NAMT
92-020

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Research Report No. 92-NA-020

June 1992

Sponsors

U.S. Army Research Office
Research Triangle Park
NC 27709

National Science Foundation
1800 G Street, N.W.
Washington, DC 20550

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AND

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June, 1992

^Partially supported by PICS 955702.

^Partially supported by NSF grant DMS-9002249 and by the Army Research Office through the Center for Nonlinear Analysis.

^Partially supported by NSF grants DMS-8801208, DMS-9024617 and DMS- 8657464 (PYI), ARO contract DAAL03-90-G-0012 and the Sloan Foundation.

[4]Part of this work was done while visiting the University of Paris — Dauphine.

Introduction

In this paper we study the connection between the *weak propagation of fronts* (closed hypersurfaces in $\mathbb{R}^N$, which propagate in the normal direction with velocity depending on the position, the normal vector and its gradient) and the *phase field theory*, as it applies to the study of the asymptotic behavior of reaction-diffusion equations. More specifically, we study the properties of the *signed distance function* to the front; we relate these properties to the level set formulation of moving fronts and we present some new, general and, in some cases, sharp results guaranteeing the uniqueness of the fronts ("no interior"). Finally, we develop a rigorous justification of the "phase field" theory.

The study of propagating fronts is very interesting from both the theoretical point of view as well as for applications (eg. phase transitions in continuum mechanics, flame propagation, pattern formation, chemical kinetics, etc). The strong geometrical formulation of the motion (which requires smoothness) faces the development of singularities; the motion can, therefore, be defined only locally in time, which is quite unsatisfactory for the applications. On the other hand, a weak geometrical formulation by Brakke [Br] for motion by mean curvature gave rise to non-uniqueness problems, which are, again, unsatisfactory for both the theory and the applications. More recently, two different approaches were introduced to deal with these issues, namely, the level set and the phase field approach. The level set approach, which was put forward by Evans, Spruck [ESp1] for motion by mean curvature and Chen, Giga, Goto [CGG] for general motions, is based on considering the front as a level set (for definiteness the zero level set) of the solution of a degenerate parabolic pde. The phase field approach, suggested by Bronsard and Kohn [BrK] and DeGiorgi [D], defines the front as the boundary of the regions where the solutions of certain (scaled) reaction diffusion equations converge to the equilibria points of the associated vector field. Both approaches have their own advantages. The level set formulation provides a large number of analytical tools to study the motion for it allows for the use of very recent developments of the theory of nonlinear degenerate parabolic pde's. The phase field formulation is very indirect but also closely related to (and very natural for) the applications. A great deal of work in this paper is devoted to justifying the "phase field" formulation. A way to related these two approaches is to study the properties of the distance function to the front; a lot of work in this paper is devoted in this direction. As a matter of fact, one could propose an alternative way to study front propagation using the distance function. This was done by Soner [So] when the normal velocity of the front does not depend on its position. We chose not to do so in this paper, although given what we prove here for the distance function one can easily develop such an approach. A very intriguing mathematical question arising with the weak formulation of moving fronts is whether such fronts are uniquely determined by their initial position (if they are described using the distance function); this is closely related to whether the level set formulation gives rise to *fat* level sets. Couple sections in this paper are devoted towards studying these questions.

The paper is organized as follows: In Section 1 we recall the level set formulation and slightly improved some of the known results. In Section 2 we discuss the "non-empty interior" difficulty and give an equivalent characterization. Section 3 is devoted to deducing some important properties of the (signed) distance to the fronts. In Section 4 we study the non-empty interior difficulty. We give some general sufficient conditions and we present some counterexamples. Section 5 provides some uniqueness properties for the distance function, which will be used later in Section 10. In Section 6 we discuss the asymptotic limits of reaction-diffusion equations and the phase field theory. Section 7 is devoted to a formal derivation of the results. In Section 8 we briefly review the theory of traveling waves of reaction-diffusion equations and we formulate our main assumptions. The main results about the phase field theory are stated in Section 9 with their proofs given in Section 10. Finally, in Section 11 we present some possible applications and state few open problems.

1 Geometrical evolution of level sets and degenerate parabolic pde's.

In this section we recall and slightly generalize the level set formulation presented in Chen, Giga and Goto [CGG] (see also Evans and Spruck [ESp1] for motion by mean curvature and Giga, Goto, Ishii and Sato [GGIS]). As mentioned in the Introduction, the underlying idea is to think of the front as the zero-level set of the solution of a pde. This type of formulation first appeared in a theoretical work of Barles [Ba1] on fronts moving with constant normal velocity. [Ba1] was motivated by the computational work of Sethian [Se1] for a simple model in flame propagation. Later Osher and Sethian [OS] used extensively this type of ideas to perform numerical computations for different types of motions and in particular motion by mean curvature. Evans and Spruck [ESp1] provided the mathematical foundation of the level set approach for motion by mean curvature and Chen, Giga and Goto [CGG] studied motions in the generality described below.

To better explain the ideas involved we first present a formal derivation: Let Γ_t be a smooth front at time $t > 0$ and assume that $\Gamma_t = \partial D_t$ where $D_t \subset \mathbb{R}^N$ is open. The normal velocity V of Γ_t at $x (\in \Gamma_t)$ is given by

$$V = v(x, t, n, Dn) \tag{1.1}$$

where v is a continuous function of its arguments, n is the exterior unit normal vector to Γ_t and Dn is its gradient. Furthermore, we assume that there exists a smooth function $u : \mathbb{R}^N \times [0, \infty) \mapsto \mathbb{R}$ such that
$D_t = \{x \in \mathbb{R}^N : u(x,t) > 0\}, \Gamma_t = \{x \in \mathbb{R}^N : u(\cdot, t) = 0\}$ and $Du \neq 0$ on Γ_t.
A classical calculation yields

$$V = \frac{u_t}{|Du|}, n = -\frac{Du}{|Du|} \text{ and } Dn = -\frac{1}{|Du|}(I - \frac{Du \otimes Du}{|Du|^2})D^2u.$$

Inserting the above formulae in (1.1) we obtain

$$u_t + F(x,t,Du,D^2u) = 0$$

where F is related to v by

$$F(x,t,p,X) = -|p|v(x,t,-\frac{p}{|p|},-\frac{1}{|p|}(I-\frac{p\otimes p}{|p|^2})X) \tag{1.2}$$

for $p \in \mathbb{R}^N$ and $X \in S^N$, the space of $N \times N$ mattrices. An immediate consequence of (1.2) is that, for all $(x,t) \in \mathbb{R}^N \times (0+\infty), p \in \mathbb{R}^N$ and $X \in S^N, F$ satisfies

$$F(x,t,\lambda p,\lambda X + \mu(p\otimes p)) = \lambda F(x,t,p,X) \quad (\lambda > 0, \mu \in \mathbb{R}). \tag{1.3}$$

Any F which satisfies (1.3) will be called *geometric.*

In order for (1.1) to be well-posed it is also necessary to assume that it is parabolic, i.e. that v is nonincreasing in the Dn argument. This translates in terms of (1.2) to F being *(degenerate) elliptic* i.e.,

$$F(x,t,p,X) \le F(x,t,p,Y) \quad \text{if } X \ge Y \tag{1.4}$$

for all $(x,t) \in \mathbb{R}^N \times (0,+\infty), p \in \mathbb{R}^N$ and $X, Y \in S^N$. The fact that F is degenerate (in fact at least in the $p\otimes p$ direction) follows from (1.3). Finally we point out that F is as smooth as v with a possible discontinuity at $p = 0$.

The level set approach to front propagations can be described as follows. Given a closed set Γ_0 in $\mathbb{R}^N$ (front at time $t = 0$), choose $u_0 : \mathbb{R}^N \to \mathbb{R}$ such that

$$\Gamma_0 = \{x \in \mathbb{R}^N : u_0(x) = 0\}, \tag{1.4}$$

solve (in the appropriate way) the pde

$$\begin{cases} u_t + F(x,t,Du,D^2u) = 0 \text{ in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) \text{ on } \mathbb{R}^N, \end{cases} \tag{1.5}$$

and, finally, define T_t (the front at time *t)* by

$$\text{(1.6)} \qquad r_t = \{x€ JR^N : u(x,t) = 0\}.$$

The main issues associated with such a program are: (i) whether (1.5) does have a global solution allowing to define T_t and (ii) whether T_t only depends on To and not the form of i*o outside IV

The first issue is settled ([ESpl], [CGG]) by considering *viscosity solutions.* Viscosity solutions, which turn out to be the correct class of generalized solutions for first- and second-order fully nonlinear pde's, were introduced by Crandall and Lions [CL] (see also [CEL] and Lions [Li] for first- and second-order equations respectively). For the precise definition as well as some of the most recent developments as well as references we refer to the "user's guide" by Crandall, Ishii and Lions [CIL]. In the sequel (unless otherwise stated) by solution we will always mean viscosity solution. In order to avoid some techniqualities we will denote by *(F)* a set of some general assumptions needed for the statement of the next theorem. We will state and discuss these assumptions at the end of this section. Finally, we will denote by *UC(O)* the set of real valued uniformly continuous functions defined on *O.*

Theorem 1.1: *Assume {F), (1.3) and (14). Then, for any* u_0 € $UC(R^N)$, *there exists a unique solution u* € $UC(IR^N$ x [0,+oo)) *of (1.5). Moreover, if u and v are respectively sub- and super-solutions of (1.5) (in* $UC\{B^N$ x[0,+oo)), *then*

$$\text{(1.7)} \qquad tz(-,O) \leq v(-,0) \text{ in } R^N \Rightarrow u \leq v \text{ in } R^N \text{ x } [0,+oo).$$

Next we discuss the issue of whether T_t depends only on IV This follows from (1.3), which yields that (1.5) is invariant by non-decreasing changes u ^ ^(u). (See [ESpl], [CGG]).

Theorem 1.2: *Assume the hypotheses of Theorem 1.1 hold and let* $u, v \in UC(\mathbb{R}^N \times [0, +\infty))$ *be solutions of (1.5) such that*

$$\{x : u(x,0) > 0\} = \{x : v(x,0) > 0\}, \{x : u(x,0) < 0\} = \{x : v(x,0) < 0\},$$

$$\{x : u(x,0) = 0\} = \{x : v(x,0) = 0\},$$

and

$$\lim_{|x| \to +\infty} |u(x,0)|, \lim_{|x| \to +\infty} |v(x,0)| > 0. \tag{1.8}$$

Then, for all $t > 0$,

$$\{x : u(x,t) > 0\} = \{x : v(x,t) > 0\}, \{x : u(x,t) < 0\} = \{x : v(x,t) < 0\}$$

and

$$\{x : u(x,t) = 0\} = \{x : v(x,t) = 0\}$$

□

This results justifies the term *equation of geometric type* for (1.5), since it yields that the evolution of the level set $\Gamma_0 \to \Gamma_t$ depends only on F and on the "signs" of the initial datum in the different regions (which in turn give a sense to the expressions "inside Γ_0" and "outside Γ_0") and not really on the choice of the initial datum. Such a result was first obtained by Evans and Souganidis [ES1] in the case where F is independent of D^2u using representation formulae from the theory of deterministic differential games. In the generality stated above the result was obtained in [CGG]. Next we present a slightly simplified proof.

Proof: Consider the functions ϕ and ψ given by

$$\phi(t) = \inf\{v(y,0) | u(y,0) \geq t\} \quad \text{and} \quad \psi(t) = \sup\{v(y,0) | u(y,0) \leq t\}.$$

It is immediate that ϕ and ψ are nondecreasing, lower- and upper-semicontinuous respectively and

$$\phi(u(\cdot,0)) \leq v(\cdot,0) \leq \psi(u(\cdot,0)) \text{ on } \mathbb{R}^N. \tag{1.9}$$

Moreover, the assumptions on $u(\cdot,0)$ and $v(\cdot,0)$ yield that ϕ and ψ are actually continuous at 0 with $\phi(0) = \psi(0) = 0$. Finally standard regularization procedures imply the existence of two sequences of nondecreasing and nonincreasing respectively smooth functions $(\phi_n)_n$ and $(\psi_n)_n$ such that

$$\phi = \sup_n \phi_n \text{ and } \psi = \inf_n \psi_n. \tag{1.10}$$

Since F is geometric, $\phi_n(u)$ and $\psi_n(u)$ are solutions of (1.5). Moreover (1.9), (1.10) and Theorem 1.1 yield

$$\phi_n(u) \leq v \leq \psi_n(u) \text{ in } \mathbb{R}^N \times [0, +\infty).$$

Letting $n \longrightarrow \infty$ we conclude easily, since, in view of the assumptions on $u(\cdot,0)$ and $v(\cdot,0)$ and the definition of ϕ and ψ, $\phi(t) > 0$ if $t > 0$ and $\psi(t) < 0$ if $t < 0$.

□

We continue by discussing some examples of motions and their related "geometrical" equations.

In the first example the hypersurface is assumed to propagate in the normal direction with velocity $v(x,t,n)$. The geometric equation in this case is

$$u_t - \alpha(x,t,\frac{Du}{|Du|})|Du| = 0 \text{ in } \mathbb{R}^N \times (0,\infty), \tag{1.11}$$

with $F(x,t,p,M) = -\alpha(x,t,\frac{p}{|p|})|p|$ satisfying (1.3). This type of propagation, when $\alpha \equiv c$ constant, was introduced by Landau as a flame front propagation model and was studied both analytically and numerically by Sethian [Se1] using (1.11). Then Barles [Ba1] showed the connections between (1.11) and (1.3).

Another very interesting example, both theoretically and from the applications point of view, is the motion of a hypersurface with normal velocity equal to its mean curvature. Here (1.5) takes the form

$$u_t - \Delta u + \frac{(D^2uDu|Du)}{|Du|^2} = 0 \text{ in } \mathbb{R}^N \times (0,\infty), \tag{1.12}$$

where $(\cdot|\cdot)$ denotes the usual inner product in $\mathbb{R}^N$. In this case (1.3) holds for every $\lambda \in \mathbb{R}$ (not only $\lambda > 0$). This yields that the equation is invariant by any change! Equation (1.12) was studied first numerically by Osher and Sethian [OS] and then analytically by Evans and Spruck [ESp1-4] (see also Chen, Giga, Goto [CGG], Soner [So] etc.).

Another example of propagations which arise very naturally in the theory of phase transitions is the case of anisotropic motion where (1.5) is of the form

$$u_t - |Du| \text{ div } \left(H(\frac{Du}{|Du|})\right) + |Du|\beta(\frac{Du}{|Du|}) = 0 \tag{1.13}$$

for some smooth functions H and β, with H convex. Equation (1.13) is studied in [So] and [CGG]. There are some very interesting models of phase transitions which yield (1.13) but with H not-convex. Following a *relaxation* process, these problems give rise to (1.3) but with F discontinuous (in addition to $p = 0$) at certain directions in the gradient space. This is the subject of Gurtin, Soner and Souganidis [GSS].

We conclude this rather long overview of the level set approach by stating and discussing assumption (F), which was necessary for the comparison result of Theorem 1.1. (F) consists of several parts, namely:

$$(F_1) \quad \begin{cases} (x,t,p,X) \mapsto F(x,t,p,X) \text{ is bounded for bounded} \\ (p,X) \text{ and continuous for } x \in \mathbb{R}^N, t \in [0,R], p \in B(0,R)\backslash\{0\} \\ \text{and } \|X\| \le R, \text{ for all } R > 0. \end{cases}$$

(F_2) $F_*(x,t,\alpha(x-y),X) - F^*(y,t,\alpha(x-y),Y) \ge -\omega(|x-y|(1+\alpha|x-y|)),$

where u>(0$^+$) = 0 and for all x,y € B^N,t 6 (0,+oo),a $\geq$ 0 and mattrices X,Y € S^N such that $\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq Ka\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$ for some constant $K > 0$. Finally,

$$(F_3) \qquad F_*(x,t,0,0) = F^*(x,t,0,0);$$

we recall that F^* and F_m denote the upper- and lower-semicontinuous envelopes of F respectively.

The proof of Theorem 1.1 can be found in [CGG], The arguments of [CGG] can be, however, slightly simplified by remarking that, since (1.5) is invariant under nondecreasing changes, it is enough to have a comparison result in $BUC(IR^N$ x [0,oo)), the space of bounded, uniformly continuous functions. This leads to an easier treatment of the unboundedness of the domain. As a matter of fact, with these assumptions, Theorem 1.1 extends easily to the case where either the sub- or the super- solution to be compared is discontinuous. Since we will use this remark throughout the paper we state it as a separate theorem. (For the definition of discontinuous sub- and super-solutions we refer to [Is].)

Theorem 1.3: ***Assume (F)$_y$ (1.3) and (14). Ifue UC{M^N x* [0,oo)) *is a subsolution of (1-5) and v : IR^N* x [0, oo) *is a discontinuous super solution, then* u(-,0) $\leq$ t>(-,0) *on R^N yields* u(-,<) $\leq$ *v(-,t) on R^N for all t>* 0. *A similar result holds if u is a discontinuous subsolution and v* 6 *UC(Ft^N* x [0,oo)) *is asupersolution.***

The final remark of this section is that assumption (1.8) in Theorem 1.2 can be relaxed to handle the case of unbounded fronts: we only need to assume that for each $a > 0$ there exists $e > 0$ such that

$$|u(z,0)|>(:r,0)|>e>0 \text{ if } d(x,r_o)>a>0.$$

□

2 The non-empty interior difficulty

The level set approach seems to avoid all the geometrical difficulties related to the onset of singularities, etc. The evolution $\Gamma_0 \to \Gamma_t$ is well defined and unique. Given this fact, the next natural questions are related to the regularity of Γ_t. When $N = 2$ this issue was completely resolved by Angenent [A1,A2] (see also the references therein). For $N \geq 3$ the issue is more complicated. In addition to a local existence result by Hamilton [H] and Evans and Spruck [ESp2] for motion by mean curvature, there only partial regularity results (only for motion by mean curvature) due to Evans and Spruck [ESp3,ESp4] and Ilmanen [Il1,Il2].

A more basic question is whether Γ_t has an empty interior for $t > 0$. In principle, one expects Γ_t to be a hypersurface in $\mathbb{R}^N$; in view of this Γ_t having interior seems rather unreasonable. This is related to the non-uniqueness features for the motion of front described by the distance function as we will explain in the next section. Before we continue discussing this difficulty we give a more precise definition.

Definition 2.1: *Let Γ_t be the evolution of Γ_0 by the level set approach. We say that Γ_t has no interior at $t > 0$ iff*

$$cl\{x : u(x,t) > 0\} = \{x : u(x,t) \geq 0\}$$

and

$$int\{x : u(x,t) \geq 0\} = \{x : u(x,t) > 0\}.$$

In most examples it can easily be shown that Γ_t has no interior in $\mathbb{R}^N$ for all $t > 0$ iff $\bigcup\limits_{t>0}(\Gamma_t \times \{t\})$ has an empty interior in $\mathbb{R}^N \times (0,\infty)$. For motion with constant normal velocity this follows from the finite speed of propagation. For motion by mean curvature it can be shown using explicit solutions of the form $\psi(|x|^2 + (N-1)t)$ as barriers. The last argument can be easily generalized to more complex equations.

In view of the above remark, we next present a new formulation of the no empty interior question in terms of whether equation (1.5) has unique discontinuous solutions, with initial datum $\mathbb{1}_{\Omega_0} - \mathbb{1}_{\Omega_0^c}$, where $\mathbb{1}_A$ denotes the characteristic function of the set A, and Ω_0 and Ω_0^c are the "inside of Γ_0" (i.e. the set where u_0 is negative) and "outside of Γ_0" (i.e. the set where u_0 is positive) respectively. (See the discussion after the statement of Theorem 1.2).

Theorem 2.1: *The set $\bigcup_{t>0}(\Gamma_t \times \{t\})$ has an empty interior in $\mathbb{R}^N \times (0, +\infty)$ iff there exists a unique solution of (1.5) with initial datum $\mathbb{1}_{\Omega_0} - \mathbb{1}_{\Omega_0^c}$.*

□

The above formulation of the non-empty interior difficulty has an interest by itself, since it provides a criterion which can be checked in some cases; we will do so in the case of first-order motions. Another important consequence of this formulation is that it is the first step to obtain the properties of the distance function to the front.

Proof of Theorem 2.1: Let $u \in UC(\mathbb{R}^N \times [0, \infty))$ be the solution of (1.5) with initial datum $d(x, \Gamma_0)$, the signed distance to Γ_0, which is normalized to be positive inside Γ_0 and negative outside. Recall that by Theorem 1.2, it suffices to use $d(x, \Gamma_0)$ as an initial datum in order to obtain Γ_t. For $\varepsilon > 0$, set

$$u^\varepsilon(x, t) = \tanh(u(x, t)/\varepsilon),$$

where $\tanh(\cdot)$ is the hyperbolic tangent function. u^ε is also a solution of (1.5) (by (1.3)). The stability results for discontinuous viscosity solutions (cf. Crandall, Ishii and Lions [CIL] yield that the limit $u_\infty = \lim_{\varepsilon \to 0} u^\varepsilon$ is a viscosity solution of (1.5). Moreover, the properties of tanh yield

$$u_\infty(x,t) = \begin{cases} 1 & \text{if} \quad u(x,t) > 0, \\ -1 & \text{if} \quad u(x,t) < 0, \\ 0 & \text{if} \quad (x,t) \in \operatorname{Int}\{u = 0\}. \end{cases}$$

For the rest of the points, the value of $u_\infty(x,t)$ depends on the lsc or usc envelope one considers in the definition of the discontinuous viscosity solution. Next pick $\alpha \in (0,1)$ and set

$$\overline{u}_\infty(x,t) = \lim_{\varepsilon \to 0} \tanh((u_\infty(x,t) + \alpha)/\varepsilon)$$

and

$$\underline{u}_\infty(x,t) = \lim_{\varepsilon \to 0} \tanh((u_\infty(x,t) + \alpha)/\varepsilon).$$

The functions $\overline{u}_\infty$ and $\underline{u}_\infty$ are again solutions of (1.5). Moreover,

$$\overline{u}_\infty(x,t) = \begin{cases} 1 & \text{if} \quad u(x,t) \geq 0 \\ -1 & \text{if} \quad u(x,t) < 0 \end{cases} \quad \text{and} \quad \underline{u}_\infty(x,t) = \begin{cases} 1 & \text{if} \quad u(x,t) > 0 \\ -1 & \text{if} \quad u(x,t) \leq 0. \end{cases}$$

If $\bigcup\limits_{t>0}(\Gamma_t \times \{t\})$ has a non-empty interior, $\overline{u}_\infty$ and $\underline{u}_\infty$ are two different discontinuous solutions of (1.5) with initial datum $1\!\!1_{\Omega_0} - 1\!\!1_{\Omega_0^c}$.

Conversely, if $\bigcup\limits_{t>0}(\Gamma_t \times \{t\})$ has empty interior, let w be a solution of (1.5) with $w(\cdot,0) = 1\!\!1_{\Omega_0} - 1\!\!1_{\Omega_0^c}$ and choose a sequence $(\phi_n)_n$ of smooth functions such that $\phi_n \equiv 1$ on $[0,+\infty), \phi_n \geq 0$ in $\mathbb{R}, \phi_n(\mathbb{R}) \subset [-1,1]$ and $\inf\limits_n \phi_n = -1$ on $(-\infty, 0]$. Since $w^*(x,0) \leq \phi_n(d(x,\Gamma_0))$ in $\mathbb{R}^N$, (1.3) and Theorem 1.3 yield $w^* \leq \phi_n(u)$ in $\mathbb{R}^N \times (0,+\infty)$ and

$$w^*(x,t) \leq -1 = \inf_n \phi_n(u(x,t)) \text{ on } \{u < 0\}.$$

On the other hand, (F_3) gives

$$F^*(x,t,0,0) = F_*(x,t,0,0) = 0;$$

hence, $+1$ and -1 are respectively sub- and super-solutions of (1.5). Therefore,

$$-1 \leq w_* \leq w^* \leq 1$$

and, finally, $w_* = -1$ on $\{u < 0\}$. The same method shows that $w_* = 1$ on $\{u > 0\}$, which, in view of the assumption that $\{u = 0\}$ has empty interior, identifies w uniquely. □

By examining the solutions $\overline{u}_\infty$ and $\underline{u}_\infty$, both equal to u_∞ in the "empty interior" case, we see that we switched from the pde formulation of the motion to a "quasi-geometric" formulation, since the notions of sub- and super-solution are only relevant on the sets $\overline{\Gamma}_t = \partial\{\overline{u}_\infty(\cdot,t) = 1\}$ and $\underline{\Gamma}_t = \partial\{\underline{u}_\infty(\cdot,t) = 1\}$. This is related to the distance function formulation for the motion, which we explain in the next section.

3 The properties of the distance function to the moving front

In this section we study the properties of the (signed) distance $d(x, \Gamma_t)$ to a front Γ_t, whose evolution has been defined by the level set approach described in Section 1. The results we present here extend the work of Soner [So], who actually used the properties of the distance function to define the evolution of fronts in the case where the velocity of the front is independent of the position. Although we could do the same here, we chose not to do so, since, once the correct definition is given, all the arguments will follow exactly as in [So]. Another motivation to study the properties of the distance function, besides the fact that this quantity intrinsically defines the front, is that the distance function plays a central role in studying the fronts generated by reaction-diffusion equations ("phase field theory") as we will explain in Sections 6-10.

As usual we begin with a closed set Γ_0 in $I\!R^N$ and assign to it a notion of inside and outside in terms of the sign of its distance function. Let $\Gamma_0 \to \Gamma_t$

be the evolution of F_o defined by the level set formulation. To state the main result we define the extinction time t^* € (0, +00] for T_t by

$$V = \sup\{< > 0 \text{ such that } T_t \wedge <f>).$$

Finally, we denote by d the signed distance function to the front T_t.

Theorem 3.1: *Assume that T_t has empty interior for all $t > 0$. Then $\underline{d} = d$ A 0 and $\overline{d} = d$ V 0 satisfy respectively*

(3.1) $$dt + F(x\text{-}dPd,i,D\&D^7d) \leq 0 \text{ in } M^N \times (0,f)$$

and

(3.2) $$\overline{d_t} + F(x - d\overline{D}d,\overline{i}, 2)3, D^23) \geq 0 \text{ in } \wedge^N \times (0,r).$$

Moreover,

(3.3) $$-(D^2\underline{d}D\underline{d}\backslash D\underline{d}) \leq 0 \text{ in } \{\underline{d} < 0\}$$

and

(3.4) $$-(2?^232?3|2?3) \geq 0 \text{ in } \{3 > 0\}.$$

□

Remark 3.2: The assumption that T_t has empty interior was made to only simplify the presentation. In fact one can show that (3.1), (3.2), (3.3) and (3.4) still hold when T_t has non-empty interior but for different solutions. Indeed let $\overline{T}_t = d\{x :$ UooOM) = 1} and $£_t = d\{x :$ ii$_{O0}$(x,<) = 1}, where $y\wedge$ and ïZoo are defined as in the proof of Theorem 2.1. Then (3.1), (3.3) and (3.2), (3.4) hold true for $d(x,T_t)$ and $d(xXt)$- This again is related to the connections between the non-empty difficulty and the nonuniqueness in the weak geometric and distance function formulations of motions. For a detailed discussion of these connections we refer to [So].

Remark 3.3: One can read the speed of the moving front from (3.1) and (3.2). Indeed if we know apriori that the front moves along its normal direction and if d is assumed to be smooth, then

$$d_t + F(x,t,Dd,D^2d) = 0 \quad \text{if} \quad d = 0,$$

which, in view of (1.1), yields $V = v(x,t,n,Dn) = -F(x,t,n,Dn)$.

Remark 3.4: One cannot expect that d will solve a pde like (1.5) as it can be observed by a direct calculation if everything is smooth. The term $x - dDd$ in (3.1) and (3.2) has a geometric meaning. Indeed, if $x \notin \Gamma_t$, then $x - dDd \in \Gamma_t$.

Proof of Theorem 3.1: We only prove (3.1) and (3.3); (3.2) and (3.4) can be obtained by similar arguments. To this end, observe that for each $k > 0$ the functions

$$w_k(x,t) = \begin{cases} 0 & \text{if} \quad u_\infty(x,t) = 1, \\ \\ -k & \text{if} \quad u_\infty(x,t) = -1, \end{cases}$$

are solutions of (1.5), where u_∞ is defined in the proof of Theorem 2.1. We next introduce the function

$$\overline{w}_k(x,t) = \sup_{y \in I\!R^N} \{w_k(y,t) - |x-y|\}.$$

An easy calculation yields

$$\overline{w}_k(x,t) = \max(\underline{d}(x,t), -k).$$

On the other hand, standard arguments from the theory of viscosity solutions (cf. Lasry and Lions [LL], Jensen, Lions and Souganidis [JLS]) yield that $\overline{w}_k$ is a subsolution of (1.5). The inequalities (3.1) and (3.3) follow then easily when $d \neq 0$. If $d = 0$, we need to observe that $\overline{w}_k \geq w_k$ in $I\!R^N \times (0,\infty)$ and if $\overline{w}_k(x,t) = w_k(x,t)$ at some point (x,t), then $D^{2,+}\overline{w}_k(x,t) \subset D^{2,+}w_k(x,t)$;

the last inclusion being exactly what is needed at $d = 0$. Letting $k \to \infty$ completes the proof. $\square$

4 When is the empty interior condition fullfilled?

It has been become, hopefully, clear by now that settling the empty interior condition is of great importance, since it may lead to some rather unintuitive situations. Unfortunately, if no conditions are imposed on Γ_0, interior may be created for $t > 0$. See for example Evans and Spruck [ESp1], Soner [So] and Ilmanen [Il1] for some simple examples in this direction for motion by mean curvature. It can, however, be argued that the interior in the examples of [ESp1] and [So] is due mainly to the fact that the initial data are not smooth which, in turn, yields that the normal direction is somehow not well defined. This, of course, raises the question of finding some necessary and sufficient conditions of Γ_0 so that no interior is created. We will address this question below for the case of first-order and second-order motions whose geometric pde's are of the form

$$u_t + \alpha(x,t)|Du| = 0 \text{ in } \mathbb{R}^N \times (0,\infty) \tag{4.1}$$

and

$$u_t + F(Du, D^2u) = 0 \text{ in } \mathbb{R}^N \times (0,\infty), \tag{4.2}$$

with initial datum

$$u(x,0) = d(x,\Gamma_0) \text{ in } \mathbb{R}^N. \tag{4.3}$$

Throughout this section we will assume that

$$\Gamma_0 = \partial\{x \in \mathbb{R}^N : d(x,\Gamma_0) < 0\} = \partial\{x \in \mathbb{R}^N : d(x,\Gamma_0) > 0\}, \tag{4.4}$$

which, in particular, implies that Γ_0 has no interior.

Theorem 4.1: *Assume (4.3), (4.4), $\alpha \in W^{1,\infty}(\mathbb{R}^N \times (0,T))(\forall T > 0)$ and that either*

(i) α *does not change sign in* $\mathbb{R}^N \times (0, +\infty)$

or

(ii) α *is independent of* t.

Then $\Gamma_t = \{x : u(x,t) = 0\}$ *does not develop interior, where* $u \in UC(\mathbb{R}^N \times (0,\infty))$ *is the solution of (4.1), (4.3).*

Theorem 4.1 is almost sharp. Indeed at the end of this section we will give an example of $\alpha(x,t)$ which changes sign and Γ_t develops interior. We do not, however, know whether interior is created if $\alpha \equiv \alpha(x, \frac{p}{|p|})$ changes sign. (The case where $\alpha(x, \dfrac{p}{|p|}) > 0$ was treated in [Sor].)

Proof of Theorem 4.1: We present here the proof only in the case of (ii) since (i) is obtained by similar and even simpler arguments. In view of Theorem 2.1 and the discussion afterwards, it suffices to prove the uniqueness of discontinuous solutions of (4.1) with the initial datum

$$u(\cdot, 0) = \mathbb{1}_{\Omega_0} - \mathbb{1}_{\Omega_0^c} \text{ in } \mathbb{R}^N, \tag{4.5}$$

where Ω_0 $\{x : d(x, \Gamma_0) > 0\}$. To this end, we first claim that we can examine the situation separately in the sets

$$O_1 = \{x \in \mathbb{R}^N | \alpha(x) > 0\} \text{ and } O_2 = \{x \in \mathbb{R}^N | \alpha(x) < 0\}.$$

A formal argument to understand why this claim is true consists in looking at the optimal control interpretation of (4.1) and in remarking that the paths of the dynamics starting from a point in O_1 (or O_2) can never reach the boundary of O_1 (or O_2). To justify this argument completely, we adapt some arguments introduced by Barron and Jensen [BJ1] (See also Barles [Ba2]).

Let u be a solution of (4.1) and consider the function $u^\varepsilon : O_1 \times [0,+\infty) \to \mathbb{R}$ given by

$$u^\varepsilon(x,t) = \inf_{y\in O_1} \{u(y,t) + e^{-\gamma t}\frac{|x-y|^2}{\varepsilon\alpha(y)}\}.$$

Combining classical, in the context of viscosity solutions, computations with the arguments of [BJ1], one shows easily that u^ε is an approximate subsolution of (4.1) in $O_1 \times (0,+\infty)$ for $\gamma > 0$ large enough. Moreover, u^ε is continuous and satisfies

$$u^\varepsilon(\cdot,0) \leq \mathbb{1}_{\Omega_0} - \mathbb{1}_{\Omega_0^c} \text{ on } O_1.$$

If v is another solution of (4.1) and (4.3), we claim that, as $\varepsilon \to 0$,

$$u^\varepsilon \leq v_* + o(1) \quad \text{in} \quad O_1 \times [0,+\infty).$$

Indeed, we perform the usual uniqueness arguments for viscosity solutions with a test function $\psi : (O_1 \times (0,+\infty)) \times (O_1 \times (0,+\infty)) \to \mathbb{R}$ given by

$$\psi(x,t,y,s) = u^\varepsilon(x,t) - v_*(y,s) - \frac{|x-y|^2}{\beta} - \theta(|x|^2 + |y|^2 + \frac{1}{\alpha(x)} + \frac{1}{\alpha(y)}),$$

where β and θ are small parameters. The only slight new point comes from the term

$$(\frac{1}{\alpha(x)} + \frac{1}{\alpha(y)})$$

which take care of the lack of boundary condition on $\partial O_1 \times (0,+\infty)$. We leave the rest of the routine but tedious details to the reader. □

Remark 4.1: An alternative way to understand the comparison result in the proof of Theorem 4.1 is to say that (4.1) holds up to the boundary of $O_1 \times (0,+\infty)$. Indeed let u by an usc subsolution of (4.1) and assume that $(x,t) \in \partial O_1 \times (0,+\infty)$ is a strict local maximum of $u - \phi$ for some smooth ϕ. The function

$$(y,s) \mapsto u(y,s) - \phi(y,s) - \frac{\theta}{\alpha(y)}$$

attains a maximum at $(y_\epsilon, s_\delta) \to (x,t)$ as $\delta \to 0$. Evaluating (4.1) at (y_ϵ, s_δ) and letting $\delta \to 0$ yields the result.

We next turn our attention to the case of the motion governed by (4.2); the typical example here being motion by mean curvature. We will be making the following additional assumption on F:

$$(4.5) \qquad F(\mu Q^t p, \mu Q^t X Q) = \mu^2 F(p, X)$$

for all $\mu > 0, p \in \mathbb{R}^n, X \in S^N$ and $Q \in O(N)$, where Q^t is the adjoint of Q and $O(N)$ is the group of $N \times N$ orthogonal matrices ($Q^* = Q^{-1}$).

Theorem 4.2: *Assume that (1.3), (1.4) and (4.5) hold and that Γ_o is of class C^2. In addition, assume that there exist nonnegative constants C_i $(i = 1,2,3)$, a skewsymmetric mattrix H and $x_0 \in \mathbb{R}^N$ such that*

$$(4.6) \qquad c_1(x-x_0)\cdot Dd(x)+c_2 H(x-x_0)\cdot Dd(x)-c_3 F(Dd(x), D^2d(x)) \geq 0 \text{ on } \Gamma_o,$$

where d is the signed distance to Γ_o. Then the set $\bigcup_{t>0}(\Gamma_t \times \{t\})$ has empty interior in $\mathbb{R}^N \times (0, +\infty)$.

The left hand side of (4.6) is the generator of rotation, dilations and translations in (x,t) evaluated at $t = 0$ on Γ_o. Condition (4.6) includes as special cases results of Ilmanen [I1] and Soner [So] for motion by mean curvature. On the other hand, (4.6) is not necessary. Indeed recent work of Soner and Souganidis [SS] (see also Altschuler, Angenent, Giga [AAG]) for bodies of rotation moving by mean curvature shows that there exist smooth Γ_o's which do not satisfy (4.6), but their evolution never develops interior. It follows, however, that (4.6) holds near the singularities of Γ_t [SS]. This is related to a conjecture of DeGiorgi [D]. A related observation is that if (4.6) hold at a later time, this again yields no interior. For the case of mean curvature, Evans and Spruck [ESp4] also showed that under some assumptions on Γ_o, almost every level set of the solution of (1.12) does not develop interior. Finally, at the end of this section we give an example where interior is created if the velocity depends on t.

Proof of Theorem 4.2: Let $u \in UC(\mathbb{R}^N \times (0,\infty))$ be the unique solution of (4.2) and (4.3) and, for $h > 0$, define the function

$$u_h(x,t) = \Phi\Big(u\Big((1+c_1h)e^{c_2hH}(x-x_0)+x_0,(1+c_1h)t+c_3h\Big)\Big),$$

where Φ is some increasing smooth function with $\Phi(0) = 0$ to be chosen later. In view of (1.3) and (4.5), u_h is also a solution of (4.2), since H being skewsymmetric yields $Q = e^{c_2hH} \in \mathcal{O}(N)$. Moreover, if h is small enough, there exists some $\eta > 0$ such that

$$|u(\cdot,0) - u_h(\cdot,0)| \geq \eta h \text{ on } \mathbb{R}^N. \tag{4.7}$$

Assuming for the moment (4.7), we observe that Theorem 1.1 yields either $u_h \leq u - \eta h$ or $u_h \geq u + \eta h$ in $\mathbb{R}^N \times (0,\infty)$. If $\cup_{t>0}(\Gamma_t \times \{t\})$ has interior, either of the above inequalities, however, yields a contradiction, for if $u = 0$ in some neighborhood of a point (x_0,t_0), then so does u_h for h sufficiently small.

We return now to the proof of (4.7). We first observe that we may choose Φ so that we only need to check (4.7) in a small neighborhood of Γ_0. But for a suitable choice of such neighborhood u is smooth. We can therefore perform the expansion

$$u((1+c_1h)e^{c_2hH}(x-x_0)+x_0,c_3h) = u(x,0) + h(c_1(x-x_0)\cdot Du(x,0)+$$

$$c_2H(x-x_0\cdot Du(x,0)+c_3u_t(x,0)) + o(h).$$

Using (4.6), that $u(x,0) = d(x)$ and the fact that the equation holds for small $t > 0$ (since Γ_0 is smooth) we conclude.

□

As a matter of fact with a slight modification of the above proof we can prove that Γ_t has no interior for $t > 0$. We leave it up to the reader to fill in the details.

We continue with an example of interior for a motion governed by (4.1).

Proposition 4.3: *Consider (4.1) in $\mathbb{R} \times (0, \infty)$ with $\alpha(x,t) = x - t$. There exists an interval $I = (\beta, \gamma)$ such that the evolution $\Gamma_0 \to \Gamma_t$ has nonempty interior at some $t_0 > 0$, where $\Gamma_0 = \partial I$.*

Proof. In view of Theorem 2.1 if suffices to show that there exists I such that the equation

$$\begin{cases} u_t + (x-t)|u_x| = 0 \text{ in } \mathbb{R} \times (0,\infty), \\ u(x,0) = (1\!\!1_I - 1\!\!1_{I^c})(x) \text{ on } \mathbb{R}, \end{cases} \tag{4.8}$$

has more than one solutions. To this end, choose $x_0 > 0$, solve the forward and backward ode's

$$\dot{X}_\pm(t) = \pm\alpha(X_\pm(t), t) \quad \text{with } X_\pm(x_0) = x_0,$$

and set $\beta = X_+(0), \eta = X_-(0)$ and $I = (\beta, \eta)$. We will compute the minimal and maximal solution of (4.8), using the control interpretation of this equation. Indeed consider the dynamics given by

$$\dot{y}_x(s) = \alpha(y_x(s), s)v(s) \quad , \quad y_x(t) = x,$$

where $v(\cdot) \in L^\infty((0,+\infty),[-1,1])$ is the control process. Following Barles and Perthame [BaP] or Barron and Jensen [BJ2], one can prove easily that the minimal and maximal solution of $u_t + (x-t)|u_x| = 0$ in $\Omega_1 = \{x > t\}$ are respectively

$$u_*(x,t) = \inf_{v(\cdot)} u_*(y_x(0),0) \text{ and } u^*(x,t) = \inf_{v(\cdot)} u^*(y_x(0),0),$$

where $u(x,0) = (\mathbb{1}_I - \mathbb{1}_{I^c})(x)$. It is easy to see from the above formulae that $u_* \equiv -1$ on $\{(x,t) : x = t\}, u^* = -1$ on $\{(x,t) : x = t\}\backslash\{(x_0,x_0)\}$ and $u^*(x_0,x_0) = 1$. We now turn our attention to $\Omega_2 = \{(x,t) : x < t\}$. Here the maximal and minimal solutions are respectively given by

$$\overline{u}(x,t) = \sup_{v(\cdot)}\{-\mathbb{1}_{\{\tau=t\}} + u^*(y_x(\tau),\tau)\mathbb{1}_{\{\tau>t\}}\}$$

and

$$\underline{u}(x,t) = \sup_{v(\cdot)}\{-\mathbb{1}_{\{\tau=t\}} + u_*(y_x(\tau),\tau))\mathbb{1}_{\{\tau>t\}}\},$$

where, for each $v(\cdot), \tau$ is the exit time from Ω_2. It follows that, while $\underline{u} \equiv -1$ in $\Omega_2, \overline{u}$ equals 1 at each point $(x,t) \in \Omega$ for which the trajectory y_x may reach the point (x_0,x_0). It is easy to check that the set of these points is exactly the region $\{(x,t) \in \Omega_2$: $X_+(t) \leq x \leq X_-(t)\}$ which has a non-empty interior.

Since (4.8) has a non-uniqueness feature, we conclude by Theorem 2.1.

□

The next example of non-uniqueness corresponds to volume preserving mean curvature flow. The derivation of this motion and its significance for applications is discussed in Section 11.

Let Γ_0 be the union of three disjoints circles in $\mathbb{R}^2$, i.e. $\Gamma_0 = \partial B(x_1,R_0) \cup \partial B(x_2,R_0) \cup \partial B(x_3,r_0)$ with $x_i \in \mathbb{R}^2 (i = 1,2,3)$ to be chosen later and $0 < r_0 < R_0$. We consider the motion of Γ_0 with normal velocity

$$V = -\operatorname{div}(Dn) + \alpha(t) \qquad (t > 0)$$

where $\alpha(t) = 2\pi N(t)L^{-1}(t), N(t)$ and $L(t)$ being the number of disjoint parts of Γ_t and its length respectively. In view of this explicit formula, at least for small time,

$$\Gamma_t = \partial B(x_1,R_t) \cup \partial B(x_2,R_t) \cup \partial B(x_3,r_t),$$

where R_t, r_t satisfy the ode's

$$\dot{R}_t = -R_t^{-1} + \alpha(t) \text{ and } \dot{r}_t = -r_t^{-1} + \alpha(t) \text{ with } \alpha(t) = 3(2R_t + r_t)^{-1}.$$

Let $t_1 = \sup\{t > 0 \text{ such that } r_t > 0\}$. The form of Γ_t above is valid for all $t \in (0, t_1)$. Since t_1 is independent of the choice of the x_i's we can choose x_1 and x_2 so that $|x_1 - x_2| = 2R_{t_1}$. In view of this choice,

$$\Gamma_{t_1} = \partial B(x_1, R_{t_1}) \cup \partial B(x_2, R_{t_1}),$$

with the two circles touching at a point. There are two possible evolutions for $t \geq t_1$ depending on whether we think of Γ_{t_1} as one set or two separate ones. In the first case Γ_t moves with $\alpha(t) = 2\pi$ (length $(\Gamma_t))^{-1}$ and actually converges to $\partial B(\frac{x_1+x_2}{2}, R_\infty)$, as $t \to \infty$, where $R_\infty = (2R_0^2 + r_0^2)^{\frac{1}{2}}$. In the second case, Γ_t remains stationary (i.e., $\alpha(t) \equiv R_{t_1}^{-1}$ for $t > t_1$).

We conclude the discussion about the "non-empty interior" difficulty with a general comment for the t-dependent velocities. It appears that one cannot hope to have a general theorem guaranteeing no interior without making very severe restrictions on the t-dependence of the normal velocity. The reason for this claim is the following. In principle, all motions have some "pathological" situations, where interior develops. One can take any such a motion, perturb its velocity by a time dependent forcing term so that to drive the front to the pathological situation and then simply turn off the time.

5 Uniqueness results for the distance function formulation

As mentioned in Section 3, one can have a weak formulation of the propagation of a front in terms of whether the signed distance to the front satisfies the inequalities (3.1) and (3.2). A natural question to ask is whether (3.1) and (3.2) are enough to identify the distance function uniquely, i.e. if z satisfies (3.1) and (3.2) and $z(x,0) = d(x, \Gamma_0)$, is it true that $z \equiv d$? In addition to being a natural mathematical question to ask, having such information simplifies a lot some of the analysis of the "phase field" theory.

In the sequel, and only in order to considerably simplify the presentation, we will only consider the equation

(5.1) tit - 6(Au - ^ p $\overset{D}{\underset{u|}{+?}}$ ^) + a(x,f)|Du| = 0 in R^N x (0,oo)

with the initial datum

(5.2) u(x,0) = d(x,r_o) in $2R^N$,

with *6* > 0 and *a* € W^{l}*°(R^N x (0,oo)). (Some of the arguments and the conclusions below hold if *6* = 0(x,t) (under some assumptions) as well as for anisotropic motions. We will discuss these situations elsewhere).

As before we denote by T_t = {x : u(x,t) = 0}. Theorem 3.1 and the discussion following it says that the functions *d* = *d(x,*$\tilde{T}_t$*)* and d_2 = d(x,$£_t$) (where T_t = *d{x* : *u(x,t)* > 0} and £< = *d{x* : w(x,t) > 0}) satisfy the inequalities

(5.3) z_t - *6Az* + *a{x* - *zDz,t)* < 0,1 - *\Dz* = 0 in *{z* < 0}

and

(5.4) $*_f$ - *B&z* + a(x - ^Dr, *t)* > 0, |X?^| - 1 = 0 in *{z* > 0}.

Of course, if the no-interior condition holds for every *t* > 0, (5.3) and (5.4) are satisfied by *d* = <f(x,I\). The inequalities in (5.3) and (5.4) are a combination of (3.1) and (3.3) and (3.2) and (3.4) respectively as they apply to (5.1). On the other hand, the equalities in (5.3) and (5.4) follow from the differentiability properties of the distance function and the definition of viscosity solutions.

Next we look into the converse of Theorem 3.1, i.e. we are interested in whether (5.3) and (5.4) identify *z* as the distance function.

Theorem 5.1: *If the usc (resp. lsc) function z satisfies (5.2) and (5.3) (resp. (5.2) and (5.4)), then*

$$z \le d_2 \text{ in } \{z < 0\} \supset \{d_2 < 0\} \tag{5.5}$$

(resp.

$$z \ge d_1 \text{ in } \{z > 0\} \supset \{d_1 > 0\}.) \tag{5.6}$$

If z satisfies (5.2), (5.3) and (5.4) and Γ_t does not develop interior for all $t > 0$, then

$$z(x,t) = d(x,\Gamma_t) \text{ in } I\!R^N \times [0,\infty). \tag{5.7}$$

Proof: The proof is based on the following two lemmas .

Lemma 5.2: *If z is usc (resp. lsc) and satisfies (5.3) (resp. (5.4)), then z is a subsolution (resp. supersolution) of*

$$z_t - \theta(\Delta z - \frac{(D^2zDz|Dz)}{|Dz|^2}) + \alpha(x - zDz, t)|Dz| = 0 \tag{5.8}$$

in $\{z < 0\}$ (resp. $\{z > 0\}$).

Lemma 5.3: *If an usc (resp. lsc) function z satisfies (5.5) (resp. (5.6)), then for C large enough, $\underline{z} = e^{ct}(z \wedge 0)$ (resp. $\overline{z} = e^{ct}(z \vee 0)$) is a subsolution (resp. supersolution) of (5.1).*

We first conclude the proof of the theorem and then prove the lemmas. We proceed by proving (5.5), since (5.6) follows in a similar way. To this end observe that, since $\underline{z}$ (defined in Lemma 5.2) is a subsolution of (5.1), Theorem 1.2 yields $\underline{z} \le u \wedge 0$ in $I\!R^N \times (0,\infty)$; recall that $u \wedge 0$ is still a solution of (5.1), since $\Phi(u) = u \wedge 0$ is an increasing change of u. So, if $u < 0$ (or equivalently if $d_2 < 0$), $z < 0$ and the proof of (5.5) is complete.

Finally, if Γ_t has no interior for all $t > 0$, then $d_1 = d_2 = d$ and (5.5) and (5.6) yield

$$\{z < 0\} = \{d < 0\}, \{z > 0\} = \{d > 0\} \text{ and } \{z = 0\} = \{d = 0\},$$

therefore, $z = d$ by the uniqueness results for the equations $|Dz| - 1 = 0$ and $1 - |Dz| = 0$ respectively in $\{z > 0\} = \{d > 0\}$ and $\{z < 0\} = \{d < 0\}$.

□

We now return to the proofs of the lemmas.

Proof of Lemma 5.2: We only treat the case of an usc z which satisfies (5.6); the other case is proved similarly. Since z is usc, the set $\Omega = \{z < 0\}$ is open. Moreover, z being a solution of $1 - |Dz| = 0$ in $\Omega_t = \{x : z(x,t) < 0\}$ for all $t > 0$, yields

$$z(x,t) = \sup\{z^*(y,t) - |x-y| : y \in \Omega_t\},$$

where $z^*(y,t) = \limsup_{\Omega_t \ni y' \to y} z(y',t)$. This formula implies that z is locally semi-convex with respect to x, *i.e.* $\frac{\partial^2 z}{\partial \chi^2} \geq -C$ in Ω, for all unit vectors $\chi \in \mathbb{R}^N$. Next we define the ε-supconvolution z^ε of z in Ω with respect to t by

$$z^\varepsilon(x,t) = \sup_{(x,s)\in\Omega} \{z(x,s) - \frac{(t-s)^2}{\varepsilon}\}.$$

It follows easily that, for (x,t) belonging to compact subset V of Ω and $\varepsilon > 0$ small enough, the supremum is actually achieved in Ω (and not on $\partial\Omega$) and that z_ε satsifies

$$\text{(5.9)} \qquad 1 - |Dz^\varepsilon| = 0 \text{ and } z^\varepsilon_t - \theta\Delta z^\varepsilon + \alpha(x - z^\varepsilon Dz^\varepsilon, t) \leq C\varepsilon \text{ in } V,$$

where C depends only on the Lipschitz bound of α. Let $(x_0, t_0) \in \Omega$ be a strict local maximum of $z - \phi$ in Ω for some smooth ϕ and take $V' \subset\subset \Omega$ in

(5.9) to be a neighborhood of (x_0, t_0). Since $z^\varepsilon \to z$, there exists $(x_\varepsilon, t_\varepsilon) \in V$ maximum points of $z^\varepsilon - \phi$, such that $(x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$ as $\varepsilon \to 0$. Now we use Alexandrov's Maximum Principle type arguments, brought in the theory of viscosity solutions by Jensen [J]. More precisely, Lemma A.3 of [CIL] implies the existence of $X_\varepsilon \in S^N$ such that

$$
(5.10) \qquad \begin{cases} (\phi_t(x_\varepsilon, t_\varepsilon), D\phi(x_\varepsilon, t_\varepsilon), X_\varepsilon) \in J^{2,+} z^\varepsilon(x_\varepsilon, t_\varepsilon) \\ \text{and} \\ -K \le X_\varepsilon \le D^2_{xx}\phi(x_\varepsilon, t_\varepsilon), \end{cases}
$$

for some constant K, which is related to semiconvexity constant of z and, therefore, of z^ε in V; the upper bound on X_ε comes from the Maximum Principle. (We refer to [CIL] for the definition of $J^{2,+}$). Last but not least, note that

$$X_\varepsilon D\phi(x_\varepsilon, t_\varepsilon) = 0.$$

Indeed, since $|Dz^\varepsilon| = 1$ almost everywhere, $D^2_{xx} z^\varepsilon D z^\varepsilon = 0$ at any point where z^ε is twice differentiable. On the other hand (cf. Lemma A.3 [CIL]) $X_\varepsilon D\phi(x_\varepsilon, t_\varepsilon)$ is obtained as a limit of $D^2_{xx} z^\varepsilon D z^\varepsilon$ evaluated at nearby points. Finally, recall that $D\phi(x_\varepsilon, t_\varepsilon) = Dz^\varepsilon(x_\varepsilon, t_\varepsilon)$, since z^ε is differentiable at maximum points of $z^\varepsilon - \phi$ (again due to the semiconvexity).

Inserting all the information in (5.9) we obtain

$$\phi_t - \theta\Big(\Delta\phi - \frac{(D^2\phi D\phi | D\phi)}{|D\phi|^2}\Big) + \alpha(x_\varepsilon - z^\varepsilon D\phi, t_\varepsilon) \le$$

$$\phi_t - \theta\Big(Tr(X_\varepsilon) - \frac{(X_\varepsilon D\phi | D\phi)}{|D\phi|^2}\Big) + \alpha(x_\varepsilon - z^\varepsilon D\phi, t_\varepsilon) \le C\varepsilon,$$

where in the two inequalities above z^ε and ϕ and its derivatives are evaluated at $(x_\varepsilon, t_\varepsilon)$. Letting $\varepsilon \to 0$ we conclude. □

Proof of Lemma 5.3: We again only present the proof in the case that z is an usc subsolution.

If c is larger than the Lipschitz constant of a, it is immediate that $e^{ci}z$ is a subsolution of (5.1), since \Dz\ = 1 yields

$$\mathbf{a(x\ -\ zDz,t)\ \geq\ a(x,<)\ -\ cz\ =\ a\{x,t)\backslash Dz\backslash\ -\ cz.}$$

To conclude let $(xj>_n)n$ be a sequence of smooth functions such that $rl>_n(t) = 0$ if $t \geq$ —1-,VC ≥ 0 and V» —• 1 uniformly on compact subsets of (—oo,0]. Using the preceeding lemma, it is easy to check that $x/;_n(e^{ct}z)$ is a subsolution of (5.1). Letting n —• oo we conclude, since $xp_n(e^{ct}z)$ —* $e^{ct}(z$ A 0). Q

6 Asymptotic limits of Reaction-Diffusion equations-Phase field theory.

Reaction-Diffusion equations of the form

(6.1) $$<£_t\text{-}A<£ + /(z,t,<£) = 0 \text{ in } 1R^N x(0,oo)$$

arise naturally in many areas of applications like phase transitions, flame propagations, pattern formations, chemical kinetics etc. In most of these applications fronts develop for large times as the boundaries of the regions where the solution <j> of (6.1) converges to the different equilibria of the vector field / (cf. Fife [Fi]). For a discussion of some cases where the solutions of (6.1) converge to the different equilibria of / we refer to Aronson, Weinberger [ArW], Fife, McCleod [FiM] etc. The main issue is to identify the rate at which cf> converges to the different equilibria. For this, one needs to have a better understanding of the fronts and in particular, the way they propagate. In the case /(x,/, </>) = f(<f>), formal results of Fife [Fi] and Caginalp [Cal,2,3] imply that the fronts propagate with normal velocity

(6.2) $$V = 0 + \frac{I}{t} c + 0\left(\frac{1}{t^2}\right) \qquad (t >> 1),$$

when K denotes the curvature.

Our goal here is to justify (6.2) rigorously in the generality of (6.1). A way to do this is to scale ϕ so that to capture the different terms in the asymptotic expansion (6.2). To obtain the first term the appropriate scaling is $(x/\varepsilon, t/\varepsilon)$. If $\alpha = 0$, we then go to the next scaling $(x/\varepsilon, t/\varepsilon^2)$. These considerations give rise to singular perturbation problems of the form

$$\phi_t^\varepsilon - \varepsilon\Delta\phi^\varepsilon + \frac{1}{\varepsilon} f^\varepsilon(x,t,\phi^\varepsilon) = 0 \text{ in } \mathbb{R}^N \times (0,+\infty), \tag{6.3}$$

and

$$\phi_t^\varepsilon - \Delta\phi^\varepsilon + \frac{1}{\varepsilon^2} f^\varepsilon(x,t,\phi^\varepsilon) = 0 \text{ in } \mathbb{R}^N \times (0,+\infty), \tag{6.4}$$

with initial data

$$\phi^\varepsilon(\cdot,0) = \phi_0^\varepsilon(\cdot) \text{ on } \mathbb{R}^N. \tag{6.5}$$

Here ϕ_0^ε is a given function, which initializes the front and f^ε is some approximation of f. Singular perturbation problems of the form (6.3) and (6.4) are of independent interest for they also arise in models with slow diffusion and fast reaction, in phase transitions etc.

In the sequel we study the behavior, as $\varepsilon \to 0$, of (6.3) and (6.4) under the assumption that $\phi \mapsto f^\varepsilon(x,t,\phi)$ is a "cubic" type nonlinearity, i.e. it has two stable and one unstable equilibria. Typical examples of f^ε are:

$$f^\varepsilon(x,t,q) = 2(q - \varepsilon\mu(x,t))(q^2-1), \tag{6.6}$$

$$f^\varepsilon(x,t,q) = 2(q - \mu(x,t))(q^2-1), \tag{6.7}$$

and

$$f^\varepsilon(x,t,q) = 2(q-\mu)(q^2-1) + \varepsilon\theta^\varepsilon(x,t), \tag{6.8}$$

where $\theta^\varepsilon, \mu \in W^{1,\infty}(\mathbb{R}^n \times [0,+\infty))$ are given and μ takes values in $(-1,1)$.

To simplify the presentation we restrict ourselves to problems where the second order operator is the Laplacian, although all the arguments can be modified to apply to more general elliptic operators (under of course suitable hypotheses). This will be addressed in the future. Finally, we remark that the case where f^ε is of "quadratic" type (i.e. f^ε has one stable and one unstable equilibrium) has been studied by probabilistic methods by Freidlin [Fr] and, in greater generality, by pde-type techniques by Evans and Souganidis [ES2,3] and Barles, Evans and Souganidis [BaES]. The latter work actually studies a general system of reaction-diffusion equations.

We conclude this section with a brief disucssion of the "phase field" approach to study propagating fronts. This consists of studying first the behavior of ϕ^ε as $\varepsilon \to 0$ in (6.3) and (6.4) and then define the propagating front as the boundary of the regions where the ϕ^ε's converge to the different equilibria of the vector field. The advantage of this approach, which is rather indirect, is that it avoids any discussion of the empty interior and the non-uniqueness difficulties at least at first glance provided of course that such a convergence can be proved. It will become, however, apparent below that the convergence is closely related to the interior issue. Another perhaps advantage of the phase field approach is that it allows other numerical methods. This way to study motion by mean curvature was proposed by Bronsard and Kohn [BrK] and DeGiorgi [D]. A byproduct of our analysis in the following sections is that the phase field formulation is equivalent to the level set and distance function ones, taking into account the non-empty interior difficulty.

7 Formal discussion

In this section we discuss, in a formal way, the essential mathematical difficulties involved in the study of (6.3) and (6.4). To simplify the arguments, we consider the special case

$$(7.1) \quad f^\varepsilon(x,t,q) = f_0(q) - \varepsilon\theta = 2(q-\mu)(q^2-1) - \varepsilon\theta \quad (\theta \in I\!R).$$

We begin observing that, for sufficiently small $\varepsilon > 0$, there exists $h_-^\varepsilon(\theta) < h_0^\varepsilon(\theta) < h_+^\varepsilon(\theta)$ such that

$$f^\varepsilon(x,t,h_-^\varepsilon(\theta)) = f^\varepsilon(x,t,h_0^\varepsilon(\theta)) = f^\varepsilon(x,t,h_+^\varepsilon(\theta)) = 0.$$

Set

$$(7.2)\quad \begin{cases} m^\varepsilon(\theta) = h_+^\varepsilon(\theta) - h_-^\varepsilon(\theta), \\ q^\varepsilon(r,\theta) = h_-^\varepsilon(\theta) + m^\varepsilon(\theta)(1+\exp(-m^\varepsilon(\theta)[r + r^\varepsilon(\theta)]))^{-1} \ (r \in \mathbb{R}), \\ c^\varepsilon(\theta) = 2h_0^\varepsilon(\theta) - h_+^\varepsilon(\theta) - h_-^\varepsilon(\theta), \end{cases}$$

where $r^\varepsilon(\theta)$ is chosen so that

$$q^\varepsilon(0,\theta) = h_0^\varepsilon(\theta).$$

A straightforward calculation yields

$$(7.3)\qquad q_{rr}^\varepsilon + c^\varepsilon(\theta) q_r^\varepsilon = f_0(q^\varepsilon) - \varepsilon\theta$$

with

$$(7.4)\qquad \lim_{r\to\pm\infty} q^\varepsilon(r,\theta) = h_\pm^\varepsilon(\theta);$$

in other words, q^ε is the *traveling wave* corresponding to the nonlinearity $f_0 - \varepsilon\theta$ which travels with speed $c^\varepsilon(\theta)$. Indeed if we set

$$\Phi^\varepsilon(\xi,t) = q^\varepsilon(\xi - c^\varepsilon(\theta)t) \text{ in } \mathbb{R}\times(0,\infty),$$

then

$$\Phi_t - \Phi_{\xi\xi} = f_0(\Phi) - \varepsilon\theta \qquad \text{in } \mathbb{R}\times(0,\infty).$$

In fact, for any "cubic type" nonlinearity there exists a unique pair of traveling wave and speed satisfying (7.3) and (7.4). A detailed discussion of this fact as well as references will be given in the next section.

We now return to (6.3) and write the solution ϕ^ε as

$$\phi^\varepsilon = q^\varepsilon\Big(\frac{z^\varepsilon}{\varepsilon},\theta\Big) \text{ in } I\!\!R^N \times (0,\infty).$$

A simple calculation yields

$$\frac{1}{\varepsilon}q^\varepsilon_r[z^\varepsilon_t - \varepsilon\Delta z^\varepsilon + c^\varepsilon(\theta)] - \frac{1}{\varepsilon}q^\varepsilon_{rr}(|Dz^\varepsilon|^2 - 1) = 0 \text{ in } I\!\!R^N \times (0,\infty),$$

where q^ε_r and q^ε_{rr} are evaluated at $(z^\varepsilon/\varepsilon,\theta)$.

Analyzing the two terms in the above equation separately, as $\varepsilon \to 0$, we formally conclude that $|Dz^\varepsilon| \cong 1$ and, therefore,

$$z^\varepsilon(x,t) \cong \text{ signed distance function of } x \text{ to } \Gamma_t,$$

where Γ_t is the interface, and

$$z^\varepsilon_t - \varepsilon\Delta z^\varepsilon + c^\varepsilon(\theta) \cong 0 \text{ on } \Gamma_t. \;,$$

Since $h^\varepsilon_0(\theta) \cong \mu + \varepsilon\theta(f_0'(\mu))^{-1}$ and $h^\varepsilon_\pm(\theta) \cong \pm 1 + \varepsilon\theta(f_0'(\pm 1))^{-1}$, (7.2) yields

$$\lim_{\varepsilon\to 0} c^\varepsilon(\theta) = 2\mu.$$

Therefore, always formally, Γ_t moves with normal velocity

$$V = -2\mu.$$

The geometric pde which gives Γ_t as the zero level set of its solutions is

$$u_t + 2\mu|Du| = 0 \text{ in } I\!\!R^N \times (0,\infty).$$

In view of the discussion in Section 6 to consider (6.4) with the vector field f^ε given by (7.1), we need to assume $\mu = 0$, i.e.

$$f^\varepsilon(x,t,q) = 2q(q^2-1) - \varepsilon\theta.$$

Proceeding as for (6.3) above, we write

$$\phi^\varepsilon = q(\frac{z^\varepsilon}{\varepsilon},\theta) \text{ in } I\!\!R^N \times (0,\infty)$$

and find

$$\frac{1}{p^*_r}[zi - Az^* + c\sim^l<f\{\$)) - \wedge q^\epsilon_{rr}(\backslash Dz<\backslash^2 - 1) = 0 \text{ in } \mathbb{R}^N \times (0,\infty),$$

where g£ and $<f_{TT}$ are evaluated at $(z^e/e,O)$. Argueing as before we find (formally) that $z^\epsilon(x,t)$ =~signed distance function from x to F_t, where F_t is the interface, and

$$_z\backslash \_ A\wedge +_e{}^{-l}c^\epsilon\{e) \text{ a } 0 \text{ on } r_t.$$

Using the expressions for /fQ(^), J*±(0) and (7.2) we find

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} c\{6) = -le.$$

Therefore, always formally, T_t moves with normal velocity

$$V = \text{mean curvature } + \frac{3}{2}0.$$

The corresponding geometric pde is

$$u_t - \left(\Delta u - \frac{(D^2uDu|Du)}{|Du|^2}\right) - \frac{3}{2}\theta|Du| = 0 \text{ in } \mathbb{R}^N \times (0,\infty).$$

8 Traveling waves

Here we discuss the existence and the general properties of traveling waves for functions tt H» $/^e(i,t,u)$, which have the property that, for a and e small, the function u »» $/^e(z,2,u)$ — ea behaves like a "cubic function" of u. More precisely, we assume that, for a and e sufficiently small, the equation

$$f^\varepsilon(x,t,u) - \varepsilon a = 0$$

has exactly three zeroes: $h_-^\varepsilon(x,t,a) < h_0^\varepsilon(x,t,a) < h_+^\varepsilon(x,t,a)$. Moreover, we assume that

$$(8.1)\qquad \begin{cases} f^\varepsilon(x,t,\cdot) - \varepsilon a > 0 \text{ in } (h_-^\varepsilon, h_0^\varepsilon) \cup (h_+^\varepsilon, +\infty), \\ f^\varepsilon(x,t,\cdot) - \varepsilon a < 0 \text{ in} (-\infty, h_-^\varepsilon) \cup (h_0^\varepsilon, h_+^\varepsilon), \\ f_u^\varepsilon(x,t,h_\pm^\varepsilon) \geq \gamma > 0, \end{cases}$$

with γ independent of (x,t,a,ε).

Since, for fixed (x,t,a,ε), the function $u \mapsto f^\varepsilon(x,t,u) - \varepsilon a$ satisfies the hypotheses of Aronson, Weinberger [ArW] and Fife, McLeod [FiM], there exists a unique pair $(q^\varepsilon(r,x,t,a), c^\varepsilon(x,t,a))$ such that

$$(8.2)\qquad q_{rr}^\varepsilon(r,x,t,a) + c^\varepsilon(x,t,a) q_r^\varepsilon(r,x,t,a) = f^\varepsilon(x,t,q^\varepsilon(r,x,t,a)) - \varepsilon a,$$

and

$$(8.3)\qquad \lim_{r\to\pm\infty} q^\varepsilon(r,x,t,a) = h_\pm^\varepsilon(x,t,a) \text{ and } q^\varepsilon(0,x,t,a) = h_0^\varepsilon(x,t,a);$$

the second part of (8.3) is necessary to fix q^ε since (8.2) is invariant under translation in r.

We continue listing a set of technical assumptions that we will be making on $(q^\varepsilon, c^\varepsilon)$. We then verify these assumptions for a particular class of f^ε's, which arise naturally in applications. To this end, we assume that, as $\varepsilon \to 0$,

$$(8.4)\qquad q^\varepsilon \text{ and } c^\varepsilon \text{ depend smoothly on } (x,t,a),$$

$$(8.5)\qquad h_\pm^\varepsilon(x,t,a) \to h_\pm(x,t,a), \quad h_0^\varepsilon(x,t,a) \to h_0(x,t,a)$$

and, either

$$(8.6)\qquad c^\varepsilon(x,t,a) \to \alpha(x,t,a),$$

or

$$(8.7)\qquad -\varepsilon^{-1} c^\varepsilon(x,t,a) \to \alpha(x,t,a), \text{ if } c^\varepsilon(x,t,a) \to 0,$$

with all the limits local uniform in (x,t,a). Moreover, if

$$\alpha(x,t) = \alpha(x,t,0), h_{\pm}(x,t) = h_{\pm}(x,t,0) \text{ and } h_0(x,t) = h_0(x,t,0),$$

we assume that there exists $K > 0$, independent of (x,t), such that, for ε and a small enough and all (x,t),

$$|\alpha(x,t) - \alpha(y,t)| \leq K|x-y|. \tag{8.8}$$

If (8.7) holds, we also assume:

$$\begin{cases} (i) & |h_{\pm t} - \Delta h_{\pm}| \leq K \\ (ii) & \lim\limits_{\varepsilon\to 0} \sup\limits_{(x,t,r,a)} [\varepsilon|q_t^\varepsilon| + \varepsilon|\Delta q^\varepsilon| + |Dq_r^\varepsilon|] = 0 \\ (iii) & \frac{1}{\varepsilon}|q_{rr}^\varepsilon(r,x,t,a)| + \frac{1}{\varepsilon^2}|q_r^\varepsilon(r,x,t,a)| \leq Ke^{-\frac{K\delta}{\varepsilon}} \text{ for all } |r| \geq \delta. \end{cases} \tag{8.9}$$

Finally, for all (x,t) and ε, a sufficiently small, we assume

$$q_r^\varepsilon \geq 0 \text{ and } q_a^\varepsilon \geq 0. \tag{8.10}$$

Next we present an example where the above hypotheses hold true. Indeed consider

$$f^\varepsilon(x,t,q) = 2(q - \mu^\varepsilon(x,t))(q^2-1) - \varepsilon\theta^\varepsilon(x,t), \tag{8.11}$$

where $\theta^\varepsilon : I\!R^N \times (0,\infty) \to I\!R$ is a given function. Let $h_0^\varepsilon, h_{\pm}^\varepsilon, q^\varepsilon$ and c^ε be as in Section 7 for each (x,t) and define

$$h_0^\varepsilon(x,t,a) = h_0^\varepsilon(\theta^\varepsilon(x,t)+a), h_{\pm}^\varepsilon(x,t,a) = h_{\pm}^\varepsilon(\theta^\varepsilon(x,t)+a), q^\varepsilon(r,x,t,a) = q^\varepsilon(r,\theta^\varepsilon(x,t)+a)$$

and

$$c^\varepsilon(x,t,a) = c^\varepsilon(\theta^\varepsilon(x,t)+a).$$

It is immediate that (8.4) holds (if θ^ε is smooth) and that (8.5) holds with $h_{\pm}(x,t,a) = \pm 1$ and $h_0(x,t,a) = \mu$; (8.6) holds with $\alpha(x,t,a) = 2\mu(x,t)$

where $\mu = \lim_{\varepsilon \to 0} \mu^\varepsilon$. If $\mu(x,t) = 0$, then (8.7) yields $\alpha(x,t,a) = \frac{3}{2}(\theta(x,t) + a)$, provided that $\theta^\varepsilon(x,t) \to \theta(x,t)$ uniformly. In view of the above, (8.8) needs

$$(8.12) \qquad |\theta(x,t) - \theta(y,t)| \leq K|x-y| \text{ or } |\mu(x,t) - \mu(y,t)| \leq K|x-y|.$$

To conclude, using the explicit formulae in (7.2) we compute

$$D_t q^\varepsilon = q_\theta^\varepsilon \theta_t^\varepsilon, D_x q^\varepsilon = q_\theta^\varepsilon D_x \theta^\varepsilon, \Delta_x q^\varepsilon = q_\theta^\varepsilon \Delta_x \theta^\varepsilon + q_{\theta\theta}^\varepsilon |D_x \theta^\varepsilon|^2.$$

Since $|q_\theta| \leq \varepsilon K$ and $|q_{\theta\theta}| \leq \varepsilon^2 K$ for some $K > 0$, (8.9)(ii) holds if θ^ε is such that

$$(8.13) \qquad \lim_{\varepsilon \to 0} \varepsilon[\sup_{(x,t)}(\varepsilon|\theta_t^\varepsilon| + \varepsilon|\Delta\theta^\varepsilon| + |D\theta^\varepsilon|)] = 0.$$

For (8.12) and (8.13) to hold, it suffices to assume that

$$(8.14) \qquad (\theta^\varepsilon)_{\varepsilon>0} \text{ is uniformly bounded in } C^{2,1}(\mathbb{R}^N \times [0,\infty)).$$

Finally, (8.9)(iii) and (8.11) hold provided $4\varepsilon|\theta^\varepsilon| \leq 1$ which, follows from (8.14) for ε small.

We conclude this section observing that similar computations are possible for

$$(8.15) \qquad f^\varepsilon(x,t,q) = 2(\theta^\varepsilon(x,t)q - \mu^\varepsilon(x,t))((\theta^\varepsilon(x,t)q)^2 - 1).$$

$\square$

9 Asymptotic behavior of reaction-diffusion equations; the main results.

We next state our main theorem about the behavior of the solution ϕ^ε of (6.3) and (6.4). To study (6.3) we consider f^ε's which satisfy (8.1) to (8.6) and (8.8) and (8.10). For (6.4) we will consider f^ε's such that (8.1) to (8.5) and (8.7) to (8.10) hold. In either case we will denote by *$(q^e(r,x,t),<f(x,t))$* the pair of traveling wave and speed which corresponds to f^e and we will assume

(9.1) $$a\{x,t,a) \geq a(x_yt) \text{ for all } a > 0.$$

Throughout the discussion below we will be assuming that

(9.2) $$<j><(x,0) = {}_g e(\wedge £ \frac{l E^{\circ}}{\varepsilon}),_M) \text{ on } JR^N,$$

where F_o is a closed set in M^N. The last assumption can be removed at the expense of rather lengthy arguments. We will address this issue elsewhere.

In view of the (formal) discussion in Section 7 we expect that the limiting behavior of $<t>^{\epsilon}$ will be governed by the geometric pde's

(9.3) $$u_t - a\{x,t)\backslash Dn\backslash = 0 \text{ in } M^N \text{ x } (0,oo)$$

for (6.3) and

(9.4) $$u_t - (A, \text{—} \{D2U \wedge U)) \text{—} a(x,I)|\wedge| = 0 \text{ in } «" \text{ x } (0,oo)$$

for (6.4), with (by (9.2)),

(9.5) $$ti(z,0) = <f(ar,r_o) \text{ on } R^N.$$

Theorem 9.1: *Let $<f>^e$ be the solution of (6.3), (9.2) with /* satisfying (8.1) to (8.6) and (8.8), (8.10) and (9.1). Ifu is the solution of (9.3), (9.5), then, as $\varepsilon \to 0$,*

(9.6) $$\begin{cases} \qquad t) > 0, \\ \text{(it)} \quad <ft'(x,t) \text{-*} \; h.\{x,t) \quad \text{if} \quad \mathbf{u(x,<)<0,} \end{cases}$$

with the limits local uniform in $\{(x,t) : u(x,t) \neq 0\}$.

Theorem 9.2: *Let* ϕ^ε *be the solution of (6.4), (9.2) with* f^ε *satisfying (8.1) to (8.5), (8.7) to (8.10) and (9.1). If* u *is the solution of (9.4), (9.5), then, locally uniformly in* $\{(x,t) : u(x,t) \neq 0\}$, *as* $\varepsilon \to 0$

$$\begin{cases} (i) \quad \phi^\varepsilon(x,t) \to h_+(x,t) \quad \text{if} \quad u(x,t) > 0, \\ (ii) \quad \phi^\varepsilon(x,t) \to h_-(x,t) \quad \text{if} \quad u(x,t) < 0. \end{cases} \tag{9.7}$$

In the special case where $f^\varepsilon(x,t,u) = 2(u-\mu)(1-u^2)$, Barles, Bronsard and Souganidis [BaBS] studied the limiting behavior or of the solutions ϕ^ε of (6.3). Gãrtner [G] also studied the same problem when $f^\varepsilon(x,t,u) = f(x,t,u)$ by a combination of probabilistic and analytic techiques. Evans, Soner and Souganidis [ESS] studied the limiting behavior of ϕ^ε in (6.4) when $f(u) = 2u(1-u^2)$; this problem was first studied in the context of radially symmetric functions by Bronsard and Kohn [BrK]. Finally, Chen [Ch] and DeMottoni and Shatzman [DS] obtained results similar to Theorems 9.1 and 9.2 (for special cases of f) assuming, however, that Γ_t is a smooth surface. No such assumption is made here.

We conclude this section remarking that actually one can obtain more precise results than (9.6) and (9.7). Indeed, it is possible to obtain WKB-type expressions for ϕ^ε of the form

$$\phi^\varepsilon(x,t) = q^\varepsilon\left(\frac{d(x,\Gamma_t) + o(1)}{\varepsilon}, x, t\right).$$

This is done below in Section 10.1 for some simple case. The arguments for the general case are, however, rather complicated and will be presented elsewhere.

10 Proofs.

Instead of presenting a general proof for Theorems 9.1 and 9.2, we will first give some less general but more direct arguments utilizing the results of Section 5. At the end we will turn to the general case. The reason for doing this is that in the less general cases it is possible to work directly at the $\varepsilon = 0$ level, as opposed to the general case where we need to build super- and subsolutions for $\varepsilon > 0$. The latter approach is tying us down to cases where the Maximum Principle holds.

10.1 The (x,t)-independent case.

Here we assume that f^ε and therefore q^ε is independent of (x,t) and is given by (6.6) for (6.4) and (6.7) for (6.3). As a matter of fact the traveling wave in either case is $q(r) = \tanh(r)(r \in I\!R)$ and the speed $2\varepsilon\mu$ and 2μ for (6.6) and (6.7) respectively.

Following the discussion in Section 7, if

$$\phi^\varepsilon = q(\frac{z^\varepsilon}{\varepsilon}) \text{ in } I\!R^N \times (0,\infty), \tag{10.1}$$

then z^ε solves

$$z_t^\varepsilon - \varepsilon\Delta z^\varepsilon + 2q(\frac{z^\varepsilon}{\varepsilon})(|Dz^\varepsilon|^2 - 1) + 2\mu = 0 \text{ in } I\!R^N \times (0,\infty) \tag{10.2}$$

in the case of (6.3), and

$$z_t^\varepsilon - \Delta z^\varepsilon + \frac{2}{\varepsilon}q(\frac{z^\varepsilon}{\varepsilon})(|Dz^\varepsilon|^2 - 1) + 2\mu = 0 \text{ in } I\!R^N \times (0,\infty) \tag{10.3}$$

in the case of (6.4), with, in either case,

$$z^\varepsilon(x,0) = d(x,\Gamma_0) \text{ on } I\!R^N. \tag{10.4}$$

We want to study the behavior of z^ε as $\varepsilon \to 0$. To this end, we assume for the moment that $(z^\varepsilon)_{\varepsilon>0}$ is locally uniformly bounded in $I\!R^N \times (0,T)$ for some

$T > 0$ and we proceed. Since (10.2) and (10.3) are translation invariant with respect to x, it is immediate that

$$|Dz^\varepsilon| \leq 1 \text{ in } I\!R^N \times (0,\infty). \tag{10.5}$$

On the other hand, the form of (10.2) and (10.3) makes any kind of estimate z_t^ε hopeless. To circumvent this difficulty we use the, by now classical, half-relaxed limit techniques described in Barles and Perthame [BaP] (see also [CIL]), i.e. we consider the functions

$$z^*(x,t) = \limsup_{\substack{\varepsilon\to 0\\ s\to t}} z^\varepsilon(x,s) \text{ and } z_*(x,t) = \liminf_{\substack{\varepsilon\to 0\\ s\to t}} z^\varepsilon(x,s). \tag{10.6}$$

We begin with (10.3) which can be rewritten as

$$z_t^\varepsilon - \Delta z^\varepsilon + 2\mu = -\frac{2}{\varepsilon}q(\frac{z^\varepsilon}{\varepsilon})(|Dz^\varepsilon|^2 - 1). \tag{10.7}$$

The form of q and (10.5) yield

$$-\frac{2}{\varepsilon}q(\frac{z^\varepsilon}{\varepsilon})(|Dz^\varepsilon|^2 - 1) \geq 0 \text{ if } z^\varepsilon > 0$$

and

$$-\frac{2}{\varepsilon}q(\frac{z^\varepsilon}{\varepsilon})(|Dz^\varepsilon|^2 - 1) \leq 0 \text{ if } z^\varepsilon < 0.$$

Using (10.5), (10.7) and the above inequalities we get that z^* is an usc subsolution of (9.4) with $\alpha = -2\mu$ and a solution of $1 - |Dz| = 0$ in $\{z < 0\}$ and that z_* is a lsc supersolution of (9.4) with $\alpha = -2\mu$ and a solution of $|Dz| - 1 = 0$ in $\{z > 0\}$. That z^* (resp. z_*) is a subsolution (resp. supersolution) of (9.4) in $\{z < 0\}$ (resp. $\{z > 0\}$) follows from (10.7) and the above inequalities, that z^* (resp. z_*) solves $1 - |Dz| = 0$ (resp. $|Dz| - 1 = 0$) in $\{z < 0\}$ (resp. $\{z > 0\}$) follows the passage to the limit in both (10.5) and (10.7).

Theorem 5.1 implies that $z^* \leq d_2$ in $\{z^* < 0\} \supset \{u < 0\}$ and $z_* \geq d_1$ in $\{z_* > 0\} \supset \{u > 0\}$, where u is the solution of (9.4) with $\alpha = -2\mu$. Moreover if the "empty interior" condition holds, Theorem 5.1 yields

$$z^*(\cdot,t) = z_*(\cdot,t) = d(\cdot,\Gamma_t)$$

therefore the result.

In the case of (10.2), we rewrite the equation as

$$z_t^\varepsilon - \varepsilon\Delta z^\varepsilon + 2\mu|Dz^\varepsilon| = -(|Dz^\varepsilon| - 1)(2\mu + 2q(\frac{z^\varepsilon}{\varepsilon}))(|Dz^\varepsilon| + 1))$$

and pass to the limit using sign type arguments, similar to the first case but for the limiting equation. Indeed, since $\mu \in (-1,1)$, we obtain

$$z_t + 2\mu|Dz| \le 0 \text{ in } \{z < 0\} \text{ and } z_t + 2\mu|Dz| \ge 0 \text{ in } \{z > 0\},$$

where above we have suppressed the z^* and z_* notation. The arguments of the proofs of Theorem 5.1 and Lemmas 5.2 and 5.3 yield

$$\{z^* < 0\} \supset \{u < 0\} \text{ and } \{z_* > 0\} \supset \{u > 0\};$$

we conclude as before.

It remains to prove the uniform local bound on z^ε. Such a bound is easy for (10.2) and we leave it up to the reader; here we concentrate on (10.3). Let $\psi : \mathbb{R} \to \mathbb{R}$ be a C^2 function such that $\psi \equiv 0$ in $[0, +\infty)$ and $\psi(-\infty) = -1$ with $\psi' > 0$ in $(-\infty, 0)$ and ψ'' bounded and consider the function $\overline{\omega}_\varepsilon$ defined by $\overline{\omega}_\varepsilon = \psi(z^\varepsilon)$. In view of the choice of ψ, it is clear that $-1 \le \overline{\omega}_\varepsilon \le 0$, i.e. $\overline{\omega}_\varepsilon$ is bounded. Next we define

$$\overline{w}^*(x,t) = \limsup_{\substack{\varepsilon\to 0\\ s\to t}} \ \overline{\omega}_\varepsilon(x,s);$$

$\overline{w}^*$ is well-defined and $\overline{w}^* = -1$ if $z^* = -\infty, \overline{w}^* = \psi(z^*)$ if $z^* \in (-\infty, 0)$ and $\overline{w}^* = 0$ if $z^* \ge 0$. Combining the above with arguments from the proof of Theorem 5.1, it can be shown that $\overline{w}^*$ is a subsolution of the two sided variational inequality

$$\max(w, \min(w+1, w_t - \left(\Delta w - \frac{(D^2 w Dw|Dw)}{|Dw|^2}\right) + 2\mu|Dw|)) = 0.$$

A direct modification of the usual comparison results yields

$$\overline{w}^* \le \psi(u) \text{ in } \mathbb{R}^N \times (0, \infty),$$

where u is the solution of (9.4) with a = —2/z. Argueing in exactly the same way with xx^. = —\f)(—ze), we find

$$w_*(x,t) = \underline{\lim}\, w_\epsilon(x,t) \geq -\psi(-u) \text{ in } \mathbb{R}^N \times (0,\infty).$$

We conclude as follows: Let f* = sup{i > 0: there exists x G $\mathbb{R}^N$ such that u(x,*) > 0}. Iff < f% the sets $\{u > 0\}$ and $\{u < 0\}$ and, therefore, $\{z^* < 0\}$ and {z. < 0} are nonempty. Then there exists points in a bounded region of $\mathbb{R}^N$ (depending only on u) such that $z^m < 0$ and z. > 0. The local uniform bound then follows from (10.5) for all $T < t\backslash$ If $t >$ f, then $z^m < 0$ and therefore <£* —* —1 at any such points.

D

10.2 The (x, t)-dependent case

We now study (6.3) and (6.4) in the case where f^ϵ is given by (6.6), (6.7) and (6.8). We only give the proof for (6.4) for f^ϵ given by (6.6); the other cases can be treated similarly. First we recall that the traveling wave q^ϵ associated with (6.6) is still $q^\epsilon = q =$ tanh. As before we perform the change <}>€ = 9(7-) and find the z^ϵ satisfies

(10.7) 4 - A z c + 2/Ka:,0 + $\frac{2}{\varepsilon}$ 9 $(\frac{z^\epsilon}{\varepsilon})$ (| ^ e | 2 - l) = 0 in R^N x(0,oo)

with

$$r^e(.T,0) = <f(x,r_o) \text{ on } R^N.$$

The main difference between this case and the (x,t)-independent one is that (10.7) is not translation invariant with respect to x. Instead of (10.5) here we have

(10.8) $$|Dz^\epsilon| \leq e^{Ct} \text{ in } \mathbb{R}^N \times (0,\infty),$$

where C is the Lipschitz constant of 2/* with respect to x.

Next we introduce the function $\underline{z}^\varepsilon$ defined by

$$\underline{z}^\varepsilon(x,t) = \inf_{y\in \mathbb{R}^N} (\eta(z^\varepsilon(y,t)) + |x-y|),$$

where $\eta \in C^2$ is such that: $\eta(0) = 0, \eta' > 1$ in $(0,\infty), \eta' < 1$ in $(-\infty, 0)$ and $\beta > \eta'' > 0$ and $\eta \geq -\beta^{-1}$ on $\mathbb{R}$ for some $\beta > 0$. Since η is bounded from below, it is clear that that infimum in the definition of $\underline{z}^\varepsilon$ is achieved for some $y^\varepsilon(x)$; y^ε also depends on t but we suppress this here. We now perform the usual arguments for this type of inf-convolution. If $y^\varepsilon(x) \neq x$, then

$$Dz^\varepsilon(y^\varepsilon(x),t) = \frac{1}{\eta'(z^\varepsilon(y^\varepsilon(x),t))} \frac{x-y^\varepsilon(x)}{|x-y^\varepsilon(x)|};$$

hence

$$q(\frac{z^\varepsilon}{\varepsilon})(|Dz^\varepsilon|^2 - 1) = q(\frac{z^\varepsilon}{\varepsilon})(\frac{1}{\eta'(z^\varepsilon)^2} - 1) \leq 0 \text{ at } (y^\varepsilon(x),t). \tag{10.9}$$

On the other hand, if $y^\varepsilon(x) = x$ and $z^\varepsilon(x,t) > 0$, then $|Dz^\varepsilon(x,t)| \leq 1$ and

$$q(\frac{z^\varepsilon}{\varepsilon})(|Dz^\varepsilon|^2 - 1) \leq 0. \tag{10.10}$$

Combining the last two inequalities we obtain

$$\underline{z}^\varepsilon_t - \Delta \underline{z}^\varepsilon + \beta|D\underline{z}^\varepsilon|^2 + 2\mu(y^\varepsilon(x),t) \geq 0 \text{ in } \{\underline{z}^\varepsilon > 0\}. \tag{10.11}$$

As in the previous section we assume that the z^ε's (and therefore the $\underline{z}^\varepsilon$'s) are locally uniformly bounded in $\mathbb{R}^N \times (0,\infty)$ and we consider

$$z_*(x,t) = \liminf_{\substack{\varepsilon\to 0\\ s\to t}} z^\varepsilon(x,s) \text{ and } \underline{z}(x,t) = \liminf_{\substack{\varepsilon\to 0\\ s\to t}} \underline{z}^\varepsilon(x,s).$$

Letting $\varepsilon \to 0$ in (10.7) we get

$$\text{sgn } (z_*)(|Dz_*| - 1) \geq 0 \text{ in } \mathbb{R}^N \times (0,\infty). \tag{10.12}$$

We also need to send $\varepsilon \to 0$ in (10.11). To do so we assume that $y^\varepsilon(x) \to y(x)$ for some $y(x)$ as $\varepsilon \to 0$ (since the family $(y^\varepsilon(x))_\varepsilon$ is bounded, $y^{\varepsilon_n}(x) \to y(x)$ for some $y(x)$ at least along some subsequence); hence

$$\underline{z}_t - \Delta \underline{z} + \beta |D\underline{z}|^2 + 2\mu(y(x), t) \geq 0 \text{ in } \{\underline{z} > 0\}.$$

Now we remark that

$$\underline{z}(x,t) = \eta(z_*(y(x),t)) + |x - y(x)|;$$

the definitions of z^ε and $\underline{z}$ together with (10.9), (10.10) and (10.12) and the properties of η yield $z_*(y(x),t) = 0$ and, therefore, $\underline{z}(x,t) = d(x, \{z_* = 0\})$. We conclude combining the arguments of the previous section and the ones of the proof of Theorem 5.1 and letting $\beta \to 0$.

10.3 The general case

Unfortunately we cannot prove Theorems 9.1 and 9.2 in the case of general f^ε by a direct passage to the limit; one of the main difficulties being the lack of an explicit formula for the traveling wave q^ε and its speed c^ε. Here we will proceed by constructing sub- and super-solutions for (6.3) and (6.4) following ideas introduced in Evans, Soner and Souganidis [ESS]. As before we will only present the proof of Theorem 9.2; Theorem 9.1 is proved in a similar way with some modifications noted below. We begin with some preliminary facts.

For fixed $\delta, a > 0$, let $u^{\delta,a}$ be the solution of

$$(10.13) \qquad \begin{cases} u_t^{\delta,a} + F(x,t,Du^{\delta,a}, D^2u^{\delta,a}) = (\alpha(x,t,a) - \alpha(x,t))|Du^{\delta,a}| \\ u^{\delta,a}(x,0) = d(x,\Gamma_0) + \delta \text{ on } I\!R^N \end{cases}$$

where

$$F(x,t,p,X) = -tr(X) + \frac{(Xp|p)}{|p|^2} - \alpha(x,t)|p|.$$

If

$$d^{\delta,a}(x,t) = d(x, \{y : u^{\delta,a}(y,t) = 0\}),$$

Theorem 3.1 yields that

$$d_t^{\delta,a} - \Delta d^{\delta,a} - \alpha(x - d^{\delta,a} Dd^{\delta,a}, t, a) \geq 0 \text{ in } \{d^{\delta,a} > 0\}. \tag{10.14}$$

Following the proof of Lemma 3.1 of [ESS], we define

$$w^{\delta,a}(x,t) = \eta_\delta(d^{\delta,a}(x,t)), \tag{10.15}$$

where, as in [ESS], $\eta_\delta : I\!R \to I\!R$ is a smooth function satisfying

$$\begin{cases} \eta_\delta(z) = -\delta \text{ if } z \leq \delta/4, \\ \eta_\delta(z) = z - \delta \text{ if } z \geq \delta/2, \\ \eta_\delta(z) \leq -\frac{\delta}{2} \text{ if } z \leq \delta/2, \\ 0 \leq \eta_\delta' \leq C \text{ and } |\eta_\delta''| \leq C\delta^{-1} \text{ on } I\!R \end{cases} \tag{10.16}$$

where $C > 0$ is independent of δ. A straightforward modification of Lemma 3.1 of [ESS] together with (10.15) yields the following Lemma.

Lemma 10.1: *There exists a constant C, independent of δ and a, such that*

$$\begin{cases} (i) \quad w_t^{\delta,a} - \Delta w^{\delta,a} - \alpha(x,t,a)|Dw^{\delta,a}| \geq -\frac{C}{\delta} \text{ in } I\!R^N \times [0,t^*), \\ (ii) \quad w_t^{\delta,a} - \Delta w^{\delta,a} - \alpha(x - w^{\delta,a} Dw^{\delta,a}, t, a) \geq 0 \text{ on } \{d^{\delta,a} > \frac{\delta}{2}\}, \end{cases}$$

and

$$|Dw^{\delta,a}| = 1 \text{ in } \{d^{\delta,a} > \frac{\delta}{2}\}. \tag{10.18}$$

where t^* is the extinction time of $\{u^{\delta,a} = 0\}$.

□

Finally, we define

(10.19) $\Phi^\varepsilon(x,t) = q^\varepsilon(\frac{w^{\delta,a}}{\varepsilon}; x,t,a)$ on $\mathbb{R}^N \times [0,\infty)$,

where, for notational simplicity, we do not exhibit the dependence of Φ^ε on δ and a.

Proposition 10.2: *Assume that f^ε satisfies the hypotheses of Theorem 9.2. Then, for every $a > 0$, Φ^ε is a supersolution of (6.4), if $\varepsilon \leq \varepsilon_0(\delta,a)$ and $\delta \leq \delta_0(a)$.*

The proof of the proposition is similar to the one of Theorem 3.2 of [ESS]. The form, however, of Φ^ε is different than the one used in [ESS]. As usual we will present the proof as if $w^{\delta,a}$ has actual derivatives, keeping in mind that actually everything has to be checked in the viscosity sense; we will leave it up to the reader to do so.

Proof: We need to show that

$$\Phi^\varepsilon_t - \Delta\Phi^\varepsilon + \frac{1}{\varepsilon^2} f^\varepsilon(x,t,\Phi^\varepsilon) \geq 0$$

for $\varepsilon \leq \varepsilon_0(\delta,a)$ and $\delta \leq \delta_0(a)$. Using the equation for $q^\varepsilon(\xi; x,t,a)$, we calculate:

$$(10.22)\qquad \begin{cases} \Phi^\varepsilon_t - \Delta\Phi^\varepsilon + \dfrac{1}{\varepsilon^2} f^\varepsilon(x,t,\Phi^\varepsilon) = \dfrac{1}{\varepsilon} + J^\varepsilon - \dfrac{q^\varepsilon_{rr}}{\varepsilon^2}(|Dw^{\delta,a}|^2 - 1) \\[2ex] \qquad q^\varepsilon_r(w^{\delta,a}_t - \Delta w^{\delta,a} + \dfrac{-}{\varepsilon}) + a) \end{cases}$$

where q^ε_r and q^ε_{rr} are evaluated at $(\frac{w^{\delta,a}}{\varepsilon}, x, t, a)$, with

$$J^\varepsilon(x,t) = (q^\varepsilon_t + \frac{2}{\varepsilon} Dq^\varepsilon_r Dw^{\delta,a} + \Delta q^\varepsilon)(\frac{w^{\delta,a}}{\varepsilon}, x,t,a).$$

In view of its definition, it is immediate that $|Dw^{\delta,a}| \leq C$ for some $C > 0$. Therefore, by (5.9)(ii),

(10.23) $J^\varepsilon = \dfrac{o(1)}{\varepsilon}$ as $\varepsilon \to 0$ uniformly in (x,t,δ,a).

We proceed by examining the next three cases.

Case 1: $\frac{\delta}{2} < d^{\delta,a} < 2\delta$

Using (10.18), the Lipschitz continuity of α with respect to x, the fact that $d^{\delta,a} < 2\delta$ and the form of η_δ, we get

$$w_t^{\delta,a} - \Delta w^{\delta,a} - \alpha(x,t,a) \geq -C\delta \text{ and } |Dw^{\delta,a}| = 1.$$

Substituting in (10.22) and employing (10.23) we obtain

(10.24)

$$\Phi_t^\varepsilon - \Delta\Phi^\varepsilon + \frac{1}{\varepsilon^2} f^\varepsilon(x,t,\Phi^\varepsilon) \geq \frac{1}{\varepsilon}[q_r^\varepsilon(-C\delta + \frac{c^\varepsilon(x,t,a)}{\varepsilon} + \alpha(x,t,a)) + a + o(1)]$$

where again q_r^ε is evaluated at $(\frac{w^{\delta,a}}{\varepsilon}, x, t, a)$. Since $\varepsilon^{-1}c^\varepsilon(x,t,a) \to -\alpha(x,t,a)$ as $\varepsilon \to 0$, uniformly in (x,t,a), we see that the right side of (10.24) is positive if ε and δ are sufficiently small.

Case 2: $d^{\delta,a} \leq \delta/2$

In this case the choice of η_δ yields

$$w^{\delta,a} \leq -\delta/2.$$

Consequently, (8.9)(iii) yields that

$$\frac{1}{\varepsilon}|q_r^\varepsilon(\frac{w^{\delta,a}}{\varepsilon}, x, t, a)| + \frac{1}{\varepsilon^2}|q_{rr}^\varepsilon(\frac{w^{\delta,a}}{\varepsilon}, x, t, a)| \leq Ke^{-\frac{K\delta}{\varepsilon}}$$

for some appropriate constant K. Using that $|Dw^{\delta,a}| \leq C$ as well as (10.17) in (10.22) we obtain

$$\Phi_t^\varepsilon - \Delta\Phi^\varepsilon + \frac{1}{\varepsilon^2} f^\varepsilon(x,t,\Phi^\varepsilon) \geq Ke^{-\frac{K\delta}{\varepsilon}}[-\frac{c}{\delta} - c] + o(1) + \frac{a}{\varepsilon}, \text{ as } \varepsilon \to 0;$$

for ε small enough the right hand side of the above inequality is again positive.

Case 3. $d^{\delta,a} > 2\delta$.

In this case we have $w^{\delta,a} > \delta$. Using (10.17) and (8.9)(iii) we conclude as in the previous case.

$\square$

We are now ready to give the proof of Theorem 9.2.

Proof of Theorem 9.2: Fix $(x_0, t_0) \in I\!R^N \times [0, t^*)$ such that $u(x_0, t_0) = -\beta < 0$. The stability of solutions of the geometric pde's yields that $u^{\delta,a} \to u$, as $\delta, a \to 0$, uniformly in (x,t). We choose, therefore, sufficiently small a and δ so that

(10.25) $$u^{\delta,a}(x_0, t_0) < -\frac{\beta}{2} < 0.$$

Let Φ^ε be given by (10.19). In addition to being a supersolution of (6.4) for sufficiently small $\varepsilon > 0, \Phi^\varepsilon$ satisfies

$$\Phi^\varepsilon(x,0) \geq q^\varepsilon(\frac{d(x,\Gamma_0)}{\varepsilon}, x, 0) \text{ on } I\!R^N,$$

where the last inequality follows from the fact that

$$w^{\delta,a}(x,0) = \eta_\delta(d(x,\Gamma_0) + \delta) \geq d(x,\Gamma_0).$$

It follows by the standard comparison theorem for viscosity solutions and (8.10) that

$$\Phi^\varepsilon \geq \phi^\varepsilon \text{ in } I\!R^N \times [0, t^*).$$

On the other hand, (10.25) yields $d^{\delta,a}(x_0, t_0) < 0$; hence

$$\limsup_{\varepsilon \to 0} \phi^\varepsilon(x_0, t_0) \leq \limsup_{\varepsilon \to 0} \Phi^\varepsilon(x_0, t_0) = h_-(x_0, t_0).$$

To prove the reverse inequality we consider $\hat{\Phi}(x,t) = h_-(x,t) - \gamma$ for some $\gamma > 0$. Since $h_- \in C^{2,1}$,

$$\frac{\partial \hat{\Phi}}{\partial t} - \Delta \hat{\Phi} + \frac{1}{\varepsilon^2} f^\varepsilon(x,t,\hat{\Phi}) \leq K + \frac{1}{\varepsilon^2}[-\gamma f_q^\varepsilon(x,t,h_-(x,t)) + o(\gamma)].$$

By (8.1), the right hand side is negative for small ε and γ. Hence by the maximum principle

$$\liminf_{\varepsilon \to 0} \phi^\varepsilon(x,t) \geq h_-(x,t) - \gamma \qquad \text{for all } (x,t) \text{ and } \gamma > 0.$$

We conclude by letting $\gamma \to 0$. A simple modification of the above arguments yields that $\phi^\varepsilon \to h_-$ locally uniformly in $\{u < 0\}$.

The fact that $\phi^\varepsilon \to h_+$ in $\{u > 0\}$ follows in a similar way, provided we construct a subsolution of (6.4).

To prove Theorem 9.1 we need to consider the traveling waves associated by $f^\varepsilon - a$ and to argue about a lower bound on $-\varepsilon\Delta w^{a,\delta}$. The latter follows from the facts that $w^{a,\delta} \neq 0$ iff $d^{a,\delta} \geq \frac{\delta}{4}$ and $\Delta d^{a,\delta} \geq -\frac{C}{d^{a,\delta}}$ in $\{d^{a,\delta} > 0\}$.

□

11 Possible applications

In this section we briefly discuss two applications where (6.4) arises naturally, with f^ε of the form

$$f^\varepsilon(x,t,q) = 2q(q^2-1) - \varepsilon\theta^\varepsilon(x,t), \tag{11.1}$$

which in view of the discussion in Section 8 satisfies the desired properties, provided $(\theta^\varepsilon)_\varepsilon$ is bounded in $C^{2,1}$. On the other hand, we do not know whether $(\theta^\varepsilon)_\varepsilon$ satisfies this necessary condition.

Example 1: *Volume constraint*

Let Ω be a bounded domain in $\mathbb{R}^N$ with an outward normal vector $n(x), x \in \partial\Omega$ and consider the reaction-diffusion equation

$$\begin{cases} \phi_t^\varepsilon - \Delta\phi^\varepsilon + \frac{2}{\varepsilon^2}\phi^\varepsilon((\phi^\varepsilon)^2 - 1) = a^\varepsilon(t) & \text{in } \Omega, \\ \dfrac{\partial\phi^\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{11.2}$$

where

$$(11.3) \qquad a^{\varepsilon}(t) = \lambda^{\varepsilon}(\phi^{\varepsilon}(\cdot,t)) = \frac{1}{\varepsilon^2}\frac{1}{\mathrm{meas}(\Omega)}\int_{\Omega} 2\phi^{\varepsilon}((\phi^{\varepsilon})^2 - 1)dx.$$

If we set

$$\theta^{\varepsilon}(x,t) = \varepsilon\lambda^{\varepsilon}(\phi^{\varepsilon}(\cdot,t)),$$

(8.13) reduces to

$$\lim_{\varepsilon\downarrow 0} \varepsilon^2 \sup_t |\theta^{\varepsilon}_t(t)| = 0.$$

We do not know whether this estimate holds. Formally the limiting equation is

$$V = \text{ mean curvature } + \alpha(t) \text{ in } \Omega,$$

with Neumann boundary condition on $\partial\Omega$ (see Giga, Sato [GS]). If Γ_t is a solution of this equation, then

$$\text{Volume enclosed by } \Gamma_t = \int_{\{u(\cdot,t)>0\}} dx = \frac{1}{2}\lim_{\varepsilon\downarrow 0}\int_{\Omega}[\phi^{\varepsilon}(x,t)+1]dx.$$

Moreover,

$$\frac{d}{dt}\int_{\Omega}(\phi^{\varepsilon}+1)dx = \int_{\Omega}[\Delta\phi^{\varepsilon} - \frac{1}{\varepsilon^2}f_0(\phi^{\varepsilon}) + \lambda^{\varepsilon}]dx = 0,$$

i.e. the volume of the region enclosed by Γ_t is constant in time. For a formal detailed analysis of this problem be refer to Rubinstein and Sternberg [RS].

The pair $(\Gamma_t, \alpha(t))$ is called a volume preserving mean curvature flow. The associated geometric pde in $I\!R^N$ is

$$u_t - \left(\Delta u - \frac{(D^2uDu|Du)}{|Du|^2}\right) - \alpha(t)|Du| = 0 \text{ in } I\!R^N \times (0,\infty).$$

When $N=2$, Lagrange multipler $\alpha(t)$ is given by the explicit formula

$$\alpha(t) = 2\pi N(t)/L(t),$$

where $N(t)$ is the number of disjoint of connected parts of $\Gamma(t)$ and $L(t)$ is the length of Γ_t. This formula indicates that the Lagrange multiplier may in general be discontinuous in time. If however we do not insist that Γ_t is the boundary of a region and replace $a(t)$ by the above formula, then a complete theory is available. In this framework the solution may develop self-intersections, which are not desirable in a physical problem.

In Section 4 we presented an example of nonuniqueness for the volume preserving flow by mean curvature.

Example 2: ***Supercooled Stefan Problem.***

We consider the problem of a melting or growing crystal in a melt. Let $\theta(x,t)$ be the appropriately scaled temperature and $C(t) \subset \mathbb{R}^N$ be the region occupied by the crystal. Gurtin [Gu] derived the equation

$$(11.5) \qquad \frac{\partial}{\partial t}\left(\theta(x,t) + \ell\chi_{C(t)}(x)\right) = \Delta\theta(x,t) \quad \text{in } (0,\infty) \times \mathbb{R}^N,$$

with the free boundary condition

$$(11.6) \qquad \text{normal velocity of } \Gamma_t = \text{curvature} - \theta(x,t) \quad \text{on } \Gamma_t,$$

where the latent heat $\ell > 0$ is a given quantity and $\chi_{C(t)}$ is the characteristic function of the set $C(t)$. In general anisotropic versions of the above equation are more appropriate and we refer to Gurtin-Soner [GuS] for a discussion of the generalizations of (11.5)-(11.6), as well as appropriate notion of solution and the underlying physics. Luckhaus [Lu] and Almgren, Wang [AlW] also studied a similar problem in which (11.6) is replaced by the Gibbs-Thompson relation

$$0 = \text{curvature} - \theta \quad \text{on } \Gamma_t.$$

The system (11.5) and (11.6) can be approximated by the reaction diffusion equations

$$(11.7) \qquad \theta^\varepsilon_t + \frac{\ell}{2}\varphi^\varepsilon_t = \Delta\theta^\varepsilon \quad \text{in } \mathbb{R}^N \times (0,\infty),$$

and

$$\phi_t^\varepsilon - \Delta\phi^\varepsilon + \frac{1}{\varepsilon^2}[f_0(\phi^\varepsilon) - \frac{2}{3}\varepsilon\theta^\varepsilon] = 0 \text{ in } I\!R^N \times (0,\infty). \tag{11.8}$$

The above approximation was first proposed by Caginalp [Ca1,2,3]. The convergence of this system was proved by Caginalp-Chen [CC] in the radial case by a method based on knowing that the limiting motion is classical. Indeed in the radial case the interface Γ_t is a sphere and (11.6) reduces to an ordinary differential equation. In general, we do not expect Γ_t to be a smooth, classical solution of (11.6). The convergence of the system (11.7)-(11.8) is an open problem.

References

[AAG] Altschuler S., Angenent S. & Giga Y.: Generalized motion by mean curvature for surfaces of rotation, Hokkaido U. preprint series # 119 (1991).

[A1] Angenent S. : Parabolic equations for curves on surfaces (I) : curves with p-integrable curvature, Ann. Math. **132** (1990), 171-217.

[A2] Angenent S. : Parabolic equations for curves on surfaces (II) : Intersections, blow-up and generalized solutions, Ann. Math. **133** (1990), 171-217.

[AlW] Almgren F.J. & Wang L.: Mathematical existence of crystall growth with Gibbs-Thompson curvature effects, preprint.

[ArW] Aronson D.G. & Weinberger H.: Multidimensional nonlinear diffusion arising in population genetics, Adv. Math. **30** (1978), 33-76.

[Ba1] Barles G.: Remark on a flame propagation model. Rapport 464 (1985).

[Ba2] Barles G.: Discontinuous viscosity solutions of first-order Hamilton-Jacobi equations: a guided visit, Nonlinear Analysis TMA, to appear.

[BaBS] Barles G., Bronsard L. & Souganidis P.E.: Front propagation for reaction-diffusion equations of bistable type, Ann. I.H.P., Analyse Nonlineaire, to appear.

[BaES] Barles G., Evans L.C. & Souganidis P.E.: Wavefront propagation for reaction-diffusion systems of PDE, Duke U. Math J. **61** (1990).

[BaP] Barles G. & Perthame B.: Discontinuous solutions of deterministic optimal stopping problems, Math. Modelling Numer. Anal. **21** (1987).

[BJ1] Barron E.N. & Jensen R.: Semicontinuous viscosity solutions for Hamilton-Jacobi Equations with convex Hamiltonians, Comm. in PDE, to appear.

[BJ2] Barron E.N. & Jensen R.: Optimal control and semicontinuous viscosity solutions. Preprint.

[Br] Brakke K.A.: *The motion of a surface by its Mean Curvature* Princeton University Press, Princeton, NJ, (1978).

[BrK] Bronsard L. & Kohn R.: Motion by mean curvature as the singular limit of Ginzburg-Landau model, J. Diff. Eq'ns, **90** (1991), 211-237.

[Ca1] Caginalp C.: An analysis of a phase field model of a free boundary, Arch. Rat. Mech., **92** (1986), 205-245.

[Ca2] Caginalp G.: Mathematical models of phase boundaries, in Material Instabilities in Continuum Mechanics and related mathematical Problems, ed. J. Ball, Clarendon Press, Oxford (1988), 35-52.

[Ca3] Caginalp C.: Stefan and Hele-Shaw type models as asymptotic limits of phase field equations, Physics Review A, **39** (1989), 887-896.

[CC] Caginalp G. & Chen X.Y.: Phase field equations in the singular limit of sharp interface problems, preprint.

[CGG] Chen Y.-G, Giga Y. & Goto S.: Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, J. Diff. Geom 33 (1991), 749-786.

[Ch] Chen X.-Y.: Generation and propagation of the interface for reaction-diffusion equation, J. Diff. Eq'ns, **96** (1992), 116-141.

[CEL] Crandall M., Evans L.C. & Lions P.-L.: Some properties of viscosity solutions of Hamilton-Jacobi Equations, Trans. Amer. Math. Soc. **282** (1984), 487-502.

[CIL] Crandall M., Ishii H. & Lions P.-L.: User's guide to viscosity solutions of second order partial differential equations, Cahier du CEREMADE na 9039 (1990) and Bull. AMS, to appear.

[CL] Crandall M. & Lions P.-L.: Viscosity solutions of Hamilton-Jacobi Equations, Trans. Amer. Math. Soc. **277** (1983), 1-43.

[D] DeGiorgi E., Some conjectures on flow by mean curvature, Proc. Capri Workshop, 1990, Benevento-Bruno-Sbardone editors.

[DS] DeMottoni P. & Schatzman M: Development of interfaces in $\mathbb{R}^N$, Proc. Royal Soc. Edin, **116A** (1990), 207-220.

[ES1] Evans L.C. & Souganidis P.E.: Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations, Ind. U. Math. J. **33** (1984), 773-797.

[ES2] Evans L.C. & Souganidis P.E.: A PDE approach to geometric optics for certain reaction-diffusion equations, Ind. U. Math. J. **38** (1989), 141-172.

[ES3] Evans L.C. & Souganidis P.E.: A PDE approach to certain large deviation problems for systems of parabolic equations, *Analyse Non Linéaire, Contributions en l'honneur de J.-J. Moreau*, Gauthier-Villars, 1989.

[ESS] Evans L.C., Soner H.M. & Souganidis P.E.: Phase transitions and generalized motion by mean curvature, CPAM, to appear.

[ESp1] Evans L.C. & Spruck J.: Motion of level sets by mean curvature, J. Diff. Geometry, **33** (1991), 635-681.

[ESp2] Evans L.C. & Spruck J.: Motion of level sets by mean curvature II, Trans. AMS, to appear.

[ESp3] Evans L.C. & Spruck J.: Motion of level sets by mean curvature III, J. Geom. Anal., to appear.

[ESp4] Evans L.C. & Spruck J.: Motion of level sets by mean curvature IV, preprint.

[Fi] Fife, P.C., Nonlinear Diffusive Waves, CBMS Conf., Utah 1987, CMBS Conf. Series (19S9).

[FiM] Fife, P.C. & McLeod B.: The approach of solutions of nonlinear diffusion equations to travelling solutions, Arch. Rat. Mech. An., 65 (1977), 335-361.

[Fr] Freidlin M.I.: ***Functional Integration and Partial Differential Equations,*** Annals of Math. Studies 109, Princeton University Press, Princeton, 19S5.

[Ga] Gärtner, J.: Bistable Reaction-Diffusion Equations and excitable media, Math. Nachr. 112 (19S3), 125-152.

[GGIS] Giga Y., Goto S., Ishii H. *k* Sato M.H. : Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains, Ind. Un. Math. J. 40 (1991).

[GS] Giga Y. & Sato M.-H., Generalized interface evolution with the Neumann boundary condition, Proc. Japan Acad., Ser. A, 67 (1991), 263-266.

[GSS] Gurtin M., Soner H.M. & Souganidis P.E.: Arisotropic planar motion of an interface relaxed by the formation of infinitesimal wrinkles, in preparation.

[GT] Gilbarg D. & Trudinger N.S. ***Elliptic Partial Differential Equations of*** *Second-Order* 2nd Edition, Springer-Verlag, New-York, 1983.

[Gu] Gurtin M.: Multiphase thermomechanics with interfacial structure. I Heat conduction and the capilary balance law, Arch Rat. Mech. An. **104** (1988), 195-221.

[GuS] Gurtin M. & Soner H.M. : Some remarks on the Stefan problem with surface structure, Quar. Applied Math., in press.

[H] Hamilton R.S.: Three manifolds with positive Ricci-curvature, J. Dif. Geo. **17** (1982), 255-306.

[Il1] Ilmanen T., Ph.D. thesis, University of California, Berkeley (1991).

[Il2] Ilmanen T., Generalized flow of sets by mean curvature on a manifold, Ind. U. Math. J., to appear.

[Is] Ishii H., Hamilton-Jacobi Equations with discontinuous Hamiltonians on arbitrary open sets, Bull. Fac. Sci. Eng. Chuo Univ. **26** (1985), 5-24.

[J] Jensen R.: The maximum principle for viscosity solutions of second-order fully nonlinear partial differential equations. Arch. Rat. Mech. An., **101** (1988), 1-27.

[JLS] Jensen R., Lions P.-L. & Souganidis P.E.: A uniqueness result for viscosity solutions of second-order fully nonlinear pde's, Proc. AMS **102** (1988), 975-978.

[LL] Lasry J.M. & Lions P.-L.: A remark on regularization in Hilbert spaces, Isr. J. Math. **55** (1986), 257-266.

[Li] Lions P.-L.: *Generalized Solutions of Hamilton-Jacobi Equations*, Research Notes in Mathematics 69, Pitman, Boston 1982.

[Lu] Luckhaus S.: Solutions for the two-phase Stefan problem with Gibbs-Thompson law for the modeling temperature, Eur. J. Appl. Math **1** (1990), 101-111.

[OS] Osher S. & Sethian J.A.: Fronts moving with curvature-dependent speed: Algorithms based on Hamilton-Jacobi equations. J. Comput. Phy. **79**, 12-49.

[RS] Rubinstein & Sternberg P.: Nonlocal reaction-diffusion equations and nucleation, J. of I.M.A., to appear.

[Se1] Sethian J.A.: PhD Thesis Berkeley (1984).

[Se2] Sethian J.A.: Recent numerical algorithms for hypersurfaces moving with curvature-dependent speed: Hamilton-Jacobi equations and conservation laws, J. Diff. Geom. **31** (1990), 131-162.

[Se3] Sethian J.A.: Curvature and evolution of fronts, Comm. Math. Physics, **101** (1985), 487-499.

[So] Soner H.M.: Motion of a set by the mean curvature of its boundary. J. Dif. Eqs, to appear.

[SS] Soner H.M. & Souganidis P.E.: Uniqueness and singularities of cylindrically symmetric domains moving by mean curvature, preprint.

[Sor] Soravia P.: Generalized motion of a front along its normal direction: A differential games approach, preprint.