NAMT
92-023

A Dynamic Programming Approach to Nonlinear Boundary Control Problems of Parabolic Type

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Research Report No. 92-NA-023

July 1992

Sponsors

U.S. Army Research Office
Research Triangle Park
NC 27709

National Science Foundation
1800 G Street, N.W.
Washington, DC 20550

A dynamic programming approach to nonlinear boundary control problems of parabolic type

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Abstract. **In** this paper we study a Hamilton-Jacobi equation related to the boundary control of a parabolic equation with Neumann boundary conditions. The state space of this problem is a Hilbert space and the equation is defined classically only on a dense subset of the state space. Moreover the Hamiltonian appearing in the equation contains fractional powers of an unbounded operator. These facts render the problem difficult. In this paper we give a revised definition of a viscosity solution to accommodate the unboundedness of the Hamiltonian. We then obtain existence .and uniqueness results for viscosity solutions. In particular we show that under suitable assumptions the value function of the boundary control problem is the unique viscosity solution of the related Hamilton-Jacobi equation.

Key words. Optimal control, value fuction, Hamilton-Jacobi equation, boundary control, viscosity solutions, parabolic equations.

AMS (MOS) subject classification. 49cl5, 49c20, 46cO5.

1. Introduction.

This paper is concerned with the Hamilton-Jacobi equation,

(1.1) $$Xu(x) + H(x,CVu\{x)) - (Ax + F(x), Vu(x)> = 0 \qquad x\ 6\ X,$$

where X is a real Hilbert space with norm $|\bullet|$ and scalar product $(\bullet,\bullet)$, A is a positive real number, $F : X$ —• X, while if, and u are real valued and defined, respectively, on X x I, and X. The operator A is the generator of an analytic semigroup in X. We assume that A is self-adjoint and strictly dissipative (see [20]) and has a dense domain. Operator C in (1.1) is a fractional power of —A. More precisely we confine our analysis to the case of $C = (-A)^{\beta}$, with $\beta \in]\frac{1}{4}, \frac{1}{2}[$.

Equation (1.1) is a generalization of the dynamic programming equation related to the boundary control of a parabolic equation with Neumann boundary conditions. We continue with a brief describtion of this boundary control problem. Let fl C H^N be a bounded, open

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domain with smooth boundary. For a given initial condition $x_0 \in L^2(\Omega)$ and a control process $\gamma \in L^2(0,T;L^2(\partial\Omega))$ consider the state equation

$$(1.2)\qquad \begin{cases} \frac{\partial x}{\partial t}(t,\xi) = \Delta_\xi x(t,\xi) + f(x(t,\xi)) & \text{on } (0,+\infty)\times\Omega \\ x(0,\xi) = x_0(\xi) & \text{on } \Omega \\ \frac{\partial x}{\partial n}(t,\xi) = \gamma(t,\xi) & \text{on } (0,+\infty)\times\partial\Omega \end{cases}$$

where $f:\mathbf{R}\to\mathbf{R}$ is a given function (see [17] and [18] for partial differential equations with boundary conditions of this kind). Let a continuous function $L : L^2(\Omega)\times L^2(\partial\Omega)\to\mathbf{R}$, and a bounded subset Γ of $L^2(\partial\Omega)$ be given. Then the control problem is to chose a control $\gamma:\mathbf{R}^+\to\Gamma$ so as to minimize the functional,

$$(1.3)\qquad J(x_0,\gamma) = \int_0^{+\infty} e^{-\lambda t} L(x(t,\cdot),\gamma(t,\cdot))dt$$

over all measurable controls taking values in Γ. In (1.3), $x(t,\cdot)$ is the solution of (1.2). To establish the connection between this boundary control problem and the equation (1.1), we define the value function by:

$$(1.4)\qquad v(x_0) = \inf_{\gamma:\mathbf{R}^+\to\Gamma \text{ measurable}} J(x_0,\gamma).$$

If v is differentiable on $L^2(\Omega)$, then it is well known that v satisfies the dynamic programming equation which is an equation of type (1.1) with $X = L^2(\Omega)$. See Section 2 for a rigorous derivation of the dynamic programming equation.

An important special case is obtained when f is linear and L is quadratic. In the control literature these type of problems are known as the linear quadratic boundary control problems. Due to the elegant feedback form of its optimal controls, linear quadratic boundary control problems have been studied extensively. We refer the reader to Lasiecka and Triggiani [16] and to the forthcoming book by Bensoussan, Da Prato, Delfour and Mitter [5]. Also Hamilton-Jacobi equations in infinite dimensions have been studied by Barbu and Da Prato ([2] and [3]) when the running cost is convex and the state equation is linear.

In this paper we study the boundary control problem (1.2)-(1.3) with general f and L. In fact, more generally, we study the Hamilton-Jacobi equation (1.1). We then treat the dynamic programming equation related to the boundary control problem as a special case of (1.1). The main purpose of this paper is to obtain a suitable notion of a viscosity solution which will allow us to prove uniqueness and existence results for (1.1).

In finite dimensions viscosity solutions to Hamilton-Jacobi equations were first defined by Crandall and Lions [8] (also see Crandall, Evans and Lions [9]). Then several infinite dimensional problems were studied by Crandall and Lions [10], Ishii [15] Soner [22], and Tataru [23]. In all these papers the operator C was assumed to be bounded. The chief contribution of this paper is to extend the viscosity theory to equations with unbounded Hamiltonians.

As we discussed earlier the main difficulty in analyzing (1.1) is to choose suitable relaxations of the unbounded terms $\langle Ax, \nabla u(x)\rangle$ and $H(x, C\nabla u(x))$ appearing in (1.1). These relaxations will then be used to define the notion of a viscosity subsolution and a supersolution of (1.1).

We treat the linear unbounded term $\langle Ax, \nabla u(x)\rangle$ in (1.1) as in Tataru [23]. We then follow Ishii's ideas [15] to relax the term $H(x, C\nabla u(x))$. For problems with a bounded C, Ishii defines the term $H(x, C\nabla u(x))$ roughly as the "limit" of $H(y, C\nabla u(y))$. This "limit" is taken on sequences y converging to x and and at which $H(y, C\nabla u(y))$ is defined. However due to the unboundedness of C, we have to further smoothen the term ∇u, see Section 2.5 below. This smoothening is achieved by integral operators from an interesting class related to the operator A. Of course as one would expect, it is the existence part of the theory which forces us to introduce this further approximation.

Our uniqueness proof is related to the one of [10]. We also systematically apply interpolation inequalities on fractional powers of unbounded operators.

When the value function of our optimal control problem happens to be Lipschitz continuous with respect the negative fractional powers of $(-A)$, a simpler existence and uniqueness theory is available, as the semidifferentials of the value function v enjoy a useful spatial regularity property, i.e.

$$D_x^{\pm} v \subset D(-A)^{\alpha}, \qquad \forall \alpha \in]0,1[. \tag{1.5}$$

Such a regularity property was first obtained in [6] for distributed control problems (or equivalently when C is bounded). Also for boundary control problems a similar continuity result holds under suitable assumptions. For example (1.5) holds, when the discount factor λ in (1.3) is greater than the Lipschitz norm of the nonlinear term f or when the state equation contains a distributed control z as well as the boundary control γ, i.e.

$$\begin{cases} \frac{\partial x}{\partial t}(t,\xi) = \Delta_\xi(t,\xi) + f(x(t,\xi)) + z(t,\xi) & \text{on } (0,+\infty)\times\Omega \\ x(0,\xi) = x_0(\xi) & \text{on } \Omega \\ \frac{\partial x}{\partial n}(t,\xi) = \gamma(t,\xi) & \text{on } (0,+\infty)\times\partial\Omega \end{cases} \tag{1.6}$$

(see [15] and section 6 below).

The paper is organized as follows. In section 2 we recall some basic results on evolution equations, fractional powers, generalized differentials and boundary control. In particular, we recall the connection between problem (1.3) and equation (1.1). In section 3, we define viscosity solutions of equation (1.1) and prove a comparison result for continuous sub and super solutions. In section 4 we study the value function v of problem (1.3) and show that it is a viscosity solution of equation (1.1). In the same section we prove a Lipschitz regularity result for v with respect to the negative fractional powers of $(-A)$. In section 5 we outline a simplified version of our existence and uniqueness results for solutions that has the property (1.5). In section 6 we use the results of the previous sections to study a control problem associated to (1.6).

2. Notation and preliminaries.

2.1 Notation

Let X and Y be two Hilbert spaces (or subsets of them). We denote by $C(X;Y)$ the space of all continuous functions $f : X \to Y$ and by $C^1(X;Y)$ the space of all continuously Fréchet differentiable functions $g : X \to Y$. We denote by $BUC(X;Y)$ the set of all functions $w : X \to Y$ that are bounded and uniformly continuous, with norm:

$$\|w\|_\infty = \sup\{|w(x)|_Y \ ; \quad x \in X\}$$

and by $Lip(X,Y)$ the set of all Lipschitz continuous functions $w : X \to Y$, with the usual seminorm

$$|w|_{Lip} = \sup\left\{ \frac{|w(x_1) - w(x_2)|_Y}{|x_1 - x_2|_X} \ , \quad x_1, x_2 \in X; \ x_1 \neq x_2 \right\}.$$

The set of all continuous linear operators B from X to Y will be denoted by $\mathcal{L}(X;Y)$ with norm $|\cdot|$ Finally, if X is finite dimensional, $L^2(X,Y)$ stands for the space of all measurable functions $\gamma(\cdot) : X \to Y$ such that $|\gamma(\cdot)|^2$ is integrable.

When the space Y is the real line $\mathbf{R}$, we will suppress it in our notation. So, for example, $C(X;\mathbf{R})$ will be replaced by $C(X)$, and so on. We set

$$C^{1,1}(X) = \left\{ f \in C^1(X;\mathbf{R}) \ : \ \nabla f \in Lip(X;X) \right\},$$

and denote by $C^{1,1}_A(X)$ the subspace of $C^{1,1}(X)$ which consists af all functions f satisfying, for every $\alpha \in [0,1[$ and for every $x \in X$,

$$\begin{cases} \text{i) } \nabla f \in Lip(D((-A)^\alpha); D((-A)^\alpha)), \\ \text{ii) } \nabla f(x) \in D((-A)^\alpha) \Leftrightarrow x \in D((-A)^\alpha). \end{cases}$$

Finally $C_\omega(X)$ is the set of all weakly sequentially continuous functions $f : X \to \mathbf{R}$.

Let $A : D(A) \subset X \to X$ be a densely defined closed linear operator satisfying

$$(2.1.2) \qquad \begin{cases} \text{i) } A = A^*, \\ \\ \text{ii) } A \text{ is strictly disssipative, i.e.} \\ \qquad \exists \omega > 0 \text{ such that } \langle Ax, x\rangle \leq -\omega |x|^2, \ \forall x \in D(A). \end{cases}$$

It is well known that i) and ii) imply that A is a generator of an analytic semigroup of operators e^{tA} in X, for $t \geq 0$.

Moreover the fractional powers of $(-A)$, $(-A)^\alpha$ with $\alpha \in \mathbf{R}$, have the following properties (see [20]).

i) For $\alpha \geq 0$, $(-A)^\alpha$ is a closed unbounded operator on X with a dense domain $D((-A)^\alpha)$ and

$$D((-A)^0) = X \; ; \qquad D((-A)^1) = D(A) \; ;$$
$$\alpha \leq \beta \Longrightarrow D((-A)^\beta) \subset D((-A)^\alpha)$$

ii) For $\alpha > 0$, $(-A)^{-\alpha}$ is a continuous linear operator. Moreover

$$(-A)^{-\alpha} \in \mathcal{L}(X; D((-A)^\alpha))$$

In particular, when $(-A)^{-1}$ is compact, $(-A)^{-\alpha} : X \to X$ is a compact operator for every $\alpha > 0$.

iii) For every $\alpha \in [0,1]$ there exists a positive constant M_α such that:

$$|(-A)^\alpha e^{tA} x| \leq \frac{M_\alpha}{t^\alpha} |x| \qquad \forall x \in X. \tag{2.1.3}$$

iv) Let $\alpha \in]0, \frac{1}{2}[$. Then for every $\sigma > 0$ there exists $C_\sigma > 0$ such that

$$|(-A)^\alpha x| \leq \sigma |(-A)^{\frac{1}{2}} x| + C_\sigma |x|, \qquad \forall x \in D((-A)^{\frac{1}{2}}) \tag{2.1.4}$$

$$|(-A)^{\alpha-1} x|^2 \leq \sigma |x|^2 + C_\sigma |(-A)^{-\frac{1}{2}} x|^2, \qquad \forall x \in X \tag{2.1.5}$$

2.2 The State Equation for Boundary Control Problems.

Let U be a Hilbert space, Γ be a bounded subset of U and set $\|\Gamma\| = \sup \{ |\gamma|_U \, ; \, \gamma \in \Gamma \}$. Let $\gamma : \mathbf{R} \to \Gamma$ and consider the following integral equation,

$$x(t) = e^{tA} x_0 + \int_0^t e^{(t-s)A} F(x(s)) ds + (-A)^\beta \int_0^t e^{(t-s)A} B\gamma(s) ds, \qquad x_0 \in X \tag{2.2.1}$$

where

$$\begin{cases} \text{i) } A \text{ satisfies (2.1.2)} \\ \text{ii) } \beta \in]\frac{1}{4}, \frac{1}{2}[\\ \text{iii) } F \in Lip(X; X). \end{cases} \tag{2.2.2}$$

and

$$\begin{cases} \text{i) } \gamma : \mathbf{R}^+ \to \Gamma \text{ is measurable and } \Gamma \subset U \text{ is bounded.} \\ \text{ii) } B \in \mathcal{L}(U; X). \end{cases} \tag{2.2.3}$$

Formally, equation (2.2.1) can be rewritten in the following way:

$$\begin{cases} x'(t) = Ax(t) + F(x(t)) + (-A)^{\beta} Bu(t) \\ x(0) = x_0 \end{cases}$$

Equation (2.2.1) is the abstract form of (1.2). We continue by explaining this fact. Let $X = L^2(\Omega)$, $U = L^2(\partial\Omega)$. **Define the Neumann map** $N : U \to X$ **by**

$$(2.2.4) \qquad N\phi = w \iff \begin{cases} \Delta w = w & \text{in } \Omega \\ \frac{\partial w}{\partial n} = \phi & \text{in } \partial\Omega. \end{cases}$$

Notice that $N : L^2(\partial\Omega) \longrightarrow H^{\frac{3}{2}}(\Omega)$ **(the Sobolev space of fractional order). We now define an unbounded operator** A **by**

$$(2.2.5) \qquad \begin{cases} D(A) = \{\phi \in H^2(\Omega) \ : \ \frac{\partial \phi}{\partial n} = 0\} \\ Ax = \Delta x - x. \end{cases}$$

It is well known that (see [18])

$$(2.2.6) \qquad D((-A)^{\theta}) = \begin{cases} H^{2\theta}(\Omega) & \text{for } \ 0 \le \theta < \frac{3}{4} \\ \{\phi \in H^{2\theta}(\Omega) \ : \ \frac{\partial \phi}{\partial n} = 0\} & \text{for } \ \frac{3}{4} < \theta < 1. \end{cases}$$

Therefore, the Neumann map defined in (2.2.4) satisfies:

$$(2.2.7) \qquad N \in \mathcal{L}(U, D((-A)^{\frac{3}{4}-\epsilon}))$$

for every $\epsilon > 0$. **Set now**

$$N_{\beta} = (-A)^{1-\beta} N.$$

Then it is easy to see that

$$(2.2.8) \qquad N_{\beta} \in \mathcal{L}(U, X).$$

We now define a nonlinear map $F : X \longrightarrow X$ **by**

$$(2.2.9) \qquad F(x)(t) = f(x(t)) + x(t).$$

Then by elementary computations, we can rewrite the state equation (1.2) in the mild form:

$$(2.2.10) \qquad x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A} F(x(s))\,ds + (-A)^{\beta} \int_0^t e^{(t-s)A} N_{\beta} u(s)\,ds \qquad t \geq 0$$

Clearly the above equation is a special case of (2.2.1) with $B = N_{\beta}$.

The following proposition contains well known results on (2.2.1), (see [13], [14], and [18] for similar results).

PROPOSITION 2.2.2 *Let $x_0 \in X$ and $\gamma : \mathbf{R}^+ \to \Gamma$ measurable. Under assumptions (2.2.2) and (2.2.3) there exists a unique mild solution $x(\cdot; x_0, \gamma)$ of the equation (2.2.1). Moreover for any small $\sigma, \varepsilon > 0$ we have:*

$$x(\cdot; x_0, \gamma) \in C(\mathbf{R}^+; X) \cap C([\sigma, +\infty[; D((-A)^{1-\beta-\varepsilon})) \tag{2.2.11}$$

Finally, we have, for some $C > 0$,

$$|x(t; x_0, \gamma) - e^{tA} x_0| \leq C t^{1-\beta} \qquad \forall x_0 \in X. \tag{2.2.12}$$

We give the proof of (2.2.12) for the reader's convenience.

PROOF. Set $x(t) = x(t; x_0, \gamma)$. By (2.2.1) we have:

$$\begin{aligned} |x(t) - e^{tA} x_0| &\leq \\ \leq +\left|\int_0^t e^{(t-s)A} F(x(s)) ds\right| & \qquad \textbf{(A)} \\ +\left|\int_0^t (-A)^\beta e^{(t-s)A} B\gamma(s) ds\right| & \qquad \textbf{(B)} \end{aligned} \tag{2.2.13}$$

We study every single term of (2.2.13).

(A) Due to the continuity of $F(\cdot)$ and $x(\cdot)$

$$\left|\int_0^t e^{(t-s)A} F(x(s)) ds\right| \leq t \sup_{s \in [0,t]} |F(x(s))|$$

(B) Using the boundedness of Γ, inequality (2.1.3), and the fact that $\frac{1}{4} < \beta < \frac{1}{2}$ (see assumption (2.1.1) and (2.2.2)ii) respectively) we obtain

$$\begin{aligned} \left|\int_0^t (-A)^\beta e^{(t-s)A} B\gamma(s) ds\right| &\leq \\ \leq \int_0^t \frac{M_\beta}{(t-s)^\beta} \|B| \, \|\Gamma\| ds &\leq \\ \leq M_\beta |B| \, \|\Gamma\| t^{1-\beta} & \end{aligned}$$

which concludes the proof of (2.2.12).

Q.E.D.

2.3 The control problem and the Hamilton - Jacobi equation.

We assume that the running cost $L(\cdot,\cdot) : X \times U \to \mathbf{R}$ satisfies

(2.3.1)
$$\begin{cases} i) & L \text{ is continuous and bounded i.e.} \\ & L \in C(X \times U);\ |L(x,\gamma)| \le L_\infty, \text{ for all } (x,\gamma) \in X \times U \text{ and some constant } L_\infty > 0 \\ ii) & |L(x,\gamma) - L(y,\gamma)| \le L_0 |x-y|,\ \forall x,y \in X,\ \forall \gamma \in \Gamma; \text{ for some } L_0 > 0 \end{cases}$$

For $\lambda > 0$, and a control function

$$\gamma \in \mathcal{A} \stackrel{def}{=} \left\{ \gamma : \mathbf{R}^+ \to \Gamma \ : \ \gamma(\cdot) \text{ is measurable} \right\},$$

we take the mild solution $x(\cdot; x, \gamma)$ of the state equation (2.2.1) and the pay off functional given by

$$J(x,\gamma) = \int_0^{+\infty} e^{-\lambda t} L(x(t;x,\gamma),\gamma(t))dt \tag{2.3.2}$$

which we seek to minimize overall $\gamma \in \mathcal{A}$.

Then, under assumptions (2.3.1), (2.2.2) and (2.2.3), the value function

$$v(x) = \inf_{\gamma \in \mathcal{A}} J(x,\gamma) \tag{2.3.3}$$

satisfies the Dynamic Programming Principle (see [11] and [19]): for every $x_0 \in X$ and $t > 0$

$$v(x) = \inf_{\gamma \in \mathcal{A}} \left\{ \int_0^t e^{-\lambda s} L(x(s;x,\gamma),\gamma(s))ds + e^{-\lambda t} v(x(t)) \right\} \stackrel{def}{=} \inf_{\gamma \in \mathcal{A}} J_t(x,\gamma) \tag{2.3.4}$$

REMARK 2.3.2. Let $\varepsilon > 0$. If $\gamma_\varepsilon \in \mathcal{A}$ is an ε−optimal control, i.e.

$$v(x) > J(x,\gamma_\varepsilon) - \varepsilon.$$

Then, for every $t \in \mathbf{R}^+$ we have

$$v(x) > J_t(x,\gamma_\varepsilon) - \varepsilon, \tag{2.3.5}$$

(see [12] ch. 1). Formula (2.3.5) easily follows from the fact that

$$J_t \le J \qquad \forall t \in \mathbf{R}^+.$$

The Hamilton-Jacobi equation related to problem (2.3.2) and (2.3.3) is

$$\lambda u(x) + H(x, (-A)^\beta \nabla u(x)) - \langle Ax + F(x), \nabla u(x) \rangle = 0 \qquad x \in X. \tag{2.3.6}$$

where

$$H(x,p) = \sup_{\gamma\in\Gamma}\{-\langle B\gamma, p\rangle - L(x,\gamma)\} \qquad \forall x,p\in X \tag{2.3.7}$$

If the value function v is continuously differentiable on X and ∇u is contained in $D((-A)^\beta)$, then the fact that v is a solution of (2.3.6) on $D(A)$ is well known and can be proved exactly as in the finite dimensional case (see for instance [19]).

REMARK 2.3.3. Hypotheses (2.2.3) and (2.3.1) implies that the Hamiltonian H given by (2.3.7) satisfies:

$$\begin{cases} (i) \quad |H(x,p) - H(x,q)| \le H_0|p-q| \\ (ii) \quad |H(x,p) - H(y,p)| \le H_0|x-y| \end{cases} \tag{2.3.8}$$

for some $H_0 > 0$. To prove comparison results, we only need to assume (2.3.8) on H. So, in this context, equation (2.3.6) is not necessarily related to a control problem.

2.4. Semidifferentials

Let Ω be an open subset of X and $\psi : \Omega \to \mathbf{R}$. For any $x_0 \in \Omega$, the *super- and sub-differentials* $D^+\psi(x_0)$, $D^-\psi(x_0)$ are defined as follows (see e.g. [9])

$$D^+\psi(x_0) = \left\{p\in X \,|\, \limsup_{x\to x_0} \frac{\psi(x)-\psi(x_0)-\langle p, x-x_0\rangle}{|x-x_0|} \le 0\right\} \tag{2.4.1}$$

$$D^-\psi(x_0) = \left\{p\in X \,|\, \liminf_{x\to x_0} \frac{\psi(x)-\psi(x_0)-\langle p, x-x_0\rangle}{|x-x_0|} \ge 0\right\} \tag{2.4.2}$$

The function ψ is Fréchet differentiable at x_0, if and only if $D^+\psi(x_0)$ and $D^-\psi(x_0)$ are both non-empty. Moreover in this case:

$$D^+\psi(x_0) = D^-\psi(x_0) = \{D\psi(x_0)\} \tag{2.4.3}$$

where $D\psi$ denotes the Fréchet derivative.

The following fact is well known and will be used in section 5.

LEMMA 2.4.1. *Let $\alpha \in]0,1[$, and $v : X \to \mathbf{R}$ be such that:*

$$|v(x) - v(y)| \le C|(-A)^{-\alpha}(x-y)| \qquad \forall x,y\in X \tag{2.4.4}$$

for an appropriate constant $C > 0$. Then:

$$D^\pm v(x) \subset D((-A)^\alpha) \qquad \forall\alpha \in]0,1[\tag{2.4.5}$$

and

$$\sup_{x\in X}\ \sup_{p\in D^\pm v(x)} |(-A)^\alpha p| < C \tag{2.4.6}$$

2.5. A class of integral operators.

We introduce a class of convolution operators that will be used in the definition of viscosity solutions.

Given an unbounded operator A that satisfies (2.1.2), we denote by $M(A)$ the set of all maps $M : [0,1] \to C(X)$ such that

(2.5.1)
$$\begin{cases} i) & |M_t x| \leq |x| \quad \forall x \in X \\ ii) & M_t x \in D(A) \quad \forall x \in X, \forall t > 0 \\ iii) & M_t x \xrightarrow{t \to 0} x \\ iv) & M_t^* = M_t \\ v) & AM_t^* = M_t A \end{cases}$$

Examples of operators in $M(A)$ are the following.

1) Consider a function $s : [0,1] \to [0,1]$ such that $s(t) \leq t$ for every $t \in [0,1]$ and define:

$$\overline{M}_t x = e^{s(t)A} x.$$

2) For $t > 0$ and $x \in X$ set

(2.5.2) $$\widehat{M}_t x \stackrel{def}{=} - f e^{sA} x ds = \mathbf{i} \, f e^{\wedge - \wedge} x ds = \wedge \frac{tA - I}{} \mathrm{i4}''^2 x$$

Then it is easy to show that both $\overline{M}_t$ and $\widehat{M}_t$ belongs to $M(A)$.

The following lemma will be useful in the proof of the comparison result.

LEMMA 2.5.1. *For every* $x \in X$ *we have the following*

(2.5.3)
(i) $$0 \leq \langle -A\widehat{M}_t x, \widehat{M}_t x\rangle \leq \langle -A\widehat{M}_t x, x\rangle$$
(ii) $$\langle -A e^{M} x, x\rangle \leq \langle -A\widehat{M}_t x, x\rangle$$

PROOF. By the definition of $\widehat{M}_t$

$$\langle -A\widehat{M}_t x, \widehat{M}_t x\rangle = \frac{1}{t^2} \int_0^t \int_0^t \langle -Ae^{rA}x, e^{sA}x\rangle dr ds$$

and

$$\langle -A\widehat{M}_t x, x\rangle = \frac{1}{t^2} \int_0^t \int_0^t \langle -Ae^{sA}x, x\rangle dr ds.$$

Hence,

(2.5.4) $$\langle -A\widehat{M}_t x, \widehat{M}_t x\rangle - \langle -A\widehat{M}_t x, x\rangle = \frac{1}{t^2} \int_0^t \int_0^t \langle -Ae^{sA}[e^{rA} - I]x, x\rangle dr ds.$$

Moreover for every $y \in X$ we have:

(2.5.5)
$$\langle -A[e^{rA} - I]y, y\rangle = - \int_0^r \langle A^2 e^{\rho A} y, y\rangle d\rho =$$
$$= - \int_0^r |Ae^{\rho A/2} y|^2 d\rho \leq 0.$$

Now use the above estimate with $y = e^{sA}x$ in (2.5.5) to arrive at the first of (2.5.3). The second inequality is easily proved with similar arguments.

Q.E.D.

3. Definition of viscosity solution and comparison results.

Let us now consider the Hamilton-Jacobi equation:

$$\lambda u(x) + H(x, (-A)^{\beta} \nabla u(x)) - \langle Ax + F(x), \nabla u(x) \rangle = 0 \tag{3.1}$$

where $\lambda > 0$ and H is a Hamiltonian satisfying (2.5.3), but otherwise not necessarily related to a control problem. Throughout this section we will assume that (2.2.2), (2.2.3) and (2.3.8) hold true.

Let $M(A)$ be the set defined in subsection 2.5 (formula (2.5.1)). We now define the viscosity sub- and super-solutions of (3.1). In the following $x \in \operatorname{argmax} u - \phi$ (argmin) is an abbreviation to say that $u(x) - \phi(x) = \max\{u(y) - \phi(y), y \in X\}$ (min).

DEFINITION 3.1. *Let* $u \in BUC(X) \cap C_{\omega}(X)$.

i) u is a viscosity subsolution of (3.1) if for every $\phi \in C^{1,1}(X)$ *we have,*

$$\begin{aligned} &\lambda u(x) - \langle F(x), \nabla \phi(x) \rangle + \\ &+ \inf_{\mathcal{M} \in M(A)} \liminf_{t \downarrow 0} \left\{ \frac{\phi(x) - \phi(e^{tA}x)}{t} + H\big(x, (-A)^{\beta} \mathcal{M}_t \nabla \phi(e^{tA}x)\big) \right\} \leq 0 \end{aligned} \tag{3.2}$$

at every $x \in \operatorname{argmax}(u - \phi)$.

ii) u is a viscosity supersolution of (3.1) if for every $\phi \in C^{1,1}(X)$ *we have,*

$$\begin{aligned} &\lambda u(x) - \langle F(x), \nabla \phi(x) \rangle + \\ &+ \sup_{\mathcal{M} \in M(A)} \limsup_{t \downarrow 0} \left\{ \frac{\phi(x) - \phi(e^{tA}x)}{t} + H\big(x, (-A)^{\beta} \mathcal{M}_t \nabla \phi(e^{tA}x)\big) \right\} \geq 0 \end{aligned} \tag{3.3}$$

at every $x \in \operatorname{argmin}(u - \phi)$.

Finally, $u \in BUC(X) \cap C_{\omega}(X)$ *is a viscosity solution of (3.1) if it is both viscosity subsolution and supersolution of (3.1).*

REMARK 3.2. In the above definition we have considered two approximations of the terms containing derivatives of u, namely the ratio:

$$\frac{\phi(x) - \phi(e^{tA}x)}{t}$$

to approximate the term $\langle Ax, \nabla \phi(x) \rangle$ (see also [23]), and the regularized term:

$$H(x, (-A)^{\beta} \mathcal{M}_t \nabla \phi(e^{tA}x))$$

to approximate $H(x, (-A)^{\beta} \nabla \phi(x))$.

Should the maximum (resp. minimum) point x in (3.2) (resp. (3.3)) belong to $D(A)$, the above inequalities would be equivalent to:

$$\lambda u(x) - \langle F(x), \nabla\phi(x)\rangle - \langle Ax, \nabla\phi(x)\rangle + \inf_{\mathcal{M}\in M(A)} \liminf_{t\downarrow 0}\left\{ H\big(x, (-A)^{\beta}\mathcal{M}_t\nabla\phi(e^{tA}x)\big)\right\} \leq 0$$

and

$$\lambda u(x) - \langle F(x), \nabla\phi(x)\rangle - \langle Ax, \nabla\phi(x)\rangle + \sup_{\mathcal{M}\in M(A)} \limsup_{t\downarrow 0}\left\{ H\big(x, (-A)^{\beta}\mathcal{M}_t\nabla\phi(e^{tA}x)\big)\right\} \geq 0$$

The above inequalities could be simplified further, provided that $\nabla\phi(x) \in D(A)$ for all $x \in D(A)$. In that case we would have:

$$\lambda u(x) - \langle F(x), \nabla\phi(x)\rangle - \langle Ax, \nabla\phi(x)\rangle + H\big(x, (-A)^{\beta}\nabla\phi(x)\big) \leq 0$$

and

$$\lambda u(x) - \langle F(x), \nabla\phi(x)\rangle - \langle Ax, \nabla\phi(x)\rangle + H\big(x, (-A)^{\beta}\nabla\phi(x)\big) \geq 0.$$

We shall use this observation in the proof of Theorem 3.3.

THEOREM 3.3. *Assume that (2.2.2), (2.2.3) and (2.3.8) hold true. Let $u, v \in BUC(X) \cap C_\omega(X)$ be a viscosity sub and super solution of the Hamilton-Jacobi equation (3.1), respectively. Then*

$$u(x) \leq v(x), \qquad \forall x \in X. \tag{3.4}$$

PROOF. For simplicity we take $\lambda = 1$. For $\varepsilon > 0$ consider the function $\Psi_\varepsilon : X \times X \to \mathbf{R}$ defined as:

$$\Psi_\varepsilon(x,y) := u(x) - v(y) - \frac{1}{2\varepsilon}\langle(-A)^{-1}(x-y), x-y\rangle \tag{3.5}$$

and, for $\mu > 0$ define the test function $\Phi_{\varepsilon,\mu} : X \times X \to \mathbf{R}$:

$$\Phi_{\varepsilon,\mu}(x,y) := \Psi_\varepsilon(x,y) - \frac{\mu}{2}[|x|^2 + |y|^2]. \tag{3.6}$$

Observe that our assumptions on A and the weak continuity and boundedness of u and v imply that

1) Ψ_ε is sequentially weakly lower semicontinuous on $X \times X$,
2) $\Phi_{\varepsilon,\mu}(x,y) \leq \|u\|_\infty + \|v\|_\infty - \mu[|x|^2 + |y|^2]$.

Therefore there exists a point $(x_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \in X \times X$ such that

$$\Phi_{\varepsilon,\mu}(x_{\varepsilon,\mu}, y_{\varepsilon,\mu}) = \max_{X\times X} \Phi_{\varepsilon,\mu}. \tag{3.7}$$

Set now $z_{\varepsilon,\mu} = x_{\varepsilon,\mu} - y_{\varepsilon,\mu}$ and, for $r > 0$,

$$m(r) = \sup_{|x-y|\leq r} |u(x) - u(y)| + |v(x) - v(y)|.$$

It is clear that

$$m(r) \leq 2(\|u\|_\infty + \|v\|_\infty). \tag{3.8}$$

We complete the proof in several steps.

Step I.

We claim that

$$\frac{1}{\varepsilon}\langle(-A)^{-1}z_{\varepsilon,\mu}, z_{\varepsilon,\mu}\rangle \le m(|z_{\varepsilon,\mu}|) \tag{3.9}$$

Indeed by (3.7) we have the inequality:

$$\Phi_{\varepsilon,\mu}(x_{\varepsilon,\mu}, x_{\varepsilon,\mu}) + \Phi_{\varepsilon,\mu}(y_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \le 2\Phi_{\varepsilon,\mu}(x_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \tag{3.10}$$

Then

$$\begin{aligned}&[u(x_{\varepsilon,\mu}) - v(x_{\varepsilon,\mu})] + [u(y_{\varepsilon,\mu}) - v(y_{\varepsilon,\mu})] - \mu[|x_{\varepsilon,\mu}|^2 + |y_{\varepsilon,\mu}|^2] \le \\ &\le 2u(x_{\varepsilon,\mu}) - 2v(y_{\varepsilon,\mu}) - \frac{1}{\varepsilon}\langle(-A)^{-1}z_{\varepsilon,\mu}, z_{\varepsilon,\mu}\rangle - \mu[|x_{\varepsilon,\mu}|^2 + |y_{\varepsilon,\mu}|^2].\end{aligned}$$

Hence

$$\frac{1}{\varepsilon}\langle(-A)^{-1}z_{\varepsilon,\mu}, z_{\varepsilon,\mu}\rangle \le [u(x_{\varepsilon,\mu}) - u(y_{\varepsilon,\mu})] + [v(x_{\varepsilon,\mu}) - v(y_{\varepsilon,\mu})] \le m(|z_{\varepsilon,\mu}|). \tag{3.11}$$

Step II. In this Step we will prove that

$$\lim_{\mu\to 0+} \mu[|x_{\varepsilon,\mu}|^2 + |y_{\varepsilon,\mu}|^2] = 0. \tag{3.12}$$

Since Ψ_ε is bounded above, for every $\delta > 0$ there are x_δ, $y_\delta \in X$ such that:

$$\Psi_\varepsilon(x_\delta, y_\delta) \ge \Psi_\varepsilon(x, y) - \delta \qquad \forall x, y \in X.$$

Then the inequality

$$\Phi_{\varepsilon,\mu}(x_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \ge \Phi_{\varepsilon,\mu}(x_\delta, y_\delta)$$

implies that

$$\begin{aligned}\Phi_{\varepsilon,\mu}(x_{\varepsilon,\mu}, y_{\varepsilon,\mu}) &= \Psi_\varepsilon(x_{\varepsilon,\mu}, y_{\varepsilon,\mu}) - \frac{\mu}{2}[|x_{\varepsilon,\mu}|^2 + |y_{\varepsilon,\mu}|^2] \ge \\ \ge \Phi_{\varepsilon,\mu}(x_\delta, y_\delta) &= \Psi_\varepsilon(x_\delta, y_\delta) - \frac{\mu}{2}[|x_\delta|^2 + |y_\delta|^2] \ge \\ &\ge \Psi_\varepsilon(x_{\varepsilon,\mu}, y_{\varepsilon,\mu}) - \delta - \frac{\mu}{2}[|x_\delta|^2 + |y_\delta|^2]\end{aligned}$$

Hence,

$$\limsup_{\mu\to 0+} \frac{\mu}{2}[|x_{\varepsilon,\mu}|^2 + |y_{\varepsilon,\mu}|^2] \le \delta + \lim_{\mu\to 0+} \frac{\mu}{2}[|x_\delta|^2 + |y_\delta|^2] = \delta.$$

Step III. We claim that for every $t_n \xrightarrow{n\to\infty} 0$, $\mathcal{M} \in M(A)$, we have:

$$\forall\varepsilon > 0 \qquad \limsup_{\mu\downarrow 0}\ \limsup_{n\to\infty}\ h_n(\mu, \varepsilon) \le 0 \tag{3.13}$$

where

$$(3.14)\qquad \begin{aligned} h_n(\mu,\varepsilon) \overset{def}{=} & -\frac{\mu}{2t_n}(|x_{\varepsilon,\mu}|^2 - |e^{t_nA}x_{\varepsilon,\mu}|^2) + H_0\mu|(-A)^\beta \mathcal{M}_{t_n} e^{t_nA}x_{\varepsilon,\mu}| + \\ & + \frac{H_0}{\varepsilon}|(-A)^{\beta-1}\mathcal{M}_{t_n}(e^{t_nA} - I)x_{\varepsilon,\mu}| + \mu\Big(|F(0)||x_{\varepsilon,\mu}| + |F|_{Lip}|x_{\varepsilon,\mu}|^2\Big) \end{aligned}$$

Indeed since all operators are bounded and $e^{t_nA}x_{\varepsilon,\mu} \overset{n\to\infty}{\longrightarrow} x_{\varepsilon,\mu}$, we have

$$\frac{H_0}{\varepsilon}|(-A)^{\beta-1}\mathcal{M}_{t_n}(e^{t_nA} - I)x_{\varepsilon,\mu}| \overset{n\to\infty}{\longrightarrow} 0$$

.

Also since $\beta < \frac{1}{2}$ the interpolation inequality (2.1.4) yields that for every $\sigma > 0$ there exists C_σ such that

$$(3.15)\qquad \begin{aligned} |(-A)^\beta \mathcal{M}_{t_n} e^{t_nA}x_{\varepsilon,\mu}| &\le \sigma|(-A)^{\frac{1}{2}}\mathcal{M}_{t_n} e^{t_nA}x_{\varepsilon,\mu}| + C_\sigma|x_{\varepsilon,\mu}| \le \\ \le \sigma|(-A)^{\frac{1}{2}} e^{t_nA}x_{\varepsilon,\mu}| + C_\sigma|x_{\varepsilon,\mu}| &\le \sigma|(-A)^{\frac{1}{2}}\widehat{\mathcal{M}}_{t_n}x_{\varepsilon,\mu}| + C_\sigma|x_{\varepsilon,\mu}| \le \\ &\le \sigma\Big(\langle(-A)\widehat{\mathcal{M}}_{t_n}x_{\varepsilon,\mu}, x_{\varepsilon,\mu}\rangle\Big)^{\frac{1}{2}} + C_\sigma|x_{\varepsilon,\mu}| \end{aligned}$$

where we have used property v) of the operator $\mathcal{M}_t$ (see (2.5.1)) and inequality (2.5.3)(ii). Thus

$$(3.16)\qquad H_0\mu|(-A)^\beta \mathcal{M}_{t_n} e^{t_nA}x_{\varepsilon,\mu}| \le \mu H_0\sigma\Big(\langle(-A)\widehat{\mathcal{M}}_{t_n}x_{\varepsilon,\mu}, x_{\varepsilon,\mu}\rangle\Big)^{\frac{1}{2}} + \mu H_0 C_\sigma|x_{\varepsilon,\mu}|$$

Moreover

(3.17)

$$\begin{aligned} -\frac{1}{t}(|x_{\varepsilon,\mu}|^2 - |e^{tA}x_{\varepsilon,\mu}|^2) &= -\frac{1}{t}\langle x_{\varepsilon,\mu} + e^{tA}x_{\varepsilon,\mu}, x_{\varepsilon,\mu} - e^{tA}x_{\varepsilon,\mu}\rangle = \\ = +\langle x_{\varepsilon,\mu} + e^{tA}x_{\varepsilon,\mu}, \frac{1}{t}\int_0^t Ae^{sA}x_{\varepsilon,\mu}ds\rangle &= +\langle x_{\varepsilon,\mu}, A\widehat{\mathcal{M}}_t x_{\varepsilon,\mu}\rangle + \langle e^{\frac{t}{2}A}x_{\varepsilon,\mu}, A\widehat{\mathcal{M}}_t e^{\frac{t}{2}A}x_{\varepsilon,\mu}\rangle. \end{aligned}$$

By inequality (2.5.3)(i), both terms in the right hand side of (3.17) are negative. Hence, for every $\mu > 0$

$$(3.18)\qquad -\frac{\mu}{2t}(|x_{\varepsilon,\mu}|^2 - |e^{tA}x_{\varepsilon,\mu}|^2) \le \frac{\mu}{2}\langle A\widehat{\mathcal{M}}_t x_{\varepsilon,\mu}, x_{\varepsilon,\mu}\rangle.$$

Using (3.16) and (3.18) we obtain for every $\sigma > 0$

$$\begin{aligned} h_n(\mu,\varepsilon) \le \frac{\mu}{2}\langle A\widehat{\mathcal{M}}_{t_n}x_{\varepsilon,\mu}, x_{\varepsilon,\mu}\rangle &+ \mu H_0\sigma\Big(\langle -A\widehat{\mathcal{M}}_{t_n}x_{\varepsilon,\mu}, x_{\varepsilon,\mu}\rangle\Big)^{\frac{1}{2}} + \\ &+ \mu H_0 C_\sigma|x_{\varepsilon,\mu}| + \mu\Big(|F(0)||x_{\varepsilon,\mu}| + |F|_{Lip}|x_{\varepsilon,\mu}|^2\Big). \end{aligned}$$

Now, choosing $\sigma = \sigma_0 = \frac{1}{2H_0}$,

$$\begin{aligned} h_n(\mu,\varepsilon) \le \frac{\mu}{2}\langle A\widehat{\mathcal{M}}_{t_n}x_{\varepsilon,\mu}, x_{\varepsilon,\mu}\rangle &+ \frac{\mu}{2}\Big(\langle -A\mathcal{M}_{t_n}x_{\varepsilon,\mu}, x_{\varepsilon,\mu}\rangle\Big)^{\frac{1}{2}} + \\ &+ \mu H_0 C_{\sigma_0}|x_{\varepsilon,\mu}| + \mu\Big(|F(0)||x_{\varepsilon,\mu}| + |F|_{Lip}|x_{\varepsilon,\mu}|^2\Big). \end{aligned}$$

Since $\frac{\mu}{2}(w - w^2) \le \mu$ for every $w \in \mathbf{R}$, by (3.12) we easily derive (3.13).

Step IV. Set

(3-19) $$r_{\varepsilon,\mu} = \frac{1}{\varepsilon}(-A)^{-1}(x_{\varepsilon,\mu} - y_{\varepsilon,\mu})$$

and

$$h(\mu,\varepsilon) = \limsup_{n\to\infty} h_n(\mu,\varepsilon)$$

Then we claim that for every $p > 0$ there exists $t_n \xrightarrow{n\to\infty} 0$, and $M \in \mathcal{M}(A)$ such that

(3.20) $$u(x_{\varepsilon,\mu}) - \langle F(x_{\varepsilon,\mu}), r_{\varepsilon,\mu}\rangle + \frac{1}{\varepsilon}\langle x_{\varepsilon,\mu}, x_{\varepsilon,\mu} - y_{\varepsilon,\mu}\rangle + \\ + \lim H\{x_{\varepsilon,\mu}, (-A)^{\beta} \ldots$$

Indeed let

(3.21) $$\phi(x) = v(y_{\varepsilon,\mu}) + \frac{1}{2\varepsilon}\langle(-A)^{-1}(x - y_{\varepsilon,\mu}), x - y_{\varepsilon,\mu}\rangle + \frac{\mu}{2}\left[|x|^2 + |y_{\varepsilon,\mu}|^2\right]$$

or equivalently:

(3.22) $$\phi(x) = u(x) - \Phi_{\varepsilon,\mu}(x, y_{\varepsilon,\mu})$$

Hence

(3.23) $$(u - \phi)(x_{\varepsilon,\mu}) = \max_{x} (u - \phi)(x)$$

Then from definition 3.1 we have that $\forall p > 0$, $\exists t_n \downarrow 0$, and $\exists M \in \mathcal{M}(A)$, such that

(3.24) $$u(x_{\varepsilon,\mu}) - \langle F(x_{\varepsilon,\mu}), \nabla\phi(x_{\varepsilon,\mu})\rangle + \\ + \lim_{n\to\infty}\left\{\frac{\phi(x_{\varepsilon,\mu}) - \phi(e^{t_n A}x_{\varepsilon,\mu})}{t_n} + H\left(x_{\varepsilon,\mu}, (-A)^{\beta}\mathcal{M}_{t_n} \nabla\phi(e^{t_n A}x_{\varepsilon,\mu})\right)\right\} \le p$$

Recalling the definition of $r_{\varepsilon,\mu}$ we have

(3.25) $$\nabla\phi(e^{tA}x_{\varepsilon,\mu}) = r_{\varepsilon,\mu} + \frac{1}{\varepsilon}(-A)^{-1}(e^{tA}x_{\varepsilon,\mu} - x_{\varepsilon,\mu}) + \mu e^{tA}x_{\varepsilon,\mu}$$

and

(3.26) $$\nabla\phi(x_{\varepsilon,\mu}) = r_{\varepsilon,\mu} + \mu x_{\varepsilon,\mu}.$$

So,

(3.27) $$\frac{\phi(x_{\varepsilon,\mu}) - \phi(e^{tA}x_{\varepsilon,\mu})}{t} = \\ = \frac{1}{2\varepsilon t}\left[\langle(-A)^{-1}x_{\varepsilon,\mu}, x_{\varepsilon,\mu}\rangle - \langle(-A)^{-1}e^{tA}x_{\varepsilon,\mu}, e^{tA}x_{\varepsilon,\mu}\rangle\right] + \\ + \frac{1}{\varepsilon}\langle(-A)^{-1}\frac{1}{t}(e^{tA} - I)x_{\varepsilon,\mu}, y_{\varepsilon,\mu}\rangle + \frac{\mu}{2t}\left(|x_{\varepsilon,\mu}|^2 - |e^{tA}x_{\varepsilon,\mu}|^2\right).$$

Observe that

$$\lim_{t\downarrow 0}\frac{1}{2\varepsilon t}\Big[\langle(-A)^{-1}x_{\varepsilon,\mu},x_{\varepsilon,\mu}\rangle-\langle(-A)^{-1}e^{tA}x_{\varepsilon,\mu},e^{tA}x_{\varepsilon,\mu}\rangle\Big]=$$
$$=\lim_{t\downarrow 0}\frac{1}{2\varepsilon}\langle x_{\varepsilon,\mu}+e^{tA}x_{\varepsilon,\mu},(-A)^{-1}\frac{(I-e^{tA})}{t}x_{\varepsilon,\mu}\rangle=\frac{1}{\varepsilon}\langle x_{\varepsilon,\mu},x_{\varepsilon,\mu}\rangle$$

and

$$\lim_{t\downarrow 0}\frac{1}{\varepsilon}\langle(-A)^{-1}\frac{e^{tA}-I}{t}x_{\varepsilon,\mu},y_{\varepsilon,\mu}\rangle=-\frac{1}{\varepsilon}\langle x_{\varepsilon,\mu},y_{\varepsilon,\mu}\rangle.$$

Therefore

$$\begin{aligned}&\frac{\phi(x_{\varepsilon,\mu})-\phi(e^{tA}x_{\varepsilon,\mu})}{t}=\\&=\frac{1}{\varepsilon}\langle x_{\varepsilon,\mu},x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\rangle+\frac{\mu}{2t}(|x_{\varepsilon,\mu}|^2-|e^{tA}x_{\varepsilon,\mu}|^2)+\omega_0(t)\end{aligned}\tag{3.28}$$

where $\lim_{t\to 0}\ \omega_0(t)=0$. Moreover, by (3.25) and assumption (3.2) v), for every $\mathcal{M}\in M(A)$

$$\begin{aligned}&|H(x_{\varepsilon,\mu},(-A)^{\beta}\mathcal{M}_t\nabla\phi(e^{tA}x_{\varepsilon,\mu}))-H(x_{\varepsilon,\mu},(-A)^{\beta}\mathcal{M}_t r_{\varepsilon,\mu})|\le\\&\le H_0\left|(-A)^{\beta}\mathcal{M}_t\Big[\mu e^{tA}x_{\varepsilon,\mu}+\frac{1}{\varepsilon}(-A)^{-1}(e^{tA}x_{\varepsilon,\mu}-x_{\varepsilon,\mu})\Big]\right|\le\\&\le\frac{H_0}{\varepsilon}|(-A)^{\beta-1}\mathcal{M}_t(e^{tA}-I)x_{\varepsilon,\mu}|+H_0\mu|(-A)^{\beta}\mathcal{M}_te^{tA}x_{\varepsilon,\mu}|.\end{aligned}$$

Therefore for every $\mathcal{M}\in M(A)$

$$\begin{aligned}&H(x_{\varepsilon,\mu},(-A)^{\beta}\mathcal{M}_t\nabla\phi(e^{tA}x_{\varepsilon,\mu}))\ge H(x_{\varepsilon,\mu},(-A)^{\beta}\mathcal{M}_t r_{\varepsilon,\mu})-\\&-\frac{H_0}{\varepsilon}|(-A)^{\beta-1}\mathcal{M}_t(e^{tA}-I)x_{\varepsilon,\mu}|-H_0\mu|(-A)^{\beta}\mathcal{M}_te^{tA}x_{\varepsilon,\mu}|.\end{aligned}\tag{3.29}$$

Finally we easily estimate the term containing F in (3.24) by (3.26) and the Lipschitz continuity of F:

$$\begin{aligned}&-\langle F(x_{\varepsilon,\mu}),\nabla\phi(x_{\varepsilon,\mu})\rangle\ge\\&\ge-\langle F(x_{\varepsilon,\mu}),r_{\varepsilon,\mu}\rangle-\mu(|F(0)||x_{\varepsilon,\mu}|+|F|_{Lip}|x_{\varepsilon,\mu}|^2)\end{aligned}\tag{3.30}$$

Using (3.28), (3.29), (3.30) into (3.24) (for $t=t_n$) and the definition of $h_n(\mu,\varepsilon)$ we conclude that

$$\begin{aligned}&u(x_{\varepsilon,\mu})-\langle F(x_{\varepsilon,\mu}),r_{\varepsilon,\mu}\rangle+\\&+\frac{1}{\varepsilon}\langle x_{\varepsilon,\mu},x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\rangle+\lim_{n\to\infty}\Big\{H(x_{\varepsilon,\mu},(-A)^{\beta}\mathcal{M}_{t_n}r_{\varepsilon,\mu})+\omega_0(t)-h_n(\mu,\varepsilon)\Big\}\le 0\end{aligned}$$

which directly implies the claim (3.20).

Step V. For $s_n \to 0$ and $\mathcal{N} \in M(A)$ define the function $k_n(\mu,\varepsilon)$ similarly to h_n (exchanging $x_{\varepsilon,\mu}$ by $y_{\varepsilon,\mu}$) and set

$$k(\mu,\varepsilon) \stackrel{def}{=} \limsup_{n\to\infty} k_n(\mu,\varepsilon)$$

Then, using the same arguments of step III and IV we conclude that

$$\limsup_{\mu\downarrow 0} k(\mu,\varepsilon) \le 0 \tag{3.31}$$

and for every $\rho > 0$ there exists $s_n \xrightarrow{n\to\infty} 0$, and $\mathcal{N} \in M(A)$ such that

$$\begin{aligned} &v(y_{\varepsilon,\mu}) - \langle F(y_{\varepsilon,\mu}), r_{\varepsilon,\mu}\rangle + \frac{1}{\varepsilon}\langle y_{\varepsilon,\mu}, x_{\varepsilon,\mu} - y_{\varepsilon,\mu}\rangle + \\ &+ \lim_{n\to\infty} H(y_{\varepsilon,\mu}, (-A)^{\beta}\mathcal{N}_{s_n} r_{\varepsilon,\mu}) \ge -\rho - k(\mu,\varepsilon) \end{aligned} \tag{3.32}$$

Step VI. *Conclusion.*

Fix $\rho > 0$ and $\varepsilon > 0$. Subtract (3.32) from (3.20) to obtain

$$\begin{aligned} &u(x_{\varepsilon,\mu}) - v(y_{\varepsilon,\mu}) + \frac{|z_{\varepsilon,\mu}|^2}{\varepsilon} \le \\ &\le \langle F(x_{\varepsilon,\mu}) - F(y_{\varepsilon,\mu}), \frac{1}{\varepsilon}(-A)^{-1} z_{\varepsilon,\mu}\rangle \qquad \text{(I)} \\ &+ \lim_{n\to\infty} \left\{ H\left(y_{\varepsilon,\mu}, (-A)^{\beta}\mathcal{N}_{s_n} r_{\varepsilon,\mu}\right) - H\left(x_{\varepsilon,\mu}, (-A)^{\beta}\mathcal{M}_{t_n} r_{\varepsilon,\mu}\right)\right\} \qquad \text{(II)} \\ &+ 2\rho + h(\mu,\varepsilon) + k(\mu,\varepsilon) \end{aligned} \tag{3.33}$$

Next, we estimate the first two terms appearing in the right-hand side of (3.33).

(I) First recall the elementary fact that, given $a, b > 0$, for every $\sigma > 0$ we have:

$$2ab \le \sigma a^2 + \frac{1}{\sigma} b^2. \tag{3.34}$$

Since $F \in Lip(X;X)$, we estimate

$$\begin{aligned} &\langle F(x_{\varepsilon,\mu}) - F(y_{\varepsilon,\mu}), \frac{1}{\varepsilon}(-A)^{-1} z_{\varepsilon,\mu}\rangle \le \\ &\le \frac{1}{\varepsilon}|F|_{Lip}|z_{\varepsilon,\mu}||(-A)^{-1} z_{\varepsilon,\mu}| \le \\ &\le \frac{1}{2\varepsilon}|F|_{Lip}\left[\frac{1}{2|F|_{Lip}}|z_{\varepsilon,\mu}|^2 + 2|F|_{Lip}|(-A)^{-1} z_{\varepsilon,\mu}|^2\right] \le \\ &\le \frac{1}{4\varepsilon}|z_{\varepsilon,\mu}|^2 + |F|^2_{Lip}|(-A)^{-\frac{1}{2}}|\frac{1}{\varepsilon}\langle(-A)^{-1} z_{\varepsilon,\mu}, z_{\varepsilon,\mu}\rangle \end{aligned} \tag{3.35}$$

(II) Using the definition of $r_{\varepsilon,\mu}$ in (3.19) and the assumptions (2.3.8) we obtain

$$
\begin{aligned}
&H(y_{\varepsilon,\mu},(-A)^{\beta}\mathcal{N}_s r_{\varepsilon,\mu}) - H(x_{\varepsilon,\mu},(-A)^{\beta}\mathcal{M}_t r_{\varepsilon,\mu}) \leq \\
&\leq H_0\frac{1}{\varepsilon}|(-A)^{\beta-1}(\mathcal{M}_t-\mathcal{N}_s)z_{\varepsilon,\mu}| + H_0|z_{\varepsilon,\mu}|\Big(1+\frac{1}{\varepsilon}|(-A)^{\beta-1}\mathcal{M}_t z_{\varepsilon,\mu}|\Big)
\end{aligned} \tag{3.36}
$$

We now estimate the right hand side of (3.36). First by the properties of $M(A)$ we have

$$\lim_{s,t\to 0} H_0\frac{1}{\varepsilon}|(-A)^{\beta-1}(\mathcal{M}_t-\mathcal{N}_s)z_{\varepsilon,\mu}| = 0. \tag{3.37}$$

Moreover

$$H_0|z_{\varepsilon,\mu}|\Big(1+\frac{1}{\varepsilon}|(-A)^{\beta-1}\mathcal{M}_t z_{\varepsilon,\mu}|\Big) \leq H_0|z_{\varepsilon,\mu}| + H_0\frac{1}{\varepsilon}|z_{\varepsilon,\mu}||(-A)^{\beta-1}\mathcal{M}_t z_{\varepsilon,\mu}|. \tag{3.38}$$

Observe that, by (3.34) and (2.1.5), for any $\sigma>0$ there exists $\overline{C}_\sigma>0$ satisfying

$$|z_{\varepsilon,\mu}||(-A)^{\beta-1}z_{\varepsilon,\mu}| \leq \sigma|z_{\varepsilon,\mu}|^2 + \overline{C}_\sigma\langle(-A)^{-1}z_{\varepsilon,\mu},z_{\varepsilon,\mu}\rangle. \tag{3.39}$$

Use (3.39) with $\sigma=\sigma_1=\frac{1}{4H_0}$ to estimate the last term in (3.38)

$$
\begin{aligned}
&H_0|z_{\varepsilon,\mu}|\frac{1}{\varepsilon}|(-A)^{\beta-1}z_{\varepsilon,\mu}| \leq \frac{|z_{\varepsilon,\mu}|^2}{4\varepsilon}+ \\
&+\frac{1}{\varepsilon}H_0\overline{C}_{\sigma_1}\langle(-A)^{-1}z_{\varepsilon,\mu},z_{\varepsilon,\mu}\rangle
\end{aligned} \tag{3.40}
$$

This concludes our estimate of the second term in (3.33). In sum, setting $t=t_n$ and $s=s_n$, and letting n go to $+\infty$ by (3.36) — (3.40) we get

$$
\begin{aligned}
&\lim_{n\to\infty}\Big\{H(y_{\varepsilon,\mu},(-A)^{\beta}\mathcal{N}_{s_n}r_{\varepsilon,\mu}) - H(x_{\varepsilon,\mu},(-A)^{\beta}\mathcal{M}_{t_n}r_{\varepsilon,\mu})\Big\} \leq \\
&\leq \frac{|z_{\varepsilon,\mu}|^2}{4\varepsilon} + H_0|z_{\varepsilon,\mu}| + \frac{1}{\varepsilon}H_0\overline{C}_{\sigma_1}\langle(-A)^{-1}z_{\varepsilon,\mu},z_{\varepsilon,\mu}\rangle
\end{aligned} \tag{3.41}
$$

We can now substitute (3.35) and (3.41) into (3.33) to obtain

$$
\begin{aligned}
&u(x_{\varepsilon,\mu}) - v(y_{\varepsilon,\mu}) + \frac{|z_{\varepsilon,\mu}|^2}{2\varepsilon} \leq \\
&\leq 2\rho + h(\mu,\varepsilon) + k(\mu,\varepsilon) + \frac{1}{\varepsilon}C_2\langle z_{\varepsilon,\mu},(-A)^{-1}z_{\varepsilon,\mu}\rangle + H_0|z_{\varepsilon,\mu}|
\end{aligned} \tag{3.42}
$$

where we have set $C_2 = \Big(H_0\overline{C}_{\sigma_1} + |F|^2_{Lip}|(-A)^{-\frac{1}{2}}|\Big)$.

Now we recall that, by (3.14), /i(/x,e) + fc(/x,e) is bounded in a neighbourhood of 0. Moreover, by (3.9) and (3.8) $(z_{ty}\wedge\{-Ay^{l}z_{t}\wedge)$ is bounded, and u and v are bounded by hypothesis. Then, inequality (3.42) yields

(3.43) $$\frac{\text{k a d i}}{2\varepsilon} \leq \#_0|2_{\text{£)M}}| + c_z$$

for some C3 > 0, for every sufficiently small /x and e. Clearly this implies that

(3.44) $$t\,|\,|\,£ \leq C_4$$

for some $C_4 > 0$. Substitute (3.44) in (3.42) and use (3.9) to conclude that

(3.45) $$u\{x_e\wedge) - v\{y_e\wedge) \leq 2p + H_0\sqrt{2C_4e} + 4C_2m(\sqrt{2C_4£}) + /i(/x,e) + k(\mu,\varepsilon).$$

Since $(x_{£,M},y_{e>/i})$ is a maximum point for \$£,/!, for every x € X we have

(3.46) $$u(x) - t/(ar) - /x|x|^2 = *_{Ct/1}(x,x) \leq \$e,/i(^xe,M\wedge\wedge M) \wedge + k(\mu,\varepsilon).$$
$$\leq u\{x_e\wedge) - v(y_{CfM}) \leq 2p + H_Qy/2C_4e + 4C_2m(y/2C_4e) + /i(/x,e)$$

Let $(i \longrightarrow 0$ in (3.46) to obtain

$$?x(x) - v(x) \leq 2p + H_0y/2Cke + 4C_2m(y/2C\wedge e) \qquad \text{V x G l}$$

Since p and £ are arbitrary, (3.5) follows from the above inequality.

Q.E.D.

The following uniqueness result is a straightforward consequence of the above comparison theorem.

COROLLARY 3.4. *Assume that (3.2) hold true. Then equation (3.1) has at most one viscosity solution in the sense of Definition 3.1.*

REMARK 3,5. By the proof of theorem 3.3 it follows that the comparison result (3.5) still hold true if we take test functions <f> e $C\backslash^{l}(X)$ in definition 3.1. Indeed the test function appearing in the proof of theorem 3.3 clearly belong to $C)\wedge^{l}\{X)$. We will use this fact in sections 5 and 6.

4. Properties of the value function v and existence results

In this section we show that the value function of the problem (2.3.2) is the only viscosity solution of the Hamilton-Jacobi equation

(4.1) $$Xu(x) + F(x, (-A)^*Vu(x)) - (Ax + F(x), Vu(x)) = 0$$

where the Hamiltonian H is as in (2.5.2). Since the Hamiltonian H defined in (2.3.7) satisfies hypotheses of Theorem 3.3 (c.f. Remark 2.3.3), the uniqueness of viscosity solutions to (4.1) follows from the results of previous section. In this section we will prove that, if the operator $(-A)^{-1}$ is compact, then the value function $v \in BUC(X) \cap C_\omega(X)$ and is a viscosity solution of (4.1) . Throughout this section we assume that (2.2.2),(2.2.3) and (2.3.1) hold true and take $\lambda > 0$.

We first study the regularity properties of the value function v.

PROPOSITION 4.1. *The value function v is Holder continuous on X with respect to the norm of $(-A)^{-\theta}$ for every $\theta \in [0,1[$. More precisely, for any $\sigma < \frac{\lambda}{|F|_{Lip}}$ and $\sigma \leq 1$ we have:*

$$|v(x) - v(y)| \leq C_{\theta,\sigma} |(-A)^{-\theta}(x-y)|^\sigma \qquad \forall x, y \in X; \quad \forall \theta \in [0,1[\tag{4.3}$$

Therefore, if the operator A has a compact resolvent, then v is weakly sequentially continuous.

PROOF. We adapt some ideas of [6] and [15].
Let $x, y \in X$ and $\gamma \in \mathcal{A}$ be given. Set $x(t) := x(t; x, \gamma)$ and $y(t) := x(t; y, \gamma)$. Then by (2.1.3) we have, for every $t \in [0, T]$

$$\begin{aligned} |x(t) - y(t)| &\leq |e^{tA}(x-y)| + |F|_{Lip} \int_0^t |x(s) - y(s)| ds \leq \\ &\leq \frac{M_\theta}{t^\theta} |(-A)^{-\theta}(x-y)| + |F|_{Lip} \int_0^t |x(s) - y(s)| ds. \end{aligned} \tag{4.4}$$

Now set $\eta(t) = \int_0^t |x(s) - y(s)| ds$ and integrate the above inequality to obtain

$$\eta(t) \leq \frac{M_\theta}{1-\theta} |(-A)^{-\theta}(x-y)| t^{1-\theta} + |F|_{Lip} \int_0^t \eta(s) ds.$$

Then, by Gronwall's inequality we can estimate $\eta(t)$. Substituting this estimate into (4.4) yields

$$|x(t) - y(t)| \leq \left[\frac{C_6}{t^\theta} + C_7 e^{|F|_{Lip} t} t^{1-\theta} \right] |(-A)^{-\theta}(x-y)| \tag{4.5}$$

for some $C_6, C_7 > 0$. For every $\sigma \in]0, \frac{\lambda}{|F|_{Lip}}[$, $\sigma \leq 1$ (4.5) implies that

$$|x(t) - y(t)|^\sigma \leq 2^\sigma \left[\frac{C_6^\sigma}{t^{\theta\sigma}} + C_7^\sigma t^{(1-\theta)\sigma} e^{\sigma |F|_{Lip} t} \right] |(-A)^{-\theta}(x-y)|^\sigma \tag{4.6}$$

We also have

$$|L(x(t),\gamma) - L(y(t),\gamma)| \leq (2L_\infty)^{1-\sigma} |L(x(t),\gamma) - L(y(t),\gamma)|^\sigma \leq \widehat{L} |x(t) - y(t)|^\sigma \tag{4.7}$$

where $\widehat{L} = (2L_\infty)^{1-\sigma} L_0^\sigma$. Choose $T > 0$ satisfying

$$\frac{2e^{-\lambda T} L_\infty}{\lambda} \le |(-A)^{-\theta}(x-y)|^\sigma. \tag{4.8}$$

By the Optimality Principle, Proposition 2.3.1, there exists $\gamma \in \mathcal{A}$ such that

$$v(y) > \int_0^T e^{-\lambda t} L(y(t), \gamma(t))dt + e^{-\lambda T} v(y(T)) - |(-A)^{-\theta}(x-y)|^\sigma. \tag{4.9}$$

Again, the principle of optimality, formulas (4.7) and (4.9), the fact that $|v(x)| \le L_\infty$ for every $x \in X$ and the choice of T in (4.8) yield

$$\begin{aligned} v(x) - v(y) &< \int_0^T e^{-\lambda t}[L(x(t), \gamma(t)) - L(y(t), \gamma(t))]dt + \\ &+ e^{-\lambda T}[v(x(T)) - v(y(T))] + |(-A)^{-\theta}(x-y)|^\sigma \le \\ &\le \widehat{L} \int_0^T e^{-\lambda t} |x(t) - y(t)|^\sigma dt + 2|(-A)^{-\theta}(x-y)|^\sigma \end{aligned} \tag{4.10}$$

Finally, by (4.6) and (4.10) we obtain

$$\begin{aligned} &|v(x) - v(y)| \le 2|(-A)^{-\theta}(x-y)|^\sigma + \\ &+ \widehat{L} 2^\sigma \int_0^T \left[C_6^\sigma \frac{e^{-\lambda t}}{t^{\theta\sigma}} + C_7^\sigma t^{(1-\theta)\sigma} e^{(\sigma|F|_{Lip} - \lambda)t} \right] dt \; |(-A)^{-\theta}(x-y)|^\sigma. \end{aligned}$$

Since $\sigma < \frac{\lambda}{|F|_{Lip}}$, (4.3) follows from the above inequality.

Q.E.D.

Motivated by Proposition 4.1 in the remainder of this paper we assume that

$$(-A)^{-1} : X \to X \text{ is a compact operator} \tag{4.11}$$

This assumption, together with proposition 4.1, guarantees the weak continuity of the value function v. The main result of this section is the following.

THEOREM 4.4. *Assume that (2.2.2), (2.2.3), (2.3.1) and (4.11) hold true, and let* $\lambda > 0$. *Then the value function v is a viscosity solution of (4.1), in the sense of Definition 3.1.*

PROOF. Take $\lambda = 1$.

Step I. In this step we will show that the value function is a subsolution of (4.1). Fix $\overline{x} \in X$, $\overline{\gamma} \in \Gamma$. Consider a constant control function $\overline{\gamma}(t) \equiv \overline{\gamma}$ for every $t \ge 0$. Set $\overline{x}(t) = x(t; \overline{x}; \overline{\gamma})$. Then by the Principle of Optimality (2.3.4) we have

$$v(\overline{x}) \le \int_0^t e^{-s} L(\overline{x}(s), \overline{\gamma})ds + e^{-t} v(\overline{x}(t))$$

which implies that

$$
(4.12)\qquad \begin{aligned} &v(\overline{x}) - L(\overline{x},\overline{\gamma}) + \frac{v(\overline{x}) - v(\overline{x}(t))}{t} \leq \\ &\leq \frac{1}{t}\int_0^t e^{-s}L(\overline{x}(s),\overline{\gamma})ds - L(\overline{x},\overline{\gamma}) + v(\overline{x}) - \frac{1-e^{-t}}{t}v(\overline{x}(t)) \stackrel{def}{=} \omega_2(t) \xrightarrow{t\to 0^+} 0 \end{aligned}
$$

due to the continuity of $\overline{x}(\cdot)$, $v(\cdot)$ and $L(\cdot,\overline{\gamma})$. Now suppose that, for some $\phi \in C^{1,1}(X)$,

$$
(4.13)\qquad v(\overline{x}) - \phi(\overline{x}) = \max(v-\phi) \geq v(\overline{x}(t)) - \phi(\overline{x}(t).
$$

Then, for every $t > 0$, (4.12) and (4.13) yield

$$
(4.14)\qquad v(\overline{x}) - L(\overline{x},\overline{\gamma}) + \frac{\phi(\overline{x}) - \phi(\overline{x}(t))}{t} \leq \omega_2(t) \xrightarrow{t\to 0^+} 0
$$

Moreover

$$
(4.15)\qquad \frac{\phi(\overline{x}) - \phi(\overline{x}(t))}{t} = \frac{\phi(\overline{x}) - \phi(e^{tA}\overline{x})}{t} + \frac{\phi(e^{tA}\overline{x}) - \phi(\overline{x}(t))}{t}.
$$

Since ϕ is differentiable

$$
(4.16)\qquad \begin{aligned} &\frac{\phi(e^{tA}\overline{x}) - \phi(\overline{x}(t))}{t} = \frac{1}{t}\langle\nabla\phi(\xi(t)), e^{tA}\overline{x} - \overline{x}(t)\rangle = \\ &= \langle\nabla\phi(e^{tA}\overline{x}), (-A)^{\beta}\widehat{\mathcal{M}}_t B\overline{\gamma}\rangle - \langle\nabla\phi(e^{tA}\overline{x}), \frac{1}{t}\int_0^t e^{(t-s)A}F(\overline{x}(s))ds\rangle - \\ &- \frac{1}{t}\langle\nabla\phi(\xi(t)) - \nabla\phi(e^{tA}\overline{x}), \overline{x}(t) - e^{tA}\overline{x}\rangle \end{aligned}
$$

where $\xi(t) = \lambda(t)e^{tA} + (1-\lambda(t))\overline{x}(t)$ for some $\lambda : \mathbf{R}^+ \to]0,1[$, and $\widehat{\mathcal{M}}_t = \frac{1}{t}\int_0^t e^{sA}xds$ as in (2.6.2).

¿From formula (2.2.11), we have

$$
|\frac{1}{t}\langle\nabla\phi(\xi(t)) - \nabla\phi(e^{tA}\overline{x}), \overline{x}(t) - e^{tA}\overline{x}\rangle| \leq |\nabla\phi|_{Lip}\frac{|\overline{x}(t) - e^{tA}\overline{x}|^2}{t} \xrightarrow{t\to 0^+} 0,
$$

uniformly with respect to $\overline{\gamma} \in \Gamma$. Moreover,

$$
\begin{aligned} &\langle\nabla\phi(e^{tA}\overline{x}), \frac{1}{t}\int_0^t e^{(t-s)A}F(\overline{x}(s))ds\rangle - \langle\nabla\phi(\overline{x}), F(\overline{x})\rangle = \\ &= \langle\nabla\phi(e^{tA}\overline{x}) - \nabla\phi(\overline{x}), \frac{1}{t}\int_0^t e^{(t-s)A}F(\overline{x}(s))ds\rangle + \\ &+ \langle\nabla\phi(\overline{x}), \frac{1}{t}\int_0^t e^{(t-s)A}[F(\overline{x}(s)) - F(\overline{x})]ds\rangle + \langle\nabla\phi(\overline{x}), (\widehat{\mathcal{M}}_t - I)F(\overline{x})\rangle \stackrel{def}{=} I_1 + I_2 + I_3 \end{aligned}
$$

where

$$
\begin{aligned} |I_1| &\leq \sup_{s\in[0,t]} |F(\overline{x}(s))||\nabla\phi|_{Lip}|e^{tA}\overline{x} - \overline{x}| \\ |I_2| &\leq |\nabla\phi(\overline{x})| \sup_{s\in[0,t]} |F(\overline{x}(s)) - F(\overline{x})| \\ |I_3| &\leq |\nabla\phi(\overline{x})||\widehat{\mathcal{M}}_t F(\overline{x}) - F(\overline{x})|. \end{aligned}
$$

Therefore,

$$\langle \nabla\phi(e^{tA}\overline{x}), \frac{1}{t}\int_0^t e^{(t-s)A}F(\overline{x}(s))ds\rangle = \langle \nabla\phi(\overline{x}), F(\overline{x})\rangle + \omega_3(t) \xrightarrow{t\to 0^+} 0, \tag{4.17}$$

uniformly with respect to $\overline{\gamma} \in \Gamma$. By (4.14) — (4.17) we thus obtain

$$\begin{aligned} &v(\overline{x}) - \langle \nabla\phi(\overline{x}), F(\overline{x})\rangle + \frac{\phi(\overline{x}) - \phi(e^{tA}\overline{x})}{t} - L(\overline{x},\overline{\gamma}) - \\ &- \langle (-A)^\beta \widehat{\mathcal{M}}_t \nabla\phi(e^{tA}\overline{x}), B\overline{\gamma}\rangle \le \omega_4(t) \xrightarrow{t\to 0^+} 0 \end{aligned} \tag{4.18}$$

where $\omega_4(\cdot)$ is independent of $\overline{\gamma} \in \Gamma$. In the above estimate we have used the equality

$$\langle (-A)^\beta \widehat{\mathcal{M}}_t \nabla\phi(e^{tA}\overline{x}), B\overline{\gamma}\rangle = \langle \nabla\phi(e^{tA}\overline{x}), (-A)^\beta \widehat{\mathcal{M}}_t B\overline{\gamma}\rangle.$$

So, taking the supremum over all $\overline{\gamma} \in \Gamma$ in (4.18) we conclude that

$$\begin{aligned} &v(\overline{x}) + \langle \nabla\phi(\overline{x}), F(\overline{x})\rangle + \frac{\phi(\overline{x}) - \phi(e^{tA}\overline{x})}{t} + \\ &+ H(\overline{x}, (-A)^\beta \widehat{\mathcal{M}}_t \nabla\phi(e^{tA}\overline{x})) \le \omega_5(t) \xrightarrow{t\to 0^+} 0. \end{aligned} \tag{4.19}$$

Letting $t \downarrow 0$ (4.19) yields

$$\begin{aligned} &v(\overline{x}) + \langle \nabla\phi(\overline{x}), F(\overline{x})\rangle + \\ &+ \limsup_{t\downarrow 0} \left\{ \frac{\phi(\overline{x}) - \phi(e^{tA}\overline{x})}{t} + H(\overline{x}, (-A)^\beta \widehat{\mathcal{M}}_t \nabla\phi(e^{tA}\overline{x})) \right\} \le 0 \end{aligned}$$

which implies that

$$\begin{aligned} &v(\overline{x}) + \langle \nabla\phi(\overline{x}), F(\overline{x})\rangle + \\ &+ \inf_{\mathcal{M}\in M(A)} \limsup_{t\downarrow 0} \left\{ \frac{\phi(\overline{x}) - \phi(e^{tA}\overline{x})}{t} + H(\overline{x}, (-A)^\beta \widehat{\mathcal{M}}_t \nabla\phi(e^{tA}\overline{x})) \right\} \le 0 \end{aligned}$$

Q.E.D. STEP I

Step II. In this step we will now prove that the value function is a supersolution of (4.1). Let $\phi \in C^{1,1}(X)$ and $\overline{x} \in X$ be a (local) minimum point for $v - \phi$. Then, for every $t \ge 0$ we have

$$v(\overline{x}) - v(\overline{x}(t)) \le \phi(\overline{x}) - \phi(\overline{x}(t)). \tag{4.20}$$

Recall that by the Bellmann Optimality Principle (2.3.4) we have

$$v(\overline{x}) = \inf_{\gamma(\cdot)\in\mathcal{A}} \left\{ \int_0^t e^{-s} L(\overline{x}(s), \gamma(s))ds + e^{-t} v(\overline{x}(t)) \right\}. \tag{4.21}$$

Therefore, (4.20) and (4.21) yield

$$(4.22)\qquad \sup_{\gamma(\cdot)\in\mathcal{A}} \left\{ -\frac{1}{t}\int_0^t e^{-s} L(\bar{x}(s),\gamma(s))ds + \frac{(1-e^{-t})}{t} v(\bar{x}(t)) + \frac{\phi(\bar{x})-\phi(\bar{x}(t))}{t} \right\} \geq 0$$

Arguing as in (4.12), we obtain

$$(4.23)\qquad \begin{aligned} -\frac{1}{t}\int_0^t e^{-s} L(\bar{x}(s),\gamma(s))ds + \frac{(1-e^{-t})}{t} v(\bar{x}(t)) &= \\ = -\frac{1}{t}\int_0^t L(\bar{x},\gamma(s))ds + v(\bar{x}) + \omega_6(t) \end{aligned}$$

where $\lim_{t\to0^+}\omega_6(t) = 0$ uniformly with respect to $\gamma(\cdot)\in\mathcal{A}$. Moreover by the same reasoning used in step I, (see (4.15), (4.16) and (4.17)):

$$(4.24)\qquad \begin{aligned} \frac{\phi(\bar{x})-\phi(\bar{x}(t))}{t} &= \frac{\phi(\bar{x})-\phi(e^{tA}\bar{x})}{t} + \frac{\phi(e^{tA}\bar{x})-\phi(\bar{x}(t))}{t} = \\ &= \frac{\phi(\bar{x})-\phi(e^{tA}\bar{x})}{t} - \langle \nabla\phi(\bar{x}), F(\bar{x})\rangle - \\ &- \langle \nabla\phi(e^{tA}\bar{x}), \frac{1}{t}(-A)^{\beta}\int_0^t e^{(t-s)A}B\gamma(s)ds\rangle + \omega_7(t) \end{aligned}$$

where $\lim_{t\to0^+}\omega_7(t) = 0$ uniformly with respect to $\gamma(\cdot)\in\mathcal{A}$. Now, from (4.22), (4.23) and (4.24) it follows that:

(4.25)

$$\begin{aligned} &v(\bar{x}) - \langle \nabla\phi(\bar{x}), F(\bar{x})\rangle + \frac{\phi(\bar{x})-\phi(e^{tA}\bar{x})}{t} + \\ &+ \sup_{\gamma(\cdot)\in\mathcal{A}} \left\{ -\frac{1}{t}\int_0^t L(\bar{x},\gamma(s))ds - \langle \nabla\phi(e^{tA}\bar{x}), \frac{1}{t}(-A)^{\beta}\int_0^t e^{(t-s)A}B\gamma(s)ds\rangle \right\} \geq -\omega_8(t) \end{aligned}$$

where $\lim_{t\to0^+}\omega_8(t) = 0$ uniformly with respect to $\gamma(\cdot)\in\mathcal{A}$.

Now observe that

$$(4.26)\qquad \begin{aligned} &\sup_{\gamma(\cdot)\in\mathcal{A}} \left\{ \frac{1}{t}\int_0^t \left[-L(\bar{x},\gamma(s)) - \langle \nabla\phi(e^{tA}\bar{x}), (-A)^{\beta}\int_0^t e^{(t-s)A}B\gamma(s)\rangle \right] ds \right\} \leq \\ &\leq \frac{1}{t}\int_0^t \sup_{\gamma\in A} \left\{ -L(\bar{x},\gamma) - \langle (-A)^{\beta}e^{(t-s)A}\nabla\phi(e^{tA}\bar{x}), B\gamma\rangle \right\} ds = \\ &= \frac{1}{t}\int_0^t H(\bar{x}, (-A)^{\beta}e^{(t-s)A}\nabla\phi(e^{tA}\bar{x}))ds \end{aligned}$$

Then, by (4.26) and (4.25) we have

$$\begin{aligned} &v(\bar{x}) - \langle \nabla\phi(\bar{x}), F(\bar{x})\rangle \frac{\phi(\bar{x})-\phi(e^{tA}\bar{x})}{t} + \\ &+ \frac{1}{t}\int_0^t H(\bar{x}, (-A)^{\beta}e^{(t-s)A}\nabla\phi(e^{tA}\bar{x}))ds \geq \omega_8(t), \end{aligned}$$

and so, for every $t > 0$ there exists $s_t \in]0,t]$ such that

$$v(\bar{x}) - \langle \nabla\phi(\bar{x}), F(\bar{x})\rangle \frac{\phi(\bar{x}) - \phi(e^{tA}\bar{x})}{t} + H(\bar{x}, (-A)^{\beta} e^{s_t A} \nabla\phi(e^{tA}\bar{x}))ds \geq \omega_8(t)$$

Let $t \downarrow 0$ to obtain

$$v(\bar{x}) - \langle \nabla\phi(\bar{x}), F(\bar{x})\rangle + \\ + \liminf_{t\downarrow 0} \left\{ \frac{\phi(\bar{x}) - \phi(e^{tA}\bar{x})}{t} + H(\bar{x}, (-A)^{\beta} e^{s_t A} \nabla\phi(e^{tA}\bar{x}))ds \right\} \geq 0.$$

As seen in section 2.6, the family of operators $e^{s_t A}, t \in]0,1]$ belongs to $M(A)$, and taking the supremum in $M(A)$ we conclude that

$$v(\bar{x}) - \langle \nabla\phi(\bar{x}), F(\bar{x})\rangle + \\ + \sup_{\mathcal{M} \in M(A)} \liminf_{t\downarrow 0} \left\{ \frac{\phi(\bar{x}) - \phi(e^{tA}\bar{x})}{t} + H(\bar{x}, (-A)^{\beta} \mathcal{M}_t \nabla\phi(e^{tA}\bar{x}))ds \right\} \geq 0,$$

Q.E.D.

5. Solution of the Hamilton-Jacobi equation in the Lipschitz case

We give in this section a simplified version of the existence and uniqueness theory for viscosity solutions of our Hamilton-Jacobi equation:

(5.1) $$\lambda u(x) + H(x, (-A)^{\beta} \nabla u(x)) - \langle Ax + F(x), \nabla u(x)\rangle = 0$$

Troughout this section we assume (2.2.2), (2.2.3), (2.3.1), (4.11) and the following:

(5.2) $$|F|_{Lip} < \lambda$$

Under the hypothesis (5.2) we have (see proposition (4.1)

(5.3) $$\begin{aligned} &\forall \alpha \in [0,1[,\ \exists C_\alpha > 0 \ \ s.t. \\ &|v(x) - v(y)| \leq C_\alpha |(-A)^{-\alpha}(x-y)| \qquad \forall x,y \in X \end{aligned}$$

This fact implies that the semidifferentials of v are contained in the domain of the fractional powers of $(-A)$ (see Lemma 2.4.1):

(5.4) $$\forall x \in X, \quad \forall \alpha \in [0,1[\qquad D^{\pm}v(x) \subset D((-A)^{\alpha})$$

In view of (5.3) and (5.4) we give the following definition.

DEFINITION 5.1. *Consider a function $u \in BUC(X) \cap C_\omega(X)$ such that:*

(5.5) $$|u(x) - u(y)| \leq C|(-A)^{-\beta}(x-y)| \qquad \forall x,y \in X$$

for some constant $C > 0$. Then, we say that u is a viscosity solution of the Hamilton-Jacobi equation (5.1) in X if $\forall \phi \in C_A^{1,1}(X)$ we have

(i) **Subsolution.** *If $u - \phi$ has a local maximum point at $x_0 \in X$, then*

$$u(x_0) + H(x_0, (-A)^\beta \nabla\phi(x_0)) - \langle F(x_0), \nabla\phi(x_0)\rangle + \liminf_{t\downarrow 0} \frac{\phi(x_0) - \phi(e^{tA}x_0)}{t} \leq 0 \tag{5.6}$$

ii) **Supersolution.** *If $u - \phi$ has a local minimum point at $x_0 \in X$, then*

$$u(x_0) + H(x_0, (-A)^\beta \nabla\phi(x_0)) - \langle F(x_0), \nabla\phi(x_0)\rangle + + \limsup_{t\downarrow 0} \frac{\phi(x_0) - \phi(e^{tA}x_0)}{t} \geq 0 \tag{5.7}$$

REMARK 5.2.

A) As one can easily see, if x_0 is a local maximum (minimum) point of $u - \phi$, then by (5.5) and Lemma 2.4.1 it follows that

$$\nabla\phi(x_0) \in D^+u(x_0) \subset D((-A)^\beta)$$

so the term $(-A)^\beta \nabla\phi(x_0)$ in (5.6) and (5.7) is well defined.

B) If, in addition we have that $D^+u(x_0) \in D((-A)^{\frac{1}{2}})$, then $\nabla\phi(x_0), x_0 \in D((-A)^{\frac{1}{2}})$ so that

$$\liminf_{t\downarrow 0} \frac{\phi(x_0) - \phi(e^{tA}x_0)}{t} = \limsup_{t\downarrow 0} \frac{\phi(x_0) - \phi(e^{tA}x_0)}{t} = \langle (-A)^{\frac{1}{2}}x_0, (-A)^{\frac{1}{2}}\nabla\phi(x_0)\rangle$$

and in such case the definition 5.1 could be simplified further replacing the lim inf and lim sup in (5.6) and (5.7) by $\langle (-A)^{\frac{1}{2}}x_0, (-A)^{\frac{1}{2}}\nabla\phi(x_0)\rangle$

C) By formula (5.5) it follows that

$$|u(x) - u(y)| \leq C|(-A)^{-\beta}||x - y| \qquad \forall x, y \in X \tag{5.8}$$

so that the function $u \in Lip(X; X)$.

The main result of this section is the following:

THEOREM 5.3. *Assume (2.2.2), (2.2.3), (2.3.1), (4.11) and (5.2). Then the value function v is the only viscosity solution of (5.1) in the sense of Definition 5.1.*

PROOF We divide the proof in two main steps proving separately existence and uniquess of a solution of (5.1).

Step 1. Existence. *Assume (2.2.2), (2.2.3), (2.3.1), (4.11) and (5.2). Then the value function v is a viscosity solution of (5.1), in the sense of Definition 5.1.*

We prove the subsolution condition. The other one follows in the same way. By theorem 4.4 the value function v satisfies equation (5.1) in the sense of definition 3.1. Then by definition 3.1 and the fact that $C_A^{1,1}(X) \subset C^{1,1}(X)$ we easily get that, given $\phi \in C_A^{1,1}(X)$, we have

$$
(5.9) \qquad \begin{aligned} &\lambda v(x) - \langle F(x), \nabla\phi(x)\rangle + \\ &+ \inf_{\mathcal{M}\in M(A)} \liminf_{t\downarrow 0} \left\{ \frac{\phi(x) - \phi(e^{tA}x)}{t} + H\big(x, (-A)^\beta \mathcal{M}_t \nabla\phi(e^{tA}x)\big) \right\} \le 0 \end{aligned}
$$

at every $x \in \operatorname{argmax}(u - \phi)$.

Let $\phi \in C_A^{1,1}(X)$ and $x_0 \in \operatorname{argmax}(u - \phi)$. By remark 5.2 A) and property (5.3) we have that $\nabla\phi(x_0) \in D((-A)^\beta)$ and therefore, $x_0 \in D((-A)^\beta)$, by definition of $C_A^{1,1}(X)$. It follows that

$$
e^{tA}x_0 \xrightarrow{t\downarrow 0} x_0 \qquad in\ D((-A)^\beta)
$$

which implies, still by definition of $C_A^{1,1}(X)$, that

$$
\nabla\phi(e^{tA}x_0) \xrightarrow{t\downarrow 0} \nabla\phi(x_0) \qquad in\ D((-A)^\beta).
$$

Finally, by the property of the class $M(A)$ and by the continuity of the function $H(x_0, \cdot)$ we obtain that for every $\mathcal{M}_t \in M(A)$ we have

$$
(5.10) \qquad \lim_{t\downarrow 0} H(x_0, (-A)^\beta \mathcal{M}_t \nabla\phi(e^{tA}x_0)) = H(x_0, (-A)^\beta \nabla\phi(x_0))
$$

and putting (5.10) in (5.9) we get that u satisfies the subsolution condition of definition 5.1.

Step 2. Uniqueness. *Assume that (2.2.2), (2.2.3), (2.3.1), (4.11) and (5.2) hold true. Let $u, v \in BUC(X) \cap C_\omega(X)$ be respectively a viscosity subsolution and a supersolution of (5.1) in the sense of definition 5.1, satisfying property (5.5). Then, for every $x \in X$ we have:*

$$
u(x) \le v(x).
$$

The uniqueness result easy follows from the above statement.

Let u be a subsolution of (5.1) in the sense of definition 5.1 and satisfying (5.4). Then consider a function $\psi \in C_A^{1,1}(X)$ and $x_0 \in \operatorname{argmax}(u - \phi)$. By formula (5.11) and definition 5.1 we obtain that

$$
(5.11) \qquad \begin{aligned} &\lambda u(x) - \langle F(x), \nabla\phi(x)\rangle + \inf_{\mathcal{M}\in M(A)} \liminf_{t\downarrow 0} \left\{ \frac{\phi(x) - \phi(e^{tA}x)}{t} + H\big(x, (-A)^\beta \mathcal{M}_t \nabla\phi(e^{tA}x)\big) \right\} = \\ &= \lambda u(x) - \langle F(x), \nabla\phi(x)\rangle + \liminf_{t\downarrow 0} \frac{\phi(x) - \phi(e^{tA}x)}{t} + H\big(x, (-A)^\beta \nabla\phi(x)\big) \le 0 \end{aligned}
$$

which implies that u is also a subsolution of (5.1) in the sense of definition 3.1, when the test functions belong to $C_A^{1,1}(X)$. In a similar way we can see that v is a supersolution of (5.1) as in definition 3.1 when the test functions belong to $C_A^{1,1}$. Now, to conclude, recall that by remark 3.5 the result of Theorem 3.3 remains true when the class of test functions is restricted to $C_A^{1,1}$.

Q.E.D.

6. A control problem with boundary and distributed control

Now we modify the control problem (2.2.1) - (2.3.3) by adding a distributed control $z(\cdot) : \mathbf{R}^+ \to X$, in the state equation (see also [15] for the study of this problem). Formally the state equation becomes

$$(6.1) \qquad \begin{cases} x'(t) = Ax(t) + F(x(t)) + z(t) + (-A)^\beta B\gamma(t) \\ x(0) = x_0 \end{cases}$$

that can be written in mild form as

$$(6.2) \quad x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}[F(x(s)) + z(s)]ds + (-A)^\beta \int_0^t e^{(t-s)A}B\gamma(s)ds, \qquad x_0 \in X.$$

The mild solution $x(t; x_0, \gamma, z)$ of (6.2) exists and is unique as one can easily see by a simple modification of Proposition 2.2.2.

Denoting by $\mathcal{Z}$ the space of measurable controls $z : \mathbf{R}^+ \to X$, we study the problem of minimizing the functional

$$(6.3) \qquad J(x, \gamma, z) = \int_0^{+\infty} e^{-\lambda t}\left[L(x(t), \gamma(t)) + \frac{1}{2}|z(t)|^2\right]dt$$

overall $\gamma \in \mathcal{A}$, $z \in \mathcal{Z}$. The value function of this problem is

$$(6.4) \qquad v_0(x) = \inf_{\gamma \in \mathcal{A};\ z \in \mathcal{Z}} J(x, \gamma, z)$$

and the Hamilton-Jacobi equation associated to this control problem is precisely

$$(6.5) \qquad \lambda u(x) + H(x, (-A)^\beta \nabla u(x)) + \frac{1}{2}|\nabla u(x)|^2 - \langle Ax + F(x), \nabla u(x)\rangle = 0$$

where H is given by (2.5.2).

PROPOSITION 6.1 *Under assumptions (2.2.2), (2.2.3), (2.3.1), the value function v_0 is Lipschitz continuous with respect to the norm induced on X by $(-A)^{-\alpha}$ for $\alpha \in [0, \frac{1}{2}[$, i.e.*

$$(6.6) \qquad \forall \alpha \in [0, \frac{1}{2}[,\ \exists C_\alpha > 0 \ \ s.t. |v_0(x) - v_0(y)| \le C_\alpha |(-A)^{-\alpha}(x - y)| \qquad \forall x, y \in X$$

PROOF. Take $A = 1$. Let $\alpha \in]0, \frac{1}{2}[$, and $x_0, x_1 \in X$. Without loss of generality we can assume that

$$\text{(6.7)} \qquad |(-A)^{-\alpha}(x_1 - x_0)| \leq \|v\|_\infty .$$

Let $\gamma_0 \in A$, $z_0 \in Z$ be such that, setting $x_0(t) := x(t; x_0, \gamma_0, z_0)$ we have

$$\text{(6.8)} \qquad \begin{aligned} &v_0(x_0) + |(-A)^{-\alpha}(x_1 - x_0)| > J(x_0, \gamma_0, z_0) \geq \\ &\geq \int_0^T e^{-t}\left[L(x(t), \gamma_0(t)) + \frac{1}{2}|z_0(t)|^2\right] dt + e^{-T} v_0(x_0(T)) \end{aligned}$$

for every $T > 0$ (see remark 2.3.2). Set

$$\text{(6.9)} \qquad z_1(t) \stackrel{def}{=} z_0(t) + F(x_0(t)) - F(x_0(t) + e^{tA}(x_1 - x_0))$$

so that

$$\text{(6.10)} \qquad x_1(t) \stackrel{def}{=} x(t; x_1, \gamma_0, z_1) = x_0(t) + e^{tA}(x_1 - x_0).$$

Then

$$\text{(6.11)} \qquad v_0(x_1) \leq \int_0^T e^{-t}\left[L(x(t), \gamma_0(t)) + \frac{1}{2}|z_1(t)|^2\right] dt + e^{-T} v_0(x(T)).$$

Subtract (6.8) from (6.11) to obtain

$$\text{(6.12)} \qquad \begin{aligned} &v_0(x_1) - v_0(x_0) \leq \\ &\leq \int_0^T e^{-t}\left\{[L(x_1(t), \gamma_0(t)) - L(x_0(t), \gamma_0(t))] + \frac{1}{2}[|z_1(t)|^2 - |z_0(t)|^2]\right\} dt + \\ &+ e^{-T}[v_0(x_1(T)) - v_0(x_0(T))] + |(-A)^{-\alpha}(x_1 - x_0)| = \\ &=: I_1 + I_2 + I_3 + |(-A)^{-\alpha}(x_1 - x_0)| \end{aligned}$$

Now we estimate every single term. For the first one we easily obtain, for every $T > 0$,

$$\text{(6.13)} \qquad \begin{aligned} |I_1| &\leq \int_0^T e^{-t} L_0 |x_1(t) - x_0(t)| dt \leq L_0 \int_0^T e^{-t} |e^{tA}(x_1 - x_0)| dt \leq \\ &\leq L_0 M_\alpha \int_0^{+\infty} \frac{e^{-t}}{t^\alpha} dt |(-A)^{-\alpha}(x_1 - x_0)| =: K_{1,\alpha} |(-A)^{-\alpha}(x_1 - x_0)| . \end{aligned}$$

where $K_{1,\alpha} < +\infty$ for $\alpha \in [0, 1[$. As fot the second term

$$\text{(6.14)} \qquad \begin{aligned} |I_2| &= \frac{1}{2}\left|\int_0^T e^{-t}\langle z_1(t) - z_0(t), z_1(t) + z_0(t)\rangle dt\right| \leq \\ &\leq \frac{1}{2}\int_0^T e^{-t}|z_1(t) - z_0(t)||z_1(t) + z_0(t)| dt \leq \\ &\leq \frac{1}{2}\int_0^T e^{-\frac{t}{2}}|F|_{Lip}|e^{tA}(x_1 - x_0)| e^{-\frac{t}{2}}\left[|z_0(t) + z_1(t)|\right] dt. \end{aligned}$$

Now we can apply Hölder inequality to obtain

$$|I_2| \le \frac{1}{2}|F|_{lip}\left(\int_0^T e^{-t}|e^{tA}(x_1-x_0)|^2 dt\right)^{\frac{1}{2}}\left(\int_0^T e^{-t}|z_0(t)+z_1(t)|^2 dt\right)^{\frac{1}{2}}. \tag{6.15}$$

The first term can be estimated as follows

$$\begin{aligned}&\left(\int_0^T e^{-t}|e^{tA}(x_1-x_0)|^2 dt\right)^{\frac{1}{2}} \le \\ &\le |(-A)^{-\alpha}(x_1-x_0)|M_\alpha\left(\int_0^{+\infty}\frac{e^{-t}}{t^{2\alpha}}dt\right)^{\frac{1}{2}} =: K_{2,\alpha}|(-A)^{-\alpha}(x_1-x_0)|\end{aligned} \tag{6.16}$$

where $K_{2,\alpha} := M_\alpha\left(\int_0^{+\infty}\frac{e^{-t}}{t^{2\alpha}}dt\right)^{\frac{1}{2}} < +\infty \qquad for \;\; \alpha \in [0, \frac{1}{2}[$..

At this point we estimate the term $|z_0(t)+z_1(t)|^2$. By (6.4) we have that $\|v\|_\infty \le L_\infty$, which implies, by formula (6.8), that

$$\frac{1}{2}\int_0^{+\infty}|z_0(t)|^2 dt \le L_\infty + |(-A)^{-\alpha}(x_1-x_0)|$$

Then, using the expression of $z_1(t)$ in (6.9), it follows that

$$\begin{aligned}|z_0(t)+z_1(t)|^2 &\le \Big[2|z_0(t)| + |F|_{Lip}|e^{tA}(x_1-x_0)|\Big]^2 \le \\ &\le 4|z_0(t)|^2 + 2|F|^2_{Lip}|e^{tA}(x_1-x_0)|^2 \le \\ \le 8L_\infty + 8|(-A)^{-\alpha}(x_1-x_0)| &+ 2|F|^2_{Lip}\frac{M_\alpha^2}{t^{2\alpha}}|(-A)^{-\alpha}(x_1-x_0)|^2\end{aligned}$$

Now, recalling that $|(-A)^{-\alpha}(x_1-x_0)| \le \|v\|_\infty \le L_\infty$,

$$\begin{aligned}&\left(\int_0^T e^{-t}|z_0(t)+z_1(t)|^2 dt\right)^{\frac{1}{2}} \le \\ &\le 16L_\infty + 2|F|_{Lip}M_\alpha L_\infty\int_0^{+\infty}\frac{e^{-t}}{t^{2\alpha}}dt =: K_{3,\alpha} < +\infty \qquad for \;\; \alpha \in [0, \frac{1}{2}[.\end{aligned} \tag{6.17}$$

By (6.16) and (6.17) into (6.15) we obtain, for every $T > 0$,

$$|I_2| \le \frac{1}{2}|F|_{Lip}K_{2,\alpha}K_{3,\alpha}|(-A)^{-\alpha}(x_1-x_0)| \tag{6.18}$$

Finally, for every $T > 0$,

$$|I_3| \le e^{-T}2\|v\|_\infty \le 2e^{-T}L_\infty \tag{6.19}$$

By (6.13), (6.18) and (6.19) in (6.12) we conclude that, for every $T > 0$ and $\alpha \in [0, \frac{1}{2}[$,

$$v(x_1) - v(x_0) \leq \left[K_{1,\alpha} + \frac{1}{2}|F|_{Lip} K_{2,\alpha} K_{3,\alpha}\right] |(-A)^{-\alpha}(x_1 - x_0)| + 2L_0 e^{-T}$$

which gives the claim by letting $T \to +\infty$.

Q.E.D.

REMARK 6.2. The main difference between Propositions 4.1, and 6.1 is the fact that we obtain the Lipschitz continuity of the value function without assuming the strong condition $|F|_{Lip} < \lambda$ to obtain the claim. This gain in regularity is due to the presence of the distributed control z in system (6.1).

Now we can repeat the theory of viscosity solutions for equation (6.5). The following theorem can be easily proved with the same method as theorem 5.3.

THEOREM 6.3. *Assume (2.2.2), (2.2.3), (2.3.1) and (4.11). Then the value function v_0 is the only viscosity solution of (6.5), in the sense of Definition 5.1.*

Theorem 6.3 is a corollary of Theorem 5.3. One has just to pay attention to the fact that the Hamiltonian term $H(x,p) + \frac{1}{2}|p|^2$ of equation (6.5) does not satisfy a Lipschitz estimate with respect to p as required by condition (3.2)v to obtain uniqueness. On the other hand, this difficulty can be easily overcome since the behaviour of the Hamiltonian at infinity is not essential when dealing with Lipschitz continuous solutions.

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