_{2^*}^{2^*}} = o(1).$$

But this is impossible since, as above, it yields:

$$\gamma \mu_f \leq (2^*-2) \| u_n \|_{2^*}^{2^*} - \int_\Omega f u_n + o(1) = o(1). \quad \blacksquare$$

At this point we are ready to establish the following:

<u>Proposition 2.1:</u> If f satisfies $\mu_f > 0$, then the minimization problem:

$$c_0 = \inf_{\Lambda} I \tag{2.4}$$

attains its infimum at a point $u_0$ which defines a critical point for I. Furthermore, $u_0 \geq 0$ for $f \geq 0$.

<u>Proof:</u> Let $f \neq 0$, since for $f = 0$ we have $c_0 = 0$ and $u_0 = 0$.

For $u \in \Lambda$ it follows that,

$$I(u) = \left[\frac{1}{2} - \frac{1}{2^*}\right] (\| \nabla u \|_2^2 + \lambda \| u \|_2^2) - (1 - \frac{1}{2^*}) \int_\Omega f u$$

from which we immediately derive that I is bounded below in $\Lambda$.

<u>Claim 1:</u> $\quad c_0 < 0 \qquad$ (2.5)

Indeed if $v \in H^1(\Omega)$ is the unique solution for:

$$-\Delta v + \lambda v = f \quad \text{in } H^1(\Omega)$$

then $\int_\Omega f v > 0$. Thus, from Lemma 2.1, there exist $0 < t^-(v) < t^+(v)$ such that $t^-(v) v \in \Lambda$ and,

$$c_0 \leq I(t^-(v)v) = \min_{t \in [0, t^+(v)]} I(tv) < 0.$$

Next, apply Ekeland's principle (cf [A–E]) to (2.4) to obtain a sequence $\{ u_n \} \subset \Lambda$ satisfying:

(a) $\qquad I(u_n) \leq c_0 + \frac{1}{n}$

(b) $\qquad I(u) \geq I(u_n) - \frac{1}{n} \| u_n - u \|, \; \forall u \in \Lambda.$

Notice that necessarily, $\| \nabla u_n \|_2^2 + \lambda \| u_n \|_2^2 - (2^* - 1) \| u_n \|^{2^*} > 0.$

As in [Ta], we show that condition (b) implies $\| I'(u_n) \| \longrightarrow 0$ as

$n \to +\infty$. In fact, in view of (2.5), for $n$ large, it follows that,

$$\int_\Omega f\,u_n \geq \frac{|c_0|}{2} \quad \text{and} \quad \|u_n\|^2 \leq 2\,\frac{(2^*-1)}{2^*-2}\int_\Omega f\,u_n.$$

Thus,

$$b_1 \leq \|u_n\| \leq b_2. \tag{2.6}$$

for suitable $b_1, b_2 > 0$.

Now, fix $n$ with $I'(u_n) \neq 0$. By Lemma 2.2 and the estimate (2.6), for $\delta > 0$ sufficiently small, we can find $t(\delta) > 0$ such that,

(1) $$u_\delta = t(\delta)\left[u_n - \delta\,\frac{I'(u_n)}{\|I'(u_n)\|}\right] \in \Lambda$$

(2) $$t(0) = 1 \quad \text{and} \quad |t'(0)| \leq \frac{c}{\|\nabla u_n\|_2^2 + \lambda\|u_n\|_2^2 - (2^*-1)\|u_n\|_{2^*}^{2^*}}$$

($c > 0$ suitable constant).

On the other hand, since $\|u_n\| \geq b_1$, from Lemma 2.3 also follows that,

$$\liminf_{n\to+\infty} \|\nabla u_n\|_2^2 + \lambda\|u_n\|_2^2 - (2^*-1)\|u\|_{2^*}^{2^*} > 0;$$

which yields

$$|t'(0)| \leq a_1 \quad \text{for a suitable } a_1 > 0.$$

Thus,

$$\frac{1}{n}\|u_\delta - u_n\| \geq I(u_n) - I(u_\delta) = \delta\|I'(u_n)\| + o(\|u_n - u_\delta\|)$$

and

$$\|u_n - u_\delta\| \leq |1-t(\delta)|\,\|u_n\| + \delta \leq b_2\,|1-t(\delta)| + \delta.$$

Therefore,

$$\|I'(u_n)\| \leq \frac{1}{n}(|t'(0)| + 1) \to 0 \quad \text{as } n \to +\infty.$$

So, if we call $u_0$ the weak limit of (a subsequence of) $u_n$ in $H^1(\Omega)$ we have that, $u_0$ solves $(1)_f$. Therefore $u_0 \in \Lambda$, and

$$c_0 \le I(u_0) = \frac{1}{N}\left( \| \nabla u_0 \|_2^2 + \lambda \| u_0 \|_2^2 \right) - \int_\Omega f u_0 \le$$

$$\le \lim_{n \to +\infty} I(u_n) = c_0.$$

Thus, $u_n \longrightarrow u_0$ strongly in $H^1(\Omega)$ and $u_0$ is the desired minimizer.

Notice that, $t^-(u_0) = 1$ ($t^-(u_0)$ as defined in Lemma 2.1). So for $f \ge 0$, we have,

$$t^-(|u_0|) \ge 1.$$

Therefore,

$$I(t^-(|u_0|)\,|u_0|) \le I(|u_0|) \le I(u_0).$$

which yields $u_0 \ge 0$. ■

Remark 2.2: Arguing as in [Ta], one can conclude that $u_0$ is a local minimum for $I$.

Set,

$$\Lambda^+ = \{ u \in \Lambda : \| \nabla u \|_2^2 + \lambda \| u \|_2^2 - (2^* - 1) \| u \|_{2^*}^{2^*} > 0 \} \subset \Lambda.$$

The argument above, shows that, $u_0 \in \Lambda^+$ and,

$$c_0 = \inf_\Lambda I = \inf_{\Lambda^+} I.$$

Thus, in the search of our second solution, it is natural to consider the second minimization problem:

$$c_1 = \inf_{\Lambda^-} I \tag{2.7}$$

where

$$\Lambda^- = \{ u \in \Lambda : \| \nabla u \|_2^2 + \lambda \| u \|_2^2 - (2^* - 1) \| u \|_{2^*}^{2^*} < 0 \}.$$

We start by describing some nice (topological) properties of $A^-$.

To this purpose set $S = \{u\ 6\ H^1(fi) : \| u \| = 1\}$. We have,

LCTnTna 2.4:

The subset A'" is closed in $H^1(ft)$. Furthermore, the map: ^ : S —» A~ given by,

$$*(u) = t^+(u)u \quad (t^+(u) \text{ as defined in Lemma 2.1})$$

defines an homomorphism.

Proof: Note that if u G A" then $\| u \| \geq b > 0$ for a suitable $b > 0$. Thus, in view of Lemma 2.3, every sequence $\{u_n\}$ in A'" satisfies

$$\bullet\lim_{n\to+\infty} \inf \ ||vu_n||_2^2 + A\| \wedge ||| - (2^* - 1) \ \| u_n \|_2^{2^*} < 0$$

which readily gives A~ closed.

The continuity of $t^+(u)$ follows immediately from its uniqueness and extremal property.

Thus, $ is continuous with continuous inverse given by:

$$\Psi^{-1}(u) = \frac{u}{\|u\|}. \qquad \blacksquare$$

We have:

Proposition 2.2: Let $N \geq 5$, then the minimization problem (2.7) attains its infinum at a critical point $u_1$ € A'" of I. In addition, $u«_1 \geq 0$ for $f \geq 0$.

Proof: First of all notice that any minimizing sequence $\{ u_n \}$ c A'" for (2.7) satisfies:

$$0 < b_x \leq \| u_n \| < b_2$$

for suitable $b_1$ and bg.

Therefore, exactly as in the proof of Proposition 2.1, via Ekeland's principle (which applies in view of Lemma 2.4) we derive a minimizing sequence $\{ u_n \}$ C A~ satisfying:

$$\| I'K) \| - 0.$$

Since I involves a nonlinearity with critical growth, to be able to carry out the final convergence argument we need some information on the value $c_1$.

Claim:

$$c_1 < c_0 + \frac{1}{N}\,\frac{S^{N/2}}{2} \tag{2.8}$$

where S is the best constant in the Sobolev inequality (cf [T]).

To establish (2.8) we follow [Ta] and note that, in view of Lemma 2.4, $\Lambda^-$ disconnects $H^1(\Omega)$ in exactly two components:

$$U^- = \{ u = 0 \text{ or } u \neq 0 : \| u \| < t^+ (\tfrac{u}{\|u\|}) \},$$
$$U^+ = \{ u : \| u \| > t^+ (\tfrac{u}{\|u\|}) \},$$

and $\Lambda^+ \subset U^-$.

As usual for this type of problem (cf.[B–N]), to obtain (2.8) we use a suitable cut off function $u_\epsilon$ of an extremum for the Sobolev inequality as given by the function:

$$U_{\epsilon,y} = \frac{(N(N-2)\epsilon)^{\frac{N-2}{4}}}{(\epsilon + |x-y|^2)^{\frac{N-2}{2}}}$$

with $\epsilon > 0$ fixed sufficiently small and $y \in \partial\Omega$ chosen so that, in a small neighborhood of y, the domain $\Omega$ lies on one side of the tangent plane of $\partial\Omega$ at y and the mean curvature of $\partial\Omega$ at y (with respect to the outward normal) is positive. The existence of such a y is guaranteed by the smoothness of $\partial\Omega$.

As well known, $\| \nabla u_\epsilon \|_2^2 = \frac{S^{N/2}}{2} + o(1)$ and $\| u_\epsilon \|_2^2 = o(1)$ as $\epsilon \to 0$. Therefore, if we let

$$A_0 = \sup_{\|u\|=1} t^+(u) < +\infty$$

for $R \geq 2S^{-N/4} A_0 + 1$ and $\epsilon > 0$ sufficiently small, we have,

$$\| \nabla(u_0 + R\, u_\epsilon) \|_2^2 + \lambda \| u_0 + R\, u_\epsilon \|_2^2 \geq \| \nabla u_0 \|_2^2 + \lambda \| u_0 \|_2^2 + \frac{S^{N/2}}{4} R^2 > A_0 \geq$$

$$\geq t^+ \left[ \frac{u_0 + R\ u_\epsilon}{\|u_0 + R\ u_\epsilon\|} \right]$$

that is, $u_0 + R\, U_\epsilon \in U^+$. Thus, we can find $t_0 \in (0,1)$ such that,

$$v_\epsilon = u_0 + t_0 R\, u_\epsilon \in \Lambda^-.$$

So, for a suitable constant $C > 0$, we find:

$$c_1 \leq I(v_\epsilon) \leq I(u_0) + I_0(t_0 R\, u_\epsilon) + C\left[ \int_\Omega |u_0|\, u_\epsilon^{2^*-1} + \int_\Omega |u_0|^{2^*-1} u_\epsilon + \int_\Omega |f| u_\epsilon \right]$$

where

$$I_0(u) = \frac{1}{2}( \|\nabla u\|_2^2 + \lambda \|u\|_2^2 ) - \frac{1}{2^*} \|u\|_{2^*}^{2^*}.$$

Direct calculations show that,

$$\int_\Omega |u_0|\, u_\epsilon^{2^*-1} + \int_\Omega |u_0|^{2^*-1} u_\epsilon + \int_\Omega |f|\, u_\epsilon = a\, \epsilon^{\frac{N-2}{4}} + o\left[ \epsilon^{\frac{N-2}{4}} \right], \quad a > 0;$$

(cf [B–N]) and $\max_{t \geq 0} I_0(t\, u_\epsilon) = \frac{1}{N} \left[ \frac{\|\nabla u_\epsilon\|_2^2 + \lambda \|u_\epsilon\|_2^2}{\|u_\epsilon\|_{2^*}^2} \right]^{N/2}$.

Therefore,

$$c_1 \leq c_0 + \frac{1}{N} \left[ \frac{\|\nabla u_\epsilon\|_2^2 + \lambda \|u_\epsilon\|_2^2}{\|u_\epsilon\|_{2^*}^2} \right]^{N/2} + 0\left[ \epsilon^{\frac{N-2}{4}} \right].$$

On the other hand, our choice of $u_\epsilon$ guarantees that, for $N \geq 4$, we have:

$$\left[\frac{\|\nabla u_\epsilon\|_2^2 + \lambda\|u_\epsilon\|_2^2}{\|u_\epsilon\|_{2^*}^2}\right]^{N/2} \leq \frac{S^{N/2}}{2} - C\,\epsilon^{1/2} + o(\epsilon^{1/2}),\ C > 0,$$

(see [A–M], [C–K], [W]). Thus, for $N \geq 5$ we conclude:

$$c_1 \leq c_0 + \frac{1}{N}\frac{S^{N/2}}{2} - C\,\epsilon^{1/2} + o(\epsilon^{1/2}) + 0\left[\epsilon^{\frac{N-2}{4}}\right] < c_0 + \frac{1}{N}\frac{S^{N/2}}{2}$$

for $\epsilon > 0$ sufficiently small.

At this point, to show that the sequence $\{u_n\}$ is precompact we use an inequality of Cherrier [Ch] which, for every $\tau > 0$, gives a constant $M_\tau > 0$ such that:

$$\left[\frac{S}{2^{2/N}} - \tau\right]\|u\|_{2^*}^2 \leq \|\nabla u\|_2^2 + M_\tau\|u\|_2^2,\ \forall u \in H^1(\Omega).$$

Since $u_n$ is uniformly bounded in $H^1(\Omega)$, after taking a subsequence (which we still call $u_n$) we find $u_1 \in H^1(\Omega)$ such that $u_n \rightharpoonup u_1$ weakly in $H^1(\Omega)$. In particular, $u_1 \in \Lambda$ and so $I(u_1) \geq c_0$.

Furthermore, if we write $u_n = u_1 + v_n$ with $v_n \rightharpoonup 0$ weakly in $H^1(\Omega)$, we derive:

$$I(u_n) = I(u_1) + \frac{1}{2}\|\nabla v_n\|_2^2 - \frac{1}{2^*}\|v_n\|_{2^*}^{2^*} + o(1) \longmapsto c_1 < c_0 + \frac{1}{N}\frac{S^{N/2}}{2}$$

which yields:

$$\lim_{n\to+\infty} \frac{1}{2}\|\nabla v_n\|_2^2 - \frac{1}{2^*}\|v_n\|_{2^*}^{2^*} < \frac{1}{N}\frac{S^{N/2}}{2}. \tag{2.9}$$

Moreover,

$$0 = (I'(u_n), u_n) = (I'(u_1), u_1) + \|\nabla v_n\|_2^2 - \|v_n\|_{2^*}^{2^*} + o(1)$$

that is,

$$\|\nabla v_n\|_2^2 - \|v_n\|_{2^*}^{2^*} = o(1) \qquad (2.10)$$

Putting together (2.9) and (2.10) we have that,

$$\lim_{n\to+\infty} \|\nabla v_n\|_2^2 := \gamma < \frac{S^{N/2}}{N} \qquad (2.11)$$

Next we show how (2.10) and (2.11) can hold simultaneously only if $\lim_{n\to+\infty} \|\nabla v_n\| = 0$, (i.e., $\gamma = 0$).

Let us argue by contradiction and assume $\gamma > 0$. Take $\tau > 0$ such that:

$$\left[\frac{S}{2^{2/N}} - \tau\right]^{N/2}$$

From (2.10) we have:

$$\|\nabla v_n\|_2^2 = \|v_n\|_{2^*}^{2^*} + o(1) \le \left[\frac{S}{2^{2/N}} - \tau\right]^{-2^*/2} \|\nabla v_n\|_2^{2^*} + o(1),$$

Since $\gamma > 0$, then $\|\nabla v_n\|_2$ is bounded below away from zero. Therefore,

$$\|\nabla v_n\|_2^2 \ge \left[\frac{S}{2^{2/N}} - \tau\right]^{N/2} + o(1)$$

which, in view of our choice of $\tau$, contradicts (2.11).

This gives $u_n \longrightarrow u_1$ strongly in $H^1(\Omega)$ and so $u$^ is the desired minimizer.

Finally, for $t \ge 0$ we have:

$$I(t\,|u_1|) \le I(t^+(|u_1|)u_1) \le \max_{t\ge 0} I(tu_1) = I(u_1)$$

which yidds $u_1 \ge 0$. ∎

Obviously $u_0 \neq u_1$. To conclude the proof of Theorem 2.1 set $u_+ = \min\{u_0, u_1\}$; we show that $f \geq 0$, $f \neq 0$ implies the existence of a solution $0 \leq u_0^* \leq u_+$. To this purpose note that when $f \geq 0$, $(f \neq 0)$ the unique solution $u_\mu$ for the problem:

$$-\Delta u + \lambda u = \mu f \text{ in } H^1(\Omega)$$

gives a positive <u>subsolution</u> for $(1)_f$ for all $\mu \in (0,1)$ and $u_\mu \longrightarrow 0$ as $\mu \longrightarrow 0$.
On the other hand, $u_+ = \min\{u_0, u_1\}$ defines a <u>supersolution</u> for $(1)_f$. So choosing $\mu > 0$ sufficiently small to guarantee $u_\mu \leq u_+$ a.e. in $\Omega$, by the method of sub–super solutions, we obtain a solution $u_0^*$ for $(1)_f$ satisfying:

$$0 < u_\mu \leq u_0^* \leq u_+.$$

This concludes the proof of Theorem 2.1.

<u>Remark 2.3:</u> Notice that, if $u_1$ is a minimizer for (2.7) and $f \neq 0$ then

$$\int_\Omega f u_1 > 0.$$

To see this, observe first that $c_1 \leq \frac{1}{N}(S_N(\lambda))^{N/2}$ with $S_N(\lambda)$ as defined in (1.2). Indeed, let $u_0$ be a minimizer for (1.2) with $\int_\Omega f\, u_0 \geq 0$ and $I_0(u_0) = \max_{t \geq 0} I(t\, u_0)$ then,

$$I_0(u_0) = \frac{1}{N}(S_N(\lambda))^{N/2} \text{ and } c_1 \leq I(t^+(u_0)u_0) \leq I_0(t^+(u_0)\, u_0) \leq I_0(u_0).$$

On the other hand, if by contradiction we assume that $\int_\Omega f\, u_1 \leq 0$, then for $t_0 > 0$ satisying $I_0(t_0\, u_1) = \max_{t \geq 0} I_0(t\, u_1)$ we have:

$$c_1 = I(u_1) \geq I(t_0\, u_1) \geq I_0(t_0\, u_1) \geq \frac{1}{N}(S_N(\lambda))^{N/2}.$$

Therefore,

$$\frac{1}{n}\| w_\delta - u_n \| \geq I(u_n) - I(w_\delta) = -(I'(u_n), w_\delta - u_n) + o(\| w_\delta - u_n \|)$$

$$= (1 - t_+(\delta)) \left[ I'(u_n), \left[ u_n - \delta \frac{I'(u_n)}{\|I'(u_n)\|} \right]^+ \right] - (1 - t_-(\delta)) \left[ I'(u_n), \left[ u_n - \delta \frac{I'(u_n)}{\|I'(u_n)\|} \right]^- \right]$$

$$+ \delta \| I'(u_n) \| + o(\| u_n - w_\delta \|).$$

On the other hand, for a suitable $C_4 > 0$ we have:

$$\| w_\delta - u_n \| \leq C_4 (| t_+(\delta) - 1 | + | t_-(\delta) - 1 | + \delta)$$

which yields,

$$\| I'(u_n) \| \leq \frac{C_4}{n} ( | t_+'(0) | + | t_-'(0) | + 1 ) + t_+'(0) (I'(u_n), u_n^+) +$$

$$+ t_-'(0) (I'(u_n), -u_n^-) \leq \frac{1}{n} C_4(1 + 2C_3) \longmapsto 0 \text{ as } n \to +\infty.$$

In conclusion the sequence $\{u_n\}$ satisfies

(i) $\quad I(u_n) \to \gamma_1 < c_1 < c_0 + \frac{1}{N}\frac{S^{N/2}}{2};$

(ii) $\quad \| I'(u_n) \| \to 0.$

Thus, as we have seen in the proof of Proposition 2.2, conditions (i) and (ii) are sufficient to guarantee a convergent subsequence for $\{u_n\}$ whose (strong) limit will give the desired minimizer. ∎

Obviously, Proposition 3.1 would yield the conclusion for Theorem 1 only if the given relations between $\gamma_1$, $\gamma_2$ and $c_1$ could be established. While it is not clear whether or not such inequalities should hold, we shall use these values to compare with another minimization problem.

Namely, set

$$\Lambda_*^- = \Lambda_1^- \cap \Lambda_2^- \subset \Lambda^-$$

and define,

$$c_2 = \inf_{\Lambda_*^-} I. \tag{3.4}$$

It is clear that $c_2 \geq c_1$. An upper bound for $c_2$ is provided by the following:

Lemma 3.1:

For fixed $\epsilon > 0$ and $y \in \partial\Omega$ there exist $s > 0$ and $\mu \in \mathbb{R}$ such that

$$s\,u_1 - \mu\, U_{\epsilon,y} \in \Lambda_*^-.$$

In particular, for $N \geq 5$,

$$c_2 \leq \sup_{s \geq 0, t} I(s\,u_1 - t\,U_{\epsilon,y}) < c_1 + \frac{1}{N}\frac{S^{N/2}}{2}$$

for $\epsilon > 0$ sufficiently small and $y$ suitably fixed in $\partial\Omega$.

Proof: We shall show that there exist $s > 0$ and $t \in \mathbb{R}$ such that

$$s(u_1 - t\,U_{\epsilon,y})^+ \in \Lambda^- \quad\text{and}\quad -s(u_1 - t\,U_{\epsilon,y})^- \in \Lambda^-. \tag{3.5}$$

To this purpose let,

$$t_2 = \max_{\bar\Omega} \frac{u_1}{U_{\epsilon,y}} \quad\text{and}\quad t_1 = \min_{\bar\Omega} \frac{u_1}{U_{\epsilon,y}}.$$

For $t \in (t_1, t_2)$ denote by $s_+(t)$ and $s_-(t)$ the positive values given by Lemma 2.1 according to which we have:

$$^8 + W(^u r^{tu}{}_f / ^e A \sim$$

and

$$-sJtX^\wedge -tU^\wedge yF \in A''.$$

Note that $s_+(t)$ is a continuous function of $t$ satisfying:

$$\lim_{t \to t_1^*} s_+(t) = {}^{t+}({}^u i - {}^t i^u{}_{c\,y}) < +\infty \text{ and } \lim_{t \to t_2^-} s_+(t) = +\infty.$$

Similarly, $s_-(t)$ is continuous and,

$$\lim_{t \to t_1^+} s(t) = +\infty \text{ and } \lim_{t \to t_2^-} s(t) = t^+(t_9 U_v - u_1) < +\infty.$$

Therefore, by the continuity of $s_\pm(t)$ we find a value $t_0 \in (t_1, t_2)$ such that

$$VV = Ut_0) = {}^s o > {}^l o-$$

This gives (3.5) with $t = t_0$ and $s = s_0$.

At this point we only need to estimate $I(su_1 - tU\ \cdot)$ for $s > 0$ and $t \in K$. To this purpose we fix $y \in 0Q$ as in the proof of proposition 2.2 and let $u_c = U_{c,y}$. The structure of $I$ guarantees the existence of $R > 0$ (independent of $e$) such that $^\wedge su^\wedge - tu_\varepsilon) < c^\wedge$

for all $s^2 + t^2 \geq R^2$. On the other hand, for $s^2 + t^2 \leq R^2$, we have:

$$I(s\,u_1 - t\,u_\epsilon) \leq I(s\,u_1) + I_0(t\,u_\epsilon) + O(\epsilon^{\frac{N-2}{4}}) \leq$$

$$\leq \max_{s \geq 0} I(s\,u_1) + \max_{t \in \mathbb{R}} I_0(t\,u_\epsilon) + O(\epsilon^{\frac{N-2}{4}}) =$$

$$= I(u_1) + \frac{1}{N}\left[\frac{\|\nabla u_\epsilon\|_2^2 + \lambda \|u_\epsilon\|_2^2}{\|u_\epsilon\|_{2^*}^2}\right]^{N/2} + O(\epsilon^{\frac{N-2}{4}}) \leq$$

$$\leq c_1 + \frac{1}{N}\frac{S^{N/2}}{2} - C\,\epsilon^{1/2} + o(\epsilon^{1/2}) + O(\epsilon^{\frac{N-2}{4}});\; C > 0$$

where we have used the estimates in [A–M], [C–K] and [W]. Hence, for $N \geq 5$ and $\epsilon > 0$ sufficiently small we readily obtain,

$$c_2 \leq \sup_{\substack{s \geq 0 \\ t \in \mathbb{R}}} I(s\,u_1 - t\,u_\epsilon) < c_1 + \frac{1}{N}\frac{S^{N/2}}{2}. \qquad \blacksquare$$

<u>Proposition 3.2:</u> Assume that $\gamma_1 \geq c_1$ and $\gamma_2 \geq c_1$. The minimization problem,

$$c_2 = \inf_{\Lambda_*^-} I$$

attains its infimum at $u_2 \in \Lambda_*^-$ which defines a (changing sign) critical point for I.

<u>Proof:</u> Exactly as in Proposition 3.1, by means of Ekeland's principle, we derive a minimizing sequence $\{u_n\} \subset \Lambda_*^-$ satisfying:

$$I(u_n) \longrightarrow c_2$$
$$\|I'(u_n)\| \longrightarrow 0.$$

In particular, we have:

$$0 < a_1 \le \| u_n^{\pm} \| \le a_2 \tag{3.6}$$

for suitable constant $a_1$ and $a_2$. Thus, after taking a subsequence, we obtain

$$u_n^{\pm} \rightharpoonup u^{\pm} \in H^1(\Omega) \text{ weakly in } H^1(\Omega).$$

We start by showing that $u^{\pm} \neq 0$.

Indeed, if by contradiction we assume for instance, that $u^{+} = 0$ then we would have:

(i) $\| \nabla u_n^{+} \|_2^2 - \| u_n^{+} \|_{2^*}^{2^*} = o(1)$

and

(ii) $\lim_{n\to+\infty} \frac{1}{2} \| \nabla u_n^{+} \|_2^2 - \frac{1}{2^*} \| u_n^{+} \|_{2^*}^{2^*} = \lim_{n\to+\infty} I(u_n^{+}) \le$

$$\le c_2 - \lim_{n\to+\infty} I(-u_n^{-}) \le c_2 - c_1 < \frac{1}{N}\frac{S^{N/2}}{2}.$$

But, we have already seen how condition (i) and (ii) can hold simultaneously only if $\lim_{n\to+\infty} \| \nabla u_n^{+} \| = 0$ which clearly contradicts (3.6). A similar argument applies to $u^{-}$.

Thus, $u_2 = u^{+} - u^{-} \neq 0$ is a changing sign solution for $(1)_f$ and in particular, $I(u_2) \ge c_0$.

Set $u_n^{+} = u^{+} + v_n^{+}$ and $u_n^{-} = u^{-} + v_n^{-}$ with $v_n^{\pm} \rightharpoonup 0$ in $H^1(\Omega)$. Note that,

$$\| \nabla v_n^{\pm} \|_2^2 - \| v_n^{\pm} \|_{2^*}^{2^*} = o(1) \tag{3.7}$$

In view of (2.8) and Lemma 3.1, we also have:

$$\lim_{n\to+\infty} I(v_n^+) + I(-v_n^-) = \lim_{n\to+\infty} I(u_n) - I(u_2) \le c_2 - c_0 <$$

$$< \frac{1}{N}\frac{S^{N/2}}{2} + c_1 - c_0 < \frac{1}{N} S^{N/2} \tag{3.8}$$

So, necessarily,

$$\lim_{n\to+\infty} \min\{I(v_n^+), I(-v_n^-)\} < \frac{1}{N}\frac{S^{N/2}}{2}$$

which, in view of (3.7), yields:

$$\| v_n^+ \| \longrightarrow 0 \quad \text{or} \quad \| v_n^- \| \longrightarrow 0$$

that is, $u_2 = u^+ - u^- \in \Lambda_1^+$ *or* $u_2 = u^+ - u^- \in \Lambda_2^+$.

Consequently, since we are in the situation where $\gamma_1, \gamma_2 \ge c_1$, we conclude:

$$I(u_2) \ge c_1.$$

Therefore, if we write $u_n = u_2 + w_n$ with $w_n \longrightarrow 0$ in $H^1(\Omega)$ we obtain,

$$\| \nabla w_n \|_2^2 - \| w_n \|_{2^*}^{2^*} = o(1)$$

and

$$\lim_{n\to+\infty} \frac{1}{2}\| \nabla w_n \|_2^2 - \frac{1}{2^*}\| w_n \|_{2^*}^{2^*} = \lim_{n\to+\infty} I(u_n) - I(u_2) =$$

$$\le c_2 - c_1 < \frac{1}{N}\frac{S^{N/2}}{2};$$

This, in the usually way, yields $\| w_n \| \longrightarrow 0$. Thus $u_n \longrightarrow u_2$ strongly in $H^1(\Omega)$ and $u_2 \in \Lambda_*^-$ gives the desired minimizer. ∎

The proof of Theorem 1:

For $f \geq 0$ ($f \leq 0$) Theorem 1 is a direct consequence of Theorem 2.1 and Proposition 3.1 and 3.2. In case $f$ changes sign, note that, if the situation of Proposition 3.1 occurs then we would be done since $\mu_1 > c_0$ and $\mu_2 > c_0$. So assume that we are in the situation of Proposition 3.2. To conclude it suffices to show that $u_2 \neq u_1$ (since, obviously, $u_2 \neq u_0$).

In fact, argue by contradiction and assume that $u_2 = u_1$. Then, $c_2 = c_1$, $u_1 \in \Lambda_1^- \cap \Lambda_2^-$ and $\gamma_1 = c_1 = \gamma_2$. On the other hand, $\int_\Omega f\, u_1 > 0$ (see Remark 2.3), thus $\int_\Omega f\, u_1^+ > 0$ or $-\int_\Omega f\, u_1^- > 0$. Assume, for instance, that $\int_\Omega f\, u_1^+ > 0$. From Lemma 2.1 then we obtain a $t^- > 0$ such that

$$t^- u_1^+ \in \Lambda^+ \text{ and } I(u_1^+) > I(t^- u_1^+).$$

This is clearly impossible since, $t^- u_1^+ - u_1^- \in \Lambda_2^-$ and

$$\gamma_2 = I(u_1) = I(u_1^+) + I(-u_1^-) > I(t^- u_1^+) + I(-u_1^-) = I(t^- u_1^+ - u_1^-) \geq \gamma_2.$$

Thus, in all circumstances, a third changing sign solution for $(1)_f$ is guaranteed. ∎

Sketch of the proof of Theorem 2

The proof of Theorem 2 follows by considering the (dual) functional,

$$F(w) = \frac{N+2}{2N} \int_\Omega | w |^{\frac{2N}{N+2}} - \frac{1}{2}\int_\Omega wKw + \int_\Omega wKf, \quad w \in E$$

with E and K as defined in (1.4) and (1.5).

As above, the idea is to consider,

$$A_+^* = \{w \in E, (F'(w),w) = 0 \text{ and } \int_\Omega |w|^{\frac{2N}{N+2}} - \int_\Omega wKw > 0\},$$

and

$$A_-^* = \{w \in E, (F'(w),w) = 0 \text{ and } \int_\Omega |w|^{\frac{2N}{N+2}} - \int_\Omega wKw < 0\}.$$

One shows that the condition $\mu_1^* > 0$ ($\mu_1^*$ as defined in (1.3)), implies that the corresponding minimization problems:

$$c_0^* = \inf_{A_+^*} F \tag{4.1}$$

$$c_1^* = \inf_{A_-^*} F \tag{4.2}$$

yield two distinct critical values for F, hence two (distinct) solutions for $(2)_\lambda$.

For the minimization problem (4.1) this follows exactly as for Proposition 2.1 with the obvious modifications.

The minimization problem (4.2) is treated similarly to that in (2.7) and the corresponding compactness argument follows by providing an appropriate upper bound on $c_1^*$. This can be derived using the estimates contained in [C—K]. We leave the details to the interested reader.

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