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Research Report No. 92-NA-019

June 1992

Sponsors

U.S. Army Research Office
Research Triangle Park
NC 27709

National Science Foundation
1800 G Street, N.W.
Washington, DC 20550

DEFORMATIONS AND STRESSES WITH AND WITHOUT MICROSLIP

by

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ABSTRACT

Kinematical definitions of deformations with and without microslip are presented. Transformation properties for such deformations are shown to follow directly from their definitions, and Burgers vector is related to the deformation without microslip. A limit procedure provides a concept of stress without microslip and leads to a natural concept of elastic response. Various decompositions of local deformation into elastic and plastic parts proposed in the literature are shown to be compatible with this kinematical setting.

INTRODUCTION

One of the principal methods for incorporating the inelastic behavior of metals into continuum models of materials is to employ classical notions of deformation and stress in describing the kinematics and dynamics of an inelastic body, but to introduce additional measures of deformation, usually called elastic and plastic deformation, into the constitutive equations for the material under consideration. Although generally successful, this approach has drawbacks that merit consideration: 1) The variety of possible choices of elastic and plastic deformation and the difficulty in comparing models based on different choices has led to a lingering controversy in the literature; 2) the underlying classical kinematics does not directly incorporate the physical processes of slip at the microscopic level.

My goal in this paper is to present a different method for incorporating the inelastic behavior of metals into a continuum model that is not subject to these drawbacks. This utilizes the research still in progress of Del Piero and Owen (forthcoming) on the kinematics of fractured continua in which classes of deformations broad enough to include explicitly slip at the microscopic level are defined and developed.

In the second and third sections I describe a collection of deformations called invertible structured deformations that includes classical deformations, and I show how the results of Del Piero and Owen (forthcoming) lead to natural notions of deformation without microslip and deformation due to microslip. All of the considerations in the second and third sections are purely kinematical, so that these new notions of deformation are not variables that appear for the first time in constitutive equations. Among the kinematical properties of invertible structured deformations summarized in the second section is the Approximation Theorem, which shows that each invertible structured deformation is a limit of "piecewise classical deformations", or "simple deformations". It is this property of invertible structured deformations that permits one to interpret their effect on a body in terms of complicated slip mechanisms occurring in the approximating simple deformations. The other results in the second section motivate and justify not only multiplicative, but also additive decompositions of local deformation into parts without microslip and parts due to microslip. All of the terms and factors in these decompositions have definite transformation properties under changes in observer and reference configuration that are direct consequences of the kinematical definitions.

In the fourth section, new results are given that lead to a notion of stress without microslip for an invertible structured deformation, This stress takes into account differences between a given element of surface area in the deformed configuration and a corresponding element of surface generated by an approximation to the given invertible structured deformation. A passage to the limit yields a simple formula for the stress without microslip in terms of the Cauchy stress and the left microslip tensor introduced in the third section.

The discussion in the fifth section shows how the availability of both deformation without microslip aiid stress without microslip provides a natural concept of elastic response for a body undergoing invertible structured deformations: the stress without microslip is a function of deformation without microslip. If the body undergoes a classical deformation and, in particular, undergoes no microslip, then the elastic response reduces to the classical relation: the Cauchy stress is a function of the deformation gradient. The relation between stress without microslip and deformation without microslip is equivalent to an equation that gives the Cauchy stress as a function of deformation without microslip and of deformation due to microslip in which the dependence on deformation due to microslip enters through a multiplicative factor (the transpose of the left microslip tensor).

The final section of the paper is devoted to showing how various decompositions of deformation into elastic and plastic parts that have been proposed in the literature (Lee and Liu, 1967), (Clifton, 1972), (Nemat–Nasser, 1979), (Green and Naghdi, 1965) fit naturally into the present framework of invertible structed deformations. If for each proposed decomposition the elastic deformation in that decomposition is identified as deformation without microslip, then the plastic deformation in that decomposition is easily related to one of the measures of deformation due to microslip introduced in the third section, and the transformation properties of both elastic and plastic deformation immediately follow. Thus, in the setting of invertible structured deformations, all the decompositions considered in the last section are consistent with one another: none has any special status, and each one identifies a particular measure of deformation due to microslip in the present theory.

The present approach has some connections with theories of continuous distributions of dislocations and theories of defective crystals. The emergence in a natural way of Burgers vector, as shown in relation (10), shows that important physical ideas incorporated into theories of continuous distribution of dislocations (Kröner, 1958) have counterparts in the kinematics of fractured continua (Del Piero and Owen, forthcoming). Within the class of invertible structured deformations described here one can identify a smaller collection of deformations, the neutral deformations discussed by (Davini and Parry, 1991) (Fonseca and Parry, forthcoming): these correspond in the present theory to the invertible structured deformations whose right microslip tensor is the gradient of a vector field.

INVERTIBLE STRUCTURED DEFORMATIONS

Let a region $\mathcal{A}$ in space represent a reference configuration for a body. An *invertible structured deformation from* $\mathcal{A}$ is specified by giving a deformation g and a tensor field G. The fields g and G are required to satisfy

$$\det G = \det \nabla g \qquad (1)$$

throughout $\mathcal{A}$, where det denotes the determinant and ∇ the gradient. For reasons to be discussed in detail in the next section g is called the *macroscopic deformation,* G is called the *local deformation without microslip,* and (1) then has the interpretation: local volume changes associated with macroscopic deformation are accounted for entirely by local volume changes without microslip.

Let s denote a simple shear of a rectangular block $\mathcal{R}$ with unit height; in Cartesian coordinates,

$$s(x_1, x_2, x_3) = (x_1 + a\, x_3, x_2, x_3) \qquad (2)$$

with $a > 0$, and consider two invertible structured deformations: $(s, \nabla s)$, called a *classical simple shear*, and (s, I), called a *simple shear due to microslip*. Here, I is the identity tensor. Results presented below justify the following description: the simple shear due to microslip is accomplished as if the body were sliced into infinitely many, infinitely thin parallel slabs, each of which is rigidly displaced an infinitesimal amount parallel to itself, whereas the classical simple shear is accomplished by smooth deformation without slicing.

The collection of invertible structured represents a broadening of the collection of classical deformations used in continuum mechanics. In fact, classical deformations are invertible structured deformations of the form $(g, \nabla g)$, i.e., deformations in which the local deformation ∇g is the same as the deformation without microslip G. Invertible structured deformations were introduced by Del Piero and Owen (forthcoming). We here need only consider a few of the principal results of that study.

Approximation Theorem: Every invertible structured deformation is a limit of simple deformations.

A simple deformation is determined by giving a "piecewise classical deformation" f which fractures the body into pieces along crack sites κ in $\mathcal{A}$ and then deforms each piece via a classical deformation. For example, the rectangular block $\mathcal{R}$ undergoes a simple deformation (σ_m, s_m) if it is sliced into m slabs by $m-1$ equally spaced parallel planes with the slabs then displaced by parallel translations giving relative displacement of magnitude a / m to adjacent slabs. Here, the crack site σ_m is the union of the slicing planes, and the mapping s_m displaces each slab by a translation. It is natural to think of s_m as a shearing displacement of a deck of cards, where each card represents one of the m slabs. The term "limit of simple deformations" in the statement of the Approximation Theorem means that there is a sequence $m \longmapsto (\kappa_m, f_m)$ of simple deformations such that the given invertible structured deformations (g, G) satisfies

$$g = \lim_{m \to \infty} f_m \tag{3}$$

$$G = \lim_{m \to \infty} \nabla f_m \tag{4}$$

along with the following condition on the sequence of crack sites $m \longmapsto \kappa_m$: no point x in $\mathcal{A}$ occurs in all the crack sites κ_j for all $j \geq m$ for some $m = m(x)$. The limits in (3) and (4) are taken in the sense of L^∞ – convergence of deformations and tensor fields. Relation (3) means that the sequence $m \longmapsto f_m$ of piecewise classical deformations approximates the macroscopic deformation g to within any desired accuracy, and relation (4) means that the gradients ∇f_m (away from the crack sites κ_m) approximate the local deformation without microslip G as accurately as desired. For example, the deck of cards sequence $m \longmapsto (\sigma_m, s_m)$ satisfies

$$\lim_{m\to\infty} s_m = s \tag{5}$$

and

$$\lim_{m\to\infty} \nabla s_m = I, \tag{6}$$

the latter because $\nabla s_m = I$ for all m. In fact, *the simple shear due to microslip* (s, I) *is the limit of the "deck of cards" sequence* $m \longmapsto (\sigma_m, s_m)$ *of simple deformations.* This conclusion justifies our earlier description of (s, I) in terms of an "infinite deck of infinitesimally thin cards."

It is important to realize that no crack sites are present in the description of an invertible structured deformation (g, G), even though crack sites do occur in the description of the approximating simple deformations (κ_m, f_m). This fact is mathematically accounted for by the condition on $m \longmapsto \kappa_m$ described above, and it is best understood by regarding the crack sites κ_m as vehicles for implementing slip at the microscopic level: the vehicles κ_m leave no macroscopic trace as fractures but do leave a macroscopic trace through the difference

$$\nabla g - G = \nabla \lim_{m\to\infty} f_m - \lim_{m\to\infty} \nabla f_m, \tag{7}$$

when this difference is non–zero. This fact is expressed by the following result from (Del Piero and Owen, forthcoming).

Characterization of the zone without microslip: the relation $\nabla g = G$ *holds throughout a given region* $\mathcal{B}$ *in* $\mathcal{A}$ *if and only if the invertible structured deformation* (g, G) *is a limit of a sequence* $m \longmapsto (\kappa_m, f_m)$ *of simple deformations whose crack sites* κ_m *all are disjoint from the given region* $\mathcal{B}$

The largest region in $\mathcal{A}$ on which $\nabla g = G$ holds is called the *zone without microslip for* (g, G), and (7) tells us that this region is characterized by the relation

$$\lim_{m\to\infty} \nabla f_m = \nabla \lim_{m\to\infty} f_m. \tag{8}$$

For the classical simple shear $(s, \nabla s)$ of $\mathcal{R}$, the zone without microslip is all of $\mathcal{R}$, while for the simple shear due to microslip (s, I), the zone without microslip is empty.

The next result from (Del Piero and Owen, forthcoming) gives a further relation between $\nabla g - G$ and the presence of crack sites.

Fundamental Theorem of Calculus for Structured Deformations: On line segments $[x_1, x_2]$

$$\int_{x_1}^{x_2} (\nabla g(x) - G(x))dx = \lim_{m\to\infty} \sum_{z_m} [f_m(z_m)] \tag{9}$$

where the integral is a line integral over the segment $[x_1, x_2]$ *and the sum* $\sum_{z_m} [f_m(z_m)]$ *is the (finite) sum of the jumps of the deformation* f_m *at points* z_m in $[x_1, x_2]$ *where* f_m *is discontinuous.*

Each jump $[f_m(z_m)]$ occurs only when the crack site κ_m intersects the segment $[x_1, x_2]$, so the line integral in (9) provides a further trace of the crack sites $m \longmapsto \kappa_m$. When (9) is applied to a closed polygonal path p, the line integral of ∇g vanishes and one obtains

$$-\oint_p G(x)dx = \lim_{m\to\infty} \sum_{z_m} [f_m(z_m)]. \tag{10}$$

This relation identifies $-\oint_p G(x)dx$ as an analogue of the Burgers vector employed in materials science and in theories of continuous distributions of dislocations in continua (Kröner, 1958).

DEFORMATION WITH AND WITHOUT MICROSLIP

The Approximation Theorem discussed above tells us that, for each invertible structured deformation (g, G), there is a sequence $m \longmapsto (\kappa_m, f_m)$ of simple deformations satisfying (3) and (4). The latter relation tells us that G is a limit of deformation gradients ∇f_m computed away from the crack sites κ_m and, hence, not affected by the jumps in f_m that can occur only across the crack sites. For example, each card of the "deck of cards" example, translates rigidly relative to the others, so that $\nabla f_m = I$ and $G = \lim_{m\to\infty} \nabla f_m = I$; thus, G is not affected by the relative slip between the cards in the deck at any stage in the approximations as $m \longrightarrow \infty$. For these reasons we earlier have described G as (*local*) *deformation without microslip.*

Because G is a limit of deformation gradients Vf_m, the following transformation laws for G are immediate consequences of those for Vf_m:

Transformation law for **G** *under change in observer.*

$$G \rightarrow QG \qquad (11)$$

where Q is the orthogonal tensor associated with the change in observer.

Transformation law for **G** *under change in reference configuration:*

$$G \rightarrow GH \qquad (12)$$

where H is the tensor associated with the change in reference configuration.

The deformation $g = \lim_{m \to \infty} f_m$ will be called the ***macroscopic deformation,*** **and Vg will be called** ***(macroscopic) local deformation*** **associated with (g, G). Relations (11), (12), and the usual transformation laws for deformation gradients tell us that** ***the local deformation*** **Vg** ***and the local deformation without microslip*** **G** ***transform in the same way under changes in observer and under changes in reference configuration.***

It is important to use both the local deformation Vg and the local deformation without microslip G associated with (g, G) to define and study various measures of "deformation due to microslip". We here shall consider three such measures M_l^*, M_r, and M defined in terms of Vg and G via the relations

$$Vg = M^{\wedge} G \qquad (13)$$

$$Vg = G M_r \qquad (14)$$

and

$$Vg = G + M. \qquad (15).$$

Although the relation (15) is an additive, rather than a multiplicative relation of the type (13) and (14), relation (15) is a natural decomposition of local deformation for two reasons. First of all, because M = Vg — G and because Vg and G have the same transformation laws, it follows that Vg, G, ***and*** **M** ***dU transform in the same way under changes in observer and under changes in reference configuration.*** **Second of all, the relations**

(9) and (15) yield the formula

$$\int_{x_1}^{x_2} M(x)dx = \lim_{m\to\infty} \sum_{z_m} [f_m(z_m)] \tag{16}$$

and we recall that $\lim_{m\to\infty} \sum_{z_m} [f_m(z_m]$ is the limit of the sum of the jumps in f_m on the line segment $[x_1, x_2]$. Therefore, integrating (15) from x_1 to x_2 along $[x_1, x_2]$ yields the relation

$$g(x_2) - g(x_1) = \int_{x_1}^{x_2} G(x)dx + \int_{x_1}^{x_2} M(x)dx, \tag{17}$$

and (16) then tells us that $\int_{x_1}^{x_2} M(x)dx$ *represents the portion of the relative deformation* $g(x_2) - g(x_1)$ *that is due to microslip.* Consequently, we are led to call M the (local) *deformation due to microslip.*

We turn now to the multiplicative relations (13) and (14) that define M_ℓ and M_r:

$$M_\ell = \nabla g \, G^{-1} \tag{18}$$

$$M_r = G^{-1} \, \nabla g. \tag{19}$$

The tensors M_ℓ and M_r will be called the *left–microslip tensor* and the *right–microslip tensor*, respectively, for (g, G). They arise naturally in the following decompositions for the given invertible structured deformation (g, G):

$$(g, G) = (i, M_\ell^{-1}) \circ (g, \nabla g) \tag{20}$$

$$(g, G) = (g, \nabla g) \circ (i, M_r^{-1}) \tag{21}$$

Relation (20) depicts (g, G) as being carried out first by performing the classical deformation $(g, \nabla g)$, followed by the invertible structured deformation (i, M_ℓ^{-1}), where i denotes the identity deformation: $i(x) = x$, for all points x. The invertible structured deformation (i, M_ℓ^{-1}) is called a *pure microslip*, because no material points are moved, and yet, for (i, M_ℓ^{-1}), the deformation due to microslip need not be zero. Similarly, the invertible structured deformation (i, M_r^{-1}) in (21) that precedes the classical deformation $(g, \nabla g)$ is also a pure microslip. Thus, we see that the microslip tensors M_ℓ and M_r determine the pure microslips that occur in relations (20) and (21). Moreover, there are simple transformation laws for M_ℓ and M_r that follow from those for ∇g and G:

Transformation laws for M_ℓ and M_r *under changes in observer:*

$$M_\ell \longrightarrow Q\, M_\ell\, Q^T \tag{22}$$

$$M_r \longrightarrow M_r. \tag{23}$$

Transformation laws for M_ℓ *and* M_r *under changes in reference configuration:*

$$M_\ell \longrightarrow M_\ell \tag{24}$$

$$M_r \longrightarrow H^{-1} M_r H \tag{25}$$

It is worth recording here the microslip tensors M_ℓ, M_r and the deformation due to microslip M for (s, I), the simple shear due to microslip:

$$M_\ell = M_r = \nabla s \tag{26}$$

$$M = \nabla s - I. \tag{27}$$

Finally, we note the following consequence of (1), (18) and (19):

$$\det M_r = \det M_\ell = 1, \tag{28}$$

so that there is no volume change associated with the pure microslips (i, M_ℓ^{-1}) and (i, M_r^{-1}) in (20) and (21). Indeed, this justifies our use of the term "microslip" in place of the term "microfracture" used in the discussion of general structured deformations (Del Piero and Owen, forthcoming).

STRESSES WITH AND WITHOUT MICROSLIP

We wish now to explore how the microslip that is included in the kinematics of invertible structured deformations can affect the measures of stress that enter into constitutive equations that describe the inelastic behavior of materials. The following simple consideration suggests that stresses do need to be reexamined when microslip occurs. Imagine slicing by a plane $x_1 = \text{constant}$ the image $s(\mathcal{R})$ of the rectangular block $\mathcal{R}$ under the simple shear due to microslip (s, I), with s given by (2). The smooth surface $\mathcal{S}$ in $s(\mathcal{R})$ so obtained is a rectangular region in a plane $x_1 = \text{constant}$ whose preimagine $s^{-1}(\mathcal{S})$ is a slanted rectangular region in a plane no longer parallel to $\mathcal{S}$. To examine the effects of microslip, we approximate (s, I) by a piecewise–rigid deformation (σ_m, s_m) (the shearing of a deck of cards). Within the j^{th} card, $s^{-1}(\mathcal{S})$ determines a smaller slanted rectangular region which, under the piecewise rigid deformation s_m, remains slanted, i.e., not parallel to the plane $x_1 = \text{constant}$. Thus, the image under s_m of the slanted rectangular region $s^{-1}(\mathcal{S})$ is a parallel collection of m smaller, slanted rectangular regions that approximate the rectangular region $\mathcal{S}$, much as the parallel slanted slats of a partially opened venetian blind approximate the flat, vertical rectangular region defined by the outside border of the blinds. Moreover, the angle of slant is independent of m, and so represents a correction to the normal of $\mathcal{S}$ due to the presence of microslip. Thus, away from the slip planes, i.e., within one of the cards, the contact surface $s_m(s^{-1}(\mathcal{S}))$ would differ from $\mathcal{S}$.

These considerations lead to the following problem: given an invertible structured deformation (g, G), and a smooth surface $\mathcal{S}$ in $g(\mathcal{A})$ with normal vector field n, determine how the contact force $\int_{\mathcal{S}} T(y)n(y)da_y$ associated with a given (Cauchy) stress field T should be corrected for the presence of microslip. A natural outgrowth of the considerations in the previous paragraph is the following solution to this problem: find a sequence $m \longmapsto (\kappa_m, f_m)$ of simple deformations that approximate (g, G) and use the following limit

$$\lim_{m\to\infty} \int_{f_m(g^{-1}(\mathcal{S})\setminus\kappa_m)} T(z)N_m(z)dA_z. \tag{29}$$

as a correction to $\int_{\mathscr{S}} T(y)n(y)da_y$. The surface $f_m(g^{-1}(\mathscr{G}) \setminus \mathscr{K}_m)$ represents the material points $g^{-1}(\mathscr{G})$ that are away from the crack site $\mathscr{K}_m$, placed in the positions determined by the deformation f_m that approximates g. Thus, each point z in $f_m(g^{-1}(\mathscr{G}) \setminus \mathscr{K}_m)$ is away from the images of the crack sites, and the corresponding normal vector $N_m(z)$ to $f_m(g^{-1}(\mathscr{G}) \setminus \mathscr{K}_m)$ will differ from the normal vector $n(y)$ to $\mathscr{G}$ at y, with y and z related by

$$z = f_m(g^{-1}(y)). \tag{30}$$

This relation tells us that

$$N_m(z)dA_z = \det(\nabla(f_m \circ g^{-1})(y))(\nabla(f_m \circ g^{-1}))^{-T}(y)n(y)da_y \tag{31}$$

where we have used the transformation law for elements of area. By using the chain rule we obtain from (31), with $x = g^{-1}(y)$,

$$\begin{aligned}N_m(z)dA_z &= \det(\nabla f_m(g^{-1}(y))(\nabla g)^{-1}(y))(\nabla f_m(g^{-1}(y))(\nabla g)^{-1}(y))^{-T}n(y)da_y \\ &= \frac{\det \nabla f_m(x)}{\det \nabla g(x)} \nabla f_m(x)^{-T}\nabla g(x)^T n(y)da_y \\ &\to \frac{\det G(x)}{\det \nabla g(x)} G(x)^{-T}\nabla g(x)^T n(y)da_y\end{aligned}$$

as $m \to \infty$. This relation, together with (1) and (18), tells us that

$$\lim_{m\to\infty} (N_m(z)dA_z) = M^{\wedge}(g^{-1}(y))n(y)da_y, \tag{32}$$

so that

$$\lim_{m\to\infty} \int_{f_m(g^{-1}(\mathscr{S}) \setminus \kappa_m)} T(z)N_m(z)dA_z = \int_{\mathscr{S}} T(y)M_\ell^T(g^{-1}(y))n(y)da_y. \qquad (33)$$

We may now identify the tensor field

$$T_\setminus := T(M_\ell^T \circ g^{-1}) \qquad (34)$$

as the *stress without microslip* for the invertible structured deformation (g, G) and stress field T. It is convenient to omit the compositional factor g^{-1} in (34) and write more briefly

$$T_\setminus = TM_\ell^T, \qquad (35)$$

with the understanding that, when T is evaluated at a point y in $g(\mathscr{A})$, M_ℓ^T is to be evaluated at the corresponding point $x = g^{-1}(y)$ in $\mathscr{A}$.

Of course, the relation (32) also permits us to identify

$$(nda)_\setminus := M_\ell^T \, nda \qquad (36)$$

(with the same understanding about evaluation) as the *element of area without microslip* for a surface $\mathscr{S}$ in $g(\mathscr{A})$, where in (36) nda is the actual element of area for $\mathscr{S}$.

The relation (33) and the definition (34) permit us to regard $T_\setminus$ and $(nda)_\setminus$ as the stress field and element of area that would be felt if one could isolate a material element from the microslip that generally occurs throughout the body. We note that the actual element of area nda agrees with $(\mathrm{nda})_\backslash$ if and only if

$$M_\ell^T n = n \Leftrightarrow (M_\ell^T - I)\, n = 0$$

$$\Leftrightarrow (M_\ell - I)^T\, n = 0 \Leftrightarrow G^{-T} M^T\, n = 0$$

$$\Leftrightarrow M^T\, n = 0, \qquad (37)$$

where we have used the relation

$$M_\ell G = G + M$$

and the invertibility of G in the last two steps of the above argument. For $M^T n = 0$ to hold, it is equivalent to have

$$n \cdot Mv = 0 \qquad (38)$$

for all vectors v which means that the relative deformation Mv due to microslip for line elements parallel to v has zero component in the direction n. Thus, *a surface with normal* n *satisfies*

$$(\mathrm{nda})_\backslash = \mathrm{nda} \qquad (39)$$

if and only if the relative deformation due to microslip is perpendicular to n, *i.e.*, *is tangent to the given surface.* For the simple shear due to microslip (s,I), relations (27) and (37) easily yield the conclusion: (39) holds if and only if $n \cdot e_1 = 0$, with e_1 the unit vector in the positive x_1 – direction. In particular, if $\mathscr{S}$ is any plane parallel to the x_1 – axis, then relation (39) holds on $\mathscr{S}$.

It is natural to define the *stress due to microslip*

$$T_m := T - T_\backslash = T\,(I - M_\ell^T). \qquad (40)$$

In general, $T_\backslash$ and T_m need not be symmetric, but there are important situations for applications in which one can deduce the symmetry of T_m and $T_\backslash$ as noted in the next section. In spite of the lack of symmetry of T_m and $T_\backslash$, relations (22), (24) and (35) tell us that each transforms in the same way as T under changes in observer and changes in reference configuration:

$$\left.\begin{aligned} T_\backslash &\to QT_\backslash Q^T \\ T_m &\to QT_m Q^T \end{aligned}\right\} \tag{41}$$

and

$$\left.\begin{aligned} T_\backslash &\to T_\backslash \\ T_m &\to T_m. \end{aligned}\right\} \tag{42}$$

A DEFINITION OF ELASTIC RESPONSE

In classical continuum mechanics, the notions of an elastic material and of elastic response are direct generalizations of Hooke's Law that allow for non–linear and anisotropic response. Thus, for a classical deformation $(g, \nabla g)$, elastic response is defined by the constitutive equation

$$T = \mathcal{T}(\nabla g) \tag{43}$$

relating the Cauchy stress T and the macroscopic local deformation ∇g. In the present non–classical setting of invertible structured deformations (g, G), a natural definition of *elastic response* emerges immediately:

$$T_\backslash = \mathcal{T}_\backslash(G). \tag{44}$$

This relation gives the stress without microslip as a function of the deformation without microslip. We call $\mathcal{T}_\backslash$ the *response without microslip*: $\mathcal{T}_\backslash$ is a function that can be determined by subjecting the body to classical deformations $(g, \nabla g)$ alone, because, for classical deformations, there holds $G = \nabla g$, so that $M_\ell = I$ and $T_\backslash = T$.

The transformation laws (41), (42) for $T_{\backslash}$ and (11), (12) for G permit us to write *the condition of independence of observer for the response without microslip* as:

$$\mathcal{T}_{\backslash}(QG) = Q\mathcal{T}_{\backslash}(G)Q^{T} \tag{45}$$

for all orthogonal tensors Q and all G, and permit us to write the *condition that* H *be a symmetry transformation for the response without microslip* as

$$\mathcal{T}_{\backslash}(GH) = \mathcal{T}_{\backslash}(G) \tag{46}$$

for all G. In particular, we say that *the response without microslip is isotropic* if (46) holds for all orthogonal tensors H.

Let us suppose that the response without microslip is isotropic. Relations (45) and (46) then tell us that

$$\mathcal{T}_{\backslash}(G) = \mathcal{T}_{\backslash}(V_{\backslash}), \tag{47}$$

where $G = V_{\backslash}R_{\backslash}$ is the polar decomposition of G, and also that

$$\mathcal{T}_{\backslash}(QV_{\backslash}Q^{T}) = Q\mathcal{T}_{\backslash}(V_{\backslash})Q^{T} \tag{48}$$

for all orthogonal tensors Q. It is not difficult to show from (48) that $\mathcal{T}_{\backslash}$ is symmetric–valued, i.e., $\mathcal{T}_{\backslash}(G)$ is a symmetric tensor for all choices of G, and we may conclude that *the stress without microslip is symmetric when the response without microslip is isotropic*:

$$T_{\backslash}^{T} = T_{\backslash}. \tag{49}$$

In addition, (35), (49) and the symmetry of the Cauchy stress then yield

$$TM_{\ell}^{T} = M_{\ell}T. \tag{50}$$

In the case of isotropic response without microslip, relations (35), (44), (47), and (50) then imply

$$T = Uf\ \wedge(V\wedge) = \wedge(V\wedge^{T}, \tag{51}$$

where £\ satisfies the condition (48).

Even in the case of anisotropic response without microslip, relations (44) and (35) permit us to relate T, G, and M^:

$$T = \wedge(G)MJ^{T}. \tag{52}$$

Thus, *when the body has elastic response in the sense* o/(44), *the Cauchy stress is given explicitly as a Junction of the deformation without microslip and the left microslip tensor.* Using relations (13) — (15) we may rewrite (52) in the equivalent forms

$$T = jr_{\backslash}(G)\ (GM_{I}^{"1}\ G^{"1})^{1} \tag{53}$$

and

$$T = jr_{\backslash}(G)\ (MG^{"4} + I)^{\sim T}. \tag{54}$$

RELATIONSHIP TO PHENOMENOLOGICAL THEORIES OF PLASTICITY

When classical notions of deformation and stresses are employed to describe the inelastic behavior of materials, it is necessary to introduce additional "interval variables" into constitutive relations in order to describe deviations from elastic behavior. Such variables can be introduced in a variety of ways, many of which give a natural physical interpretation that facilitates their use in detailed models of elastic—plastic behavior. Nevertheless, the variety of possible choices of variables such as "elastic deformation" and "plastic deformation" has created a lingering controversy in the literature over the choice of internal variables and over the choice of transformation laws for such variables. One reason that this controversy persists lies in the magnitude of the task of using each choice of variables to give detailed quantitative and qualitative predictions for specific materials and of comparing the different predictions that arise from different choices of the variables.

I believe that the conceptual framework described here can contribute to settling this controversy by offering a natural choice of the variable used to describe "elastic deformation". Thus, I propose to use invertible structured deformations (g, G) to describe the deformations possible in elastic—plastic bodies and to identify vg as the macroscopic deformation gradient and G as the elastic deformation. The advantages of doing so rest not only on the results from Sections 2 and 3, which show that G has the definite, purely kinematical identity of deformation without microslip and has definite transformation laws, but also on the fact that, once a common notion of elastic deformation has been identified, apparent conflicts among the variety of choices of "plastic deformation" and of transformation laws for plastic deformation disappear.

To illustrate this last advantage, let us indicate how a variety of proposed decompositions of deformation into elastic and plastic parts all can be expressed within the framework of invertible structured deformations.

The decomposition of Lee and Liu (1967)

If we put $F^e := G$, $F := \nabla g$, then the relation

$$F = F^e F^p \tag{55}$$

proposed by Lee and Liu implies that

$$F^p = M_r, \tag{56}$$

i.e., the plastic deformation F^p equals the right microslip tensor defined in (19), and the transformation laws for F^p become those for M_r:

$$F \longrightarrow QF \Longrightarrow F^p \longrightarrow F^p \tag{57}$$

$$F \longrightarrow FH \Longrightarrow F^p \longrightarrow H^{-1} F^p H. \tag{58}$$

The decomposition of Clifton (1972)

If we put $\bar{F}^e := G$, $F := \nabla g$, then the relation

$$F = \bar{F}^p \bar{F}^e \tag{59}$$

proposed by Clifton implies that

$$\bar{F}^p = M_\ell, \tag{60}$$

i.e., the plastic deformation $\bar{F}^p$ equals the left microslip tensor defined in (18), and the transformation laws for $\bar{F}^p$ become those for M_ℓ:

$$F \longrightarrow QF \Longrightarrow \bar{F}^p \longrightarrow Q\,\bar{F}^p\,Q^T \tag{61}$$

$$F \longrightarrow FH \Longrightarrow \bar{F}^p \longrightarrow \bar{F}^p. \tag{62}$$

<u>The decomposition of Green and Naghdi (1965)</u>

If we put $E_e := \frac{1}{2}(G^T G - I)$ and $E := \frac{1}{2}(\nabla g^T \nabla g - I)$, then the relation

$$E = E_e + E_p \tag{63}$$

proposed by Green and Naghdi implies that

$$E_p = \tfrac{1}{2}(\nabla g^T \nabla g - G^T G) \tag{64}$$

and yields the transformation laws

$$F \longrightarrow QF \Longrightarrow E_p \longrightarrow E_p \tag{65}$$

$$F \longrightarrow FH \Longrightarrow E_p \longrightarrow H^T E_p H. \tag{66}$$

<u>The decomposition of Nemat–Nasser (1979)</u>

If we put $\bar{F}^e := G, F := \nabla g$, then the relation

$$F = \bar{F}^e + F^p - I \tag{67}$$

arising from Nemat–Nasser's considerations implies that

$$F^p = M + I \tag{68}$$

i.e., the plastic deformation F^p equals the deformation due to microslip plus the identity and yields the transformation laws

$$F \longrightarrow QF \Longrightarrow F^p \longrightarrow Q\,F^p \tag{69}$$

$$F \longrightarrow FH \Longrightarrow F^p \longrightarrow F^p\,H, \tag{70}$$

It should be emphasized that the main step in including within one framework all of the above decompositions is that of identifying the usual deformation gradient with ∇g and identifying elastic deformation with G, the deformation without microslip. Of course, the more difficult task of deciding which relations between stress and elastic and plastic deformations are most appropriate in specific contexts has been the subject of a large body of research and cannot be settled once and for all simply by writing down one set of constitutive assumptions. However, the definition of elastic response given in the previous section and its consequence (52) may provide a useful addition to the many proposals already under consideration, because it is closely related to the more restrictive constitutive assumption

$$T = \hat{T}(G) \tag{71}$$

used by many authors and yet provides for an explicit and simple dependence of the Cauchy stress on deformation due to microslip.

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