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ON THE PROPAGATION OF SINGULARITIES OF SEMI-CONVEX FUNCTIONS

L. AMBROSIO* P. CANNARSA H.M. SONER*****

Abstract. The paper deals with the propagation of singularities of semi-convex functions. We obtain lower bounds on the degree of the singularities and on the size of the singular set in a neighborhood of a singular point. These results apply to viscosity solutions of Hamilton-Jacobi-Bellman equations. In particular, they provide sufficient conditions for the propagation of singularities, depending only on the geometry of the superdifferential at the singular point.

Key **words,** convexity, semi-concavity, propagation of singularities, Hamilton-Jacobi equations

INTRODUCTION

In a recent paper [1], ***upper*** **bounds on the dimension of singular sets of semi-convex functions were derived by measure theoretic arguments.**

To briefly describe these upper bounds, let u : R^n —• R^n be a semi-convex function (Definition 1.2 below). Define

$$S^k(u) = \{x \in R^n : dim(du(x)) = k),$$

where k € [0,n] is an integer and $du(x)$ denotes, as usual, the subdifferential of u. Clearly, $\{S^k(u)\}^\wedge_{=:0}$ is a partition of R^n and $S^\circ(u)$ is the set of all points of differentiability of u. Since we are interested in first order singularities, we call a point x singular for u if $x \in S^k(u)$ for some $k \geq 1$.

In [1] it is proved that $S^k(u)$ is countably $W^{n \sim fc}$-rectifiable. In particular,

$$H\text{-}dim(S^k(u)) \leq n\text{-}fc,$$

where $7i$ — dim is the Hausdorff dimension.

The purpose of the present work is to obtain ***lower*** **bounds on the dimension of $S^k(u)$. More precisely, we will describe the structure of $S^k(u)$ in a neighborhood of x, knowing the geometry of $du\{x)$.**

A motivating application of these results concerns the analysis of singularities of solutions to the Hamilton-Jacobi-Bellman equation

$$(1) \qquad H(x,u,Vu) = 0.$$

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In fact, if the data are smooth, viscosity solutions of such PDE's (and, in particular, the solutions that are relevant to optimal control) enjoy well known semi-concavity properties (see for instance [12], [13], [15], [16]).

The present work is related to [4] and [5], in which viscosity solutions of (1) are shown not to have any isolated singularity if *H* is strictly convex with respect to p. In [4], [5], however, no attention is paid to the dimension of *du* at such singular points, and no attempt is made to estimate the Hausdorff measure of the singular sets.

Different approaches to the analysis of singularities of Hamilton-Jacobi equations are obtained for the one dimensional case in [14] and using characteristics in [21].

Semi-convexity was the only property used in [1] to prove upper bounds on singular sets. On the contrary, to obtain lower bounds we need additional information. This fact is the essential difference between [1] and the present paper. In order to understand the nature of the additional information, let us consider the set of reachable subgradients

$$V>u(x) = \{ \lim Vu(x_h) : x_h \; G \; S^\circ(u) \setminus \{x\}, \; x_h — x\}.$$

The above set is a set of generators of *du(x)* in the sense of convex analysis. Then, we show that the strict inclusion

(2) $$V,u(z) \quad du\{x).$$

is a sufficient condition for the propagation of any singularity x G $S^k(u)$, $1 \leq k < n$ (see Example 2.1 below). The inclusion (2) is satisfied by any viscosity solution of (1) with a strictly convex Hamiltonian, as *V+u(x)* is contained in the zero level set of *H*(x,tt(x),-).

Moreover, if x is an isolated singularity, by adapting a variational argument of Tonelli (see the proof of the implicit function theorem in [20]), we show that *V+u(x)* coincides with ć?u(x), see Theorem 2.1 below.

Furthermore, inserting nonsmooth analysis into this procedure, we obtain a more detailed description of the singular sets. In Theorem 2.2 we prove that singularities propagate along directions related to the geometry of *du{x).* These directions are orthogonal to the exposed faces of *du(x).* In Theorem 2.3 we give a lower bound on the maximum integer $m \leq k$ such that x is a cluster point of

$$E"» = Q S'(u),$$

and in (2.7) we estimate from below the Hausdorff (n — fc)-dimensional measure of S^m(tx). Roughly speaking, the computation of *m* takes into account how many vectors in V*u(x) are necessary to generate *du(x).*

We conclude with an outline of the paper. The first section contains preliminary material on Hausdorff measures, semi-convex functions, and the estimates of [1]. In §2 we develop our main results on propagation of singularities of semi-convex functions. The last section is devoted to applications to Hamilton-Jacobi-Bellmann equations and to the discussion of some examples.

1. NOTATION AND PRELIMINARIES

We briefly introduce some notation. We denote by $B_p(x)$ the open ball in R^n centered in x with radius p, and we abbreviate $B_p = B_P(Q)$.

For any set A C R^n we denote by $co(A)$ the convex hull of A. Moreover, the following sets of convex combinations of points of A will be often used in the sequel.

$$I_j(A) = \left\{ \sum^{j} \lambda_i p_i : p_i \in A,\ \lambda_i \geq 0,\ \sum^{j} \lambda_i = 1 \right\}$$

for any integer j $\geq$ 1. We also define

$$\pi i(A) = \text{maxjj} \geq 0 :1;(A) \# co(A)\}.$$

Clearly $I\backslash (A)$ = A, hence $m(A) = 0$ if and only if A is a convex set. Moreover, by Carathéodory's Theorem (see for example [18, p.155]) we know that Jfc+i (A) = co(A), where k is the dimension of co(A). Therefore $m(A) \leq$ dim [co(A)]. However, the integer $m(A)$ does not depend just on the dimension of $co(A)$. For example, if A is a finite set of affinely independent points, then $m(A)$ equals the dimension of $co(A)$. On the other hand, if A is the boundary of a fc-dimensional ball, then $m(A) = 1$.

For any set S C R^n we define

$$S^{1\prime} = \{p \in R^n : q \longrightarrow (<7,p) \text{ is constant on } 5\},$$

and

$$T(5,x) = \{r0:r>0, 6= \lim_{h\to+\infty} \frac{x_h - x}{p^\wedge - x[}, x_h \in S \setminus \{x\}, x_h \to x\}.$$

The set T(5, x) defined above is the so-called *contingent* cone to S at x ([3], [6]).

For any real number r €]0,n] we denote by $H^r(B)$ the Hausdorff r-dimensional measure of B C R^n, defined by

$$\mathcal{H}^r(B) = \frac{\wedge}{2} \sup_{\delta>0} \inf \wedge \sum_{i=1}^{\infty} (\text{diam}(£_t))^r \text{ I } B \subset \bigcup_{i=1}^{\infty} B_i,\ \text{diam}(B_i) < \delta \ V,$$

where o;$_r$ is the Lebesgue measure of the unit ball in R^r if r is an integer, any positive constant otherwise. We also denote by $7i^\circ(B)$ the cardinality of B. The Hausdorff dimension of B is defined by

$$H - \dim(B) = \inf\{r > 0 : H^r\{B) = 0\}.$$

For an introduction to the properties of Hausdorff measures see for example [10], [17]. We merely recall that H^r is a Borel regular measure in R^n, and

$$(1.1) \qquad n^r(B) < +oo \quad => \quad W^m(B) = 0 \ V m > r.$$

We now recall the definition of semi-convexity and the main properties of semi-convex functions.

DEFINITION 1.2. Let ft C R^n be an open convex set, and *u :* ft -+ R. We say that *u* is semi-convex in ft if there is a non decreasing upper semicontinuous function *u :* [0, +oo[-> [0, +oo[such that *v(0)* = 0 and

$$(1.2) \qquad \begin{array}{l} tu\{x_t) + (1 - t)u\{x_2) - ufo) \geq -t(l - t)\backslash x_I - x_2\backslash u>(\backslash x_I - x_2\backslash) \\ x_t = txa + (1 - t)x_2, \ xi, \ x_2 \ € \ ft, \ t \ € \ [0,1]. \end{array}$$

We call semi-convexity modulus of *u* the least function *u)* satisfying (1.2). If *u :* ft —» R is semi-convex and *x* € ft, we say that *p* € R^n is a subgradient of *u* at *x* if

$$\liminf_{y \to x} \frac{u(y) - u(x) - \langle p, y - x\rangle}{|y - x|} \geq 0.$$

Borrowing the notation of convex analysis, we denote by *du{x)* the set of subgradients of *u* at x, call it the subdifferential of *u* at x. It is easy to see that *du(x)* is a compact, nonempty, convex set. Moreover,

$$(1.3) \quad pedu(x) \quad <=> \quad u(y)\text{-}u(x)\sim (p,y - x) \geq -\backslash y - x\backslash w(\backslash y - x\backslash), \ Vj/€ \ ft.$$

It can also be shown that *du(x)* is a singleton if and only if *u* is differentiable at *x.* Hence, the set of non differentiability points of *u* can be classified according to the dimension of the subdifferential at the singular point.

DEFINITION 1.3. Let *x e* ft, and let *k e* {0,... ,n} be an integer. We define

$$S^k(u) = \{x € ft : dim(du\{x)) = fc\},$$

and

$$£^*(u) = \bigcup^{n} S\text{^}u) = \{x € ft : dim(5u(x)) \succ ib\}.$$

In order to find sufficient conditions for the propagation of singularities, it will be useful to consider the set V*u(x) of reachable subgradients.

DEFINITION 1.4. Let *u* : ft —» R be a semi-convex function, and let *x* G ft. We define

$$V+u(x) = \{ \lim \ Vu(x_h) : x_h \ 6 \ 5°(tx), \ x_h \text{ -+ } x).$$

Then, it is known that *du(x)* is the convex hull of V«,u(x) (see e.g. [4]).

In the following theorem we list some basic properties of semi-convex functions. We recall (see [3]) that a set-valued map *S(x)* is said to be upper semicontinuous if the following implication holds:

$$Ph'€ \ S(x_h), \quad x_h\text{->}x, \quad PH\text{-^}P \quad => \quad pe \ S(x).$$

THEOREM 1.1. *Let u :* ft —* R *be a semi-convex function. Then,*

(1) *u is locally Lipschitz continuous in* ft *and*

$$\frac{du}{\partial\theta}(x) = \lim \text{-*}\frac{u(x + t6)\text{-}u(x)}{t} = \max\{\langle p, \theta\rangle : p \in \partial u(x)\}$$

for any z G *it and any 6* G $R^n \setminus \{0\}$.

(2) Tie *set-valued maps du(x),* V*tz(z) are *upper semicontinuous in x.*

(3) If z G $S^k(u)$, *then* !*+! (V.u(x)) = *8u{x).*

(4) *For any k* G {0,... ,n} and any *p* > 0 we have

$$T(Sj(u),z) \subset [du(x)]^{\pm} \qquad Vz \; G \; 5j(u),$$

where $S^k_\rho(u)$ *denotes the set of all points x* G $S^k(u)$ *such that du(x) contains* a fc-dimensional *ball of radius p.*

(5) *For any integer k* G {0,... ,n} the *set* $S^k(u)$ *is countably* $H^n\sim^k$*-recti6able, that is it can be covered, up to a* %$^n\sim^k$*-negligible set, with a countable sequence of* C^1 *hypersurfaces Th* C R^n *of dimension (n — k), i.e.*

$$n^n\text{-}^k(s^k(u)\backslash\backslash \overset{\infty}{J}r_h \backslash \; . =0.$$

Moreover,

$$\int_{s^k(u)nn^f} H^k\{du(x))d'H^n\text{-}^k(x)<+oo$$

for any open set Q^r CC ft.

Proof. (1) See [1] and [4].

(2) The upper semicontinuity of the map *du(x)* easily follows by (1.3), and the upper semicontinuity of V*rx(x) follows directly from its definition.

(3) Since V*u(x) is closed and its convex hull equals 9u(x), the assertion follows by Carathéodory's Theorem.

(4) See [1], Theorem 3.1.

(5) See [1], Theorem 4.1. |

REMARK 1.1. Note that (5) provides an upper bound on the Hausdorff dimension of $S^k(u)$, which is not greater than (n — *k).* It is easy to see that this bound is optimal. Indeed, let

$$u\{x_u \ldots \; ,x_n) = |x_1| + \ldots + |x_k|.$$

Then, $S^k(u)$ is the (n - fc)-plane of all *x* G R^n such that z,· = 0 for $1 \leq i \leq k$.

2. EXPOSED FACES AND REACHABLE SUBGRADIENTS

We want to study the structure of the singular set E^1 *(u)* in the neighborhood of a singular point z.

DEFINITION 2.1. We define the singularity degree of z G $\wedge^l(u)$ as the unique integer *k* such that z G $S^k(u)$. We say that z is an isolated singularity of degree *k* if $T(E^{fc}(tx),z)$ = 0. We say that a singularity propagates if

$$T(\Sigma^1(u), x) \neq \emptyset.$$

Moreover, all vectors *6 e* T(E^1(u),x) n *dB* are called directions of propagation of the singularity at *x.*

Clearly, a convex function may well have an isolated singularity of degree *n.* Indeed, if *x* G S£*(u)* for some *p* > 0, then *du(x)* contains an n-dimensional ball. Hence, by Theorem 1.1, *x* is not a cluster point of *S^(u).* In other words, *S%(u)* is a discrete set for any *p* > 0. Moreover, there are convex functions with isolated singularities of degree < n.

EXAMPLE 2.1. Let

$$u(x_1,\ldots,x_n) = \sqrt{\left(x_1^2+\ldots+x_k^2\right)+\left(x_{k+1}^4+\ldots+x_n^4\right)}.$$

Then, *u* is a convex function in R^n and *u* € $C^2(R^n \setminus \{0\})$. On the other hand, *du(0)* = [-1,1]* x $\{0\}^{n}$~*, so that 0 is the only point in $S^k(u)$.

Note that, in the above example *du(0)* = V*u(0). More generally, we will show that a sufficient condition for the propagation of a singularity of degree *k < n* at *x* is the strict inclusion *V+u(x) du(x).* In particular, this condition is satisfied for solutions of some Hamilton-Jacobi equations, see §3.

In the remainder of this paper we always assume that *Q* C R^n is a convex open set, *u : Q* —> R is a semi-convex function, and *u(t)* is the semi-convexity modulus of *u.* Since our statements are local, we assume that *u* is Lipschitz continuous in *Q* and we denote by $[u]_{u_p}$ its Lipschitz semi-norm.

We will see that the directions of propagation of singularities are related to the geometry of the subdifferential *du(x)* at the starting point *x.* To analyze the singular directions we introduce the following sets.

DEFINITION 2.2. Let *x* € fi and *0* € *dBi* we set

$$du\{x,6) = \{pe\ du(x) : \langle p,0) = \frac{du}{\partial a}(x) = \max_{q\,\in\,\partial u\{x)} \langle q,\theta\rangle\},$$

$$\mathbf{V.u(x,0)}=\{\lim Vu(x_h) : x_h \in S^\circ(u) \setminus \{x\},\ x_h \to x,\ \frac{x_h-x}{|x_h-x|}\to\theta\}.$$

The collection {0tt(x,0) : ^ G *dBi}* consists of all the exposed faces of the convex set *du(x).* The following theorem is the basis of our singularity propagation argument (see Theorem 2.2 and Theorem 2.3).

THEOREM 2.1. ***Let x* € £2,p € R^n and *sequences x^* —• *x,du(xh) 3 Ph —> P be given. Suppose that***

(2.1) $$\lim_{h\to\infty} \frac{x_h - x}{|x_h - x|} = 0$$

Tien, p G *du(x,6). In particular,*

$$V+u(x,6) \subset du(x,8).$$

Conversely, for any $p \in \partial u(x,\theta)$ there are sequences $x_h \to x$ satisfying (2.1), and $\partial u(x_h) \ni p_h \to p$.

Proof. We have to show that $\partial_* u(x,\theta) = \partial u(x,\theta)$, where

$$\partial_* u(x,\theta) = \{ \lim_{h\to+\infty} p_h : p_h \in \partial u(x_h),\, x_h \neq x,\, x_h \to x,\, \frac{x_h - x}{|x_h - x|} \to \theta \}.$$

Let p_h, x_h be as in the definition of $\partial_* u(x,\theta)$ and set

$$t_h = |x_h - x|, \qquad p = \lim_{h\to+\infty} p_h.$$

We know, by the upper semicontinuity of $\partial u(x)$, that $p \in \partial u(x)$. We will now show that $p \in \partial u(x,\theta)$. Indeed, by the semi-convexity of u we have

$$u(x) - u(x_h) - \langle p_h, x - x_h \rangle \geq -t_h \omega(t_h).$$

Devide both sides by t_h to obtain

$$\langle p_h, \frac{x_h - x}{t_h} \rangle \geq \frac{u(x + t_h\theta) - u(x)}{t_h} + \frac{u(x_h) - u(x + t_h\theta)}{t_h} - \omega(t_h).$$

Since

$$\frac{|u(x_h) - u(x + t_h\theta)|}{t_h} \leq [u]_{\text{Lip}} \left| \frac{x_h - x}{t_h} - \theta \right| \to 0,$$

by letting $h \to +\infty$ we get

$$\langle p, \theta \rangle \geq \frac{\partial u}{\partial \theta}(x).$$

Thus, $p \in \partial u(x,\theta)$ and $\partial_* u(x,\theta) \subset \partial u(x,\theta)$.

Next, we proceed to show the reverse inclusion. Let us denote by d the dimension of $\partial u(x,\theta)$. Since θ is orthogonal to $\partial u(x,\theta)$, d is strictly less than n. We may assume that $d > 0$, the inclusion being trivial if $\partial u(x,\theta)$ is a singleton.

Since $\partial_* u(x,\theta)$ is compact, it suffices to show that $p \in \partial_* u(x,\theta)$ for any $p \in \text{Int}\big(\partial u(x,\theta)\big)$, the relative interior of $\partial u(x,\theta)$.

Let θ_i, $1 \leq i \leq (n-d)$ be an orthonormal basis of $\big[\partial u(x,\theta)\big]^\perp$, i.e.,

$$\langle \theta_i, \theta_j \rangle = \delta_{ij}, \qquad \langle (p-q), \theta_i \rangle = 0 \quad \forall p,\, q \in \partial u(x,\theta).$$

We can also take θ_1 to be equal to θ. For r, $t > 0$ satisfying the condition $t\sqrt{1+r^2} <$ dist$(x, \partial\Omega)$, let $y(r,t)$ be a minimizer of the function

$$u\big(x + t(\theta_1 + y)\big) - t\langle p, y \rangle$$

in the compact set K_r defined by

$$K_r = \big\{ y \in \mathbf{R}^n : \langle y, \theta_i \rangle = 0 \;\forall i = 1, \ldots, (n-d),\; |y| \leq r \big\}.$$

We claim that for any r > 0 there is r > 0 (depending on r) such that for $t < r$ any minimizer *y(r,t)* satisfies the condition |y(r,£)| < r. Indeed, if the claim were not true it would be possible to find $r > 0$ and a sequence of minimizers $y_h = y(r,th)$ € K_r n dB_r corresponding to an infinitesimal sequence *th-* Passing to a subsequence, we may assume that *y^* converges to *y* € K_r n dB_r. Since *yh* is a minimizer, we have

$$u\big(x+t_h(\theta_1+y_h)\big)-t_h\langle p,y_h\rangle \le u(x+t_h\theta_1).$$

Hence,

$$\frac{u\{x + th(Oi+Vh))-u\{x)}{} - \frac{u(x + t_h 9_1)-u\{x)}{} < \langle p,y_h\rangle.$$

Recalling that

$$\left|\frac{u\big(x+t_h(\theta_1+y_h)\big)-u\big(x+t_h(\theta_1+y)\big)}{t_h}\right| \le [u]_{\mathrm{Lip}}|y_h-y| \to 0,$$

we obtain

$$\frac{\partial u}{\partial(\theta_1+y)}(x)-\frac{\partial u}{\partial\theta_1}(x)\le\langle p,y\rangle. \tag{2.2}$$

On the other hand, since the. map (-,0i) is constant on *du(x,0\),* we have that *du/ddi(x)* = (p,0i). Also, since p € Int(du(0)),

$$\frac{du}{\partial(\theta_1+y)}r(x) > (p + €?/, 0i + v) = (p, 0i) + (Pi\,v) + cr^2$$

for |e| sufficiently small. We thus obtain a contradiction with (2.2), and the claim is proved.

Now, let r > 0 and let *r(r)* > 0 be given by the claim. Returning to the definition of y(r, £), by the nonsmooth Lagrange multiplier rule (see for instance [6], 6.1.1) we conclude that for any *t* €]0,r(r)[we can find Aj(r,i) 6 R satisfying

$$0 \text{ e } t\{du(x + i\{9i + y\{r,t))) - p\} - \sum_{i=1} \lambda_i(r,t)\theta_i,$$

or, equivalently,

$$p+\sum_{i=i}^{n-d}\frac{\lambda_i(r,t)}{t}\theta_i \in \partial u\big(x+t(\theta_1+y(r,t))\big). \tag{2.3}$$

Let (r_h) C]0,+oo[and t_h €]0,r(r/$_1$)[be two sequences converging to 0. By taking scalar products in (2.3) with *6i* it is easy to see that *\\i{rh,t_h)\/t_h* is not greater than 2 $[u]_{\mathrm{Lip}}$. Hence, by passing to a subsequence if necessary, we may assume that **i(r_h,t_h)/th* converges to A* as *h* -+ +oo for *i* = 1,... , (n - *d).*

Then, by letting *h* —» +00 in (2.3) we get

$$p+\sum_{i=1}^{n-d}\bar{\lambda}_i\theta_i \in \partial_* u(x,\theta_1),$$

as \|*y{r*

$$\lim_{h\to+\infty}\frac{\theta_1+y(r_h,t_h)}{|\theta_1+y(r_h,t_h)|}=\theta_1.$$

On the other hand, since the vectors 0* are orthogonal to *du(x,6)$_y$* all Ā* are equal to 0. Thus, p € 5*u(x,0i) and the proof of the theorem is complete. |

THEOREM 2.2. *Let x* € ft,0 € &Bi, and an *integer m* € [l,n] be given. *Then,*

(2.4) J_m(V*Tz(x,0)) # *du(x,0)* =» 0 € Tan(E^m(tz),x).

Moreover, <?tz(x,0) = co(V*tz(x,0)).

REMARK 2.1. In particular, if V*u(x,0) ^ du(x,0), then 0 is a direction of propagation of the singularity at x. Moreover, (2.4) provides a lower bound on the degree of the singularity near x. Indeed, in view of definition 1.1, (2.4) implies that 0 € T(E^m(u),x), where *m* = m(V*u(x,0)). Hence, there are singular points of degree *m* near x, along the direction 0.

Proof of Theorem 2.2. Let p G *du{x,6)* \ J_m(V,tz(x,0)). We argue by contradiction. So, suppose that 0 ^ T(E^m(tt),x). By Theorem 2.1, there are a sequence (x_h) C ft \ {x}, and vectors *ph* such that p^ € ^(x/J and

lim /i-*+oo *ph* = p, lim /i—•+00 *Xh* = x, lim /i—>+c» (x ^ — x)/(Ix^ — X|) = 0.

By our assumption, for *h* large enough x^ does not belong to S^m(u). Hence, the dimension of ^(x^) does not exceed *m* — 1. By Theorem 1.1(3), there are vectors *Pi,h* € V<ti6(x/l) and non negative real numbers A^h such that

(2.5) $$p_h = \wedge^{m} \lambda_{i,h}p_{i,h}, \qquad \sum^{m}\lambda_{i,h}=1.$$

By passing to a subsequence, we may assume that for any *i* the m-tuples $\lambda_{i,h}$ converge as *h* —• +00 to A* and *pi,h* converge to p» as *h* —• +cx). Since p^ G ^^^(x_h) a diagonal argument shows that p» € V»u(x, 0). Now, let /i —* +00 in (2.5) to obtain

$$p=\sum_{i=1}^{m}\lambda_i p_i, \qquad \sum_{t=1}^{m}\lambda_i = 1.$$

Hence, p € J_m(V«tt(x, ^)) and this contradiction proves (2.4).

Finally, a similar argument (with m = n+1) shows that each vector p € *du(x, 6)* is the convex combination of at most (n + 1) points of V*tx(x, 0). |

Note that (2.4) implies that x is only a cluster point of $E^m(u)$. However, we will show that, under suitable assumptions, there is a whole continuum of singular points near x, whose size can be estimated from below.

Let 5 be any plane in R^n passing through the origin, and let *ns* be the orthogonal projection on 5. For any 7 > 0 we denote by $C_y(S)$ the cone

$$C_y(S) = \{xeR^n: |irs(*)| \leq I|\pi_{S^\perp}(x)|\}.$$

We note that $C_7(5)$ D 5^X and $C_y(S)$ approaches 5^X as 7 -» 0+.

THEOREM 2.3. *Letx* € $S^k(u)$ *with* _l<k<_n-1 *begiven.* *Setm* = m(V*ti(*)J-
Tien,

(2.6) $$T(5T»,z) \text{ D } [\wedge(x)]^{*1}.$$

In*addition,wehave*

(2.7, Ihninf $\underline{«-*(*•(«) \text{ n } B,M\ 0\ [x + C_7(S)])} \geq 1$

for any 7 > 0, wiere 5 is tie *k-plane parallel to du(x) and containing* 0.

Proof Observe that c?i/(z,0) equals *du(x)* and V>u(x,5) C V*?x(x) for any *0* € $[\partial u(x)]^\perp$. Hence, (2.6) follows from the previous theorem.

In order to simplify our proof of (2.7), we assume that x = 0. Since $E^m(u)$ = *Q* if m = 0, we may also assume that m > 0. Let us denote by S^{-1} the unit sphere in $S^\perp$.

Let us pick a vector p in the set *du(0)* \ $/_m$(V*u(0)), which is not empty. For any 2 6 S^1 and any r, t > 0 we denote by *y(r,t,z)* a minimizer of the function *u(tz + ty) —t(p,y)* in the set

$$K_r=\{yeS:\backslash y\backslash<r\}.$$

We claim that for every r > 0 there is *r(r)* > 0 such that for any t €]0, *r(r)[* and any z G S^x any minimizer *y(r,t,z)* belongs to the (essential) interior of K_r. This claim can be proved as in Theorem 2.1. Indeed, suppose that the claim is not true. Then, there exist r > 0 and a sequence of minimizers $y\wedge$ = j/(r, t^, z_h) *e* $K_rD\ dB_r$ corresponding to a sequence *th* —> 0. Passing to a subsequence, we may assume that *yh* converges to y € K_r n dB_r and *ZH* converges to z € S^x. Since y_h is a minimizer, we infer

$$u\{t_hz_h + 1/\wedge) - \wedge(p, j/fc) \leq u(t_hz_h).$$

Hence,

$$\frac{u(t_hz_h + t_hy_h) - tt(0)}{t_h} \quad \frac{u\{t_hz_h) - u(0)}{h} \leq \{p,y_h).$$

Recalling that

$$\left|\frac{u\{t_hz_h \; + \; t_hy_h) \; - \; u(t_hz \; + \; t_hy)}{t_h}\right| \leq [u]_{\mathrm{Lip}}\left(|z_h - z| + |y_h - y|\right) \to 0,$$

and

$$\left|\frac{u(t_hz_h) \; - \; u\{t_hz)}{th}\right| \leq [u]_{\mathrm{Lip}}|z_h - z| \to 0,$$

we obtain

$$\frac{du}{\partial(z+y)}(0) - \frac{\partial u}{\partial z}(0) \leq \langle p, y \rangle. \tag{2.8}$$

On the other hand, since the map (•, *z)* is constant on é)u(O), we have that

$$\frac{du}{Tz}\{(0) = \langle p, z \rangle.$$

Also, since p € Int(ćhz(0)),

$$\frac{du}{d(z \; + y)}(0) \geq (\mathrm{p} + \epsilon y, z + y) = \mathrm{Ov}z\rangle + \langle p, y \rangle + \epsilon r^2$$

for |e| sufficiently small. We thus obtain a contradiction with (2.8), and the claim is proved.

Next, we claim that there is $6 > 0$ such that if r < 6 and t < inf{r(r), 5}, then for any z € S-[1], any minimizer 2/(r,t,z) satisfies the condition

$$tz + ty(r, t, z) \in \Sigma^m(u).$$

Indeed, let us assume that the claim is not true. Then, by the variational argument used in the proof of Theorem 2.1, we construct a sequence of minimizers y_h = *y(rhith,Zh)* € K_{rh} corresponding to sequences r&, *th* —• 0 and real constants A*,i,... ,*h_y*n-k* such that

$$p_h := p + \sum_{i=1}^{\text{n-fc}} \lambda_{h,i}\theta_i \in \partial u(t_hz_h + t_hy_h), \tag{2.9}$$

$$t_hz_h + t_hy_h \notin \Sigma^m(u), \tag{2.10}$$

and

$$\lim \quad z_h = 2 \; 6 \; \mathrm{S}^1, \qquad \lim \quad \mathrm{A}_{/\mathrm{li}} = \mathrm{A}_{\mathrm{i}}\, \mathrm{e}\, \mathrm{R} \quad \mathrm{Vi} = 1, \ldots An - k).$$

Passing to the limit as h —* +oo in (2.9) we get

$$p + \sum_{i=1}^{\text{n-fc}} \lambda_i\theta_i \in \partial u(0).$$

Hence A* = 0 for any *i* = 1,... ,(n - fc) and p_h converges to *p* as *h* -> +00. Moreover, by (2.10) and Theorem 1.1(3) each vector p_h belongs to the convex hull of at most m vectors of *V+u(xh)-* Repeating the argument of Theorem 2.2 we obtain a set *A* C V*u(0) consisting of at most *m* points, such that *p* € *co(A).* Hence, *p* € $/_m$(V*u(0)), and this contradiction proves the second claim.

Finally, let *6* > 0 be given by the second claim. For any fixed 7 > 0 let r <inf{7,£}. Then,

$$\Sigma^m(u) \cap C_\gamma(S) \cap B_\rho \supset \left\{tz + ty(r,t,z) : z \in S^\perp,\ 0 \le t < \frac{\rho}{\sqrt{1+r^2}}\right\}$$

provided *p* < *y/1* + r² inf{r(r), *6}*. Since *7Ts±* does not increase the Hausdorff measure (see for instance [17], Proposition 3.5), by the inclusion

$$\text{TT}_5\text{X } (\text{E}^m(\text{tx}) \text{ n } \text{C}_7(5) \text{ n } B_p) \ D\ \{ze\ S^\pm : \backslash z\backslash < \frac{\rho}{\sqrt{1+r^2}}\}$$

we infer

$$\liminf_{\rho\to0^+} \frac{\mathcal{H}^{n-k}(\Sigma^m(u) \cap B_\rho \cap C_\gamma(S))}{\omega_{n-k}\rho^{n-k}} \quad + r^2)^{(k-n)/2}.$$

By letting r —* 0, we complete the proof. |

REMARK 2.2. By (1.1) and Theorem 1.1(5) we infer that *Ti^{n nk}(S^i(u))* = 0 for any *i* ≥ *k + 1.* Hence, (2.7) can be written in the equivalent form: for any *x* € *S^k(u)*

$$\liminf_{\rho\to} \text{f}_{i=m} \frac{\text{«-M^(-)nB,(x)n[x + c}_7\text{(5)])}}{\omega_{n-k}\rho^{n-k}} > 1,$$

where *m* = m($V_{1||}$ix(x)). In particular, if 7^(V*u(x)) 7^ *du(x)* (i.e., m = A:), we get

$$\liminf_{\rho\to0^+} \frac{\mathcal{H}^{n-k}\left(S^k(u) \cap B_\rho(x) \cap [x + C_\gamma(S)]\right)}{\omega_{n-k}\rho^{n-k}} \ge 1,$$

and coupling this estimate with Theorem 1.1(5) we conclude that *H—dim(S^k(u))* = (n-fc).

3. HAMILTON-JACOBI EQUATIONS

In this section we will apply the general results on the singularities of semiconvex functions to solutions of the Hamilton-Jacobi-Bellman equation

(3.1) $$F\big(y, u(y), \nabla u(y)\big) = 0, \quad y \in \Omega$$

where *Q* C R^ is an open domain. We will assume that

(3.2) F : f l x R x R^ -* R is continuous;

(3.3) p *-+ F(t/, s,p) is convex in K^N V(j/, *s)* € ft x R;

(3.4) n is semi-concave (i.e. —*u* is semi-convex);

(3.5) (3.1) holds at any differentiability point of *u.*

We note that, for a semi-concave function u, the interesting semidifferential is the so-called superdifFerential, defined as

$$d^+u(y) = \{p \in R^\wedge : \limsup_{*-*y} \frac{u(z) - u(y) - \langle p\ z - y\rangle}{|z \sim y|} \leq 0\}.$$

Equivalently, *d+u(y)* = —*d[-u](y).* Hence, *d+u(y)* ^ 0 for any *y* € *Q* and the following implication holds

(3.6) *du(y)* # 0 =* *u* is differentiate at j/.

Accordingly, the definitions 1.2 and 2.2 will be modified as follows for a semi-concave function *u:*

$$S^k(u) = \{x\ 6\ n : dim(d+u\{x)) = fc\},$$

$$E^*(u) = \bigcup_{n} {}^\wedge(u) = \{x6fi: dim(0+u(x)) \geq fc\},$$

$$\partial^+u(x,\theta) = \left\{p \in \partial^+u(x) : \langle p,\theta\rangle = \frac{\partial u}{}(x) = \min \langle q,\theta\rangle\right\}.$$

REMARK 3.1. From (3.2)-(3.5) it follows that *u* is a viscosity solution in the sense of [8] (see also [7]). Indeed, (3.2) and (3.5) yield

(3.7) F(»,u(y),p)=0 Vp€V.u(j/)

for any y 6fi, and so (3.3) implies that

$$F(y,u(y),p) \leq 0 \quad \forall p \in \#^+u(j/).$$

The converse inequality on the elements of *du(y)* trivially follows by (3.6).

REMARK 3.2. Semi-concavity is a natural property to expect on viscosity solutions of Hamilton-Jacobi-Bellman equations. Indeed, several existence and uniqueness results were first obtained in classes of semi-concave functions (see [15]). More recently, H-J equations have been studied in the framework of viscosity solutions (see [8] and [7]). Under suitable regularity assumptions on *F* and on the (Dirichlet) boundary data, viscosity solutions to (3.1) are known to be semi-concave (see [16] and [12]). Similar results are also available for viscosity solution of second order H-J equations, see [13]; hence the result of §2 apply to these equations as well. For the sake of simplicity we confine our statements to first order equations.

For any compact convex set *C* C R^ we denote by Ext(C) the set of extreme points of *C.* We say that a set *A* C R^ is extremal if no *p* 6 *A* can be written as a convex combination of other points of A, i.e.

$$p \notin co(A \setminus \{p\}) \quad \forall p \in A.$$

Our terminology is motivated by the following result.

LEMMA 3.1. *Any compact extremal set A coincides with Ext(co(A)).*

Proof. Let *C = co(A),* and let p € Ext(C). By Carathéodory's Theorem, we can represent p as a convex combination of *(N.*+ 1) points *pi* € *A:*

$$P = \underset{i=1}{\overset{N+l}{JZ}} A*P*' \quad Ai > 0.$$

Since *p* is an extreme point of C, p = *pi* for any i € {1,... ,7^ + 1}, hence p 6 A

Conversely, let *p e A.* By the Krein-Milman theorem (see for instance [18], page 167) we can represent *p* as a convex combination of at most *(N+l)* points *Pi* eExt(C):

$$p = \sum_{i=1}^{N+l} \lambda_i Pt, \quad A< > o, \quad \sum_{i=1}^{N+l} \lambda_i = 1.$$

In turn, each p* can be represented as a convex combination of at most *(N* +1) points *pij* € *A:*

$$p_i = \sum_{j=1}^{N+l} \lambda_{ij} p_{ij}, \quad \lambda_{ij} > 0, \quad \sum_{j=1}^{N+l} \lambda_{ij} = 1,$$

so that

$$P = \underset{t,i=i}{\overset{N+l}{22}} A»A^p^.$$

Since -4 is extremal, p = ptj for any t, j, hence p = p^ € Ext(C). |

The main result of this section is the following.

THEOREM 3.2. *Assume (3.2), (3.3), (3.4), (3.5), and let x* G *$S^k(u)$ be a singular point. Let us further assume that*

(3.8) $\{p€R^N : F(y,u(y),p) = 0\}$ *is extremal.*

Then

(1) *V+u(y) = Ext(d*u(y)), and if k < N the singularity propagates. Moreover m = m(V*u(y))* >_ 1, *and*

,3.9) $$T\big(\Sigma^m(u), y\big) \supset \big[\partial^+ u(y)\big]^{\perp}, \qquad \liminf_{\rho \to 0+} \frac{\mathcal{H}^{n-k}\big(\Sigma^m(u) \cap B_\rho(y)\big)}{\omega_{n-k}\rho^{n-k}} \geq 1.$$

(2) Let 0 € &Bi and iet us assume tiat 5+M(J/,*6) is not* a *singleton. Then, V*u(y,6) coincides with Ext(d+u(y,6)), m = m(V*u(y,6))* >_1 *and 6* € $T\big(\Sigma^m(u), y\big)$.

Proof. (1) By (3.7) and (3.8), V*u(y) satisfies the hypotheses of Lemma 3.1, so that V*u(y) = *Ext(d+u(y)).* To show (3.9), we need only to apply Theorem 2.3 to —u.

(2) As in (1), Lemma 3.1 yields *V+u(y,d)* = Ext(d+u(y,0)). The other statements follow from Theorem 2.2 and Remark 2.1. |

REMARK 3.3. **The extremality condition (3.8) cannot be dropped. In fact, let** N **= 2 and** $u(y_u y_2)$ = *-y/yj + y*<>*$_2$**as in example 2.1. Then,** u **is concave in** R^2, **and has an isolated singularity at (0,0). Moreover, it is a viscosity solution of the equation**

$$\sqrt{y_2^2 u_{y_1}^2 + \frac{1}{4} u_{y_2}^2} = |y_2|.$$

REMARK **3.4. The condition (3.8) is trivially satisfied if**

***p* >> F(y, *s,p)* is strictly convex in R^ V(y, *s)* G *ft* x R.**

Theorem 3.2 also applies to nonstationary H-J equations with strictly convex Hamiltonian. Infact, let N = n **+1,** y = **(£,x) with** t **G R and** x **G** R^n**, and** p = *(pt,p*$_x$*)* **€ R x** R^n**. Let**

$$H\big((t,x),s,p_x\big) : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$$

be a continuous function, strictly convex in p_x**. Then,**

$$F(y,s,p) = p_t + H\big((t,x),s,p_x\big)$$

satisfies (3.2) and (3.3), and any semi-convave function u **:** *ft* **—> R satisfying (3.5) is a viscosity solution of the equation**

$$u_t + H(t,x,u,\nabla u) = 0. \tag{3.10}$$

Finally, for any y **€** *ft* **and any** s **€ R**

Z{y,s) = *{{p*$_u$*p*$_x$*)* G R x R^n :p< + *H(y,s,p*$_x$*)* = 0}

is extremal, because of the strict convexity of H**. Indeed, let**

$$Z(y,s) \ni p = \sum_{i=1}^{N+1} \lambda_i p_i$$

with *pi e Z(y,* **5), Aj > 0 and S ^ 1 A* = 1* ^d *et us sh o w that p»· = p for any i. Since**

p_t + *H(y,s,p*$_x$*)* < 53(A$_t$p$_{it}$ + A^(j/,5,p$_{ix}$)J =0

unless p_x = pi_X **for any i G {1,... ,TV + 1}, we have**

pa = *-H{y,s,p*$_{ix}$*)* = *-H(y,s,p*$_x$*)* = p_t Vi G {1,... ,JV + 1}

and, in particular, p = *Pi* **for any i.**

More generally, the same argument of Theorem 3.2 shows that singularities propagate in the direction *6* **if** *d~*~u(y, 6)* **is not a singleton and if the restriction of** ***F*(*yi*^{u}*(y)i*^{m}*)* to #$^+$w(j/)0) is strictly convex, so that m(V*u(y,0)) ≥ 1.**

REMARK 3.5. In Theorem 3.2(1) it is necessary to assume that x **is not a singularity of degree** N**. In fact,** *u(y)* = —*\y* **is a solution of the eikonal equation** *\Vu(y)*^{2} **-1 = 0, and the singularity in the origin does not propagate.**

However, propagation of singularities of any degree has been proved for nonstationary H-J equations with strictly convex Hamiltonian (see [4]). Due to the special structure of the equation it has been shown in [5] that for any singularity y there is at least a direction *6 € dBi* such that *du(y, 6)* is not a singleton. Note that, once the existence of such a direction has been proved, the propagation of the singularity would follow by Theorem 3.2(2).

In [5] it is also shown that viscosity solutions of (3.10) with strictly convex H are such that any p € V*tt(y) is exposed, i.e., there exists *8 € dB* such that *d+u(y,0)* = {p}. This condition is stronger than extremality.

REMARK 3.6. We note that the lower bound in Theorem 3.2 on the maximum degree of the singularity near y depends only on the geometry of *d+u(y).* To illustrate this phenomenon, we now discuss three examples. In the first example the subdifFerential *d+u(y)* is a triangle in R^3 and the singularity propagates in singularities of degree two, as implied by Theorem 3.2.

In the second example we show that a singularity y of degree k may well propagate in singularities of degree $m < k$ when *m{V*u(y))* $< k$.

Finally, the third example shows that Theorem 3.2 provides only a sufficient condition for the propagation of singularities of high degree.

EXAMPLE 3.1. Let fi = R^3 and let

$$u(t,x,z) = \min\{t,x,z\}.$$

Then, u is a viscosity solution of the equation —*Ut* + *H(Vu)* = 0, where

$$H(p_x,p_z) = (p« -_{Pz})^2 + 2\{_{Px} + p_z - I)^2 - 1$$

is strictly convex. We note that $S^2(u)$ is equal to the line spanned by (1,1,1) and

$$\nabla_* u(s,s,s) = \{(1,0,0),(0,1,0),(0,0,1)\} \qquad \forall s \in \mathbf{R}.$$

In this case m(V*u(0,0,0)) = 2. We note that $S^l(u)$ consists of three halfplanes intersecting each other in the above singular line, with directions orthogonal to the triangle generated by V*u(0,0,0). This example describes the typical situation analyzed in Theorem 3.2.

EXAMPLE 3.2. Let $u : R^3$ -» R be the function

$$u(t,x,z) = -\sqrt{x^2 + (|z| + t^2)^2}.$$

The equality

$$V\overline{a^2 + /3^2} = \sup\{aa + bp : a \geq 0,\ b \geq 0,\ a^2 + b^2 < l\} \qquad a, /3 \geq 0$$

implies that $y/\overline{(p^2 + i>^2}$ is a convex function whenever *cp* and V are non negative convex functions. In particular, u is a concave function. The origin belongs to $S^2\{u)$ and

$$9^+u(0,0,0) = \{0\} x \bar{B}i, \qquad m\{\{0)\ x\ dBi) = I.$$

The singularity in the origin propagates in singularities of degree 1. In fact, the origin is the only point in $S^2(u)$, $S^1(u) = \{(t,x,0) : t \neq 0\}$ and

$$\partial^+ u(t,x,0) = \left\{ \left(\frac{-2t^3}{u(t,x,0)}, \frac{-x}{u(t,x,0)}, \frac{t^2\rho}{u(t,x,0)}\right) : |\rho| \leq 1 \right\} \qquad \forall (t,x,0) \in S^1(u).$$

Finally, we note that

$$\nabla u(t,x,z) = \left(\frac{-2t\big(|z| + t^2\big)}{u(t,x,z)}, \frac{-x}{u(t,x,z)}, \frac{-y\big(|z| + t^2\big)}{|z| u(t,x,z)} \right) \qquad \forall (x,z,t) \in S^0(u),$$

so that u is a solution of the equation (3.1) with

$$F(t,x,z,p_t,p_x,p_z) = -p_t + |p_x|^2 + |p_z|^2 - 1 + \frac{2t\big(|z| + t^2\big)}{\sqrt{x^2 + \big(|z| + t^2\big)^2}}.$$

The function F satisfies (3.2), (3.3) and the extremality condition (3.8).

EXAMPLE 3.3. Let $\Omega = \mathbf{R}^3$, $y = (t,x)$ with $t \in \mathbf{R}$ and $x \in \mathbf{R}^2$. The function

$$u(t,x) = \begin{cases} t/2 - |x| - 1 & \text{if } |x| + 2 \geq t; \\ \\ \dfrac{|x|^2}{2(2-t)} & \text{if } |x| + 2 < t \end{cases}$$

is a viscosity solution of the equation $-u_t + |\nabla u|^2/2 = 0$. We note that $(2,0) \in S^3(u)$, and

$$\nabla_* u(2,0) = \left\{ (p_t, p_x) \in \mathbf{R} \times \mathbf{R}^2 : |p_x| \leq 1,\ p_t = \frac{|p_x|^2}{2} \right\}.$$

Moreover, $S^2(u)$ is the halfline $(t,0)$ with $t < 2$. The unit vector $\theta = (-1,0)$ belongs to $\mathrm{T}\big(S^2(u),(2,0)\big)$ even though $m\big(\nabla_* u((2,0),\theta)\big) = 1$.

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