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Generalized Motion by Mean Curvature with Neumann Conditions and the Allen-Cahn Model for Phase Transitions

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Chen, Gigaand Goto ([14]) and, independently, Evans and Spruck ([20]) proved the existence and uniqueness of *viscosity solutions* for (3) thereby providing an alternative weak formulation for (1). The notion of viscosity solutions was introduced by M. Crandall and P, L. Lions for Hamilton-Jacobi equations ([16]) and extended to second order (degenerate) equations by P. L. Lions ([31]).

The extensive mathematical literature on the problem of evolution by mean curvature was partially motivated by questions arising in materials science. Indeed, equation (1) was introduced by Mullins in [33] to study the motion of grain boundaries. A more general equation that accounts for anisotropy has been derived by Angenent and Gurtin ([4],[27]) from thermodynamical considerations. An alternative derivation of (1) was obtained by Allen and Cahn in the context of antiphase boundaries. In [1], they introduced the free energy functional

$$E^*(v) = j\,(F(v) + <?\backslash Dv\backslash^2)\ dx \quad (e < 1) \qquad \textbf{(4)}$$

for an order parameter *v*. Here *F* is an even function with exactly two local minima, say at $v = r\backslash$ and t> = r2, and e is proportional to the antiphase boundary thickness. After rescaling the corresponding gradient flow becomes

$$v\backslash \;=\; Av^{€} \;-\; {}^\wedge f(v^{€}) \qquad \text{where } / = F\backslash \qquad (5)$$

By considering (5), Allen and Cahn formally established (1) as the correct limiting law of motion for antiphase boundaries. Their analysis was later mathematically justified by several authors using different techniques: the solutions $v^{€}$ were rigorously shown to converge, as e —> 0, to a piecewise constant function whose surfaces of discontinuities move according to (1). Such results were established in [9],[17] and [12] for smooth mean curvature flow. The first global-in-time results, past the geometric singularities, were proved in [19] and later generalized in [7], using the viscosity formulation of (3).

In the work cited above however, the interaction of the walls of a container Ω holding the two-phase mixture with the interface was neglected. In [11], Cahn proposed a free energy functional modified so as to take into account this phenomenon. This functional has the form

$$E^\epsilon(v) = \int_\Omega \left(F(v) + \epsilon^2|Dv|^2\right) dx + \epsilon \int_{\partial\Omega} c(v)\, d\sigma\,, \tag{6}$$

where the last term is the *contact energy*. The modified gradient flow becomes, again after rescaling,

$$\begin{aligned} v_t^\epsilon &= \Delta v^\epsilon - \frac{1}{\epsilon^2} f(v^\epsilon) \quad x \in \Omega\,, \\ \frac{\partial v^\epsilon}{\partial \nu} &= -\frac{1}{2\epsilon} c'(v^\epsilon), \quad x \in \partial\Omega\,. \end{aligned} \tag{7}$$

As a first attempt to study this problem, we shall consider here the case of constant contact energy, i.e.

$$c'(v^\epsilon) = 0\,.$$

Thus, the equation (7) is subject to the Neumann boundary condition

$$\frac{\partial v^\epsilon}{\partial \nu} = 0\,, \quad x \in \partial\Omega\,. \tag{8}$$

In [35] Rubinstein, Sternberg and Keller formally established the convergence of the solutions v^ϵ of (7), (8) to a function taking only the values r_1 and r_2. The "interfaces" Γ_t between the regions of constancy move according to (1) and satisfy the contact angle condition (2). A rigorous local-in-time result for the convergence problem was proved in [12] with the implicit assumption that Ω is convex (see Section 4.2). For the limiting equations (1),(2) Huisken ([28]) established the global existence and uniqueness of smooth solutions for the graph case in a cylindrical domain. Results concerning smooth solutions in two dimensions can be found in [35]; notice that the motion can develop singularities in finite time *even* in this case (see Section 2).

Motivated by the formal and classical results of [12],[28] and [35], we shall study *globally in time* both the convergence of the solutions v^ϵ of (7),(8) and the limiting problem (1),(2) using the level set approach (3). In Section 2 we present a proof of the local existence of classical solutions of (1),(2). Section 3 is concerned with global weak (viscosity) solutions of (1),(2): we discuss the issues of existence, uniqueness and behavior of the interface near the boundary of the domain. Finally, Section 4 is devoted to the convergence of the solutions v^ϵ of the Allen-Cahn equation subject to homogeneous Neumann boundary conditions to the global weak solution of (1),(2).

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2 Local existence of classical solutions

In this section we prove local (in time) existence and uniqueness of smooth hypersurfaces restricted in a domain $\Omega \subset \mathbb{R}^n$ moving with normal velocity equal to their mean curvature and meeting the boundary $\partial\Omega$ at a prescribed angle. Several approaches to this problem could be taken which would yield essentially the same result. In every case the size of the existence time-interval will depend on some smooth norm of the initial data. For example, one could try to extend the approach of Evans and Spruck [21] to deal with the boundary conditions; alternatively, the evolving surfaces could be sought as graphs over the initial manifold yielding a scalar parabolic problem (see [13]). We choose however to take a different perspective which we believe is more adaptable to more complex situations such as different conditions on the boundary of the domain or to the evolution of "networks". This formulation, a version of which was used in [10] in the context of three-phase boundaries, is a natural extension of the equations of motion for graphs. More precisely, we shall describe the evolving surfaces via a parametrization

$$p : [0,T] \times \mathcal{U} \to \Omega \quad \mathcal{U} \subset \mathbb{R}^{n-1}.$$

Then, the mean curvature κ of the surface $\Gamma_t \equiv p(\{t\} \times \mathcal{U})$ is given by

$$\kappa = \text{trace}((Dp \cdot Dp^t)^{-1} D^2 p) \cdot N$$

where

$$(Dp \cdot Dp^t)_{ij} = p_{x_i} \cdot p_{x_j} \quad (D^2 p)_{ij} = p_{x_i x_j}$$

and

$$\begin{aligned} N &\equiv N(p(t, x_1, \cdots, x_{n-1})) = \text{unit normal vector to } \Gamma_t = \frac{\tilde{N}}{|\tilde{N}|} \\ \tilde{N} &= \det \begin{bmatrix} e_1 & e_2 & \cdots & e_{n-1} & e_n \\ & & p_{x_1} & & \\ & & p_{x_2} & & \\ & & \vdots & & \\ & & p_{x_{n-1}} & & \end{bmatrix} . \end{aligned}$$

Thus, the condition that the normal velocity be equal to the mean curvature can be written in the form

$$p_t \cdot N = \text{trace}((Dp \cdot Dp^t)^{-1} D^2 p) \cdot N .$$

It is natural then to choose a tangential velocity so that the evolution of the parametrization p is governed by the *parabolic* system

$$p_t = \text{trace}((Dp \cdot Dp^t)^{-1} D^2 p) . \tag{9}$$

To complete our formulation we need to impose conditions on the boundary of Ω or, equivalently, on the boundary of $\mathcal{U}$. In order to keep the presentation simpler, we shall assume that the parametrization of the initial surface Γ_0 has the property that

$$\nu_0 - \cos(\gamma) N_0 = \mu \sum_{i=1}^{n-1} n_i^0 p_{x_i}^0 \quad \text{for } (x_1, x_2, \cdots, x_{n-1}) \in \partial\mathcal{U}$$

where

$$\begin{aligned}
\nu_0 &= \text{unit normal vector to } \partial\Omega \text{ at } p(0, x_1, x_2, \cdots, x_{n-1}), \\
n^0 &\equiv (n_1^0, n_2^0, \cdots, n_{n-1}^0) = \text{unit normal vector to } \partial\mathcal{U} \text{ at } (x_1, x_2, \cdots, x_{n-1}), \\
N_0 &= N(p(0, x_1, \cdots, x_{n-1})), \\
\gamma &= \text{contact angle at the boundary } \partial\Omega \text{ (i.e. } \nu_0 \cdot N_0 = \cos(\gamma)), \\
\mu &= \mu(x_1, \cdots, x_{n-1}) = \text{scaling factor} \\
\text{and } p_{x_i}^0 &= p_{x_i}(0, x_1, x_2, \cdots, x_{n-1}).
\end{aligned}$$

Then, if we let

$$\partial\Omega = \{ p \in \mathbb{R}^n \,/\, \mathrm{b}(\mathrm{p}) = 0 \}$$

we have

$$\nu_0 = \frac{Db(p(0, x_1, \cdots, x_{n-1}))}{|Db(p(0, x_1, \cdots, x_{n-1}))|}$$

and the condition that the surfaces remain within Ω is

$$b(p(t, x_1, \cdots, x_{n-1})) = 0 \quad \text{for } (x_1, x_2, \cdots, x_{n-1}) \in \partial\mathcal{U} \tag{10}$$

while the contact angle condition at the boundary becomes

$$\begin{aligned}
N(p(t, x_1, \cdots, x_{n-1})) \quad &\cdot \quad \frac{Db(p(t, x_1, \cdots, x_{n-1}))}{|Db(p(t, x_1, \cdots, x_{n-1}))|} \\
&= \cos(\gamma) \quad \text{for } (x_1, x_2, \cdots, x_{n-1}) \in \partial\mathcal{U}.
\end{aligned} \tag{11}$$

Now, to make the problem well-posed we still need to impose $n-2$ conditions on the boundary. These conditions, which should determine the tangential velocity of the surfaces Γ_t, can take many different forms, a fact that again shows the versatility of this formulation. The form of the boundary conditions that we choose is perhaps the one for which the well-posedness of the resulting parabolic system is easiest to check; these conditions are

$$\left(\sum_{i=1}^{n-1} n_i^0 p_{x_i}\right)\cdot \tau^j = \left(\sum_{i=1}^{n-1} n_i^0 A W\right) \quad \text{for } (*i.*2,-",*«-i) \in dU \text{ and } j = 1,\dots,n-2 \tag{12}$$

where r[1], • • •, r[n]~[2] is a basis for the tangent plane of Fondft at p(0, x\, #2, • • • * ^n-i)- We may now state a local existence and uniqueness result.

Theorem 1 *Let* 0 < *a* < 1 *and let dft* € C[2]+°, 6 € C[2]+°(R[n]) *and* p(-,0) € C[2]+[a](Z7). *Then there exists T* = T(||&||[2]+a,|b(0,-)||2+a) > 0 *such that the system* ***(9) subject to the boundary conditions (10)-(12) has a unique solution p* € C[2+a](W̄ x**

[0,TD-

Proof: The idea of the proof is to first linearize the equations and boundary conditions about the initial data, prove existence and uniqueness for the linearized system and finally use a fixed point argument to establish a local result for the full problem. Once the linearized system is shown to be well-posed, the rest of the proof follows using standard estimates (see [10]). Thus we shall only show here that, under the assumptions of the theorem, the linearized version of (9)-(12) has a unique solution.

After linearizing about the initial conditions we obtain

$$p_t = \text{trace}((Dp^\circ \cdot Dp^{oi})^{\prime\prime l} D^2 p) \quad \text{in } [0,T] \times U \tag{13}$$

with boundary conditions on [0,T] x *dU* given by

$$\begin{aligned} \nu_0 \cdot p &= \nu_0 \cdot p^0 - \left(\frac{b(p) - b(p_0)}{|Db(p_0)|} + \nu_0 \cdot (p - p^0)\right), \\ (\nu_0 - \cos(\gamma) N_0) D\tilde{N}_0 p_x &= |\tilde{N}_0 \text{Icos}(7) — \tilde{N}Q \cdot IQ + (i^\wedge o — cos(*y)No) D\tilde{N}op\% \\ &+ (-\tilde{I}V \cdot VQ + \tilde{N}Q\text{-} VQ + VQD\tilde{N}Q(P_X — p^\wedge)) \end{aligned} \tag{14}$$

$$
\begin{aligned}
&+ \quad (|\tilde N|\cos(\gamma) - |\tilde N_0|\cos(\gamma) - \cos(\gamma) N_0 D\tilde N_0 (p_x - p_x^0)) \\
&+ \quad \tilde N(\nu_0 - \frac{Db(p)}{|Db(p)|})
\end{aligned}
$$

$$
\text{and } \left(\sum_{i=1}^{n-1} n_i^0 p_{x_i}\right)\cdot \tau^j = \left(\sum_{i=1}^{n-1} n_i^0 p_{x_i}^0\right)\cdot \tau^j \quad \text{for } j = 1,\cdots,n-2.
$$

Here,

$$
D\tilde N = \begin{bmatrix} D_{p_{x_1}}\tilde N^1 & D_{p_{x_2}}\tilde N^1 & \cdots & D_{p_{x_{n-1}}}\tilde N^1 \\ D_{p_{x_1}}\tilde N^2 & D_{p_{x_2}}\tilde N^2 & \cdots & D_{p_{x_{n-1}}}\tilde N^2 \\ \vdots & \vdots & \cdots & \vdots \\ D_{p_{x_1}}\tilde N^n & D_{p_{x_2}}\tilde N^n & \cdots & D_{p_{x_{n-1}}}\tilde N^n \end{bmatrix},
$$

$$
p_x = \begin{bmatrix} p_{x_1} \\ p_{x_2} \\ \vdots \\ p_{x_{n-1}} \end{bmatrix}
$$

and

$$
D\tilde N\, p_x = \begin{bmatrix} \sum_{i=1}^{n-1} D_{p_{x_i}}\tilde N^1 p_{x_i} \\ \sum_{i=1}^{n-1} D_{p_{x_i}}\tilde N^2 p_{x_i} \\ \vdots \\ \sum_{i=1}^{n-1} D_{p_{x_i}}\tilde N^n p_{x_i} \end{bmatrix}.
$$

We want to show that the system (13) subject to the linearized conditions (14) with *given* right hand sides (which we shall denote by $F(t,x)$), has a unique solution in $C^{2+\alpha}([0,T]\times\overline{\mathcal{U}})$. For this, let $\mathcal{L}$ denote the (diagonal) matrix associated to the system (13) and let $\mathcal{B}$ be the matrix of boundary conditions so that (13), (14) can be written in the form

$$
\mathcal{L}(x,\partial_t,\partial_x)p = 0 \quad \text{in } [0,T]\times\mathcal{U},
$$

and

$$
\mathcal{B}(x,\partial_x)p = F(t,x) \quad \text{on } [0,T]\times\partial\mathcal{U}.
$$

The system above will have unique solutions in Hölder spaces provided the "complementary condition" is satisfied on the boundary $[0,T]\times\partial\mathcal{U}$, see [37]. Since both our operator and boundary conditions coincide with their principal part, the complementary condition can be stated in the following way: let $L = \det \mathcal{L}$, $\hat{\mathcal{L}} = L\mathcal{L}^{-1}$ and let $\lambda_1, \cdots, \lambda_n$ denote the roots of $L(x, \zeta, i(\theta - \lambda n^0))$ (as a polynomial in λ) with positive imaginary part; then the *complementary condition* is satisfied if

$$\begin{gathered}\text{for every } x \in \partial\mathcal{U} \text{ and every } \theta \text{ in the tangent plane of } \partial\mathcal{U} \text{ at } x,\\ \text{the rows of } \mathcal{B}\hat{\mathcal{L}}(x, \zeta, i(\theta - \lambda n^0)) \text{ are linearly}\\ \text{independent modulo the polynomial } \textstyle\prod_{j=1}^{n}(\lambda - \lambda_j)\\ \text{whenever } \Re(\zeta) \geq 0\,, |\zeta| > 0\,.\end{gathered} \tag{15}$$

Let $\mathcal{B}_0 \equiv \mathcal{B}_0(x, \zeta, \lambda_1, \cdots, \lambda_n)$ denote the matrix $\mathcal{B}(x, i(\theta - \lambda n^0))$ when in the $j-$th column λ is replaced by λ_j = root of $\mathcal{L}_{jj}(x, \zeta, i(\theta - \lambda n^0))$ with positive imaginary part. Then, it is easy to check that the condition (15) is equivalent to

$$\det \mathcal{B}_0 \neq 0\,. \tag{16}$$

From (14), we see that the matrix $\mathcal{B}(x, \partial_x)$ is given by

$$\begin{bmatrix} \nu_0^1 & \cdots & \nu_0^n \\ \sum_{i,j}(\nu_0^i - \cos(\gamma)N_0^i)D_{p_{x_j}^1}\tilde{N}_0^i\partial_{x_j} & \cdots & \sum_{i,j}(\nu_0^i - \cos(\gamma)N_0^i)D_{p_{x_j}^n}\tilde{N}_0^i\partial_{x_j} \\ \tau_1^1\sum_j n_j^0\partial_{x_j} & \cdots & \tau_n^1\sum_j n_j^0\partial_{x_j} \\ \tau_1^2\sum_j n_j^0\partial_{x_j} & \cdots & \tau_n^2\sum_j n_j^0\partial_{x_j} \\ \vdots & \vdots & \vdots \\ \tau_1^{n-2}\sum_j n_j^0\partial_{x_j} & \cdots & \tau_n^{n-2}\sum_j n_j^0\partial_{x_j} \end{bmatrix}$$

and, therefore, the matrix $\mathcal{B}_0$ takes the form

$$\begin{bmatrix} \nu_0^1 & \cdots & \nu_0^n \\ i\sum_j(\theta_j - \lambda_1 n_j^0)\sum_i(\nu_0^i - \cos(\gamma)N_0^i)D_{p_{x_j}^1}\tilde{N}_0^i & \cdots & i\sum_j(\theta_j - \lambda_n n_j^0)\sum_i(\nu_0^i - \cos(\gamma)N_0^i)D_{p_{x_j}^n}\tilde{N}_0^i \\ i\tau_1^1\sum_j n_j^0(\theta_j - \lambda_1 n_j^0) & \cdots & i\tau_n^1\sum_j n_j^0(\theta_j - \lambda_n n_j^0) \\ i\tau_1^2\sum_j n_j^0(\theta_j - \lambda_1 n_j^0) & \cdots & i\tau_n^2\sum_j n_j^0(\theta_j - \lambda_n n_j^0) \\ \vdots & \vdots & \vdots \\ i\tau_1^{n-2}\sum_j n_j^0(\theta_j - \lambda_1 n_j^0) & \cdots & i\tau_n^{n-2}\sum_j n_j^0(\theta_j - \lambda_n n_j^0) \end{bmatrix}.$$

Now, since $\nu_0 - \cos(\gamma)N_0 = \mu\sum_{i=1}^{n-1} n_i^0 p_{x_i}^0$ and

$$\sum_i(\nu_0^i - \cos(\gamma)N_0^i)D_{p_{x_j}}\tilde{N}^i = \det\begin{bmatrix} & & (\nu_0 - \cos(\gamma)N_0) & & \\ & & p_{x_1} & & \\ & & \vdots & & \\ & & p_{x_{j-1}} & & \\ e_1 & e_2 & \cdots & e_{n-1} & e_n \\ & & p_{x_{j+1}} & & \\ & & \vdots & & \\ & & p_{x_{n-1}} & & \end{bmatrix} = -\mu n_j^0\tilde{N}$$

we obtain that

$$\mathcal{B}_0 = \begin{bmatrix} \nu_0^1 & \nu_0^2 & \cdots & \nu_0^n \\ i\mu\sum_j(\theta_j - \lambda_1 n_j^0)(-n_j^0\tilde{N}_0^1) & i\mu\sum_j(\theta_j - \lambda_2 n_j^0)(-n_j^0\tilde{N}_0^2) & \cdots & i\mu\sum_j(\theta_j - \lambda_n n_j^0)(-n_j^0\tilde{N}_0^n) \\ i\tau_1^1\sum_j n_j^0(\theta_j - \lambda_1 n_j^0) & i\tau_2^1\sum_j n_j^0(\theta_j - \lambda_2 n_j^0) & \cdots & i\tau_n^1\sum_j n_j^0(\theta_j - \lambda_n n_j^0) \\ i\tau_1^2\sum_j n_j^0(\theta_j - \lambda_1 n_j^0) & i\tau_2^2\sum_j n_j^0(\theta_j - \lambda_2 n_j^0) & \cdots & i\tau_n^2\sum_j n_j^0(\theta_j - \lambda_n n_j^0) \\ \vdots & \vdots & \vdots & \vdots \\ i\tau_1^{n-2}\sum_j n_j^0(\theta_j - \lambda_1 n_j^0) & i\tau_2^{n-2}\sum_j n_j^0(\theta_j - \lambda_2 n_j^0) & \cdots & i\tau_n^{n-2}\sum_j n_j^0(\theta_j - \lambda_n n_j^0) \end{bmatrix}.$$

Thus,

$$\mathcal{B}_0 = \begin{bmatrix} \nu_0^1 & \nu_0^2 & \cdots & \nu_0^n \\ i\mu\lambda_1\tilde{N}_0^1 & i\mu\lambda_2\tilde{N}_0^2 & \cdots & i\mu\lambda_n\tilde{N}_0^n \\ -i\lambda_1\tau_1^1 & -i\lambda_2\tau_2^1 & \cdots & -i\lambda_n\tau_n^1 \\ -i\lambda_1\tau_1^2 & -i\lambda_2\tau_2^2 & \cdots & -i\lambda_n\tau_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ -i\lambda_1\tau_1^{n-2} & -i\lambda_2\tau_2^{n-2} & \cdots & -i\lambda_n\tau_n^{n-2} \end{bmatrix}$$

and, if we define

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad \text{and} \quad \tilde{\mathcal{B}}_0 = \begin{bmatrix} \lambda_1^{-1}\nu_0^1 & \lambda_2^{-1}\nu_0^2 & \cdots & \lambda_{n-1}^{-1}\nu_0^{n-1} & \lambda_n^{-1}\nu_0^n \\ & & i\mu\tilde{N}_0 & & \\ & & -i\tau^1 & & \\ & & -i\tau^2 & & \\ & & \vdots & & \\ & & -i\tau^{n-2} & & \end{bmatrix},$$

we can write

$$\mathcal{B}_0 = \tilde{\mathcal{B}}_0\Lambda\,.$$

Hence, to establish (16), it suffices to show that

$$(\lambda_1^{-1}\nu_0^1, \lambda_2^{-1}\nu_0^2, \cdots, \lambda_n^{-1}\nu_0^n)\cdot\nu_0 \neq 0\,. \tag{17}$$

But (17) follows immediately from the fact that the imaginary parts of the roots λ_j are positive. □

The local character of the existence statement in Theorem 1 is not just a result of the method of proof employed. Indeed, hypersurfaces moving by curvature will, in general, develop singularities in finite time at which

$$||p(t,\cdot)||_{2+a} \longrightarrow \infty \qquad \text{as } t \to T$$

(a notable exception is the case in which the hypersurfaces are graphs over an n — 1 dimensional domain, [28]). A sphere, for example will shrink to a point in time proportional to the square of its radius, while the "dumbbell"-shaped region in Figure 1 is, by now, the classical example of a surface that, when evolving by mean curvature, will develop a singularity *before* it becomes extinct.

The presence of the domain boundary $\partial\Omega$, on the other hand, gives rise to other kinds of singularities. For example, we can imagine the surfaces Γ_t (smoothly) advancing towards a nonconvex part of $\partial\Omega$, until they finally meet, as shown in Figure 2. At that point in time, where the contact angle condition is not satisfied, we expect that the evolving manifolds will "break up" to immediately satisfy the boundary condition. This, of course, cannot be established using (9)-(12), as these equations implicitly assume that the interior of the initial manifold Γ_0 does not intersect the boundary $\partial\Omega$. However, a similar formulation can be derived in that case, rendering a free boundary problem for the surface and its intersection with the boundary of the domain (with an incompatibility condition at $t = 0$). The study of this free boundary problem and, in general, of the behavior of the evolving manifolds near a singularity will, at any rate, be left for future work.

3 Global motions

3.1 Existence and uniqueness

As discussed above, singularities are expected to develop for a surface evolving by its mean curvature. Classical solutions to (1),(2) will not exist in general, so that the concept of a "weak" solution to these equations should be introduced. As we stated in the introduction, we shall use here the level set approach to define the motion *in the viscosity sense.*

For smooth surfaces Γ_t, the evolution can be written in terms of the level sets of a function u in the form

$$\Gamma_t = \{\, x \,:\, u(t,x) = 0\,\}\,, \tag{18}$$

with

$$(NP) \quad \begin{cases} u_t = \left(\delta_{ij} - \frac{u_{x_i}u_{x_j}}{|Du|^2}\right) u_{x_i x_j} & (t,x) \in \Omega_T \equiv (0,T) \times \Omega\,, \\ Du \cdot \nu = 0 \quad \text{on } (0,T) \times \partial\Omega\,, \\ u(0,x) = g(x) \quad x \in \Omega \end{cases}$$

where the initial surface is given by $\Gamma_0 = \{\, x \,:\, g(x) = 0\,\}$. To further motivate the need for a weak formulation of (NP), first notice that the parabolic equation is degenerate along the normal direction to Γ_t, $N = \frac{Du}{|Du|}$. This means that there is no diffusion along N so that the zero level set of u moves without using any information about other level sets. In particular, the interior of Γ_t does not "see" the boundary of the domain until it hits it; at this instance, the Neumann condition is not satisfied (see Figure 2). Thus, not only do we need to interpret the equation in a weak sense but also the boundary condition. The correct viscosity formulation for degenerate second order boundary value problems was first developed motivated by questions in optimal control theory, see [31],[6],[15]. We shall now present this formulation as it applies to the Neumann Problem (NP) and discuss existence and uniqueness.

Definition 1 *Let u be a bounded upper (resp. lower) semicontinuous function on*

$\overline{\Omega}_T$. *The function u is a viscosity subsolution (resp. supersolution) of (NP) provided* $u(0,x) \le g(x)$ *(resp.* $\ge$*) and for each* $\phi \in C^2([0,\infty) \times \overline{\Omega})$ *we have:*

(i) if $u-\phi$ *has a local maximum (resp. minimum) at a point* $(t_0, x_0) \in (0,+\infty) \times \Omega$, *then*

$$\begin{cases} \phi_t \le (\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2})\phi_{x_i x_j}, \ (resp. \ge) \ at \ (t_0, x_0) \\ if \ D\phi(t_0,x_0) \ne 0, \end{cases} \tag{19}$$

and

$$\begin{cases} \phi_t \le (\delta_{ij} - \eta_{x_i}\eta_{x_j})\phi_{x_i x_j}, \ (resp. \ \ge) \ at \ (t_0, x_0) \\ for \ some \ \eta \in I\!R^n \ with \ |\eta| \le 1, \ if \ D\phi(t_0,x_0) = 0, \end{cases} \tag{20}$$

(ii) if $u-\phi$ *has a local maximum (resp. minimum) at a point* $(t_0, x_0) \in (0,+\infty) \times \partial\Omega$ *and* $D\phi(t_0,x_0) \ne 0$, *then*

$$\min\left(\phi_t - (\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2})\phi_{x_i x_j}, \ D\phi \cdot \nu\right) \le 0, \ \ at \ \ (t_0, x_0), \tag{21}$$

$$(\,resp. \ \max\left(\phi_t - (\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2})\phi_{x_i x_j}, \ D\phi \cdot \nu\right) \ge 0, \ at \ (t_0, x_0)\,), \tag{22}$$

where ν *is the outward unit normal to* $\partial\Omega$.

The function u is a viscosity solution of (NP) if it is a sub- and supersolution of (NP) and satisfies the initial condition $u(0,x) = g(x), \ x \in \Omega$.

The following comparison theorem has been proved in [36] for Ω convex and in [25] for a general domain.

Theorem 2 *Let* Ω *be a smooth bounded domain in* $I\!R^n$. *Let u and v be, respectively, viscosity sub- and supersolutions of* (NP). *If* $u(0,x) \le v(0,x)$, *then* $u \le v$ *on* $\overline{\Omega}_T$.

The following existence lemma was also proved in [36].

Lemma 1 *For every function* $g \in C(\overline{\Omega})$ *there exists a lower semicontinuous subsolution u and an upper semicontinuous supersolution v of (NP) satisfying* $u(t,x) \le g(x) \le v(t,x)$ *in* Ω_T *and* $u(0,x) = g(x) = v(0,x)$ *in* Ω.

Generalized motion by mean curvature with Neumann conditions and the Allen-Cahn model for phase transitions

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Abstract

We study a ***sharp-interface*** model for phase transitions that incorporates the interaction of the phase-boundaries with the walls of a container Ω. In this model the interfaces move by their mean curvature and are normal to $\partial\Omega$. We first establish local-in-time existence and uniqueness of smooth solutions for the mean curvature equation with a normal contact angle condition. We then discuss global solutions by interpreting the equation and the boundary condition in a weak (viscosity) sense. Finally, we investigate the relation of the aforementioned model with a *transition-layer* model. We prove that if Ω is *convex*, the transition-layer solutions converge to the sharp-interface solutions as the thickness of the layer tends to zero. We conclude with a discussion of the difficulties that arise in establishing this result in non-convex domains.

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1 Introduction

In this paper we study some aspects of the evolution of hypersurfaces by their mean curvature inside a domain flcR^n, subject to a normal contact angle condition on *dn.* More precisely, we axe concerned with the evolution problem for hypersurfaces T_t satisfying

$$V = \kappa \tag{1}$$

$$N \cdot v = 0 \tag{2}$$

where V is the normal velocity, K the mean curvature and N the unit normal vector of Ft, and v denotes the unit normal vector to *dSl.*

Equation (1) has been extensively studied in recent years. Results for n = 2, in which case solutions remain smooth until they become extinct, can be found for example, in [26],[24],[3]. Also, global existence and uniqueness of classical solutions in higher dimensions was proved in [18] for the evolution of graphs. However, if $n \geq 3$ singularities may appear as the surfaces evolve, so that (1) should be understood in a "weak" sense. Brakke in [8] provided a weak formulation in terms of varifolds. A different approach was inspired by the work of Osher and Sethian [34], who represented I\ as the zero level set of a function *u* in this case, equation (1) can be written in the form

$$\frac{u_t}{\quad} \qquad Du.$$

or equivalently,

$$* \; \left(6ii \; \ll_{Cj} u_c A \; W\$ \right)^{Utiti} = \text{trace} \; f_{fT} \; V \; " \sim W \; \frac{Du \circledR Du}{) \, D \, u^2} \Big) \tag{3}$$

Notice that equation (3) is not only nonlinear and degenerate parabolic but it is also singular when $Du = 0$.

Using Lemma 1 and Perron's method (see [29]) we can construct at least one viscosity solution of (NP) which by Theorem 2 is unique. Therefore, we have the following theorem.

Theorem 3 *There exists a unique continuous viscosity solution of* (NP).

Remark 1 If we assume that the initial data $g \in W^{2,\infty}$ we can establish the *Lipschitz* continuity of the viscosity solution. For this, consider the solutions of the following approximating problems:

$$\begin{cases} u_t^{\varepsilon,\sigma} = \left((1+\sigma)\delta_{ij} - \frac{u_{x_i}^{\varepsilon,\sigma} u_{x_j}^{\varepsilon,\sigma}}{|Du^{\varepsilon,\sigma}|^2+\varepsilon^2}\right) u_{x_i x_j}^{\varepsilon,\sigma} \quad \text{in} \quad \Omega_T, \\ Du^{\varepsilon,\sigma} \cdot \nu = 0 \quad \text{on } (0,T) \times \partial\Omega, \\ u^{\varepsilon,\sigma}(0,x) = g(x) \quad x \in \Omega. \end{cases} \tag{23}$$

Using Bernstein's method and standard maximum principle arguments (see [20]) we can obtain uniform estimates for $||u^{\varepsilon,\sigma}||_{L^\infty}$ and $||u_t^{\varepsilon,\sigma}||_{L^\infty}$. We can also get uniform estimates for $||Du^{\varepsilon,\sigma}||_{L^\infty}$: first notice that $Du \cdot \nu = 0$ implies $D|Du|^2 \cdot \nu \le C_0|Du|^2$, where C_0 is a positive constant depending on the curvature of $\partial\Omega$ (see [32]). Therefore, for any positive constants k, λ and if $\rho(x) = dist(x, \partial\Omega)$, the function $z(t,x) = k\exp(C_0\rho(x))|Du^{\varepsilon,\sigma}(t,x)|^2 - \lambda u^{\varepsilon,\sigma}(t,x)$, satisfies $Dz \cdot \nu \le 0$. It follows that for an appropriate choice of k, λ, the maximum of z occurs at $t = 0$. This in turn, implies a uniform bound for $||Du^{\varepsilon,\sigma}||_{L^\infty}$. By extracting a convergent subsequence, we obtain $u^{\varepsilon',\sigma'} \to u$, locally uniformly as $\varepsilon', \sigma' \to 0$, where u is the unique viscosity solution of (NP).

The solution of (NP) describes a *geometric* evolution of level sets. Therefore the evolution should be invariant under any arbitrary relabeling of the initial level set. Indeed, the following proposition, which will be used subsequently, can be proved along the lines of [20, Theorem 2.8].

Proposition 1 *Assume u is the viscosity solution of (NP) and $\Psi : I\!R \to I\!R$ is continuous. Then*

$$v \equiv \Psi(u)$$

is the viscosity solution of (NP) with $v(0,x) = \Psi(g(x))$.

3.2 Geometric properties of the viscosity solution

In this section we present some results concerning the interaction of Γ_t with the boundary of Ω and we establish a geometric condition for Γ_t to vanish in finite time at the boundary.

We first state the following lemma which describes the properties of the sub and superdifferentials of the solution on $\partial\Omega$.

Lemma 2 *If u is a viscosity solution of (NP) and $u - \phi$ has a local maximum (resp. minimum) at a point $(t, y) \in (0, +\infty) \times \partial\Omega$, then*

$$\textit{either } D\phi \cdot \nu \leq 0 \ (\textit{resp.} \geq 0) \ \textit{ or } D\phi = |D\phi|\,\nu \ (\textit{resp.} - |D\phi|\,\nu) \ \textit{ at } \ (t, y).$$

In the latter case

$$\phi_t - \mathrm{trace}\Big((I - \nu \otimes \nu)D^2\phi\Big) \leq 0 \ (\textit{resp.} \geq 0).$$

Proof: Assume that $u - \phi$ has a local maximum at the point (t, y), $y \in \partial\Omega$ and $D\phi \cdot \nu > 0$. Then (see [15])

$$(D\phi - l\,\nu, X + m\nu \otimes \nu) \in J^{2,+}u(t, y), \quad l > 0, \ \ m \in I\!R,$$

where $J^{2,+}u(t, y)$ is the superjet of u at y for t fixed, i.e.,

$$u(\cdot, x) - \phi(\cdot, x) \leq u(\cdot, y) - \phi(\cdot, y) + < D\phi, x - y > -l < \nu, x - y >$$
$$+ \frac{1}{2} < (X + m\nu \otimes \nu)(x - y), x - y > + o(|x - y|^2) \text{ as } x \to y, \ x \in \overline{\Omega}.$$

The matrix X is any $n \times n$ symmetric matrix satisfying the inequality

$$P\,D\phi\,P \le P\,X\,P - l\,S,$$

where P is the projection onto the tangent plane to $\partial\Omega$ at y and S the symmetric operator in the tangent plane corresponding to the second fundamental form of $\partial\Omega$ at y.

Since u is the viscosity solution and the boundary condition is not satisfied, letting $l \longrightarrow 0$ we have that

$$\begin{aligned}\phi_t - \operatorname{trace}\Big((I - \frac{D\phi \otimes D\phi}{|D\phi|^2})(X + m\nu \otimes \nu)\Big) \\ = \phi_t - \operatorname{trace}\Big((I - \frac{D\phi \otimes D\phi}{|D\phi|^2})X\Big) - m\Big(1 - \frac{(D\phi \cdot \nu)^2}{|D\phi|^2}\Big) \le 0.\end{aligned}$$

The last inequality is valid for every m only if $D\phi \cdot \nu = |D\phi|$, which concludes the proof of the lemma. □

The above lemma describes the interaction of the interface with the boundary. According to the lemma a smooth interface either meets the boundary orthogonally or "touches" the boundary tangentially. In the first case we expect that the classical Neumann condition is satisfied. In the latter case the equation has to be satisfied at the touching time: for any test function ϕ,

$$\frac{D\phi}{|D\phi|} \cdot \nu = \pm 1,$$

that is, the normal unit vector to Γ_t is collinear with the unit normal vector to Ω. As we explained at the beginning of the section, the touching of the boundary by the interface $\Gamma_t = \{\, x : u(t,x) = 0 \,\}$ at a time $t = t^*$ occurs due to the degeneracy of the equation along $N = \frac{Du}{|Du|}$. If Γ_{t^*} has positive curvature at the touching point (with respect to ν), it will have to split apart for $t > t^*$ (see Figure 2). Then due to the nondegeneracy of the equation in the tangential directions of Γ_t, the Neumann condition is immediately taken into account, and a right angle condition is imposed between the interface and the boundary. This kind of discontinuities on *Du* exclude in general, the existence of classical solutions even in two dimensions.

We now consider a smooth surface So such that the interior of So does not intersect *dil.* Let *a* be the signed distance from *So-* We make the following assumption:

(A1) The surface So has positive mean curvature with respect to the unit normal field directed towards *{a* > 0} n *SI.* Moreover, for any point $x_0 \in \{\sigma > 0\} \cap \partial\Omega$,

<*r(x_0)* = |x_0 - z_o\, 2_0 € S_o \ *dQ* and Xo — *o • *v(x_0)* > 0.

For example this is true if the boundary has the geometry depicted in Figure 3. In particular (Al) is valid if *{a* > 0} n *dQ,* is convex. We have the following lemma.

Lemma 3 ***Let v(t,x) = a a(x) — t. Then, under assumption (Al), there exists a constant a* > 0 *such that***

vt - (t>a -]|- $\frac{v_{x_j}}{v}$ p K*> ≥ 0, /or x € fl n {a > 0}, **(24)**

and

***Dv'U≥0forxedCin{cT>* 0}.** **(25)**

Proof: Assume that t; — *4>* has a minimum at (fo>£o) and *v{to,xo)* = $\phi(t_0, x_0)$. If *Xo* € *dCt* n *{a* > 0}, let *a(x_0)* = |xo — z_o\, z_0 € So. We define the function *w(x)* = *a\x — 2o* — to- Since *w(x)* ≥ *aa(x)* — to» we have that

w(x) - <f>(t_o,x) ≥ (aa(x) -1_0) - <t>(t_Q,x) ≥ w{x_0) - <t>(t_{Oj}x_o).

Thus xo is a local minimum for *w* — *<f>*. Hence, *D<p* is in the subdifferential of *w* and (see [15, equation (2.15)])

$$D\phi(t_0, x_0) = a\frac{x_0 - z_0}{|x_0 - z_0|} + l\,\nu, \quad l > 0.$$

In view of assumption (A1),

$$\frac{\partial \phi(t_0, x_0)}{\partial \nu} \geq 0.$$

On the other hand, if $x_0 \in \Omega \cap \{\sigma > 0\}$ then, $\phi_t = -1$ and we have to prove that

$$-(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2})\phi_{x_i x_j} - 1 \geq 0. \tag{26}$$

Arguing as in [19, Theorem 2.2], we obtain

$$|D\phi| = a.$$

Define $\psi(x) = \phi(t_0, x + x_0 - z_0) - v(t_0, x_0)$. Then,

$$\{\psi > 0\} \subset \{\sigma > 0\}. \tag{27}$$

Indeed let x be such that $\psi(x) > 0$ and assume $\sigma(x) = 0$. Then, since $\psi(x) > 0$

$$a\,\sigma(x + x_0 - z_0) - t_0 \geq \phi(t_0, x + x_0 - z_0) > a\,\sigma(x_0) - t_0.$$

Thus, since $\sigma(x) = 0$, we obtain the contradiction

$$a|x_0 - z_0| - t_0 \geq a\sigma(x + x_0 - z_0) - a\sigma(x) - t_0 > a\sigma(x_0) - t_0.$$

Now, $|D\psi(z_0)| = a$ and (27) yield that the mean curvature of the surface $\{x : \psi(x) = 0\}$ is greater than that of S_0 at z_0. Let κ denote the mean curvature of the level set $\{x : \psi(x) = 0\}$ at z_0 with respect to $\frac{D\psi}{|D\psi|}$. Since S_0 has positive mean curvature, κ is bounded from below by a positive number. Thus,

$$-1 - (\delta_{ij} - \frac{\psi_{x_i}\psi_{x_j}}{|D\psi|^2})\psi_{x_i x_j} = -1 + a\kappa > 0, \text{ at } (t_0, z_0). \tag{28}$$

for a sufficiently large. By using

$$D\psi(z_0) = D\phi(t_0, x_0) \text{ and } D^2\psi(z_0) = D^2\phi(t_0, x_0),$$

(27) and (28) we obtain (26). □

A direct consequence of the above lemma and Proposition 1 is the following:

Proposition 2 *Let S_0 satisfy (A1). Then, $(a\sigma(x)-t)^+$ is a supersolution of (NP) with initial data $(a\sigma(x))^+$.*

Proof: By Proposition 1, $v^+(x) = (a\sigma(x) - t)^+$ satisfies (24), (25). However for $t > 0$, $\{v^+ > 0\} \subset\subset \{\sigma > 0\}$ and consequently v^+ is a supersolution of (NP) in Ω. $\Box$

Using the above proposition we can prove the following:

Theorem 4 *Let σ and S_0 be as in Proposition 2 and let the initial interface $\Gamma_0 = \{x : g(x) = 0\} \subset \{x : \sigma > 0\}$ (see Figure 3). Then, Γ_t vanishes in finite time.*

Indeed, since $\{x : g > 0\} \subset \{x : (a\sigma(x))^+ > 0\}$, an application of Theorem 2 yields

$$\{x : u(t,x) > 0\} \subset \{x : v(t,x) > 0\}.$$

Therefore, there exists a time $\bar{t}$ such that $\{x : u(t,x) > 0\} = \emptyset$ for $t > \bar{t}$.

A consequence of the above theorem is the following corollary which has appeared in [35] for the case of a smooth plane curve Γ_0.

Corollary 1 *If Γ_0 is contained in a convex subdomain of Ω, then Γ_t will vanish in finite time.*

4 The Allen-Cahn Model with Neumann boundary conditions

In this section we are concerned with the limiting behavior as $\epsilon \to 0$ of the solutions to the Allen-Cahn equation subject to Neumann boundary conditions:

$$\begin{cases} v_t^\epsilon - \Delta v^\epsilon + \frac{1}{\epsilon^2} f(v^\epsilon) = 0 & \text{in} \quad (0,\infty) \times \Omega, \\ v^\epsilon(0,x) = g^\epsilon(x) & \text{in} \quad \Omega, \\ \frac{\partial v^\epsilon}{\partial \nu}(t,x) = 0 & \text{on} \quad (0,\infty) \times \partial\Omega. \end{cases} \tag{29}$$

Here, Ω is a smooth bounded domain and $f = F'$ where F is a W-shaped potential with wells of equal depth. We assume for simplicity (see Remark 2) that $F(r) = \frac{1}{2}(r^2-1)^2$, so that $f(r) = 2(r^3-r)$. A formal analysis in [35] suggests that, as $\epsilon \to 0$, Ω separates into two regions P_t and N_t where $v^\epsilon \approx 1$ and $v^\epsilon \approx -1$ respectively and the separating front Γ_t moves by its mean curvature with a normal contact at $\partial\Omega$. Moreover, this analysis yields the asymptotic formula

$$v^\epsilon \approx q\left(\frac{d(t,x)}{\epsilon}\right) \quad as \ \epsilon \to 0$$

for the solutions v^ϵ, where $q(z) = \tanh(z)$ is the traveling wave associated to the nonlinearity f (see [5]) and $d(t,x)$ is the signed distance function to Γ_t. Thus, we shall assume

$$g^\epsilon(x) = q\left(\frac{d(0,x)}{\epsilon}\right)$$

where $d(0,x)$ denotes the distance to Γ_0. The above assumption can be removed by studying the *generation* of the front (see [12],[7]), but we shall not pursue this here.

In Section 4.1 we rigorously establish, for *convex* Ω, that in the limit as $\epsilon \to 0$ the hypersurfaces Γ_t evolve by their mean curvature, in the generalized sense of Definition 1. In the final section we discuss the difficulties encountered when trying to extend our global-in-time results to general nonconvex domains. (Local convergence of the solutions can be established as long as the motion is classical; see Section 4.2).

The first rigorous results for (29) in more than 1-dimension were obtained in [9] where it was proved, using energy methods, that a subsequence $v^{\epsilon'} \to \pm 1$ a.e. in $(0,\infty) \times \Omega$. Global convergence results have been recently established by Evans, Soner and Souganidis in [19] and by Barles, Soner and Souganidis in [7], for the case $\Omega = I\!R^n$.

4.1 The convex domain case

Theorem 5 *Let Ω be a convex domain in $\mathbb{R}^n$ and v^ϵ the solution of (29). Then*

$$\begin{cases} v^\epsilon \to 1 & \text{in } P = \{(t,x) : u(t,x) > 0\} \\ v^\epsilon \to -1 & \text{in } N = \{(t,x) : u(t,x) < 0\}, \end{cases} \tag{30}$$

locally uniformly on compact subsets of P and N.

In our analysis we essentially use the method introduced in [19]: we "trap" v^ϵ between super and subsolutions of (29) of the form $q(\frac{d(t,x)}{\epsilon})$ where d is the signed distance from $\Gamma_t = \{x : u(t,x) = 0\}$ and u is the viscosity solution of (NP). In the sequel we shall only discuss the construction of a supersolution, since the arguments can be easily modified to yield subsolutions.

Since

$$q'' - f(q) = 0$$

substituting $q(\frac{d(t,x)}{\epsilon})$ into (29) we see that, to obtain a supersolution, d has to satisfy

$$\begin{cases} \frac{1}{\epsilon}q'(\frac{d}{\epsilon})(d_t - \Delta d) - \frac{1}{\epsilon^2}q''(\frac{d}{\epsilon})(|Dd|^2 - 1) \geq 0 & \text{in} \quad (0,\infty) \times \Omega, \\ \frac{\partial d}{\partial \nu}(t,x) \geq 0 & \text{on} \quad (0,\infty) \times \partial\Omega. \end{cases} \tag{31}$$

The above inequalities suggest that d should be taken to be a distance function, so that $|Dd| = 1$, and that it should also be a supersolution of the heat equation with Neumann conditions. The following lemma shows that in a convex domain the distance function to Γ_t satisfies the required inequalities.

Lemma 4 *Let $d(t,x) = dist(x,\Gamma_t)$ where Γ_t is given by* (18) *and u is the viscosity solution of (NP). If Ω is a convex domain then,*

$$d_t - \Delta d \geq 0 \quad in \quad (0,t^*) \times (\Omega \cap \{d > 0\}) \tag{32}$$

$$(resp. \;\; d_t - \Delta d \leq 0 \quad in \quad (0,t^*) \times (\Omega \cap \{d < 0\})$$

and

***Dd-v>Qin(OX)x(dnn{d>Q})* (*resp.* *Dd-v* < *0 in* (0,**) x *(dQn{d<* 0})).** (33)

when V = inf*{t* > **0** : **T,** = **0}** .

Proof: We shall first prove (32). Assume x_0 € ft n *{d* > 0} and let ^ be a smooth function such that *d*— </> has a minimum at (t_o,x_o) € (0,f) x H. We will show that

$$\phi_t - \Delta\phi \geq 0 \quad \text{at} \quad (t_o,x_o). \tag{34}$$

Let ^o € T_{io} be such that *d(to,xo)* = |XQ — 2o|- Then either *ZQ* € n or *z$* € $\partial\Omega$.

Case 1: *p € fl. Then the arguments in [19, Theorem 2.2] yield (34). The idea is to first translate *<j>* along *XQ*—*ZQ* to obtain a test function for ^(u) at (to, *o)> where $ is an appropriate monotone function. Then, (34) follows from the semiconcavity of the distance function.

Case 2: *z0 e dfl.* Define *%}>{t,x)* = *<f>{t,x+*x_o-z_o*)-6* where $d(t_o,x_o)$ = **|x0—^o|** = ^, and let *w(x)* = |x — zo|. Since *w(x)* ≥ *d(to,x),* we have that

w(x) - ^**(to,a:)** ≥ ***d(to,x)*** - ^(t0^) >***d(to,xo)*** - ***<f>(tOyxo)*** = **t^(x0)** - ***<t>{tOyxQ).***

Thus xo is a local minimum for u; — </> and

$$D\phi(t_0,x_0) = \frac{x_0 - z_0}{|x_0 - z_0|} = D\psi(t_0,z_0).$$

Since *Q* is convex we have

$$\frac{x_0 - z_0}{|x_0 - z_0|} \cdot \nu(z_0) < 0$$

and therefore

$$\frac{\partial\psi(t_0,z_0)}{\partial\nu} < 0 \ . \tag{35}$$

As in [19] we can construct a nondecreasing continuous function $\Psi : [0,\infty) \to [0,\infty)$ such that $\Psi(0) = 0$, $\Psi(z) > 0$ if $z > 0$ and

$$\psi(t,z) \leq \Psi(u(t,z)) \text{ for all } (t,z) \text{ near } (t_0, z_0).$$

Therefore, $\Psi(u) - \psi$ has a local minimum at (t_0, z_0) and by Proposition 1 and (35),

$$\psi_t - (\delta_{ij} - \frac{\psi_{x_i}\psi_{x_j}}{|D\psi|^2})\psi_{x_i x_j} \geq 0, \text{ at } (t_0, z_0). \tag{36}$$

By rotating coordinates we may assume that $x_0 = z_0 + \delta e_n$, where $e_n = (0, \ldots, 0, 1)$, and prove as in [19] that

$$\phi_{x_n x_n}(t_0, x_0) \leq 0\,. \tag{37}$$

Since $D\phi(t_0, x_0) = e_n$, (36),(37) imply

$$\phi_t - \Delta\phi = \phi_t - (\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2})\phi_{x_i x_j} - \phi_{x_n x_n} \geq 0 \text{ at } (t_0, x_0),$$

which concludes the proof of (32).

In order to prove (33) assume now that $x_0 \in \partial\Omega \cap \{d > 0\}$. As above if $d - \phi$ has a local minimum at (t_0, x_0) so does $w - \phi$. Thus (cf. Lemma 3)

$$\frac{\partial\phi(t_0, x_0)}{\partial\nu} = \frac{x_0 - z_0}{|x_0 - z_0|} \cdot \nu + l\,\nu \geq 0, \;\; l > 0$$

which implies (33). □

In order to construct supersolutions to (29) consider the viscosity solutions of

$$\begin{cases} u_t^{\delta,a} - (\delta_{ij} - \frac{u_{x_i}^{\delta,a} u_{x_j}^{\delta,a}}{|Du^{\delta,a}|^2}) u_{x_i x_j}^{\delta,a} = \frac{3}{2} a |Du^{\delta,a}|, & \text{for } x \in \Omega \\ Du^{\delta,a} \cdot \nu = 0, & \text{on } \partial\Omega \\ u^{\delta,a}(0,x) = d(x, \Gamma_0) + \delta. \end{cases}$$

The proof of existence and uniqueness of viscosity solutions for this problem is similar to that for (NP). Now set $d^{\delta,a}(t,x) = d(x, \Gamma_t^{\delta,a})$, where $\Gamma_t^{\delta,a}$ is the zero level set of

$u^{\delta,a}$. Following the proof of Lemma 3.1 of [19], we define $w^{\delta,a}(t,x) = \eta_\delta(d^{\delta,a}(t,x))$ where η_δ is a real smooth function such that

$$\begin{cases} \eta_\delta(z) = -\delta \text{ for } z \le \delta/4, \\ \eta_\delta(z) = z - \delta \text{ for } z \ge \delta/2, \\ \eta_\delta(z) \le -\delta/2 \text{ for } z \le \delta/2, \\ 0 \le \eta'_\delta \le C \text{ and } |\eta''_\delta| \le C\delta^{-1} \end{cases}$$

and C is a positive constant independent of δ.

By the semiconcavity of the distance function and a simple adaptation of Lemma 4 we obtain (see [19], where $\Omega = \mathbb{R}^n$):

Lemma 5 *There is a constant K, independent of a, δ such that*

$$w_t^{\delta,a} - \Delta w^{\delta,a} - \frac{3}{2}a|Dw^{\delta,a}| \ge -\frac{K}{\delta} \text{ in } \Omega_T \tag{38}$$

and

$$w_t^{\delta,a} - \Delta w^{\delta,a} - \frac{3}{2}a \ge 0 \text{ in } \Omega_T \cap \{d^{\delta,a} > \delta/2\}, \tag{39}$$

$$|Dw^{\delta,a}| = 1 \text{ in } \Omega_T \cap \{d^{\delta,a} > \delta/2\} \tag{40}$$

and

$$Dw^{\delta,a} \cdot \nu \ge 0 \text{ on } ((0,T] \times \partial\Omega) \cap \{d^{\delta,a} > \delta/2\}. \tag{41}$$

We now define

$$Z^\epsilon(x,t) = q^\epsilon(\frac{w^{\delta,a}(t,x)}{\epsilon}, a),$$

where $q^\epsilon(z,a)$ is the traveling wave associated to $f^\epsilon(r) = f(r) - \epsilon a$. From [5] we know that there is a unique pair $(q^\epsilon(z,a), c^\epsilon(a))$ such that

$$q^\epsilon_{zz} + c^\epsilon(a)q^\epsilon_z = f^\epsilon(q), \tag{42}$$

$$\frac{c^\epsilon(a)}{\epsilon} \to \frac{3}{2}a \text{ as } \epsilon \to 0.$$

Lemma 6 *For every $a > 0$, there is $\delta_0(a)$ and $\epsilon_0(\delta, a)$ such that Z^ϵ is a viscosity supersolution of (29).*

The proof of $Z^\epsilon_t - \Delta Z^\epsilon + \frac{1}{\epsilon^2} f(Z^\epsilon) \geq 0$ appears in [7] (Proposition 10.2). It only remains to prove that $DZ^\epsilon \cdot \nu \geq 0$. However, this is immediate from Lemma 5 (equation (41)).

We conclude with the proof of Theorem 5.

Proof of Theorem 5: Let (t_0, x_0) be such that $u(t_0, x_0) \leq -\theta < 0$. By the stability properties of the viscosity solutions,

$$u^{\delta,a}(s,y) \leq -\frac{\theta}{2} < 0 \tag{43}$$

for all δ, a sufficiently small and (s,y) in a neighborhood of (t_0, x_0). Furthermore since $q(z,a)$ is increasing in a (see [5]), we have:

$$Z^\epsilon(0,x) \geq g^\epsilon(x) = q(\frac{d(x,\Gamma_0)}{\epsilon}, 0). \tag{44}$$

By standard comparison arguments for viscosity solutions, we obtain that $Z^\epsilon \geq v^\epsilon$ in $\overline{\Omega}_T$. On the other hand by the classical maximum principle, $v^\epsilon \geq -1$.

From (43), (44) we have

$$\limsup_{\substack{\epsilon \to 0 \\ (s,y)\to(t_0,x_0)}} v^\epsilon(s,y) \leq \limsup_{\substack{\epsilon \to 0 \\ (s,y)\to(t_0,x_0)}} Z^\epsilon(s,y) = -1.$$

The result follows since

$$\liminf_{\substack{\epsilon \to 0 \\ (s,y)\to(t_0,x_0)}} v^\epsilon(s,y) \geq -1.$$

In a similar way we obtain $v^\epsilon \to 1$ locally uniformly in P. □

Remark 2 Theorem 5 can be extended to the case in which $f = f(t,x,r)$ (see [7]). Namely, if $k_-(t,x) < k(t,x) < k_+(t,x)$ are the roots of $f(t,x,r)$ and $\int_{k_-}^{k_+} f(t,x,r)dr = 0$, then Theorem 5 holds true with k_-, k_+ in the place of -1 and 1 respectively.

4.2 The nonconvex domain case

When Ω is nonconvex the proof of Lemma 4, and consequently that of Theorem 5, fail. On the one hand, the distance function d does not always satisfy $Dd \cdot \nu \geq 0$, see Figure 4, and hence DZ^ϵ is not necessarily greater than or equal to 0. On the other hand, the distance function is no longer a supersolution of the heat equation on $\{d > 0\}$. For example, in Figure 4 the interface does not move, so that $d_t = 0$, and $\Delta d = 1$ in Sector A of Ω.

The question arises then as to what is the appropriate choice for "d" in $Z^\epsilon \approx q(\frac{d}{\epsilon})$ that will allow us to overcome the above mentioned difficulties. Freidlin in [22, 23] deals with the so-called Kolmogorov-Petrovskii-Piskunov equation (KPP)

$$\begin{cases} v_t^\epsilon - \epsilon\Delta v^\epsilon + \frac{1}{\epsilon} f(v^\epsilon) = 0 & \text{in} \quad (0,\infty) \times \Omega, \\ v^\epsilon(0,x) = g^\epsilon(x) & \text{in} \quad \Omega, \\ \frac{\partial v^\epsilon}{\partial \nu}(t,x) = 0 & \text{on} \quad (0,\infty) \times \partial\Omega, \end{cases} \tag{45}$$

where $f(r) = r(1-r)$. Based on a result of Anderson and Orey ([2]), Freidlin proved that $u^\epsilon \approx q(\frac{d_\Omega(x,\Gamma_t)}{\epsilon})$. Here, Γ_t is an interface moving with constant speed, q is the stable traveling wave of KPP and

$$d_\Omega(x,y) = \inf\{\int_0^1 |\phi'(s)|^2 ds \ : \ \phi \in C^{0,1},\ \phi(0) = x,\ \phi(1) = y,\ \phi(s) \in \overline{\Omega},\ s \in [0,1]\}.$$

In [30] a similar result is proved for (45), when $f(r) = (r-m)(r^2-1)$ and $m \neq 0$.

We conjecture that for (29), Theorem 5 holds and $u^\epsilon \approx q(\frac{d_\Omega(x,\Gamma_t)}{\epsilon})$ where Γ_t moves according to (NP). However, it is not always true that $q(\frac{d_\Omega(x,\Gamma_t)}{\epsilon})$ is a supersolution of (29): although d_Ω satisfies the Neumann condition (33), it is not necessarily a supersolution of the heat equation; see the example of Figure 4, where $\Delta d_\Omega(x,\Gamma_t) \to +\infty$ as $x \to (0,0)$ and $x \in$ Sector A.

Finally, a word is in order regarding the local convergence of the solutions of (29) when Γ_t is smooth. This question was addressed in [12]; there, under the assumptions that (a) there exists a local smooth solution to (1),(2) and (b) that the distance function to the interfaces is smooth, a version of Theorem 5 was established. As shown in Section 2, statement (a) can be proved for smooth initial data. On the other hand, (b) is not always true in a non-convex domain: in the example in Figure 4, the distance function *is not* smooth across the line separating sectors A and B. However, the arguments in [12] can be easily modified to account for this behavior. For example one could smoothly extend the interface outside Ω and consider the distance function to this extended interface.

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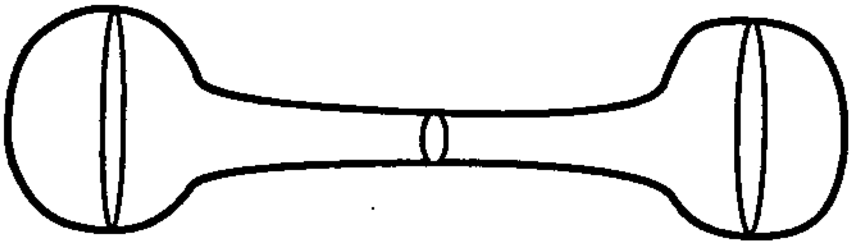

Figure 1

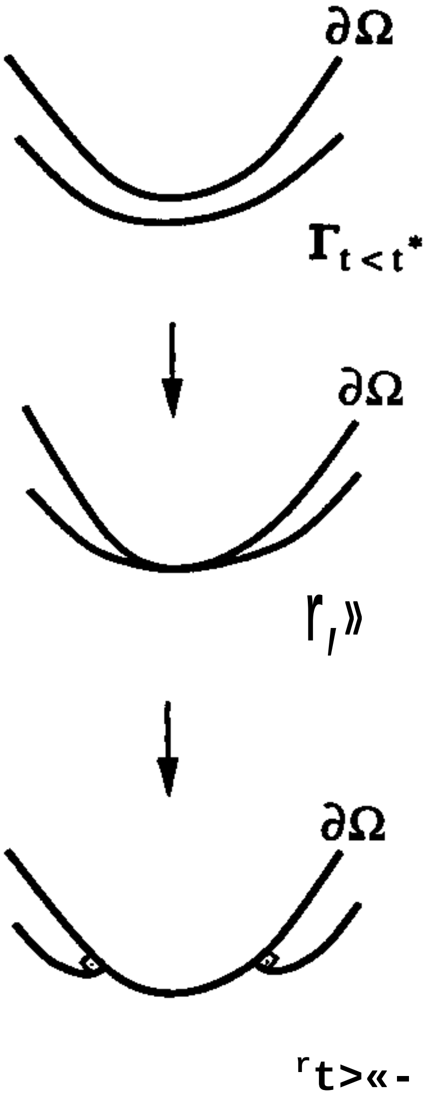

Figure 2

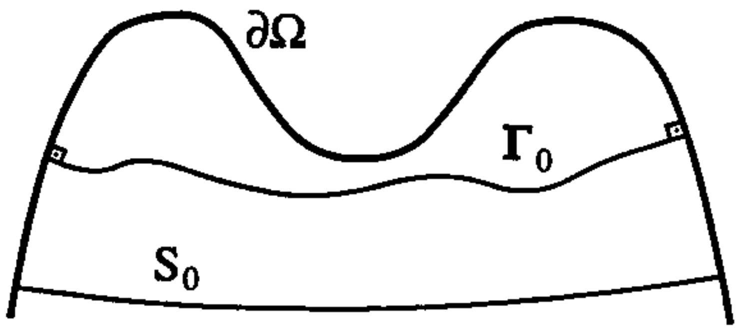

Figure 3

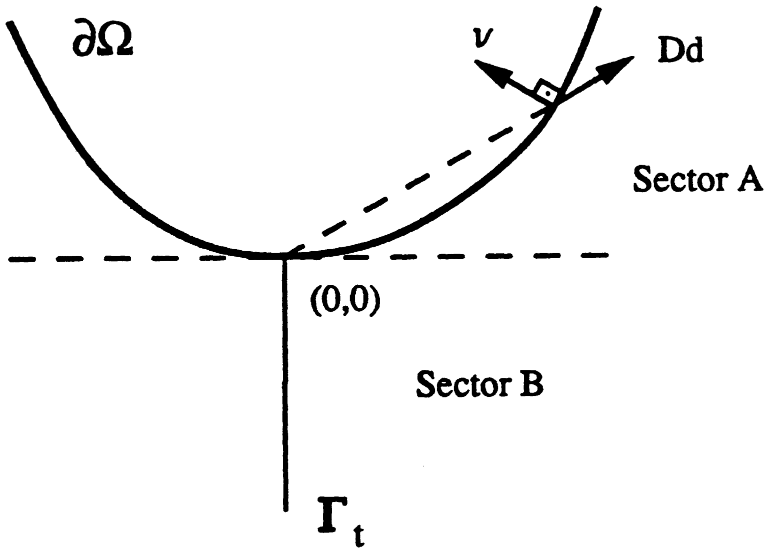

Figure 4