On Semilinear Elliptic Equations with Indefinite Nonlinearities

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On semilinear elliptic equations with indefinite nonlinearities

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Abstract. This paper concerns semilinear elliptic equations whose nonlinear term has the form $W(x)f(u)$ where W changes sign. We study the existence of positive solutions and their multiplicity. The important role played by the negative part of W is contained in a condition which is shown to be necessary for homogeneous f. More general existence questions are also discussed.

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INTRODUCTION

In this paper we study the Dirichlet problem for a class of semilinear elliptic equations involving indefinite nonlinearities. More precisely, given $\Omega \subset \mathbf{R}^N$ a bounded open set with smooth boundary $\partial\Omega$, we seek solutions for:

$$(*_\lambda) \qquad \begin{cases} -\Delta u - \lambda u = W(x)f(u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

where $W \in C(\overline{\Omega})$ changes sign in Ω and f is a nonlinear function with superquadratic growth both at zero and at infinity.

The case where $\lambda < \lambda_1$ is somewhat less interesting, since it is indifferent to the fact that the weight function W changes sign. A (positive) solution in this case can be obtained by rather standard variational techniques, provided that the Palais-Smale (PS) condition is satisfied. It should be noted, however, that the (PS) condition is a nontrivial issue here and it will be discussed below.

A similar remark applies for $\lambda >> \lambda_1$ and multiple solutions for $(*_\lambda)$ can be established if, for instance, f is even (see section 3). Still some delicate analysis is needed to obtain existence for $(*_\lambda)$, λ large, when f is not even, and we believe that this case already offers a wealth of interesting questions to be addressed. But even more appealing is the case where $\lambda \geq \lambda_1$ but close to λ_1. There the problem becomes affected by the negative part of W which contributes to push up the spectrum of the linearized problem and possibly create a "ground state" solution. Therefore, for λ in a right neighborhood of λ_1 a more interesting problem is the study of *positive* solutions for $(*_\lambda)$. As we shall see in section 2, for f behaving like $|u|^{q-2}u$ near zero with $2 < q \leq \frac{2N}{N-2}$, $N \geq 3$, the influence of the negative part of W is displayed in the following condition:

$$(**) \qquad \int_\Omega W(x)e_1^q < 0$$

with e_1 the first positive eigenfunction for $-\Delta$ in $H_0^1(\Omega)$. As it turns out, condition $(**)$ is very important, since it is also necessary when f is homogeneous, ie, $f(u) = |u|^{q-2}u$ (see appendix).

Condition $(**)$ was inspired by a corresponding necessary condition derived in [B-P-T] for a Neumann problem. It appears however, that in the context of the Neumann problem, conditions of the same type were already introduced by Kazdan and Warner [KW] as an obstruction to the solvability of the prescribed scalar curvature problem for compact Riemannian manifolds. See also [E-S].

We have also learned recently from L. Nirenberg that in [B-CD-N] condition (**) was derived as a necessary and sufficient condition for the solvability of (*A=AJ with- $u > 0$ and $f(u)$ = |u|*-^{2}u. See also our Corollary 1.2 and Corollary 2.8.

Surprisingly enough, the same condition appears also for more general /, and yields

THEOREM. *Assume that f € C^X(K)* satisfies

(i) $$\lim_{u\to 0} \frac{f(u)}{|u|^{q-2}u} = a > 0, \quad 2 < q < \frac{2N}{N-2}, \; N \geq 3,$$

(ii) $$\lim_{|u|\to\infty} \frac{f(u)}{|u|^{p-2}u} = 1, \quad 2 < p < \frac{2N}{N-2}, \; N \geq 3,$$

(iii) |/(u)u -pF(u)|_< d|u|^2 + c_2, $C_U C_2 > 0$

(iv) |/'(u)|_<C_3|ur^2 + C_4, C_3,C_4>0

IfW changes sign in Q and (**) *holds, then there exists* A > Ai such *that*

(a) *For every* A € (Ai, A), (*A) *admits at least two positive solutions.*

(b) *For* A = Ai and A = A, *problem* (*A) *admits at least one positive solution.*

(c) *For* A > A *problem* (*A) *does not admit any positive solutions.*

The case A < Ai does not require (**) and the existence of a positive solution can be established only under the assumptions (i)-(iv) above.

The critical case $p = q$ = -j ^ ^ *$^{s\ a}$ ^^{so} treated, and an analogous result is established (see Theorem 4.1). We also point out that condition (iii) may be dropped in case

$$\overline{\{x \text{ € } n : W(x) > 0\}} \text{ n } \overline{\{x \text{ € « : } W(x) < 0\}} = 0$$

and we refer to Lemma 1.5 for a more precise statement.

The Theorem stated above was prompted by an analogous result obtained by Ouyang (see [Ou]) for A > Ai (but not too far from Ai) and for $f(u)$ = |u|^{p}~^{2}ix. However, it must be said that while the approach of Ouyang motivated ours, there are considerable differences between the case of homogeneous nonlinearities and the more general case. In particular, in the homogeneous case one can use a variety of minimization problems to obtain solutions of $(*_\lambda)$ after rescaling. Also the fact that, in this situation, it is possible to relate the nonlinear term with its derivatives allows one to perform explicit calculations which yield rather precise information about the solutions of $(*_\lambda)$. For more general f this is no longer possible and even the (PS) condition becomes a delicate issue, since any inequality which relates $F(u)$ with $F'(u) = f(u)$ will not be of much help when W changes sign. In practice one faces the problem that the vanishing of W could balance the difference in homogeneity (at infinity, say) which is at the heart of any (PS) condition for problems of this type (cf [R], [St]). Here we have managed to obtain (PS) under some restrictions on f or W (see Lemma 1.5).

The paper is organized as follows: the first two sections deal with the existence of positive solutions for $\lambda = \lambda_1$ and $\lambda > \lambda_1$ respectively. The third section treats the existence of (possibly changing-sign) solutions for any λ and corresponding multiplicity results. The fourth section is concerned with nonlinearities having critical Sobolev growth and it contains the extension of the above theorem to this case.

Finally, we conclude by observing that our methods could be generalized to obtain analogous results for the Neumann problem corresponding to $(*_\lambda)$ (in which case $\lambda_1 = 0$) as well as to equations on compact Riemannian manifolds. Also the Laplace operator in the above equation can be replaced by any general second order, positive semidefinite elliptic operator, in which case the role of λ_1 should be adjusted accordingly.

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1. Positive solutions in the case $\lambda = \lambda_1$

We first establish some definitions and notations. Throughout the paper we assume that Ω is an open, bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$ and that $W \in C^\beta(\overline{\Omega})$ $(0 < \beta < 1)$ is a given function which changes sign in Ω. So, if we define

$$\Omega^+ = \{x \in \Omega : \ W(x) > 0\}$$
$$\Omega^- = \{x \in \Omega : \ W(x) < 0\}$$
$$\Omega^0 = \Omega \backslash \overline{(\Omega^+ \cup \Omega^-)}$$

we have

$$\Omega^+ \neq \emptyset, \qquad \Omega^- \neq \emptyset$$

Consider also a function F such that

$$\begin{aligned} &F \in C^2(\mathbf{R}), \qquad F'(u) = f(u) > 0 \text{ for } u > 0 \\ &F(0) = f(0) = f'(0) = 0, \\ &F(u) \leq A + B|u|^p \end{aligned} \tag{1.1}$$

for constants $A, B > 0$ and $2 < p < 2^*$, where, as usual, we denote

$$2^* = \begin{cases} \frac{2N}{N-2}, & \text{if } N \geq 3 \\ +\infty, & \text{if } N = 1,2 \end{cases}$$

Let $\lambda_1 < \lambda_2 \leq \ldots$ be the sequence of eigenvalues of $-\Delta$ on $H_0^1(\Omega)$, and denote by $e_1, e_2, \ldots$ the associated eigenfunctions normalized so that $\int_\Omega |\nabla e_j|^2 = 1$, $j = 1, 2, \ldots$ and $e_1 > 0$ in Ω.

a) The constrained case. In this section, we consider some constrained problems which give rise to the nonlinear eigenvalue equation,

$$\begin{cases} -\Delta u - \lambda_1 u = \gamma W(x) f(u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1.2}$$

with $\gamma \in \mathbf{R}$. The method we use is inspired by techniques introduced by Fleckinger-Pellé [FP] and Geztesy and Simon [GS] in solving linear eigenvalue problems with indefinite weight functions. Basically, the idea is to separate the positive and negative parts of W, and introduce a one parameter family of equations,

$$-\Delta u - \lambda_1 u + \mu(W^-(x) + a)f(u) = \gamma(\mu)(W^+(x) + a)f(u) \tag{1.3}$$

where $\mu \in \mathbf{R}$ and $a > 0$ is a fixed number. In order to solve (1.2), we then need to show that (1.3) can be solved with $\gamma(\mu) = \mu$.

THEOREM 1.1. *Suppose that F satisfies*

$$F(tu) \geq \begin{cases} t^p F(u), & \text{if } t \geq 1 \\ t^q F(u), & \text{if } t \leq 1 \end{cases} \tag{1.4}$$

with $2 < p, q < 2^*$. If for some $t_0 > 0$ we have

$$\int_\Omega W(x)F(t_0 e_1) < 0, \tag{1.5}$$

then there exists a solution pair (u, γ) to (1.2) with $u > 0$ and $\gamma > 0$.

Before going into the proof, let us mention an interesting consequence of Theorem 1.1.

COROLLARY 1.2. *Let $2 < q < 2^*$. Then the problem*

$$\begin{cases} -\Delta u - \lambda_1 u = W(x)u^{q-1} & \text{in } \Omega \\ u|_{\partial\Omega} = 0 & \\ u > 0 \quad \text{in } \Omega & \end{cases} \tag{1.6}$$

admits a solution if and only if

$$\int_\Omega W(x)e_1^q < 0 \tag{1.7}$$

The proof of the corollary follows in one direction from Theorem 1.1 (after a suitable scaling of the solution) and in the other direction as a consequence of the following necessary condition which is particular to homogeneous nonlinearities:

LEMMA 1.3. *Suppose there exists a positive solution $u \in H_0^1(\Omega)$ to*

$$-\Delta u - \lambda u = W(x)u^{q-1}$$

for $q > 2$ and $\lambda \geq \lambda_1$. Then necessarily (1.7) must hold.

The proof is technical and is deferred to an appendix.

As already mentioned, the result of Corollary 1.2 was also derived in [B-CD-N] via a different minimization problem. Also, a similar necessary condition for the corresponding Neumann problem can be found in [B-P-T]; see also [E-S], [K-W]. We shall see how (1.7) plays a crucial role also in the study of (1.6) with more general nonlinearities and $\lambda > \lambda_1$.

PROOF: Fix $\kappa > 0$ such that $\tilde{W} = \frac{1}{\kappa} W$ satisfies

$$\int_\Omega \tilde{W}^+(x) F(t_0 e_1) = \frac{1}{2}$$

Now fix $a > 0$ so that

$$\int_\Omega (\tilde{W}^+ + a) F(t_0 e_1) = 1 \tag{1.8}$$

Clearly, it is sufficient to solve (1.2) with W replaced by $\tilde{W}$, as the same u will solve the original (1.2) with μ replaced by $\frac{1}{\kappa}\mu$. So without loss of generality, we drop the tilda and assume that (1.8) holds for the original W.

Define the constraint set

$$S = \{u \in H_0^1(\Omega) : \quad \int_\Omega (W^+(x) + a) F(u) = 1\}$$

and a family of functionals

$$J_\mu(u) = F(\sqrt{\|\nabla u\|_2^2 - \lambda_1 \|u\|_2^2}) + \mu \int_\Omega (W^-(x) + a) F(u)$$

Set

$$\gamma(\mu) = \inf_{u \in S} J_\mu(u) \tag{1.9$_\mu$}$$

Note that if u is a minimizer for (1.9$_\mu$) then it satisfies

$$\frac{f(\sqrt{\|\nabla u\|_2^2 - \lambda_1 \|u\|_2^2})}{\sqrt{\|\nabla u\|_2^2 - \lambda_1 \|u\|_2^2}} (-\Delta u - \lambda_1 u) + \mu (W^-(x) + a) f(u) = \nu(\mu)(W^+(x) + a) f(u) \tag{1.10}$$

for some Lagrange multiplier $\nu(\mu)$. This would yield a solution for our problem provided that we find μ with $\nu(\mu) = \mu$ (note that in this case necessarily $\|\nabla u\|_2^2 > \lambda_1 \|u\|_2^2$). However, very little can be said about the function $\nu(\mu)$. Even its continuity is not apparent, unless F is homogeneous, in which case $\nu(\mu)$ would essentially be the same as $\gamma(\mu)$. Nevertheless, under the more general assumption (1.4) and with the help of an auxilliary functional, we still manage to show that for $\mu = \gamma(\mu)$ the minimizer of (1.9_μ) solves (1.10) with $\mu = \nu(\mu)$.

To this end we start by showing that for each $\mu \geq 0$, $\gamma(\mu)$ is attained at (at least) one function $v_\mu \in S$. Indeed, fixing $\mu \geq 0$ and choosing a minimizing sequence $u_n = u_{n,\mu} \geq 0$ for $\gamma(\mu)$, we clearly have

$$0 \leq \|\nabla u_n\|^2 - \lambda_1 \|u_n\|_2^2 \leq C \tag{1.11}$$

for suitable constant $C > 0$. Now, applying (1.4),

$$\begin{aligned} \int_\Omega |u_n|^p &\leq \int_{\{u_n(x)\geq 1\}} |u_n|^p + |\Omega| \\ &\leq \frac{1}{F(1)} \int_{\{u_n(x)\geq 1\}} F(u_n) + |\Omega| \\ &\leq \frac{1}{aF(1)} \int_\Omega (W^+ + a) F(u_n) + |\Omega| \leq C \end{aligned} \tag{1.12}$$

Putting together (1.11) and (1.12), we see that $\|u_n\|_{H_0^1(\Omega)} \leq C$ so extracting a subsequence $u_n \rightharpoonup v_\mu$ in $H_0^1(\Omega)$. As F has subcritical growth, $v_\mu \in S$ and hence $v_\mu \not\equiv 0$. Also

$$F(\sqrt{\|\nabla v_\mu\|_2^2 - \lambda_1 \|v_\mu\|_2^2}) \leq \liminf_{n\to\infty} F(\sqrt{\|\nabla u_n\|_2^2 - \lambda_1 \|u_n\|_2^2})$$

so

$$J_\mu(v_\mu) \leq \liminf_{n\to\infty} J_\mu(u_n) = \gamma(\mu) \tag{1.13}$$

Now it is easy to verify that:

$$\gamma(0) = 0 \tag{1.14}$$

$$\gamma(\mu) \ \text{ is strictly increasing for } \mu > 0 \tag{1.15}$$

$$\gamma(\mu) \ \text{ is Lipschitz continuous for } \mu \in [0,\infty) \tag{1.16}$$

Our next task is to show that there exists $\mu^* > 0$ with $\gamma(\mu^*) = \mu^*$.

First, we claim that there exists \ī > 0 so that

(1.17) 7(JI) < \i for all \i > p.

To see this, choose any tp € Cϑ°(fl) with

$$\begin{cases} v(*) \geq 0_5 \qquad \text{supp}\wedge C\ \Omega^+ \\ /(W^+(_3:) + a)F(\varphi) = 1 \\ \text{in} \end{cases}$$

Then,

$$700 \leq Jy(fp) \leq C + fi\ f(W\text{-}(x) + a)F(<p)$$

$$= C + ti\ (l - J\wedge W\{x)F\{<p)\}$$

But, as suppy> C £2$^+$ we have

$$/ W(x)F(tp) > 0$$
$$\jmath n$$

and

$$\frac{\gamma(\mu)}{\mu} \xrightarrow{\mu \to \infty} 1 - \ / W(x)F(\varphi) < 1$$

Note that when /x = 0 the unique positive minimizer is *VQ* = <o^i with to ^^ in (1.5). In fact, as /i -f 0, it is clear from (1.11), (1.12), and (1.13) that r$_M$ -+ i>$_0$ strongly in U^(fi).

In order to show that 7(/i) = /x has a solution, we will show that the curve 7(AO lies above the diagonal 7 = /x for // near zero. Indeed,

$$\frac{\gamma(\mu)}{\mu} = \frac{1}{\mu}F(\sqrt{\|\nabla v_\mu\|_2^2 - \lambda_1\|v_\mu\|_2^2}) + \left(1 - \int_\Omega W(x)F(v_\mu)\right)$$

$$\geq 1 - / W(x)F(v_\mu)$$

(1.18) - ^ * 1 - / *W(x)F(vo)* > 1

where we have used (1.5) and the fact uo = $t_0 e_1$.

Now by putting together (1.16), (1.17), and (1.18) there exists /x* > 0 with

$$\gamma(\mu *) = f$$

Denote by $v^* \in S$ the corresponding minimizer for $(1.9_{\mu=\mu^*})$.

Next we introduce an auxilliary functional in order to show that v^* actually solves equation (1.2). Let

$$\tilde{J}(u) = F(\sqrt{\|u\|_2^2 - \lambda_1 \|u\|_2^2}) - \mu^* \int_\Omega W(x)F(u), \quad u \in H_0^1(\Omega) \tag{1.19}$$

Note that

$$\tilde{J}(u) = J_{\mu^*}(u) - \mu^*, \quad \text{for all } u \in S$$

We will show that v^* is the global minimizer for $\tilde{J}$ in $H_0^1(\Omega)$. Clearly $\tilde{J}(v^*) = 0$, so it suffices to show that $\tilde{J}(u) \geq 0$ for all $u \in H_0^1(\Omega)$.

Let $u \in H_0^1(\Omega)$, $u \neq 0$, be fixed. As the function $\psi(t) = \int_\Omega (W^+ + a)F(tu)$ is continuous with $\psi(0) = 0$ and

$$\psi(t) \geq t^p \int_\Omega (W^+ + a)F(u)$$

for all $t \geq 1$ (by (1.4)), there exists $t = t(u) > 0$ such that $\psi(t(u)) = 1$, that is

$$t(u)u \in S \tag{1.20}$$

For such t, (1.4) implies that

$$\int_\Omega (W^+ + a)F(u) \leq \min\{t(u)^{-p}, t(u)^{-q}\} \tag{1.21}$$

and hence

$$\begin{aligned}
\tilde{J}(u) &= F\left(\frac{1}{t(u)}\sqrt{\|\nabla(t(u)u)\|_2^2 - \lambda_1 \|t(u)u\|_2^2}\right) - \mu^* \int_\Omega W(x)F(u) \\
&\geq \min\{t(u)^{-p}, t(u)^{-q}\} J_{\mu^*}(t(u)u) - \mu^* \int_\Omega (W^+ + a)F(u) \\
&\geq \mu^* \left(\min\{t(u)^{-p}, t(u)^{-q}\} - \int_\Omega (W^+ + a)F(u)\right) \geq 0
\end{aligned}$$

via (1.4), (1.19), (1.20), and (1.21).

Hence v^* is a nontrivial critical point of $\tilde{J}$, and therefore it satisfies

$$\alpha(-\Delta v^* - \lambda_1 v^*) = \mu^* W(x) f(v^*) \tag{1.22}$$

where

$$\alpha = \frac{f(\sqrt{\|\nabla(v^*)\|_2^2 - \lambda_1\|v^*\|_2^2})}{\sqrt{\|\nabla(v^*)\|_2^2 - \lambda_1\|v^*\|_2^2}}$$

Clearly $\alpha = 0$ only when $v^* = t_0 e_1$ for t_0 as in (1.5). However, if this were the case, (1.22) would imply that $\mu^* W(x) f(t_0 e_1) = 0$, which is impossible as $\mu^* > 0$, $e_1 > 0$, and $W \not\equiv 0$. Hence, $(v^*, \frac{\mu^*}{\alpha})$ solves (1.2). ∎

b) The unconstrained case. Here we handle the more general problem

$$\text{(1.23)} \qquad \begin{cases} -\Delta u - \lambda_1 u = W(x) f(u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \\ u > 0 & \text{in } \Omega \end{cases}$$

In the following we shall assume that f is extended as an odd function for $u < 0$ and let

$$F(u) = \int_0^u f(\xi)\, d\xi$$

Note that $F(u) = F(|u|)$. We seek critical points for the (even) functional

$$\text{(1.24)} \qquad I_{\lambda_1}(u) = \frac{1}{2}\left(\|\nabla u\|_2^2 - \lambda_1\|u\|_2^2\right) - \int_\Omega W(x) F(u), \quad u \in H_0^1(\Omega)$$

As usual for this setting, the variational techniques are helpful only if suitable information on the nonlinearity near zero and near infinity are available. So we will impose the following conditions:

$$\text{(1.25)} \qquad \lim_{u \to 0} \frac{f(u)}{|u|^{q-2}u} = a > 0$$

$$\text{(1.26)} \qquad \lim_{|u| \to \infty} \frac{f(u)}{|u|^{p-2}u} = 1$$

for some $2 < p, q \leq 2^*$.

LEMMA 1.4. *If f satisfies (1.25) and (1.26) and in addition*

$$\text{(1.27)} \qquad \int_\Omega W(x) e_1^q < 0$$

then $u_0 = 0$ is a strict local minimum for the functional I_{λ_1}.

PROOF: Set

$$A = -\int_\Omega W(x) e_1^q > 0$$

via (1.27). Decompose $u \in H_0^1(\Omega)$ as $u = te_1 + v$ for $t \in \mathbf{R}$ and $\int ve_1 = 0$. Suppose that $\|\nabla u\|_2 < \frac{1}{10\|e_1\|_\infty}$. Then, clearly $|t| \leq \frac{1}{10\|e_1\|_\infty}$. Now

$$
\begin{aligned}
I_{\lambda_1}(u) &= \frac{1}{2}\left(\|\nabla v\|_2^2 - \lambda_1\|v\|_2^2\right) - \int_\Omega W(x)F(te_1 + v) \\
&\geq \frac{1}{2}(1 - \frac{\lambda_1}{\lambda_2})\|\nabla v\|_2^2 - |t|^q \int_\Omega W(x)e_1^q + R(t,v) \\
&= \frac{1}{2}(1 - \frac{\lambda_1}{\lambda_2})\|\nabla v\|_2^2 + A|t|^q + R(t,v)
\end{aligned}
$$

where the remainder term R is

$$
\begin{aligned}
R(t,v) &= \int_\Omega W(x)\left[a|te_1|^q - F(te_1)\right] + \int_\Omega W(x)\left[F(te_1) - F(te_1 + v)\right] \\
&= \int_\Omega W(x)\left[F(te_1) - F(te_1 + v)\right] + o(|t|^q)
\end{aligned}
\tag{1.28}
$$

using (1.25). Now to estimate the remaining term in (1.28), we have for each v, t, x a number $\theta = \theta(v,t,x)$ with $0 \leq \theta \leq 1$ so that

$$
F(te_1(x)) - F(te_1(x) + v(x)) = f(te_1(x) + \theta v(x))v(x)
\tag{1.29}
$$

Putting together (1.25), (1.26), f satisfies the bound

$$
|f(u)| \leq \begin{cases} C|u|^{q-1}, & \text{if } |u| \leq 1 \\ C|u|^{p-1}, & \text{if } |u| \geq 1 \end{cases}
\tag{1.30}
$$

with some constant $C > 0$. Fixing v, t for now, we consider two cases. First if x is such that $|u(x)| = |te_1(x) + \theta v(x)| \geq 1$, we have (recalling $|t| \leq \frac{1}{10\|e_1\|_\infty}$) that $|\theta v(x)| \geq 9|t|e_1$, so applying (1.30) we see:

$$
\begin{aligned}
|f(te_1(x) + \theta v(x))v(x)| &\leq C\left|te_1(x) + \theta|v(x)|\right|^{p-1} \cdot |v(x)| \\
&\leq \frac{10}{9}C|v(x)|^p
\end{aligned}
\tag{1.31}
$$

On the other hand, if x is such that $|u(x)| = |te_1(x) + \theta v(x)| \leq 1$, then let $0 < \epsilon < \frac{A}{2}$ and apply (1.30) again to obtain

$$
\begin{aligned}
|W(x)|\,|f(te_1(x) + \theta v(x))v(x)| &\leq C\left|te_1(x) + \theta|v(x)|\right|^{q-1} \cdot |v(x)| \\
&\leq C\left[|te_1|^{q-1} + |v(x)|^{q-1}\right]|v(x)| \\
&\leq \epsilon|te_1|^q + C_\epsilon|v(x)|^q
\end{aligned}
\tag{1.32}
$$

Plugging (1.29), (1.31) and (1.32) into (1.28) (and using 2 <*p,q* ≤ 2*,) we see that

***hM* > ^(1 " ^)||V»|g + |<* + *o(t«)* + *O(\\Vv\\$_2$)* + O(||Vv|iI)**

As p, *q* > 2, we're done. |

Note that for <*p* € C£°(fi+) it is clear that

***I\\$_x$(s(p)* —• oo   as *s* —• oo**

So, if in addition *I\\$_x$* satisfied the (PS) condition we would be able to solve (1.23) via the mountain-pass lemma (cf [A-R]). Here we state some conditions on / and *W* which guarantee the (PS) condition for the family of functionals

h(u)* = *\\$_1${\\Vu* - A|M© - */ W{x)F{u),*   *u* € *HliSl)
Ja

As the proofs are technical, we will present them in the appendix. Denote by cr(fi°) the collection of eigenvalues of —A in *H$_0^1$(CI°).* (In case £2° = 0, then take cr(fi°) = 0 also.)

LEMMA 1.5. ***Suppose* A £ *cr(Q°),* and / satisfies *(1.26) with 2 <* p < 2* and *the estimate***

(1.33)   **|/'(u)|<A|ur^2+J9**

for suitable constants* A, *B* > 0. *Then I\\ satis£es (PS) if either of the following conditions hold:

(a) ?FnTF=0

(b) ***There exist constants c\\,C2* > 0 *and 2 <* p *< 2* such that for all u* € R**

\\f(u)u-$_P$F(u)\\<c$_1$\\u\\$^2$*   +   *c$_2$

REMARK: We believe that (PS) should in fact hold only under the assumption (1.26) with 2 < p < 2 * .

Note that by domain monotonicity we have Ai < mina(£2°). Now we may use the mountain pass theorem to obtain solutions to (1.23) in the subcritical case. The critical case *f(u)* = u$^{2*"1}$ will be treated separately in section 4.

THEOREM 1.6. *Assume that (1.25), (1.26), and (1.33) hold. If either of the conditions (a) or (b) in Lemma 1.5 is satisfied, then there exists a positive solution to (1.23).*

PROOF: In view of the above remarks, the value

$$c = \inf_{\gamma \in \mathcal{P}} \max_{0 \le t \le 1} I_{\lambda_1}(\gamma(t)) > 0$$
$$\mathcal{P} = \{\gamma \in C([0,1]; H_0^1(\Omega)) : \ \gamma(0) = 0, \ I_{\lambda_1}(\gamma(1)) < 0\}$$

defines a critical value for I_{λ_1}. The positivity for the corresponding (nontrivial) critical point follows from the observation that if $\gamma \in \mathcal{P}$ then $|\gamma| \in \mathcal{P}$ and

$$I_{\lambda_1}(\gamma(t)) = I_{\lambda_1}(|\gamma(t)|) \quad \text{for all } t \in [0,1]$$

We leave the details to the interested reader.

2. POSITIVE SOLUTIONS IN THE CASE $\lambda > \lambda_1$

In this section we study positive solutions to the problem

$$(2.1_\lambda) \qquad \begin{cases} -\Delta u - \lambda u = W(x) f(u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \\ u > 0 \quad \text{in } \Omega \end{cases}$$

We follow Ouyang [Ou], and base the first part of our arguments upon Perron's method of sub- and super-solutions. In fact, the existence of subsolutions for (2.1_λ) follows with only a simple hypothesis on the nonlinear term f:

LEMMA 2.1. *Suppose that there exist constants $\eta, C > 0$ and $q > 2$ so that*

$$0 \le f(u) \le C u^{q-1}$$

for all $0 \le u \le \eta$. Then for every t which satisfies

$$(2.2) \qquad 0 < t < \left(\frac{\lambda - \lambda_1}{\|W^-\|_\infty \|e_1\|_\infty^{\alpha-1}} \right)^{\frac{1}{\alpha-1}}$$

the function $u = te_1$ is a subsolution for (2.1_λ), $\lambda > \lambda_1$.

PROOF: Let $t > 0$, $\varphi \in C_0^\infty(\Omega)$ with $\varphi(x) \geq 0$. Then,

$$
\begin{aligned}
t\int_\Omega \nabla e_1 \cdot \nabla\varphi - \lambda t \int_\Omega e_1\varphi - \int_\Omega W(x) f(te_1)\varphi \\
&= -(\lambda - \lambda_1)t\int_\Omega e_1\varphi - \int_\Omega W(x) f(te_1)\varphi \\
&\leq \int_\Omega W^-(x) f(te_1)\varphi - (\lambda - \lambda_1)t\int_\Omega e_1\varphi \\
&\leq \left[\|W^-\|_\infty \|e_1\|_\infty^{q-2} t^{q-2} - (\lambda - \lambda_1)\right] t \int_\Omega e_1\varphi < 0
\end{aligned}
$$

and we see that te_1 is a subsolution when (2.2) holds. ∎

Our next observation is that positive solutions of (2.1_λ), $\lambda > \lambda_1$ cannot exist for λ too large, as the following simple calculation shows:

LEMMA 2.2. *There exist $\bar\lambda > \lambda_1$ so that (2.1_λ) does not admit a positive solution for any $\lambda \geq \bar\lambda$*

PROOF: This fact has already been observed in [Ou], where it is shown that it is enough to take $\bar\lambda = \lambda_1(\Omega^*)$ where $\Omega^* = \Omega \backslash \overline{\Omega^-}$. In fact, let $\psi > 0$ be the first eigenfunction of the Dirichlet Laplacian on Ω^*. Multiply (2.1_λ) by ψ, and integrate by parts to obtain:

$$
\int_{\partial\Omega^*} u\frac{\partial\psi}{\partial\nu} - (\lambda - \bar\lambda)\int_{\Omega^*} u\psi - \int_{\Omega^*} W^+(x) f(u)\psi = 0 \tag{2.3}
$$

On the other hand,

$$
\int_{\partial\Omega^*} u\frac{\partial\psi}{\partial\nu} < 0
$$

so (2.3) cannot hold for $\lambda \geq \bar\lambda$. ∎

Define

$$
\Lambda = \sup\{\lambda \geq \lambda_1 : \ (2.1_\lambda) \text{ admits a positive solution}\} \tag{2.4}
$$

LEMMA 2.3. *Suppose that $f \in C^1(\mathbf{R})$ and there exists p, q with $2 < p, q \leq 2^*$ and* *constants $A, B > 0$ so that:*

$$\lim_{u \to 0} \frac{f(u)}{u^{q-1}} = a > 0 \tag{2.5}$$

$$\int_\Omega W(x) e_1^q < 0 \tag{2.6}$$

$$|f(u)| \le A + B|u|^{p-1} \tag{2.7}$$

Then $\Lambda > \lambda_1$.

PROOF: Following [Ou], we shall use bifurcation theory to show that (2.1_λ) admits positive solutions for $\lambda > \lambda_1$ near λ_1. To this purpose, we define $\mathcal{F} : H_0^1(\Omega) \times \mathbf{R} \to H^{-1}$ by

$$\mathcal{F}(u, \lambda) = -\Delta u - \lambda u - W(x) f(u)$$

From (2.5), $f(0) = 0$, and so $\mathcal{F}(0, \lambda) = 0$ for all λ. Moreover, (2.5) also implies that $f'(0) = 0$, and so $\mathcal{F}_u(0, \lambda_1)v = -\delta v - \lambda_1 v$. Hence,

$$N(\mathcal{F}_u(0,\lambda_1)) = \operatorname{span}\{e_1\}, \qquad \operatorname{codim} R(\mathcal{F}_u(0,\lambda_1)) = 1,$$
$$\text{and } \ \mathcal{F}_{\lambda,u}(0,\lambda_1)e_1 = -e_1 \notin R(\mathcal{F}_u(0,\lambda_1))$$

Consequently, $(0, \lambda_1)$ is a bifurcation point for $\mathcal{F}$ (cf. [C-R]) So if we decompose

$$H_0^1(\Omega) = \operatorname{span}\{e_1\} \oplus Z$$

(here $Z = \operatorname{span}\{e_1\}^\perp$) then by the bifurcation theorem of [C-R] we obtain a neighborhood U of $(0, \lambda_1)$ in $H_0^1(\Omega) \times \mathbf{R}$, continuous functions $\varphi : (-a, a) \to \mathbf{R}$, $\psi : (-a, a) \to Z$ with $\varphi(0) = \lambda_1$, $\psi(0) = 0$ and

$$\mathcal{F}^{-1}(0) \cap U = \{(\alpha e_1 + \alpha\psi(\alpha), \varphi(\alpha)) : \alpha \in (-a, a)\} \cup \{(0, \lambda) : (0, \lambda) \in U\}$$

Set $u_\alpha = \alpha e_1 + \alpha \psi(\alpha)$.

CLAIM: $\psi(\alpha) \to 0$ in $C^{1,\beta}(\bar{\Omega})$ as $\alpha \to 0$.

This claim follows by standard elliptic regularity theory and the fact that (by continuity) $\psi(\alpha) \to 0$ in $H_0^1(\Omega)$ as $\alpha \to 0$. For the reader's convenience, we reserve the proof of the Claim for the end of the lemma.

Note that the Claim guarantees that $u_a > 0$ in fi for all a sufficiently small.

Next we show that <p(a) > Ai for all sufficiently small positive *a*. To this purpose, note that

$$\frac{}{a^{q-1}} \int_{Jo} W(x)f(u_a)e_1 \to a \int_{Jn} W(x)e\backslash$$

In fact, as ^(<*) —• 0 uniformly as *a* —* 0, (2.5) yields

$$\frac{f(<*ei (*) + atP(a)(x))}{\alpha^{q-1}[\alpha e_1(x) + \alpha\psi(\alpha)(x)]^{q-1}} \xrightarrow{a\text{-}o} u$$

uniformly in *Q*, and

$$\frac{1}{\alpha^{q-1}}\int_\Omega W(x)f(\alpha e_1 + \alpha\psi(\alpha))e_1 = \int_\Omega W(x)\frac{f(\alpha e_1 + \alpha\psi(\alpha))}{\alpha^{q-1}[\alpha e_1 + \alpha\psi(\alpha)]^{q-1}}(e_1 + \psi(\alpha))^{q-1}e_1$$

$$\to a \int_{Jn} W(x)e\backslash$$

The desired result *ip(a)* > *X* now follows easily with an argument by contradiction. In fact suppose that there is a sequence of a_n —> 0^+ with <^(a$_n$) _< Ai. Denote u_n = t*$_{an}$, which axe positive solutions to (2.1^(Qn)). We have

$$0 \le \frac{(\lambda_1 - \varphi(\alpha_n))}{a_n^{q-1}}\int_\Omega u_n e_1 = \int_\Omega W(x)\frac{f(u_n)e_1}{ah^{n-1}}$$

$$\xrightarrow{\alpha\to 0} \int_{Jn} W(x)t\backslash < 0$$

(via (2.6)) which is clearly impossible.

It remains to prove the Claim. Note first that (2.5) and (2.7) imply that there exist constants *A*, *A2* > 0 with

(2.8) $$\backslash f(u)\backslash \le AM^{*-*} + A_2\backslash ur>$$

Also, as u_a satisfies (2.1^(a)), *tf>(a)* itself satisfies the equation

(2.9) $$-\Delta\psi(\alpha) - \varphi(\alpha)\psi(\alpha) = (\varphi(\alpha) - \lambda_1)e_1 + W(x)\frac{f(\alpha e_1 + \alpha\psi(\alpha))}{\alpha}$$

Estimate (2.8) shows that the right hand side of (2.9) converges to zero as *a* -» 0 in 2/(Ω) for *s* = 2*/[max(p,g) ~ !]• K *p,q* are subcritical, then *s* > 2*/(2* — 1) and standard bootstrap arguments guarantee that

$$\|\psi(\alpha)\|_{C^{1,\beta}(\bar\Omega)} \xrightarrow{\alpha\to 0} 0$$

In the critical case, the same conclusion follows provided that we show that $\|\psi(\alpha)\|_r \to 0$ for some $r > 2^*$. To this end, we use Moser's iteration scheme as presented in [B-K]. Writing (2.9) as

$$-\Delta\psi(\alpha) = w(\alpha)\psi(\alpha) + \lambda\psi(\alpha) \tag{2.10}$$

we have $w(\alpha) \in L^{N/2}(\Omega)$ and $\|w(\alpha)\|_{N/2} \to 0$, $\|\psi(\alpha)\|_{2^*} \to 0$ as $\alpha \to 0$. For fixed $s > 0$ such that $2(s+1) = 2^*$, and $L > 1$ set

$$\varphi_L(x) = \psi(\alpha)\min\{|\psi(\alpha)|^{2s}, L^2\} \in H_0^1(\Omega)$$

Using φ_L as a test function for equation (2.10), we obtain

$$\begin{aligned}\lambda\int_\Omega \psi(\alpha)\varphi_L + \int_\Omega w(\alpha)\psi(\alpha)\varphi_L &= \int_\Omega |\nabla\psi(\alpha)|^2\min\{|\psi(\alpha)|^{2s}, L^2\} \\ &\quad + 2s\int_{\{|\psi(\alpha)|<L\}} (|\psi(\alpha)|^s\,|\nabla\psi(\alpha)|)^2 \\ &\geq \frac{2s}{(s+1)^2}\int_\Omega |\nabla(\psi(\alpha)\min\{|\psi(\alpha)|^s, L\})|^2\end{aligned}$$

This calculation yields:

$$\begin{aligned}&\frac{2s}{(s+1)^2}\int_\Omega |\nabla(\psi(\alpha)\min\{|\psi(\alpha)|^s, L\})|^2 \\ &\qquad\leq \lambda\int_\Omega |\psi(\alpha)|^{2(s+1)} + \int_\Omega |w(\alpha)|\,|\psi(\alpha)\min\{|\psi(\alpha)|^s, L\}|^2 \\ &\qquad\leq \lambda\|\psi(\alpha)\|_{2^*} + \|w(\alpha)\|_{N/2}S^{-1}\int_\Omega |\nabla(\psi(\alpha)\min\{|\psi(\alpha)|^s, L\})|^2 \qquad (2.11)\end{aligned}$$

with S the best constant in the Sobolev inequality. Since $\|w(\alpha)\|_{\frac{2^*}{2^*-2}} \to 0$ as $\alpha \to 0$, we can find $\alpha_0 > 0$ such that

$$\|w(\alpha)\|_{\frac{2^*}{2^*-2}} \leq \frac{s}{(s+1)^2}S$$

for all $\alpha < \alpha_0$. Thus we can absorb the second term of the right hand side of (2.11) to obtain

$$\int_\Omega |\nabla(\psi(\alpha)\min\{|\psi(\alpha)|^s, L\})|^2 \leq \lambda\frac{(s+1)^2}{s}\|\psi(\alpha)\|_{2^*}$$

Letting $L \to +\infty$, we see that $|\psi(\alpha)|^s\psi(\alpha) \in H_0^1(\Omega)$ and

$$\int_\Omega |\nabla(\psi(\alpha)|\psi(\alpha)|^s)|^2 \xrightarrow{\alpha\to 0} 0$$

In turn, the Sobolev inequality then implies that

$$\int_\Omega |\psi(\alpha)|^{(s+1)2^*} \xrightarrow{\alpha\to 0} 0$$

and so (as $s > 0$), we can proceed as in the subcritical case. This concludes the proof of Lemma 2.3. ▮

COROLLARY 2.4. *For each $\lambda \in (\lambda_1, \Lambda)$ problem (2.1_λ) admits a minimal positive solution u_λ. Furthermore, the map $\lambda \to u_\lambda$ is strictly monotone increasing, that is, if $\lambda < \mu$ then $u_\lambda(x) < u_\mu(x)$ for all $x \in \Omega$.*

PROOF: Let $\lambda \in (\lambda_1, \Lambda)$ be fixed. By the definition of Λ there exists a $\lambda_0 \in (\lambda, \Lambda)$ such that the problem (2.1_{λ_0}) admits a positive solution u_+. It is easy to verify that u_+ is a supersolution to (2.1_λ). Indeed for any $\varphi \in H_0^1(\Omega)$ with $\varphi \geq 0$ in Ω,

$$\int_\Omega \nabla u_+ \cdot \nabla\varphi - \lambda\int_\Omega u_+\varphi - \int_\Omega W(x)f(u_+)\varphi = (\lambda_0 - \lambda)\int_\Omega u_+\varphi \geq 0$$

Furthermore, since $-\Delta u_+ \geq 0$ in a small neighborhood of $\partial\Omega$, by the Hopf Lemma we conclude that $\frac{\partial u_+}{\partial\nu} \leq -c$ on $\partial\Omega$ for a suitable constant $c > 0$. This allows us to take $t > 0$ sufficiently small to have $u_- = te_1$ as a subsolution for (2.1_λ) with $u_- < u_+$ in Ω. The sub- and supersolution method now guarantees a solution u of (2.1_λ) with $u_- < u < u_+$.

To show that there is in fact a minimal solution for each λ we rely upon the information near $u = 0$ for λ near λ_1 as given by the Bifurcation Theorem and the fact that we can construct subsolutions for (2.1_λ) with $\lambda > \lambda_1$ as small as required (see Lemma 2.1). Define

$$u^+(x) = \inf\{u(x) : \ u \text{ positive solution for } (2.1_\lambda)\}, \quad x \in \Omega \tag{2.27}$$

First, we claim that $u^+ \not\equiv 0$. Indeed, note that the minimum of any two positive solutions of (2.1_λ) furnishes a supersolution for (2.1_λ). Hence we can construct a monotone decreasing sequence u_n^+ of positive supersolutions for (2.1_λ) with $u_n^+ \to u^+$ uniformly in Ω. If $u^+ \equiv 0$ then there would be a supersolution (and hence a solution) of ($2.1_{\lambda=\varphi(\alpha)}$) as close to zero as desired, for each $\alpha > 0$ with $\lambda_1 < \varphi(\alpha) < \lambda$. This is impossible in view of the bifurcation theorem. Hence, $u^+ \not\equiv 0$.

Thus, as above, we can find a sufficiently small $t > 0$ such that $u_- = te_1$ will be a subsolution for (2.1_λ) with $u_- < u^+$. The usual sub- and supersolution method now yields a positive solution u_λ which is minimal (it is clearly just u^+ itself). A similar sub- and supersolution argument also shows the strict monotonicity of the minimal solution family u_λ. ∎

We now explore some properties of the minimal solution family u_λ, $\lambda_1 < \lambda < \Lambda$. We would like to know whether or not these minimal solutions are actually (strict) local minima of the functional associated with (2.1_λ),

$$I_\lambda(u) = \frac{1}{2}\left(\int_\Omega |\nabla u|^2 - \lambda u^2\right) - \int_\Omega W(x)F(u), \qquad u \in H_0^1(\Omega)$$

where $F(u) = \int_0^u f(v)\,dv$. That this is the case has been shown by Ouyang in [Ou] in the special case of homogeneous nonlinearities (ie, $F(u) = \frac{1}{p}|u|^p$). There, explicit calculations can be carried out which are no longer possible for more general F's. In fact, in the general case it is reasonable to expect discontinuities for the map $\lambda \to u_\lambda$ which would naturally cause some degeneracies. However, it is still possible to provide some (second order) information on u_λ (see Lemma 2.5) which together with a continuation argument of [C-R] will enable us to identify certain solutions as strict local minima (for almost all λ).

To this purpose, we consider the variational formulation of Perron's method (see [St]). Namely, if there exist u_- a subsolution and u_+ a supersolution to (2.1_λ) with the property that $u_-(x) < u_+(x)$ for each $x \in \Omega$, then one obtains a solution u_*, $u_- \le u_* \le u_+$, to (2.1_λ) by solving the minimization problem:

$$\inf_{u_- \le u \le u_+} I_\lambda(u) \tag{2.12}$$

The advantage of this characterization of u_* is that it yields the additional information $(I''(u_*)\varphi, \varphi) \ge 0$ for all $\varphi \in H_0^1(\Omega)$ (unless $u_* \equiv u_-$ or $u_* \equiv u_+$.) This justifies the following:

LEMMA 2.5. *If u_λ is the minimal solution, then*

$$(I''(u_\lambda)\varphi, \varphi) \ge 0 \tag{2.13}$$

for all $\varphi \in H_0^1(\Omega)$.

PROOF: Choose a sequence $\lambda_n \nearrow \lambda$ and denote by $u_n = u_{\lambda_n}$ the corresponding minimal solutions. Clearly $u_n < u_\lambda$ and

$$\sup u_n = \lim u_n = u_\lambda$$

Take another sequence μ_n, $\lambda_n < \mu_n < \lambda$, and apply Perron's method to find a sequence of solutions v_n to (2.1_{μ_n}) with $u_n < v_n < u_\lambda$ and

$$I_{\mu_n}(v_n) = \inf_{u_n \le u \le u_\lambda} I_{\mu_n}(u)$$

This readily gives $(I''(v_n)\varphi, \varphi) \ge 0$ for all $\varphi \in H_0^1(\Omega)$. As $v_n \xrightarrow{n\to\infty} u_\lambda$ strongly, in the limit we arrive at (2.13). ∎

More generally, property (2.13) is inherited also by the right limit of minimal solutions (which need not be minimal).

LEMMA 2.6. *Suppose* $\lambda_0 \in (\lambda_1, \Lambda)$. *Set*

$$u_* \equiv \inf_{\lambda > \lambda_0} u_\lambda > 0$$

Then u_* *is a solution for* (2.1_{λ_0}) *which satisfies* $(I''_\lambda(u_*)\varphi, \varphi) \ge 0$ *for all* $\varphi \in H_0^1(\Omega)$.

PROOF: First of all, note that $u_{\lambda_0} \le u_*$. If $u_{\lambda_0} = u_*$, then the conclusion follows from our previous lemma. Hence, assume that in fact $u_{\lambda_0} < u_*$.

Define

$$\tilde{u}(x) = \sup_{v \in V} v(x) \tag{2.14}$$

With

$$V = \{v > 0 \text{ subsolutions for } (2.1_{\lambda_0}) \text{ with } v \le u_*,\ (I''(v)\varphi, \varphi) \ge 0\ \forall \varphi \in H_0^1(\Omega)\}$$

Note that $u_{\lambda_0} \in V$, and $\tilde{u} \le u_*$. Now, since the supremum of two solutions for (2.1_{λ_0}) is a subsolution for (2.1_{λ_0}), using u_λ as a supersolution and passing to the limit as $\lambda \searrow \lambda_0$, we construct a monotone increasing sequence of solutions w_j of (2.1_{λ_0}) with $w_j \in V$ and $\sup w_j = \tilde{u}$. Clearly $w_j \to \tilde{u}$ in $\mathcal{D}_0(\Omega)$, so $\tilde{u}$ is a solution for (2.1_{λ_0}) with

$$(2.15) \qquad (I''(\tilde{u})\cdot,\cdot) \geq 0$$

and $\tilde{u} \leq u_m$.

We now claim that $\tilde{u} = u_*$. Suppose that in fact $\tilde{u} < u^*$. By (2.15), the first eigenvalue δ of the linearized problem at $(\tilde{u}, \lambda_0)$,

$$-\Delta\varphi - \lambda_0\varphi - W(x)f'(\tilde{u})\varphi = \delta\varphi$$

is non-negative. Since $\tilde{u} < u_* = \inf_{\lambda > \lambda_0} u_\lambda$ and u_λ are minimal, the solution set of $\mathcal{F}(u, \lambda)$ near $(\tilde{u}, \lambda_0)$ can only cover a left neighborhood of λ_0. This implies that, necessarily $\delta = 0$. So we may apply Theorem 3.2 of [C-R] which ensures that the solutions of $\mathcal{F}(u, \lambda) = 0$ in a small neighborhood of $(\tilde{u}, \lambda_0)$ form a single C^1-curve $(u(s), \lambda(s))$ with $\dot{u}(s) > 0$ and $\lambda(s) \leq \lambda_0$ for all s. In particular, we can find $s_0 > 0$ sufficiently small such that $\lambda(s_0) < \lambda_0$ and $\tilde{u} < u(s_0) < u^*$ in Ω. Now using (2.12) with $u(s_0)$ as a subsolution and $u_{\lambda_0+\epsilon}$ $(\epsilon > 0)$ as a supersolution we obtain (after passing to the limit $\epsilon \to 0$) a solution $\bar{u} \in V$ with $\bar{u} > \tilde{u}$. This contradicts (2.14), and so we must have $\tilde{u} = u^*$. Hence $u^* \in V$, and the lemma is finished. |

Now that we have solutions to (2.1_λ) for $\lambda \in (\lambda_1, \Lambda)$ which are (almost) local minima, we may construct a second solution via a mountain pass.

THEOREM 2.7. *Suppose $f \in C^a(\mathbb{R})$ satisfies the hypotheses of Lemma 2.3 and Lemma 1.5, and Λ is as in (2.4). Then:*

(a) *For every $\lambda \in (\lambda_1, \Lambda)$, (2.1_λ) admits at least two positive solutions.*

(b) *For $\lambda = \lambda_1$ and $\lambda = \Lambda$, problem (2.1_λ) admits at least one positive solution.*

(c) *For $\lambda > \Lambda$ problem (2.1_λ) admits no positive solutions.*

REMARK: If $\lambda < \lambda_1$ more standard variational techniques of mountain- pass type apply and the existence for a positive solution can be established under all the hypotheses above except condition (2.6) which is no longer necessary for the existence of positive solutions for (2.1_λ) when $\lambda < \lambda_1$. We leave the details to the interested reader.

PROOF: (a) Fix $\lambda \in (\lambda_1, \Lambda)$ and let u^*_λ be the solution of (2.1_λ) given by Lemma 2.6, with

$$(I''(u^*_\lambda)\varphi, \varphi) \geq 0 \quad \text{for all} \quad \varphi \in H^1_0(\Omega)$$

In the case that $I''(u^*_\lambda)$ is in fact positive definite, (ie, there exists $\delta(\lambda) > 0$ such that $(I''(u^*_\lambda)\varphi, \varphi) \geq \delta(\lambda)\|\varphi\|^2$,) then u^*_λ is a strict local minimum for I_λ and a second solution would follow using a Mountain-Pass procedure as shown below. So we focus on the more delicate case where $\delta(\lambda) = 0$ is the first eigenvalue for $I''(u^*_\lambda)$, and we denote by $\varphi > 0$ the corresponding eigenfunction. In this situation, the continuation theorem of [C-R] gives that the solution set of $\mathcal{F}(u, \lambda) = 0$ in a sufficiently small neighborhood of (u^*_λ, λ) defines a C^1 curve $(u(s), \lambda(s))$, $s \in (-\epsilon, \epsilon)$, $\epsilon > 0$, with $u(0) = u_0$, $\dot{u}(0) = \varphi$, $\lambda(0) = \lambda$, and $\dot{\lambda}(0) = 0$. In particular, we can always assume that $\dot{u}(s) > 0$ for $s \in (-\epsilon, \epsilon)$.

By construction, there are minimal solutions for $\mathcal{F}(u, \lambda) = 0$ near u^*_λ in any right neighborhood of λ_0. Hence, $\lambda(s) < \lambda$ for $s < 0$ and we can find $s_0 \in (0, \epsilon)$ with $\lambda(s_0) > \lambda$. In particular, the range of $\lambda(s)$, $s \in [0, s_0]$ must contain the interval $[\lambda, \lambda(s_0)]$.

Set

$$\mathcal{S} = \{\mu \in \mathbf{R} : \ \text{there exists } s \in [0, s_0] \text{ with } \ \lambda(s) = \mu, \ \dot{\lambda}(s) = 0\};$$

by Sard's Theorem, we know that the Lebesgue measure of $\mathcal{S}$, $|\mathcal{S}| = 0$. For all $\mu \in (\lambda, \lambda(s_0))\backslash\mathcal{S}$ define

$$s_\mu = \max\{s \in [0, s_0] : \ \lambda(s) = \mu\}$$

Note that $0 < s_\mu < s_0$. Furthermore,

$$\dot{\lambda}(s_\mu) > 0$$

Indeed, since $\mu \notin \mathcal{S}$ clearly $\dot{\lambda}(s_\mu) \neq 0$. If $\dot{\lambda}(s_\mu) < 0$, then there would exist $s_1 \in (s_\mu, s_0)$ with $\lambda(s_1) < \mu < \lambda(s_0)$. The continuity of $\lambda(s)$ would then yield yet another value $s_2 > s_\mu$ with $\lambda(s_2) = \mu$, which would contradict the maximality of s_μ.

Now, $\dot{u}(s_\mu)$ satisfies

$$-\Delta\dot{u}(s_\mu) - \mu\dot{u}(s_\mu) - W(x)f'(u(s_\mu))\dot{u}(s_\mu) = \dot{\lambda}(s_\mu)u(s_\mu) \tag{2.16}$$

Denote by $(\delta_\mu, \varphi_\mu)$ the first eigenpair for the linearized problem

$$-\Delta\varphi_\mu - \mu\varphi_\mu - W(x)f'(u(s_\mu))\varphi_\mu = \delta_\mu\varphi_\mu \quad \text{in } H_0^1(\Omega) \tag{2.17}$$

with $\varphi_\mu > 0$ in Ω. Combining (2.16) and (2.17), we obtain

$$\delta_\mu \int_\Omega \varphi_\mu \dot{u}(s_\mu) = \dot{\lambda}(s_\mu) \int_\Omega u(s_\mu)\varphi_\mu > 0$$

that is,

$$\delta_\mu > 0 \tag{2.18}$$

This information will enable us to construct a second solution w_μ for (2.1_μ) for all $\mu \notin \mathcal{S}$, which will not lie in the curve $u(s)$ for $s \in (0, s_0)$.

To this purpose, fix $\mu \in (\lambda, \lambda(s_0))\backslash\mathcal{S}$. We seek a solution to (2.1_μ) of the form $w = \hat{u}_\mu + v$ with $v > 0$ and $\hat{u}_\mu = u(s_\mu)$. Direct calculations show that v can be characterized as a critical point for the functional

$$J_\mu(v) = \frac{1}{2}(\|\nabla v\|_2^2 - \mu\|v_+\|_2^2) - \int_\Omega G_\mu(x, v_+)$$

where $v_+(x) = \max[v(x), 0]$ and

$$\begin{aligned} G_\mu(x,\xi) &= \int_0^\xi W(x)[f(\hat{u}_\mu(x) + v) - f(\hat{u}_\mu(x))]\,dv \\ &= W(x)[F(\hat{u}_\mu(x) + \xi) - F(\hat{u}_\mu(x)) - f(\hat{u}_\mu(x))\xi] \end{aligned}$$

Next we show that, in virtue of (2.18), $v = 0$ is a strict local minimum for $J_\mu(v)$. Indeed,

$$\begin{aligned} J_\mu(v) &= \frac{1}{2}\left(\|\nabla v\|_2^2 - \mu\|v_+\|_2^2\right) - \int_\Omega W(x)f'(\hat{u}_\mu)v_+^2 + o(\|v_+\|_{H_0^1(\Omega)}^2) \\ &= \frac{1}{2}\|\nabla v_-\|_2^2 + (I''_\mu(\hat{u}_\mu)v_+, v_+) + o(\|v\|_{H_0^1(\Omega)}^2) \\ &\geq \frac{1}{2}\|\nabla v_-\|_2^2 + \frac{1}{2}\delta_\mu\|\nabla v_+\|_2^2 + o(\|v\|_{H_0^1(\Omega)}^2) \geq C\|\nabla v\|_2^2 \end{aligned}$$

for a suitable constant C, whenever $\|v\|_{H_0^1(\Omega)}$ is small. Hence $v = 0$ is a (strict) local minimum.

On the other hand, let $v_0 \in C_0^\infty(\Omega^+)$, $v_0 \geq 0$, and evaluate

$$J_\mu(tv_0) = \frac{1}{2}t^2\left(\|\nabla v_0\|_2^2 - \mu\|v_0\|_2^2\right) - \int_\Omega G_\mu(x, tv_0) \tag{2.19}$$

But for large positive t,

$$\begin{aligned} \frac{1}{t^2}\int_\Omega G_\mu(x, tv_0) &= \int_\Omega W(x)\frac{F(\hat{u}_\mu + tv_0)}{t^2} + o(1) \\ &= t^{p-2}\int_\Omega W^+(x)\frac{F(t[v_0 + (\hat{u}_\mu/t)])}{t^p} + \int_\Omega W^-(x)\frac{F(\hat{u}_\mu(x))}{t^2} + o(1) \\ &= \frac{t^{p-2}}{p}\int_\Omega W^+(x)v_0^p + o(t^{p-2}) \end{aligned} \tag{2.20}$$

via (2.5), (2.7) and the dominated convergence theorem. Putting together (2.19), (2.20), we derive:

$$J_\mu(tv_0) = -\frac{t^p}{p}\int_\Omega W^+(x)v_0^p + o(t^p) \to -\infty$$

as $t \to \infty$ large.

Thus, the mountain-pass lemma (cf [A-R]) applies to $J_\mu(v)$, and gives a candidate for a critical value

$$c = \inf_{\gamma\in\mathcal{P}}\max_{0\leq t\leq 1} J_\mu(\gamma(t)) > 0$$
$$\mathcal{P} = \{\gamma \in C([0,1]; H_0^1(\Omega)) : \ \gamma(0) = 0, \ J_\mu(\gamma(1)) < 0\}$$

Since for any $\gamma \in \mathcal{P}$ we have $J_\mu(\gamma_+) \leq J_\mu(\gamma)$, it follows that $\gamma_+ \in \mathcal{P}$, and we derive the existence of a sequence v_k with

$$J_n(v_k) \to c > 0, \qquad \|J'(v_k)\| \to 0, \qquad v_k \geq 0.$$

On the other hand, $w_k = \hat{u}_\mu + v_k$ satisfies:

$$\begin{cases} I_\mu(w_k) = I_\mu(\hat{u}_\mu) + J_\mu(v_k) \to c_\mu + c \\ \|I'_\mu(w_k)\| = \|I'_\mu(\hat{u}_\mu)\| + \|J'_\mu(v_k)\| = \|J'_\mu(v_k)\| \to 0 \end{cases}$$

as $k \to \infty$. Here we denote $c_\mu = I_\mu(\hat{u}_\mu)$. Hence, we see that w_k is a Palais-Smale sequence for the original functional I_μ. On the other hand, since $\mu < \Lambda < \lambda_1(\Omega^0)$ (see Lemma

2.2), by Lemma 1.5 $\{w_k\}$ possesses a strongly convergent subsequence, $v_k \to v_\mu$ in $H_0^1(\Omega)$. Since $J_\mu(v_\mu) = c > 0$, v_μ is a non-trivial critical point of J_μ and $v_\mu \geq 0$. In particular,

$$\int_\Omega \nabla v_\mu \cdot \nabla \varphi - \lambda_n v_\mu \varphi - W(x)[f(\hat{u}_\mu + v_\mu) - f(\hat{u}_\mu)]\varphi = 0$$

for all $\varphi \in H_0^1(\Omega)$. So, by the strong maximum principle, we must have $v_\mu > 0$ in Ω and $w_\mu = \hat{u}_\mu + v_\mu > \hat{u}_\mu$ is a second positive solution to (2.1_μ).

Now, the maximality of s_μ and the fact that $\hat{u}_\mu < w_\mu$ guarantee that w_μ cannot belong to the solution curve $\{u(s): \ s \in [-s_0, s_0]\}$. In particular, for a small neighborhood U of u_λ^* we have $w_\mu \notin U$. We now conclude by an approximation argument. Indeed since the Lebesgue measure of $\mathcal{S}$, $|\mathcal{S}| = 0$, we can certainly pick a sequence $\mu_n \in (\lambda, \lambda(s_0))\backslash\mathcal{S}$ with $\mu_n \to \lambda$. Denote by $w_n = w_{\mu_n}$, $u_n = \hat{u}_{\mu_n}$, $v_n = v_{\mu_n}$, and $J_n = J_{\mu_n}$. The mountain-pass construction for J_n yields that $I_{\mu_n}(w_n)$ are uniformly bounded. Indeed, letting v_0 be as in (2.19),

$$\begin{aligned} J_n(v_n) &\leq \max_{t>0} J_n(tv_0) \\ &= \frac{1}{2}t^2\left(\|\nabla v_0\|_2^2 - \mu_n\|v_0\|_2^2\right) - \frac{t^p}{p}\int_\Omega W(x)v_0^p + O(t) \\ &\leq C \end{aligned}$$

Since $u_\lambda^* \leq u_n \leq u(s_0)$ we derive that $\|\nabla u_n\|_2 \leq C$ and

$$\begin{aligned} I_\lambda(w_n) &= I_{\mu_n}(u_n) + J_n(v_n) + (\mu_n - \lambda)\|w_n\|_2^2 \\ &\leq C + (\mu_n - \lambda)\|w_n\|_2^2 + O(1) \end{aligned} \tag{2.21}$$

Similarly,

$$(I'_\lambda(w_n), \varphi) = (I'_{\mu_n}(w_n), \varphi) + (\mu_n - \lambda)\int_\Omega w_n\varphi = (\mu_n - \lambda)\int_\Omega w_n\varphi \tag{2.22}$$

So, as $\mu_n \to \lambda$, w_n is almost (but not quite) a Palais-Smale sequence for I_λ. However, by returning to the proof of Lemma 1.5, we see that (2.21) and (2.22) are sufficient to conclude that $w_n \to w_\lambda$ strongly in $H_0^1(\Omega)$. Clearly, w_λ is a positive solution to (2.1_λ) with $w_\lambda \geq u_\lambda^* > 0$ Furthurmore, since $w_n \notin U$ we have that $w_\mu \notin U$ and $w_\mu > u_\lambda^*$.

Note that in case ixj > *u* (ie, A is a point of discontinuity for the map A —* *u\)* this would yield a *third* positive solution for (2.1A).

This ends the proof of part (a).

(b) The existence of a positive solution at A = A* has already been proven in Theorem 1.6. For the case A = A, choose a sequence A_n /* A and construct a sequence of solutions u_n to (2.1A_n) via (2.12), with subsolution *u~ = tt* and supersolution u_M with A_n < /i < A. Using as a test-function the subsolution *u~ = tt* for *t* > 0 chosen suffiently small, we see that

$$I_{\lambda_n}(u_n) \leq I_{\lambda_n}(te_1) < 0$$

and hence

(2.23) $$I_A(u_n) \leq 0$$

Also, we have

(2.24)
$$(r_A(u_nU) = (/;>„),*>) - (A - A_n) \int_{Jn} U_n<f$$
$$= -(A - A_n) \int_{Jn} u_n<p$$

Again, u_n is almost a Palais-Smale sequence for 7A, in the sense that conditions (2.23) and (2.24) are sufficient to follow the argument in Lemma 1.5 and obtain a convergent subsequence whose limit UA will solve (2.1A) with *u* > *u* for each Ai < A < A.

(c) This is just Lemma 2.2 plus the definition (2.4). |

Putting together the results of section 1 and section 2 we derive the following:

COROLLARY 2.8. *Let* 2 < *q* < 2*. Then *problem*

$$\begin{cases} -Au - Xu = W\{x)u^{q}\wedge & \text{in fi} \\ u|_{\partial\Omega} = 0 \\ u > 0 & \text{in} Q, \end{cases}$$

admits a solution if and only if X € [Ai, A] *and (2.6) holds.*

A similar result will be obtained for *p* = 2*: see Theorem 4.1.

3. Multiple solutions

Here we study multiplicity for the problem

$$(3.1_\lambda)\qquad \begin{cases} -\Delta u - \lambda u = W(x)f(u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

with $\lambda \in \mathbf{R}$ and f satisfying (1.26) with $2 < p < 2^*$. We are going to use standard min-max methods and therefore assume that f is odd. We have:

THEOREM 3.1. *Let $\lambda \notin \sigma(\Omega^0)$ and suppose that the assumptions of Lemma 1.5 hold. Then (3.1_λ) possesses infinitely many pairs of nontrivial solutions.*

As in (1.24), introduce

$$I_\lambda(u) = \frac{1}{2}\left(\|\nabla u\|_2^2 - \lambda\|u\|_2^2\right) - \int_\Omega W(x)F(u), \quad u \in H_0^1(\Omega)$$

Denote by

$$S_\rho = \{u \in H_0^1(\Omega) : \ \|\nabla u\|_2 = \rho\}$$

We shall show that the following critical point theorem for even functionals applies to I_λ. To this end, let H be a Hilbert space and $I \in C^1(H, \mathbf{R})$ be an even functional. Denote by

$$\Sigma = \{A \subset H \text{ closed, symmetric; ie, } u \in A \rightarrow -u \in A\}$$

As is well known, the Krasnoselski genus gives a well defined map $i : \Sigma \rightarrow \mathbf{N} \cup \{+\infty\}$ (see [R]). Let $\mathcal{H}$ denote the family of odd heomeomorphisms $h : H \rightarrow H$ for which both h and h^{-1} map bounded sets to bounded sets. Then define the psuedoindex

$$i^*(A) = \inf_{h \in \mathcal{H}} i(A \cap h(S_\rho))$$

where $S_\rho = \{u \in H : \|u\| = \rho\}$ and $\rho > 0$ is fixed. Note that if $Y \subset H$ is a linear subspace, then $i^*(Y) = \dim Y$. Following [Be] it is not difficult to obtain the following:

THEOREM 3.2. *Let $I \in C^1(H,\mathbf{R})$ be an even functional on a Hilbert Space H which satisfies (PS). Suppose:*

(a) *There exists a linear subspace $X \subset H$ with* $\operatorname{codim} X = k_0 < +\infty$ *and a bounded symmetric neighborhood U of 0 such that*

$$\inf I|_{X\cap\partial U} \geq c_0$$

for some constant c_0.

(b) *For every $k = 1,2,\ldots$ there exists a linear space $Y_k \subset H$ with* $\dim Y_k \geq k$ *and*

$$\sup_{Y_k} I < +\infty$$

Then for $k > k_0$ the numbers

$$c_k = \inf_{\substack{i^*(A)\geq k\\ A\in\Sigma}} \sup_A I \geq c_0$$

define critical values for I. In particular, if $c_0 > I(0)$, then I admits infinitely many pairs of critical points.

PROOF OF THEOREM 3.1: Fix $\lambda \notin \sigma(\Omega^0)$. Denote for each j

$$E_j = \operatorname{span}\{e_1,\ldots,e_j\}, \qquad X_j = E_j^{\perp} \tag{3.2}$$

We now verify the conditions of Theorem 3.2.

First, we claim that for each $j > m$ there exists $\rho = \rho_j$ and $\gamma_j > -\infty$ such that $\gamma_j \xrightarrow{j\to\infty} \infty$ and

$$I_\lambda(u) \geq \gamma_j \quad \text{for all } u \in X_j \cap S_{\rho_j}$$

Indeed, simple arguments show that for any $u \in X_j$,

$$\|u\|_p^p \leq \frac{S}{\lambda_{j+1}^{\alpha}} \|\nabla u\|_2^p \tag{3.3}$$

where $\alpha = 2[\frac{2^*-p}{2^*-2}] > 2$, and S is the best Sobolev constant. Now, (1.1) and (3.3) imply that

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{j+1}}\right)\|\nabla u\|_2^2 - C\int_\Omega |u|^p - C \\ &\geq \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{j+1}}\right)\|\nabla u\|_2^2 - \frac{C}{\lambda_{j+1}^{\alpha}}\|\nabla u\|_2^p - C \end{aligned} \tag{3.4}$$

Taking ρ_j for which the right-hand side of (3.4) is maximized with $\|\nabla u\|_2 = \rho_j$, we obtain constants $A, B > 0$ which are independent of j and such that

$$I(u) \geq A\left(1 - \frac{\lambda}{\lambda_{j+1}}\right)^{\frac{p}{p-2}} \lambda_{j+1}^{\frac{\alpha}{p-2}} - B \equiv \gamma_j$$

Clearly $\gamma_j \xrightarrow{j \to \infty} \infty$, and so the claim holds.

Now, from the claim we see that we may choose j sufficiently large such that condition (a) of Theorem 3.2 holds with $X = X_j$, $\rho = \rho_j$, and $c_0 > 0 = I_\lambda(0)$.

Next we verify condition (b) of Theorem 3.2. Fix $k \geq 1$ and let $\varphi_1, \ldots, \varphi_k \in C_0^\infty(\Omega^+)$ be any fixed collection of functions with disjoint supports. Set

$$Y_k \equiv \operatorname{span}\{\varphi_1 \ldots, \varphi_k\}$$

As the supports are disjoint, $\dim Y_k = k$. Now, for $u \in Y_k$, $u = \sum_{i=1}^k t_i \varphi_i$, we calculate

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\sum_{i=1}^k t_i^2 \left(\|\nabla\varphi_i\|_2^2 - \|\varphi_i\|_2^2\right) - \sum_{i=1}^k \int_{\Omega+} W^+(x)F(t\varphi_i) \\ &\leq \frac{C}{2}\sum_{i=1}^k t_i^2 - C\left(\sum_{i=1}^k |t_i|^2\right)^{p/2} + C \\ &\leq \Gamma_k < \infty \end{aligned}$$

where we have used the lower bound

$$F(u) \geq a|u|^p - b$$

for constants $a, b > 0$, which is a consequence of (1.26). Thus condition (b) is satisfied.

Note that our hypotheses in Theorem 3.1 guarantee that (PS) is satisfied, via Lemma 1.5. Hence, Theorem 3.2 holds, and we obtain infinitely many pairs of critical points for (3.1_λ). ∎

We conclude this section with some remarks concerning the existence of solutions for problem (3.1_λ) when f is not necessarily odd.

In this situation one could try a linking argument (cf [R], [St]) to establish existence. That is, find a manifold whose boundary Γ links with the sublevel $\{I_\lambda > \epsilon > 0\}$ and for which $I\|r \leq 0$. While the choice of such a manifold is quite natural when $W \geq 0$ (where it is enough to take care of the positivity of the quadratic part,) in the case where W changes sign the situation is more delicate since both terms in the functional need to be insured of a proper sign. We sketch here one possible way of handling this situation, which is however very much inspired by the case of a fixed-sign W, and therefore requires the availability of "good" subspaces for both the quadratic and superquadratic parts of the functional.

To this purpose, let A* < A < A*+1 and suppose that there exists a linear subspace E C $Hl\{Si)$ with dimE = k such that

$$(3.5_\lambda) \qquad \begin{cases} \sup \|Vw\|l = \backslash x < A < \lambda_{k+1} \\ \inf_{w \in E} \frac{!*J_Q W(X)F(W)}{} = a > 0 \end{cases}$$

PROPOSITION 3.3. *Let* A £ $cr(Q°)$ U {A_{19} A_2,... } and assume *that for* A 6 (A*, A*+i) *there exists a subspace E, dimE = k, which satisfies (3.5\). Let f satisfy (1.25), the assumptions of Lemma 1.5, and*

$$(3.6) \qquad f\{u)u > 0 \quad \text{for } u \wedge 0.$$

Then (3.1\) admits at least one nontrivial solution.

REMARK: The most obvious choice of the space E would be E = span{ei,..., e*}.

PROOF: We will only deal with the case A > Ai, since for A < Ai the proof will follow by even easier arguments. We show that a linking argument (see [St]) applies for the functional I_x. Assume, without loss of generality, that 0 € ft and $W(0) > 0$. Let $(p$ € $Cg°(B_p(0))$ with $p > 0$ small enough such that $\overline{B_p(0)}$ C ft^+. For $e > 0$ construct the following mollifiers:

$$\varphi_\epsilon(x) = \frac{1}{\epsilon^{N/p}}\varphi(\frac{x}{\epsilon}).$$

Set

$$E_t = \{u = w + ttp_(: \quad w \in E,\ t \geq 0\}$$

where E is the linear subspace as given in (3.5A).

CLAIM: We can fix $\epsilon_0 > 0$ sufficiently small such that for R and T sufficiently large we have

$$I_\lambda|_{\partial Q} \le 0$$

where

$$Q = \{u = w + t\varphi_{\epsilon_0} \in E_{\epsilon_0} : \ \|\nabla w\|_2 \le R, \ 0 \le t \le T\}$$

To obtain the claim we evaluate:

$$\begin{aligned}
I_\lambda(w + t\varphi_\epsilon) &= \frac{1}{2}\left(\|\nabla w\|_2^2 - \lambda\|w\|_2^2\right) - \int_\Omega W(x)F(w) \\
&\quad + \frac{t^2}{2}\left(\|\nabla\varphi_\epsilon\|_2^2 - \lambda\|\varphi_\epsilon\|_2^2\right) - \int_\Omega W(x)F(t\varphi_\epsilon) \\
&\quad + t\int_\Omega(\nabla w\cdot\nabla\varphi_\epsilon - \lambda w\varphi_\epsilon) - \int_\Omega W^+(x)[F(w + t\varphi_\epsilon) - F(w) - F(t\varphi_\epsilon)] \\
&\le -\frac{1}{2}\left(\frac{\lambda}{\mu} - 1\right)\|\nabla w\|_2^2 - \alpha\int_\Omega F(w) \\
&\quad + \frac{t^2}{2}\left(\|\nabla\varphi_\epsilon\|_2^2 - \lambda\|\varphi_\epsilon\|_2^2\right) - c_1t^q\|\varphi_\epsilon\|_q^q - c_2t^p\|\varphi_\epsilon\|_p^p \\
&\quad + Ct\|\Delta w\|_2\|\varphi_\epsilon\|_1 + C\int_\Omega W^+(x)\left(|w|^{p-2} + (t\varphi_\epsilon)^{p-2} + 1\right)t\varphi_\epsilon|w| \\
&\le -\frac{1}{2}\left(\frac{\lambda}{\mu} - 1 - \delta\right)\|\nabla w\|_2^2 - \alpha(C_1\|w\|_q^q + C_2\|w\|_p^p) \\
&\quad + \frac{t^2}{2}\left(\|\nabla\varphi_\epsilon\|_2^2 - \lambda\|\varphi_\epsilon\|_2^2\right) - c_1t^q\|\varphi_\epsilon\|_q^q - c_2t^p\|\varphi_\epsilon\|_p^p \\
&\quad + t^2C_\delta\|\varphi_\epsilon\|_1 + C\left(\|w\|_p^{p-1}\|t\varphi_\epsilon\|_1 + \|t\varphi_\epsilon\|_p^{p-1}\|w\|_1\right)
\end{aligned} \tag{3.7}$$

where, to estimate the nonlinear terms, we have used (1.33) and

$$F(u) \ge C_1|u|^q + C_2|u|^p \tag{3.8}$$

(which is a consequence of (1.25), (1.26), and (3.6)).

On the other hand, by the definition of φ_ϵ we have

$$\|\varphi_\epsilon\|_p^p = c_0, \qquad \|\varphi_\epsilon\|_q^q = C\epsilon^{N(1-\frac{q}{p})},$$
$$\|\varphi_\epsilon\|_p^{p-1} = C\epsilon^{N/p}, \qquad \|\varphi_\epsilon\|_1 = C\epsilon^{N(1-\frac{1}{p})}.$$

Consequently, (3.7) can be estimated by

$$\begin{aligned}
I_\lambda(w + t\varphi_\epsilon) &\le -\frac{1}{2}\left(\frac{\lambda}{\mu} - 1 - \delta\right)\|\nabla w\|_2^2 - (C_1\alpha - \delta)\|w\|_p^p \\
&\quad + \frac{t^2}{2}\left(\|\nabla\varphi_\epsilon\|_2^2 - \lambda\|\varphi_\epsilon\|_2^2\right) - Ct^p\left[c_0 - \epsilon^{N(p-1)} - \epsilon^{N/(p-1)}\right] - Ct^q\epsilon^{N(1-\frac{q}{p})}
\end{aligned}$$

Now fix δ sufficiently small such that $\frac{\lambda}{\mu} - 1 - \delta > 0$ and $C_1\alpha - \delta > 0$. Subsequently, choose $\epsilon_0 > 0$ sufficiently small such that

$$c_0 - \epsilon^{N(p-1)} - \epsilon^{N/(p-1)} > 0$$

We conclude that

$$I_\lambda(w + t\varphi_\epsilon) \leq -A\|\nabla w\|_2^2 - B\|\nabla w\|_2^p + \frac{C}{2}t^2 - \frac{D}{p}t^p - \frac{E}{q}t^q$$

for suitable positive constants A, B, C, D, and E.

Now we check the conditions on ∂Q. If $t = 0$ then obviously $I_\lambda(w) \leq 0$. On the other hand, for all $t \geq 0$

$$\frac{C}{2}t^2 - \frac{D}{p}t^p - \frac{E}{q}t^q \leq \left(\frac{1}{2} - \frac{1}{p}\right)\frac{C^{p/(p-2)}}{D^{2/(p-2)}}$$

Therefore if $w \in Q$ and $\|\nabla w\|_2 = R$ then

$$I_\lambda(w + t\varphi_\epsilon) \leq -(A + BR^{p-2})R^2 + \left(\frac{1}{2} - \frac{1}{p}\right)\frac{C^{p/(p-2)}}{D^{2/(p-2)}} < 0$$

for R sufficiently large. Finally, for $T > (\frac{p}{2}\frac{C}{D})^{1/(p-2)}$ we have

$$I_\lambda(w + T\varphi_\epsilon) \leq -A\|\nabla w\|_2^2 - B\|\nabla w\|_2^p + \frac{C}{2}T^2 - \frac{D}{p}T^p - \frac{E}{q}T^q \leq 0$$

This concludes the proof of the Claim.

On the other hand, standard arguments show that for a suitable $\rho_0 > 0$,

$$I_\lambda(w) \geq c_0 > 0, \quad \text{for all } w \in X_k \cap S_{\rho_0}$$

with X_k as in (3.2). Moreover, it is not difficult to show that ∂Q and $(X_k \cap S_{\rho_0})$ link. In fact, let $v_1, \ldots, v_k$ be an orthonormal basis for E, and set $A = (a_{ij})_{i,j=1,\ldots,k}$ with $a_{ij} = \int_\Omega e_i v_j$. By (3.5$_\lambda$) and the variational characterization of the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$, we have that necessarily $\det A \neq 0$. Define also the vector

$$z = \left(\int_\Omega \varphi_{\epsilon_0} e_1, \ldots, \int_\Omega \varphi_{\epsilon_0} e_k\right)$$

Via a degree argument, we will be finished if we show that the system of equations

$$A \begin{bmatrix} t_1 \\ \vdots \\ t_k \end{bmatrix} + tz = 0, \qquad \|\nabla[\sum_{i=1}^{k} t_i v_i + t\varphi_{\epsilon_0}]\|_2 - \rho_0 = 0$$

admits a unique solution for $\sum_{i=1}^{k} t_i^2 < R^2$ and $t \in [0, T]$. But this is an easy consequence of the invertibility of A, and so the linking is proven.

Since the given assumptions ensure that (PS) holds, the conclusion of the Theorem follows fromTheorem 8.4 in [St]. ∎

4. The critical case

In this section we would like to extend the results of Theorem 2.7 to include nonlinearities with critical growth. In fact, to simplify the technical details, we shall assume that $f(u) = |u|^{2^*-2}u$, $N \geq 3$. More precisely, we investigate solutions to the problem:

$$(4.1_\lambda) \qquad \begin{cases} -\Delta u - \lambda u = W(x)u^{2^*-1} & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \\ u > 0 \quad \text{in } \Omega \end{cases}$$

Set

$$(4.2) \qquad B_N = \int_{\mathbf{R}^N} \frac{(N(N-2))^{\frac{N}{N-2}}}{(1+|x|^2)^{\frac{N}{N-2}}},$$

and for $W \in C^2(\Omega)$ and $y \in \Omega$ with $W(y) \neq 0$, let

$$(4.3) \qquad A_N(y) = \frac{(N(N-2))^{\frac{N}{2}}}{2W(y)} \int_{\mathbf{R}^N} \frac{(-D^2W(y)x, x)}{(1+|x|^2)^N}$$

We have:

THEOREM 4.1. *There exist a constant $\Lambda > \lambda_1$ such that (4.1_λ) admits a solution for all $\lambda \in (\lambda_1, \Lambda]$ if and only if*

$$(4.4) \qquad \int_\Omega W(x)e_1^{2^*} < 0$$

In addition, if $W \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

$$\max_{\Omega} W(x) = W(x_0) > 0 \quad \text{for some } x_0 \in \Omega \tag{4.5}$$

then problem (4.1_λ) admits:

(i) at least two solutions if $\lambda \in (\lambda_1, \Lambda)$ and if, for $N \geq 6$, the inequality

$$\lambda B_N > \frac{2}{2^*} A_N(x_0) \tag{4.6}$$

holds with A_N, B_N defined as in (4.3), (4.2)

(ii) at least one solution if $\lambda = \lambda_1$ and if, for $N \geq 5$, (4.6) holds with $\lambda = \lambda_1$.

REMARK 4.2: It will follow from our arguments that the case $\lambda \in (0, \lambda_1)$ can be treated in a similar fashion without assuming (4.4). So for $\lambda \in (0, \lambda_1)$ the existence of a positive solution can be established provided that, for $N \geq 5$, the inequality (4.6) holds. Also notice that when $N = 3$ the argument becomes more delicate and one has to follow Lemma 5 of [Br] to show that there exists a $0 < \lambda^* < \lambda_1$ such that for $\lambda \in (\lambda^*, \lambda_1)$ there exists a positive solution to (4.1_λ). We leave the (lengthy) details to the reader.

REMARK 4.3: Condition (4.6) should be compared with the degeneracy conditions (2.1) and (3.2) of [E-S].

PROOF:

The necessity of the first part of our result just follows from the definition of Λ,

$$\Lambda = \sup\{\lambda \geq \lambda_1 : \ (4.1_\lambda) \text{ admits a solution}\} > \lambda_1 \tag{4.7}$$

and Lemma 1.3. For the sufficiency, we know from section 2 that for all $\lambda_1 < \lambda < \Lambda$ there exists a minimal solution u_λ to (4.1_λ). Combining the variational formulation of Perron's method and the fact that the minimal solution u_λ is nondegenerate (see [Ou]), we derive, for given $\lambda_- < \lambda < \lambda_+$, that

$$I_\lambda(u_\lambda) = \min_{u_- \leq u \leq u_+} I_\lambda(u) \leq 0$$

with $u_- = u_{\lambda_-} < u_+ = u_{\lambda_+}$. Choosing a sequence $\lambda_n \nearrow \Lambda$ denote the corresponding monotone sequence of minimal solutions by $u_n = u_{\lambda_n}$. We have

$$\begin{cases} I_\Lambda(u_n) = I_{\lambda_n}(u_n) - (\Lambda - \lambda_n)\|u_n\|_2^2 \leq 0 \\ I'_\Lambda(u_n) = I'_{\lambda_n}(u_n) - (\Lambda - \lambda_n)u_n = -(\Lambda - \lambda_n)u_n \end{cases} \tag{4.8}$$

As in the proof of Lemma 1.5 (see also the analysis shown below for Claim 2), (4.8) is sufficient to conclude that $||Vu_n||_2 \leq C$ and in turn $||u_n||oo \leq C$ Hence UA = $\sup u_n > 0$ gives the desired solution for (4.1A)-

We now move to the second part of Theorem 4.1. Without loss of generality, we assume that $x_0 = 0$ € n and $W(0) = 1$.

(i) This result was already claimed in [Ou], but without recourse to condition (4.6) for $N \gtrsim 6$. We believe however that without this condition his argument does not guarantee that the second solution found therein (as a weak limit of approximate solutions) is different from the first solution.

Nevertheless, from [Ou], we know that the minimal solutions $u\backslash$ for (4.1A) (constructed in section 2) are, in fact, strict local minima for the functional

$$I_\lambda(u) = \frac{1}{2}\left(\int_\Omega |\nabla u|^2 - \lambda u^2\right) - \frac{1}{2^*}\int_\Omega W(x)u^{2^*}$$

More precisely, for each A 6 (Ai, A) there exists a $6(X) > 0$ such that

(4-9) $$(I''_\lambda(u_\lambda)\varphi, \varphi) \geq \delta(\lambda)||\varphi|\&_{\dot{0}}^{1}(\Omega)$$

for all $<p$ £ $H^1_0(Q)$. Ouyang derived this information with a very nice argument which is, however, limited to homogeneous nonlinearities.

As for Theorem 2.7, we seek a second solution of the form

$$w = u_\lambda + \varphi$$

with $ip \geq 0$. This amounts to finding a (positive) critical point for the functional

$$M<P) = \backslash(\backslash\backslash V<P\backslash\backslash 1 - A||\wedge_+||^2_2) - \wedge J\wedge W(x)[(u_x + V+f - uX - 2V;^{-1}\varphi_+]$$

which, in view of condition (4.9), admits $<\!\!? = 0$ as a strict local minimum. That is,

(4.10) $$J_x\{<f) \geq co > 0(= J_A(0))$$

for $|M!n^1_0(n) = {}^r>$ some $r > 0$ sufficiently small.

To obtain our candidate for a critical value, we use the mountain-pass procedure, and define

$$c = \inf_{\gamma \in \mathcal{P}} \max_{0 \le t \le 1} J_\lambda(\gamma(t)) \ge c_0 \tag{4.11}$$

$$\mathcal{P} = \{\gamma \in C([0,1]; H_0^1(\Omega)) : \ \gamma(0) = 0, \ J_\lambda(\gamma(1)) < 0\} \tag{4.12}$$

($\mathcal{P}$ is non-empty by the structure of J_λ.) As is well-known for variational problems involving critical exponents, concentration phenomena can occur and violate the (PS) condition for certain values c. Therefore, to guarantee that this is not the case for the value constructed in (4.11), we will continue as in [B-N] and estimate c from above.

CLAIM 1:

$$c < \frac{1}{N} S^{N/2} \tag{4.13}$$

with S the best constant in the Sobolev embedding.

Define

$$U_\epsilon(x) = \frac{c_N \epsilon^{\frac{N-2}{2}}}{(\epsilon^2 + |x|^2)^{\frac{N-2}{2}}} \tag{4.14}$$

where $c_N = (N(N-2))^{\frac{N-2}{4}}$, so that U_ϵ satisfies

$$-\Delta U_\epsilon = U_\epsilon^{2^*-1} \quad \text{in } \mathbf{R}^N.$$

Fix $\rho > 0$ so that $B_\rho(0) \subset \Omega^+$, and pick a function $\eta \in C_0^\infty(B_\rho(0))$ such that $0 \le \eta(x) \le 1$ and $\eta(x) = 1$ for all $x \in B_{\rho/2}(0)$. Then set

$$v_\epsilon = \eta \cdot U_\epsilon \tag{4.15}$$

As usual, to establish (4.13) we will estimate $\max_{t \ge 0} J_\lambda(t v_\epsilon)$.

First, we introduce the familiar estimates (see [B-N], [St]):

$$\|\nabla v_\epsilon\|_2^2 = S^{N/2} + O(\epsilon^{N-2}), \qquad \|v_\epsilon\|_{2^*}^{2^*} = S^{N/2} + O(\epsilon^N)$$
$$\|v_\epsilon\|_2^2 = \begin{cases} B_N \epsilon^2 + O(\epsilon^{N-2}), & \text{if } N \ge 5 \\ O(\epsilon^2 |\log \epsilon|), & \text{if } N = 4 \\ C\epsilon + O(\epsilon^2), & \text{if } N = 3 \end{cases} \tag{4.16}$$

where B_N is as in (4.2). Then, it is easily verified that for $\epsilon_0 > 0$ chosen sufficiently small there exists a (large) $R > 0$ such that

$$J_\lambda(Rv_\epsilon) < 0 \tag{4.17}$$

holds for all $0 < \epsilon < \epsilon_0$. In other words, the path $\gamma(t) = tRv_\epsilon$, $t \in [0,1]$ belongs to $\mathcal{P}$, and hence

$$c \leq \max_{0 \leq t \leq R} J_\lambda(tv_\epsilon) \tag{4.18}$$

To take advantage of the "better" error terms involved when estimating (4.18), we distinguish a few cases according to the dimension N.

CASE $N \geq 7$: Let $0 \leq t \leq R$. Then

$$\begin{aligned} J_\lambda(tv_\epsilon) &= \frac{t^2}{2}\left(\|\nabla v_\epsilon\|_2^2 - \lambda\|v_\epsilon\|_2^2\right) - \frac{t^{2^*}}{2^*}\int_\Omega W(x)v_\epsilon^{2^*} \\ &\quad + t\int_\Omega W(x)u_\lambda^{2^*-1}v_\epsilon - \frac{1}{2^*}\int_\Omega W(x)[(u_\lambda + tv_\epsilon)^{2^*} - u_\lambda^{2^*} - (tv_\epsilon)^{2^*}] \\ &\leq \frac{t^2}{2}\left(\|\nabla v_\epsilon\|_2^2 - \lambda\|v_\epsilon\|_2^2\right) - \frac{t^{2^*}}{2^*}\int_\Omega W(x)v_\epsilon^{2^*} + CR\|v_\epsilon\|_1 + CR^{2^*-1}\|v_\epsilon\|_{2^*-1}^{2^*-1} \\ &\leq \frac{1}{N}\left[\frac{\|\nabla v_\epsilon\|_2^2 - \lambda\|v_\epsilon\|_2^2}{\left(\int_\Omega W(x)v_\epsilon^{2^*}\right)^{2/2^*}}\right]^{\frac{N}{2}} + O(\epsilon^{\frac{N-2}{2}}) \end{aligned} \tag{4.19}$$

where we have used the estimates

$$\|v_\epsilon\|_1 = O(\epsilon^{\frac{N-2}{2}}) = \|v_\epsilon\|_{2^*-1}^{2^*-1} \tag{4.20}$$

On the other hand, using our hypotheses on W, we have

$$\begin{aligned} \int_\Omega W(x)v_\epsilon^{2^*} &= \int_{B_\rho(0)} W(x)U_\epsilon^{2^*} + O(\epsilon^N) \\ &= \int_{B_{\rho/\epsilon}(0)} W(\epsilon x)\frac{c_N^{2^*}}{(1+|x|^2)^N} + O(\epsilon^N) \\ &= W(0)S^{N/2} + \int_{B_{\rho/\epsilon}(0)} (1 - W(\epsilon x))\frac{c_N^{2^*}}{(1+|x|^2)^N} + O(\epsilon^N) \\ &= W(0)S^{N/2} + \epsilon^2\frac{c_N^{2^*}}{2}\int_{\mathbf{R}^N}\frac{(-D^2W(0)x, x)}{(1+|x|^2)^N} + o(\epsilon^2) \\ &= S^{N/2} + \epsilon^2 A_N(x_0 = 0) + o(\epsilon^2) \end{aligned} \tag{4.21}$$

In the sequel we set $A_N = A_N(x_0 = 0)$. Resuming from (4.19), after applying the estimates (4.16) and (4.21) we see that

$$\begin{aligned}\max_{0\le t\le R} J_\lambda(tv_\epsilon) &\le \frac{1}{N}\left[\frac{S^{N/2}-\lambda B_N\epsilon^2+o(\epsilon^2)}{\left(S^{N/2}-A_N\epsilon^2+o(\epsilon^2)\right)^{2/2^*}}\right]^{\frac{N}{2}} + O(\epsilon^{\frac{N-2}{2}})\\ &= \frac{1}{N}S^{N/2}[1-\frac{N}{2S^{N/2}}(\lambda B_N - \frac{2}{2^*}A_N)\epsilon^2 + o(\epsilon^2)] + O(\epsilon^{\frac{N-2}{2}})\\ &< \frac{1}{N}S^{N/2} \qquad (4.22)\end{aligned}$$

for $\epsilon > 0$ sufficiently small, using (4.6). The claim is completed in the case $N \geq 7$.

CASE $N = 6$: When $N = 6$ then $2^* = 3$ and one can estimate explicitly the above expression to derive

$$J_\lambda(tv_\epsilon) = \frac{t^2}{2}\left(\|\nabla v_\epsilon\|_2^2 - \lambda\|v_\epsilon\|_2^2\right) - \frac{t^3}{3}\int_\Omega W(x)v_\epsilon^3 - t^2\int_\Omega W(x)u_\lambda v_\epsilon^2$$

Since $\int_\Omega W(x)u_\lambda v_\epsilon^2 > 0$ we conclude as in (4.22),

$$\begin{aligned}\max_{0\le t\le R} J_\lambda(tv_\epsilon) &\le \frac{1}{6}\left[\frac{\|\nabla v_\epsilon\|_2^2-\lambda\|v_\epsilon\|_2^2}{\left(\int_\Omega W(x)v_\epsilon^3\right)^{2/3}}\right]^{\frac{3}{2}}\\ &= \frac{1}{6}S^3[1-\frac{3}{S^3}(B_N\lambda - \frac{2}{3}A_N)\epsilon^2 + o(\epsilon^2)]\\ &< \frac{1}{6}S^3\end{aligned}$$

for ϵ sufficiently small, under our hypothesis (4.6). Note also that in this case we could allow equality in (4.6) because we control the sign of the error term, which is also $O(\epsilon^2)$.

CASE $N = 3, 4, 5$: Here we must be more careful with the error terms of order $\epsilon^{\frac{N-2}{2}}$, as they represent the first order correction in these dimensions. As above, let $0 \leq t \leq R$. We have

$$\begin{aligned}J_\lambda(tv_\epsilon) &= \frac{t^2}{2}\left(\|\nabla v_\epsilon\|_2^2 - \lambda\|v_\epsilon\|_2^2\right) - \frac{t^{2^*}}{2^*}\int_\Omega W(x)v_\epsilon^{2^*} - t^{2^*-1}\int_\Omega W(x)u_\lambda v_\epsilon^{2^*-1}\\ &\quad - \frac{1}{2^*}\int_\Omega W(x)\{(u_\lambda + tv_\epsilon)^{2^*} - u_\lambda^{2^*} - (tv_\epsilon)^{2^*} - 2^* t u_\lambda v_\epsilon[u_\lambda^{2^*-2} + (tv_\epsilon)^{2^*-2}]\}\\ &\le \frac{t^2}{2}\left(\|\nabla v_\epsilon\|_2^2 - \lambda\|v_\epsilon\|_2^2\right) - \frac{t^{2^*}}{2^*}\int_\Omega W(x)v_\epsilon^{2^*} - C_0u_\lambda(0)t^{2^*-1}\epsilon^{\frac{N-2}{2}} + o(\epsilon^{\frac{N-2}{2}})\\ &= \frac{t^2}{2}S^{N/2} - \frac{t^{2^*}}{2^*}S^{N/2} - C_0u_\lambda(0)t^{2^*-1}\epsilon^{\frac{N-2}{2}} + o(\epsilon^{\frac{N-2}{2}}) \qquad (4.23)\end{aligned}$$

with $C_0 > 0$ a suitable constant. Now, if we call t_ϵ the value of t which attains the maximum of the right-hand side of (4.23) over $[0, R]$, then

$$0 \leq t_\epsilon - 1 = O(\epsilon^{\frac{N-2}{2}})$$

and hence

$$\begin{aligned}
\max_{0\leq t\leq R} J_\lambda(tv_\epsilon) &\leq \frac{t_\epsilon^2}{2} S^{N/2} - \frac{t_\epsilon^{2^*}}{2^*} S^{N/2} - C_0 u_\lambda(0) t_\epsilon^{2^*-1} \epsilon^{\frac{N-2}{2}} + o(\epsilon^{\frac{N-2}{2}}) \\
&\leq \frac{1}{N} S^{N/2} - C_0 u_\lambda(0) t_\epsilon^{2^*-1} \epsilon^{\frac{N-2}{2}} \\
&< \frac{1}{N} S^{N/2}
\end{aligned}$$

for ϵ sufficiently small. This completes Claim 1 in all dimensions.

We are now ready to conclude the proof of part (i). The mountain-pass principle, (4.11), yields a sequence $w_n = u_\lambda + \varphi_n \in H_0^1(\Omega)$, with $\varphi \geq 0$ and satisfying:

$$I_\lambda(w_n) \to I_\lambda(u_\lambda) + c < I_\lambda(u_\lambda) + \frac{1}{N} S^{N/2} \tag{4.24}$$

$$\|I'_\lambda(w_n)\| \to 0 \tag{4.25}$$

Claim 2: $\|\nabla w_n\|_2 \leq C$ for suitable $C > 0$.
We argue by contradiction, and assume instead that $\|\nabla w_n\|_2 \to \infty$. We begin by observing that this implies

$$\frac{\|w_n\|_2}{\|\nabla w_n\|_2} \geq C > 0$$

In fact, combining (4.24) and (4.25) we have

$$\left(\frac{1}{2} - \frac{1}{2^*}\right)(\|\nabla w_n\|_2^2 - \lambda \|w_n\|_2^2) = (I'_\lambda(w_n), w_n) + O(1)$$

which yields

$$\frac{\|w_n\|_2^2}{\|\nabla w_n\|_2^2} \to \frac{1}{\lambda} > 0$$

Now, setting $v_n = \dfrac{w_n}{\|\nabla w_n\|_2}$, we have (along some subsequence) $v_n \rightharpoonup v_0 \neq 0$ in $H_0^1(\Omega)$. Then, (4.25) yields

$$\begin{aligned}
\int_\Omega \nabla v_n \cdot \nabla \varphi - \lambda v_n \varphi &= \int_\Omega W(x) \frac{w_n^{2^*-1}}{\|\nabla w_n\|_2} \varphi \\
&= \|\nabla w_n\|_2^{2^*-2} \int_\Omega W(x) v_n^{2^*-1} \varphi
\end{aligned}$$

for all *if* € *HQ(Q).* Necessarily,

t Jn* *W(x)v%-*[l]*ip->0 **for all $\varphi \in H_0^1(\Omega)$**

that is,

***I'* *W(x)vf~*[l]*<p = 0* for all *<p* € J5Tj(fl)**

which implies that $v_0 = 0$ in $Q\backslash Q°$. In other words, t;0 € iTj(fl[0]), t;0 ^ 0, and satisfies

***—AVQ* — *\vo* = 0**

This is clearly impossible, as A < A < Ai(ft+ U *Q°)* < Ai(ft°), and Claim 2 follows.

Passing to a subsequence, w_n —> *WQ* with *WQ* € -ffg(fi) a solution to (4.1A). Note that *wo* ≥*u\in Q.* We now claim that *wo* is the desired second solution.

CLAIM 3: $w_Q > u\backslash$.

It suffices to show that *wo* ^ *u*. Let us argue by contradiction and assume that *WQ* = *u*. As w_n = *u* + *(pn,* this would imply that ip_n —* 0 in iTg(^) anc^ ^n turn:

$$\frac{1}{2}\|\nabla\varphi_n\|_2^2 - \frac{1}{2^*}\int_\Omega W(x)|\varphi_n|^{2^*} \to c \in (0, \wedge S^{N/2})$$

$$\|\nabla\varphi_n\|_2^2 - \int_\Omega W(x)|\varphi_n|^{2^*} = o(1)$$

That is,

(4.26) -L||v¥,2||->c€(O,1 $S^{N/2}$) N N

and

(4.27) $$\|\nabla\varphi_n\|_2^2 \le \|\varphi_n\|_{2^*}^{2^*} + o(1) \le \left(\frac{1}{S}\|\nabla\varphi_n\|_2^2\right)^{\frac{2^*}{2}} + o(1)$$

From (4.26) it follows that ||V^n||2 is bounded away from zero, and hence (4.27) yields

$$\|\nabla\varphi_n\|_2^2 \ge S^{N/2} + o(1)$$

which clearly contradicts (4.26). This completes the proof of (i).

(ii) Now $\lambda = \lambda_1$, and $u_{\lambda_1} = 0$, so $J_{\lambda_1}(v) = I_{\lambda_1}(v)$. By Lemma 1.4, we have that $u = 0$ is a strict local minimum for J_{λ_1}, and we can proceed as in (4.11), (4.12) to obtain a candidate for a critical value c via the mountain-pass lemma. To obtain the estimate (4.13) in this case, we proceed as above. If $N \geq 5$ and $A_n = A_n(x_0 = 0)$, corresponding calculations yield,

$$\begin{aligned}
J_{\lambda_1}(tv_\epsilon) &= \frac{t^2}{2}\left(\|\nabla v_\epsilon\|_2^2 - \lambda_1\|v_\epsilon\|_2^2\right) - \frac{t^{2^*}}{2^*}\int_\Omega W(x)v_\epsilon^{2^*} \\
&\leq \frac{1}{N}\left[\frac{\|\nabla v_\epsilon\|_2^2 - \lambda_1\|v_\epsilon\|_2^2}{\left(\int_\Omega W(x)v_\epsilon^{2^*}\right)^{2/2^*}}\right]^{\frac{N}{2}} \\
&= \frac{1}{N}\left[\frac{S^{N/2} - \lambda_1 B_N\epsilon^2 + o(\epsilon^2)}{\left(S^{N/2} - A_N\epsilon^2 + o(\epsilon^2)\right)^{2/2^*}}\right]^{\frac{N}{2}} \\
&= \frac{1}{N}S^{N/2}[1 - \frac{N}{2S^{N/2}}(\lambda_1 B_N - \frac{2}{2^*}A_N)\epsilon^2 + o(\epsilon^2)] \\
&< \frac{1}{N}S^{N/2}
\end{aligned}$$

provided (4.6) holds at $\lambda = \lambda_1$ and $\epsilon > 0$ is sufficiently small.

When $N = 4$, the estimate (4.13) holds without condition (4.6), as

$$\begin{aligned}
J_{\lambda_1}(tv_\epsilon) &\leq \frac{1}{N}\left[\frac{\|\nabla v_\epsilon\|_2^2 - \lambda_1\|v_\epsilon\|_2^2}{\left(\int_\Omega W(x)v_\epsilon^{2^*}\right)^{2/2^*}}\right]^{\frac{N}{2}} \\
&\leq \frac{1}{N}\left[\frac{S^{N/2} - \lambda_1 C\epsilon^2|\log\epsilon| + o(\epsilon^2)}{\left(S^{N/2} - A_N\epsilon^2 + o(\epsilon^2)\right)^{2/2^*}}\right]^{\frac{N}{2}} \\
&= \frac{1}{N}S^{N/2}[1 - C'\epsilon^2|\log\epsilon| + O(\epsilon^2)] \\
&< \frac{1}{N}S^{N/2}
\end{aligned}$$

for all $\epsilon > 0$ sufficiently small, where $C, C' > 0$ are constants.

The case $N = 3$ is a simple consequence of a result of Brezis-Nirenberg [B-N] where these type of estimates were first introduced to handle problems with critical exponents. In this case there is a competition between the "good" error term given by $\lambda_1\|v_\epsilon\|_2^2$ and the "bad" error term as given by the expansion of $\|\nabla v_\epsilon\|_2^2$. Thus, as in Lemma 5 of [Br] one shows that there exists $\lambda^* \in (0, \lambda_1)$ such that

$$\max_{t\geq 0} I_\lambda(tv_\epsilon) < \frac{1}{N}S^{N/2}$$

for $\lambda \in (\lambda^*, +\infty)$, and hence (4.13) follows in our case where $\lambda = \lambda_1 \in (\lambda^*, +\infty)$.

Now, the convergence part follows exactly as in (i) to provide the solution for $\lambda = \lambda_1$. ∎

REMARK: Following the arguments of section 3, we can also derive existence for (4.1_λ) when λ is large provided that the corresponding assumptions (3.5_λ) hold. Namely, we assume that for $\lambda \notin \sigma(\Omega^0)$ and $\lambda_k < \lambda < \lambda_{k+1}$,

(i) there exists a linear subspace $E \subset H_0^1(\Omega)$ with

$$\dim E = k, \qquad \sup_{\substack{w \in E \\ \|w\|_2^2 = 1}} \|\nabla w\|_2^2 < \lambda < \lambda_{k+1}$$

(ii)

$$\inf_{\substack{w \neq 0 \\ w \in E}} \frac{\int_\Omega W(x)|w|^{2^*}}{\int_\Omega |w|^{2^*}} \equiv \alpha > 0$$

Then, exactly as for Proposition 3.3, a linking argument can be utilized by taking $\varphi_\epsilon(x) = v_\epsilon(x)$ (as defined in (4.15)) to yield a candidate critical value below the threshold $\frac{1}{N} S^{N/2}$, provided condition (4.6) is satisfied for $N \geq 5$.

5. APPENDIX

In this section, we return to some of the technical lemmata from section 1. Our first concern is the Palais-Smale condition (Lemma 1.5)

PROOF OF LEMMA 1.5: Consider a sequence $\{u_n\} \in H_0^1(\Omega)$ for which

$$I_\lambda(u_n) = \frac{1}{2}(\|\nabla u_n\|_2^2 - \lambda \|u_n\|_2^2) - \int_\Omega W(x)F(u_n) \leq c \tag{5.1}$$

$$I'_\lambda(u_n)\varphi = \int_\Omega \nabla u_n \cdot \nabla \varphi - \lambda u_n \varphi - W(x)f(u_n)\varphi = \int_\Omega \nabla z_n \cdot \nabla \varphi \tag{5.2}$$

where $\|\nabla z_n\|_2 \to 0$ as $n \to \infty$.

Note that by the (subcritical) growth condition (1.1), it suffices to show that the sequence u_n is bounded in $H_0^1(\Omega)$. We suppose the contrary, that is $\|\nabla u_n\|_2 \to \infty$. Let $v_n = \dfrac{u_n}{\|\nabla u_n\|_2}$. Then $v_n \rightharpoonup v_0$ in $H_0^1(\Omega)$. The crucial step in the proof is the following:

CLAIM: $v_0 \not\equiv 0$ in Ω (ie, $\frac{\|u_n\|_2}{\|\nabla u_n\|_2} \not\to 0$.)
Assuming the claim for the moment, we then divide (5.2) by $\|\nabla u_n\|_2$ and pass to the limit to obtain

$$\int_\Omega \nabla v_0 \cdot \nabla\varphi - \lambda v_0\varphi = \lim_{n\to\infty}\left(\|\nabla u_n\|_2^{p-2}\int_\Omega W(x)\frac{f(u_n)}{\|\nabla u_n\|_2^{p-1}}\varphi\right) \tag{5.3}$$

for all $\varphi \in H_0^1(\Omega)$. Now, using the condition at infinity (1.26) and dominated convergence, we see

(5.4)

$$\int_\Omega W(x)\frac{f(u_n)}{t_n^{p-1}}\varphi = \int_\Omega W(x)|v_0|^{p-2}v_0\varphi + \int_\Omega W(x)\left[\frac{f(\|\nabla u_n\|_2 v_n) - f(\|\nabla u_n\|_2 v_0)}{\|\nabla u_n\|_2^{p-1}}\right]\varphi + o(1)$$

Now, using (1.33), and denoting $t_n = \|\nabla u_n\|_2$, we have

$$\begin{aligned}\left|\int_\Omega W(x)\left[\frac{f(t_n v_n) - f(t_n v_0)}{t_n^{p-1}}\right]\varphi\right| &\le C\int_\Omega \frac{|f'(t_n[\theta v_n + (1-\theta)v_0])|\,|v_n - v_0|}{t_n^{p-1}}\varphi \\ &\le C\int_\Omega [|\theta v_n + (1-\theta)v_0|^{p-2} + 1]|v_n - v_0|\,|\varphi| \\ &\le C[\|\theta v_n + (1-\theta)v_0\|_p^{p-2} + C]\|v_n - v_0\|_p\|\varphi\|_p = o(1) \end{aligned} \tag{5.5}$$

Applying (5.5), (5.4), and (5.3), we have

$$\int_\Omega W(x)|v_0|^{p-2}v_0\varphi = 0$$

for all $\varphi \in H_0^1(\Omega)$. Hence $v_0 \in H_0^1(\Omega^0)$, and from (5.3) we have

$$\int_\Omega \nabla v_0 \cdot \nabla\varphi - \lambda v_0\varphi = 0 \tag{5.6}$$

for all $\varphi \in H_0^1(\Omega^0)$. As we are assuming that the claim holds, this is impossible unless $\lambda \in \sigma(\Omega^0)$. However, this contradicts our assumption, and so we see that in fact $\|\nabla u_n\|$ must be bounded, and the Lemma follows.

It remains only to prove the Claim. In fact, we shall show that it is a consequence of either of the conditions (a) or (b). We treat the two cases separately.

So suppose first that (a) holds; ie, f satisfies (1.26) with $2 < p < 2^*$ and

$$\overline{\Omega^+} \cap \overline{\Omega^-} = \emptyset \tag{5.7}$$

To prove the claim we argue again by contradiction and suppose that $v_0 = 0$. In that case, (5.2) yields

$$\lim \int W(x)\frac{f(u_n)u_n}{\|\nabla u_n\|_2^2} = 1 \tag{5.8}$$

Now, in view of (5.7) there exists $\psi \in C_0^\infty(\mathbb{R}^N)$ with $0 \le \psi(x) \le 1$, $\psi(x) = 1$ for all $x \in \Omega^-$ and $\psi(x) = 0$ for all $x \in \Omega^+$. Let $\varphi = \psi v_n \in H_0^1(\Omega)$ and apply (5.2) again to obtain

$$\int_\Omega |\nabla v_n|^2\psi + \int W^-(x)\frac{f(u_n)u_n}{\|\nabla u_n\|_2^2} = o(1) \tag{5.9}$$

In particular

$$\lim \int_\Omega W^-(x)\frac{f(u_n)u_n}{\|\nabla u_n\|_2^2} = 0 \tag{5.10}$$

On the other hand, combining (5.1) and (5.2), we have

$$\liminf_{n\to\infty} \int W(x)\left[\frac{F(u_n) - \frac{1}{2}f(u_n)u_n}{\|\nabla u_n\|_2^2}\right] \ge 0 \tag{5.11}$$

Fixing $0 < \epsilon < \frac{1}{2}\min(\frac{1}{p}, \frac{1}{2} - \frac{1}{p})$, apply (1.26) to find a constant C such that

$$\left(\frac{1}{p} - \epsilon\right) f(u)u - C \le F(u) \le \left(\frac{1}{p} + \epsilon\right) f(u)u + C \tag{5.12}$$

for all $u \in \mathbb{R}$. Now applying this estimate to (5.11), we obtain

$$\begin{aligned}\left(\frac{1}{p} - \frac{1}{2} + \epsilon\right)\int_\Omega W^+\frac{f(u_n)u_n}{\|\nabla u_n\|_2^2} &\ge \int_\Omega W^+\left[\frac{F(u_n) - \frac{1}{2}f(u_n)u_n}{\|\nabla u_n\|_2^2}\right] + o(1)\\ &\ge \int_\Omega W^-\left[\frac{F(u_n) - \frac{1}{2}f(u_n)u_n}{\|\nabla u_n\|_2^2}\right] + o(1)\\ &\ge \left(\frac{1}{p} - \frac{1}{2} - \epsilon\right)\int_\Omega W^-\frac{f(u_n)u_n}{\|\nabla u_n\|_2^2} + o(1)\\ &= o(1)\end{aligned} \tag{5.13}$$

where we have used (5.10) in the last step. By the choice of ϵ, we conclude that

$$\lim_{n\to\infty}\int_\Omega W^+\frac{f(u_n)u_n}{\|\nabla u_n\|_2^2} = 0 \tag{5.14}$$

By putting together (5.8), (5.9), and (5.14), we obtain the contradiction; hence the Claim must hold in case (a).

In case (b), we again combine (5.1) with (5.2) to obtain

$$\int_\Omega W(x)\left[\frac{1}{2}f(u_n)u_n - F(u_n)\right] \le c - \frac{1}{2}\int_\Omega \nabla z_n \cdot \nabla u_n \tag{5.15}$$

Using condition (b) and (5.15) we obtain

$$\left(\frac{1}{2} - \frac{1}{p}\right)\int_\Omega W(x)f(u_n)u_n \le c + C\|u_n\|_2^2 - \frac{1}{2}\int_\Omega \nabla z_n \cdot \nabla u_n \tag{5.16}$$

We again assume for a contradiction that $v_n \to 0$. Dividing (5.16) by $\|\nabla u_n\|_2^2$ and passing to the limit, we obtain

$$\lim_{n\to\infty}\int_\Omega W(x)\frac{f(u_n)u_n}{\|\nabla u_n\|_2^2} = 0 \tag{5.17}$$

However, if we use (5.2) with $\varphi = u_n$, we obtain a contradiction since

$$\begin{aligned} 1 &= \lambda\frac{\|u_n\|_2^2}{\|\nabla u_n\|_2^2} + \int_\Omega W(x)\frac{f(u_n)u_n}{\|\nabla u_n\|_2^2} + o(1) \\ &= o(1) \end{aligned} \tag{5.18}$$

Therefore, the claim must hold under the condition (b) as well. ∎

PROOF OF LEMMA 1.3: Suppose u solves

$$\begin{cases} -\Delta u - \lambda u = W(x)u^{p-1} & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \\ u > 0 \quad \text{in } \Omega \end{cases} \tag{5.19$_\lambda$}$$

with $\lambda \ge \lambda_1$ and $p > 2$. Replacing e_1 by te_1 if necessary ($t > 0$) we can always assume that $e_1 \le u$ in $\overline{\Omega}$. Then

$$\begin{aligned} \int_\Omega W(x)e_1^p &= -\int_\Omega \Delta u\frac{e_1^p}{u^{p-1}} - \lambda_1\int_\Omega \frac{e_1^p}{u^{p-2}} - (\lambda-\lambda_1)\int_\Omega \frac{e_1^p}{u^{p-2}} \\ &= \frac{p(p-1)}{(p-2)}\int_\Omega |\nabla e_1|^2\frac{e_1^{p-2}}{u^{p-2}} - 2\lambda_1\left(\frac{p-1}{p-2}\right)\int_\Omega \frac{e_1^p}{u^{p-2}} \\ &\quad - (p-1)\int_\Omega |\nabla u|^2\frac{e_1^p}{u^p} - (\lambda-\lambda_1)\int_\Omega \frac{e_1^p}{u^{p-2}} \end{aligned} \tag{5.20}$$

where we have integrated by parts twice to reduce the term with Δu.

Now, we calculate $\Delta(\log e_1)$ and $\Delta(\log u)$ to produce the following two identities:

$$|\nabla e_1|^2 = -e_1^2\Delta(\log e_1) - \lambda_1 e_1^2$$
$$|\nabla u|^2 = -Wu^{p-1}u - u^2\Delta(\log u) - \lambda u^2$$

After substituting into (5.20), we obtain:

$$\begin{aligned}
-(p-2)\int_\Omega W(x)e_1^p &= -\frac{p(p-1)}{(p-2)}\int_\Omega \frac{e_1^p}{u^{p-2}}\Delta(\log e_1) \\
&\quad + (p-1)\int_\Omega \frac{e_1^p}{u^{p-2}}\Delta(\log u) \\
&\quad - 4\lambda_1\frac{(p-1)}{(p-2)}\int_\Omega \frac{e_1^p}{u^{p-2}} + (\lambda-\lambda_1)(p-2)\int_\Omega \frac{e_1^p}{u^{p-2}} \\
&\geq \frac{(p-1)}{(p-2)}\left[-\int_\Omega \frac{e_1^p}{u^{p-2}}\Delta\left(\log\frac{e_1^p}{u^{p-2}}\right) - 4\lambda_1\int_\Omega \frac{e_1^p}{u^{p-2}}\right] \\
&= 4\left(\frac{p-1}{p-2}\right)\left[\int_\Omega \left|\nabla\sqrt{\frac{e_1^p}{u^{p-1}}}\right|^2 - \lambda_1\left|\sqrt{\frac{e_1^p}{u^{p-1}}}\right|^2\right] \qquad (5.21)
\end{aligned}$$

On the other hand,

$$\sqrt{\frac{e_1^p}{u^{p-1}}} \notin \operatorname{span}\{e_1\}$$

and so the right-hand side of (5.21) is strictly positive and

$$\int_\Omega W(x)e_1^p < 0.$$

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References

[A-R] A. Ambrosetti, P. Rabinowitz, *Dual Variational Methods in Critical Point Theory and Applications*, J. Funct. Anal. **14** (1973), 349–381.

[B-P-T] C. Bandle, M.A. Pozio, A. Tesei, *Existance and Uniqieness of Solutions of Nonlinear Neumann Problems*, Math. Z. **199** (1988), 257–278.

[Be] V. Benci, *On Critical point theory for indefinite functionals in the presence of symmetries*, Trans. Amer. Math. Soc. **274** (1982), 533–572.

[B-CD-N] H. Berestycki, I. Capuzzo Dolcetta, L. Nirenberg, in preparation.

[Br] H. Brezis, *Some Variational problems with lack of compactness*, "Proc. Symp. Pure Math.," Amer. Math. Soc., 1986, pp. 165-201.

[B-K] H. Brezis, T. Kato, *Remarks on the Schrödinger operator with singular complex potential*, J. Math. Pures Appl. **58** (1979), 137-151.

[B-N] H. Brezis, L. Nirenberg, *Positive Solutions of Nonlinear Elliptic Equations Involving Critical Sobolev Exponent*, Comm. Pure Appl. Math. **36** (1983), 437-477.

[C-R] M. Crandall, P. Rabinowitz, *Bifurcation, Perturbation of Simple Eigenvalues and Linearized Stability*, Arch. Rat. Mech. Anal. **52** (1973), 161-180.

[E-S] J.F. Escobar, R.M. Schoen, *Conformal metrics with prescribed scalar curvature*, Inv. Math. **86** (1986), 243-254.

[FP] J. Fleckinger-Pellé, *Asymptotics of eigenvalues for some "nondefinite" problems*, "Lecture Notes in Mathematics," Springer-Verlag, Berlin, 1985, pp. 148–156.

[G-S] F. Geztesy, B. Simon, *On a theorem of Deift and Hempel*, Comm. Math. Phys. **116** (1988), 503–505.

[K-W] J.L. Kazdan- F. Warner, *Scalar curvature and conformal deformation of Riemannian structure*, J. Diff. Geom. **10** (1975), 113–134.

[Ou] T. Ouyang, *On the positive solutions of semilinear elliptic equations* $\Delta u + \lambda u + hu^p = 0$ *on Compact Manifolds, Part II*, Indiana Univ. Jour. **40** (1991), 1083-1140.

[R] P.H. Rabinowitz, "Minimax methods in critical points theory with applications to differential equations, C.B.M.S. Regional Conf.Math. Series," A. M. S., Providence, 1986.

[St] M. Struwe, "Variational Methods and applications to Nonlinear Partial Differential Equations and Hamiltonian Systems," Springer-Verlag, Berlin, 1990.