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Cosmic String Solutions of the Einstein-Matter-Gauge Equations

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Abstract

This paper establishes the existence of string-like static solutions of the coupled Einstein-matter-gauge equations. Such solutions have important implications in cosmology and quantum physics. It is shown that for a prescribed string location, the system possesses a continuous family of distinct finite energy solution configurations. The proof relies on a two-side shooting method. Power-type decay estimates at spatial infinity are also obtained.

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1 Introduction

A fundamental problem in cosmology is the quest for a model of galaxy formation. Recent theoretical developments have provided the scenario that a sequence of phase transitions in the early universe at various critical temperatures corresponding to a series of symmetry breaking scales can lead to the production of cosmic strings which are the seeds for the accretion of matter to form galaxies [14],[22]. The Yang–Mills–Higgs theory gives the mechanism for symmetry-breaking and cosmic strings are realized as cylindrically symmetric solutions of the fully coupled Einstein–Yang–Mills–Higgs equations with a suitable gauge group G. Naturally, a first-step understanding of the model should be achieved for the case where G takes the simplest form $G = U(1)$ and the equations describe an Einstein-matter-gauge system in which the $U(1)$ symmetry is spontaneously broken (namely, a gravity-condensed matter system). In fact, most studies and progress in understanding cosmic strings have been made for the $U(1)$ system and people hope that the conclusions reached may be useful in the investigation of a more general theory. Along this direction, for example, Garfinkle [6] studied

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the existence and properties of the string solutions by a heuristic argument, Laguna-Castillo and Matzner [17] presented a numerical solution of the system, Gregory [8] made a stability analysis, Gibbons, Ortiz and Ruiz [7] proved a non-existence theorem for strings with certain prescribed asymptotic properties. However, due to the complexity of the coupled Einstein-matter-gauge equations, people have not yet been able to establish rigorously the existence of these important cosmic string solutions.

Recently, the works of Linet [15],[16] and Comtet and Gibbons [5] have shed new light on the problem. In their approach, it is shown that at a critical coupling phase, the second order Einstein-matter-gauge equations allow a reduction into a coupled Einstein-BogomoPnyi system. In particular, Comtet and Gibbons [5] further proved that the Einstein-Bogomol'nyi equations may be put into a system of two coupled nonlinear elliptic equations, one of which can be integrated exactly so that the resolution of the problem relies on the understanding of the remaining equation of a Liouville type which has a much more promising structure.

In this paper we prove the existence of cosmic string solutions of the Einstein-matter-gauge system in the above mentioned critical Bogomornyi coupling using the equation derived in the work of Comtet and Gibbons. It may be surprising to notice that under our condition for existence, there is also non-uniqueness. More precisely, we shall show that when the winding number of the string is not too large, there exist a continuous family of distinct solutions realizing a prescribed string location. We shall also obtain some power-type decay estimates for the field configurations and the gravitational metric. An interesting feature of the governing equation (see (4.7)) is that it shares some common properties with the corresponding equation derived recently by Hong, Kim, and Pac [10] and Jackiw and Weinberg [11] in their studies of the self-dual Chern-Simons Higgs theory and the string solutions of the equation here resemble the non-topological Chern-Simons vortex-lines constructed in Spruck and Yang [20]. Therefore we are able to extend the shooting method used in [20] to the existence problem of cosmic strings considered in this paper.

Here is an outline of the contents. In Section 2 we introduce the reduction of Comtet and Gibbons *from* the Einstein-matter-gauge equations to an Einstein-Bogomol'nyi system and fix most of the notation of the paper. In Section 3 we establish the equivalence of the Einstein-Bogomol'nyi system and the reduced Liouville type elliptic equations under the assumption that the 2-manifold where the strings reside is a Riemann surface. Such a condition is important (and also general enough) to ensure the use of suitable bundle isomorphisms to recover the Einstein-Bogomol'nyi equations. In Section 4 we state and prove our main existence result

(Theorem 4.6) for an isolated string solution. Detailed properties are also obtained in this section. In Section 5 we present an existence theorem for "0-string" solutions without any restriction on the range of parameters. In Section we assume that the background gravity is known and study the existence of separated strings arising from the matter-gauge sector. Section 7 contains some final remarks.

2 The Einstein–Bogomol'nyi Equations

In this section we shall follow the main line in Comtet and Gibbons [5]. First of all, the matter-gauge sector of the theory in the critical coupling phase is described by the action density

$$\mathcal{L} = -\frac{1}{4}g^{\mu\mu'}g^{\nu\nu'}F_{\mu\nu}F_{\mu'\nu'} + \frac{1}{2}g^{\mu\nu}(D_\mu\phi)(D_\nu\phi)^* - \frac{1}{8}(|\phi|^2-1)^2, \tag{2.1}$$

where $g = (g_{\mu\nu})$ is the metric tensor of a 4-dimensional Minkowski manifold (spacetime), A_μ is a 4-vector (gauge) field, ϕ is a complex scalar (matter) field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell electromagnetic field, and $D_\mu\phi = \partial_\mu\phi - \mathrm{i}A_\mu\phi$ is the gauge-covariant derivative. The local $U(1)$ gauge-invariance $\phi \mapsto e^{\mathrm{i}\omega}\phi$, $A_\mu \mapsto \partial_\mu\omega + A_\mu$ is observed in the model.

Cosmic strings are the solutions of the coupled Einstein-matter-gauge equations under the assumption that the spacetime metric takes a special form. When the strings are parallel to the z-axis, the line element is given by the expression

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}z^2 + g_{jk}\mathrm{d}x^j\mathrm{d}x^k,$$

where and in the sequel, $g = (g_{jk})$ $(j,k = 1,2)$ is the metric tensor of an unknown (or otherwise prescribed) non-compact Riemannian 2-manifold M.

It is consistent to assume that the only non-vanishing components of the gauge field are A_j $(j = 1,2)$ and that A_j and ϕ are fields on M. In this case the energy density function for the matter-gauge sector takes the form

$$\mathcal{E}_{\mathrm{mg}} = \frac{1}{4}g^{jj'}g^{kk'}F_{jk}F_{j'k'} + \frac{1}{2}g^{jk}(D_j\phi)(D_k\phi)^* + \frac{1}{8}(|\phi|^2-1)^2. \tag{2.2}$$

The field equations for ϕ and $A = (A_j)$ are the Euler–Lagrange equations of (2.2):

$$\begin{aligned} \tfrac{1}{\sqrt{g}}D_j(g^{jk}\sqrt{g}[D_k\phi]) &= \tfrac{1}{2}\phi(|\phi|^2-1), \\ \tfrac{1}{\sqrt{g}}\partial_{j'}(g^{jk}g^{j'k'}\sqrt{g}F_{kk'}) &= \tfrac{\mathrm{i}}{2}g^{jk}(\phi[D_k\phi]^* - \phi^*[D_k\phi]), \end{aligned} \qquad x \in M. \tag{2.3}$$

The full system consists of (2.3) and the Einstein equations.

On the other hand, it is easily seen that (2.2) can be rewritten in the form

$$\begin{aligned}\mathcal{E}\mathrm{mg} = \;& \tfrac{1}{4}g^{jj'}g^{kk'}(F_{jk} \pm \tfrac{1}{2}\epsilon_{jk}(|\phi|^2-1))(F_{j'k'} \pm \tfrac{1}{2}\epsilon_{j'k'}(|\phi|^2-1)) \\ & +\tfrac{1}{4}g^{jk}(D_j\phi \pm \mathrm{i}\epsilon_j^{j'}D_{j'}\phi)(D_k\phi \pm \mathrm{i}\epsilon_k^{k'}D_{k'}\phi)^* \\ & \pm\nabla_j(\epsilon^{jk}J_k),\end{aligned} \tag{2.4}$$

where ϵ_{jk} is the standard Levi–Civita skew-symmetric 2-tensor satisfying $\epsilon_{12} = \sqrt{g}$, $\epsilon_j^k = \epsilon_{jj'}g^{kj'}$, ∇_j is the covariant derivative with respect to the metric g, and J_k is the current vector defined by

$$J_k = \frac{1}{2}A_k - \frac{\mathrm{i}}{4}(\phi^*[D_k\phi] - \phi[D_k\phi]^*). \tag{2.5}$$

Since the last term on the right-hand-side of (2.4) is a total divergence, a solution of the equations

$$\begin{aligned} F_{jk} \pm \tfrac{1}{2}\epsilon_{jk}(|\phi|^2-1) &= 0, \\ D_j\phi \pm \mathrm{i}\epsilon_j^k D_k\phi &= 0, \end{aligned} \qquad x \in M \tag{2.6}$$

may be a minimizer of the energy $\int_M \mathcal{E}\mathrm{mg}\sqrt{g}\mathrm{d}^2x$. Thus a solution of (2.6) should also satisfy the equations (2.3). In fact it can be directly verified that (2.6) always imply (2.3). The system (2.6) is the curved-space version of the Bogomol'nyi equations [3].

Let K be the Gaussian curvature of the 2-manifold (M, g) and $G > 0$ the universal gravitational constant. To complete our setting of the problem, we need to add the Einstein equations to the system. In the above framework, it can be seen [5] that the Einstein equations are reduced to

$$K = 8\pi G T_{tt}, \tag{2.7}$$

where $T_{tt} = \mathcal{E}\mathrm{mg}$ is the energy component of the energy-momentum tensor $T_{\mu\nu}$ decided by the matter-gauge action density (2.1). As a consequence, we see from (2.4), (2.6), and (2.7) that the full Einstein-matter-gauge system can now be solved by the following coupled Einstein–Bogomol'nyi equations

$$\begin{aligned} D_j\phi \pm \mathrm{i}\epsilon_j^k D_k\phi &= 0, \\ F_{jk} \pm \tfrac{1}{2}\epsilon_{jk}(|\phi|^2-1) &= 0, \qquad x \in M. \\ K \mp 8\pi G\nabla_j(\epsilon^{jk}J_k) &= 0, \end{aligned} \tag{2.8$\pm$}$$

The unknown is the metric-matter-gauge triplet (g, ϕ, A). Since the equations (2.8+) and (2.8−) are equivalent under the "conjugacy" $(g,\phi,A) \mapsto (g,\phi^*,-A)$, from now on we shall only consider (2.8+).

3 Charaterization of Solutions

In this section we make a general discussion of the string-like solutions of the Einstein–Bogomol'nyi equations (2.8+).

Suppose we have found a solution triplet (g, ϕ, A) of (2.8+) so that (M, g) is a Riemannian mainfold. Since M is two-dimensional, it can be covered by isothermal coordinate charts. Namely, for given $p \in M$, there is a local coordinate system $(U, (x^j))$ so that $p \in U \subset M$, $x^j(p) = 0$, and $g_{jk} = \Omega_U(x)\delta_{jk}$ around p, where $\Omega_U(x)0$ is a smooth function defined on U. In this local chart, the first equation in (2.8+) becomes

$$D_1\phi + \mathrm{i}D_2\phi = 0,$$

which says in view of the ∂^*-Poincaré lemma (see [9]) that, if p is a zero of ϕ, then there holds the expression

$$\phi(z) = z^n h(x^1, x^2), \qquad z = x^1 + \mathrm{i}x^2 \tag{3.1}$$

in a neighborhood of $z(p) = x(p) = 0$, where h is a non-vanishing smooth function and the multiplicity $n > 0$ is an integer. In this case people say there is a string passing through p with the winding number n, or simply say there are n strings at p.

The fundamental existence problem is this: Given a 2-manifold M and $p_1, ..., p_m \in M$, $n_1, ..., n_m \in \mathbb{N}$ (the set of positive integers), does the system (2.8+) have a solution triplet (g, ϕ, A) so that (M, g) is a Riemannian manifold, $p_1, ..., p_m$ are exactly the zeros of ϕ with corresponding multiplicities $n_1, ..., n_m$, and the total energy (the energy per unit length of strings)

$$E = \int_M \mathcal{E}\mathrm{mg}\sqrt{g}\,\mathrm{d}^2x + \int_M K\sqrt{g}\,\mathrm{d}^2x \tag{3.2}$$

is finite? Such a solution describes $N = n_1 + \cdots + n_m$ cosmic strings located at $p_1, ..., p_m$.

To see the structure of the above problem, we need a further reduction as in [5]. Since ϕ has the local representation (3.1) around each point $p = p_\ell$ (with $n = n_\ell$), $\ell = 1, ..., m$, one sees that the substitution $u = \ln|\phi|^2$ puts the first two equations in (2.8+) into the form

$$\Delta_g u = e^u - 1 + 4\pi\sum_{\ell=1}^{m} n_\ell \delta_{p_\ell}, \tag{3.3}$$

where Δ_g is the Laplace-Beltrami operator with respect to the metric g:

$$\Delta_g u = \frac{1}{\sqrt{g}}\partial_j(\sqrt{g}g^{jk}\partial_k u),$$

and δ_p is the Dirac distribution on (M, g) concentrated at p.

However, in view of the first two equations in (2.8+), we can rewrite the current vector (2.5) in the form

$$J_k = \frac{1}{2}A_k - \frac{1}{4}\epsilon_k^j \partial_j |\phi|^2. \tag{3.4}$$

Thus using the first two equations in (2.8+) again, we get

$$\nabla_j(\epsilon^{jk}J_k) = -\frac{1}{4}(|\phi|^2 - 1) + \frac{1}{4}\Delta_g|\phi|^2.$$

As a consequence, the third equation in (2.8+) is reduced to

$$K + 2\pi G([|\phi|^2 - 1] - \Delta_g|\phi|^2) = 0, \tag{3.5}$$

or

$$K + 2\pi G([e^u - 1] - \Delta_g e^u) = 0. \tag{3.6}$$

In summary, (2.8+) is now reduced to the coupled equations (3.3) and (3.6) with unknown (g, u), which looks difficult to tackle. To go on, we introduce the following standard device. Suppose g_0 is a prescribed Riemannian metric on M and the unknown metric g is related to g_0 by a pointwise conformal deformation $g = e^\eta g_0$ where η is an unknown function on M. Let K_0 denote the Gaussian curvature of (M, g_0). Then it is well-known that K_0 and K are related through the two-dimensional Yamabe equation (see Aubin [1] or Kazdan and Warner [13])

$$-\Delta_{g_0}\eta + K_0 = Ke^\eta. \tag{3.7}$$

On the other hand, since $\Delta_g F = e^{-\eta}\Delta_{g_0}F$ for any $F \in C^2(M)$, we have by using (3.7) in (3.6) that

$$-\Delta_{g_0}\eta + K_0 + 2\pi G(e^\eta[e^u - 1] - \Delta_{g_0}e^u) = 0. \tag{3.8}$$

Besides, (3.3) becomes

$$\Delta_{g_0}u = e^\eta(e^u - 1) + 4\pi\sum_{\ell=1}^{m} n_\ell \delta_{p_\ell}, \tag{3.9}$$

where, now, δ_p is the Dirac distribution on (M, g_0).

The equations (3.8)–(3.9) are the reduced form of (2.8+) for an $N = n_1 + \cdots + n_m$ cosmic string solution (g, ϕ, A) with strings located at $p_1, ..., p_m$. So we are led to the following important questions.

(Q_1) Under what condition on (M, g_0), can a solution of (3.8)–(3.9) be used to construct a desired N-string solution of (2.8+)?

(Q2) For what (M,<7o), can the coupled equations (3.8)-(3.9) have a solution?

In Section 4 we shall solve a special case of (Q2)- In the rest of this section, we give an answer to (Qi).

Theorem S.I. ***If*** *M is a non-compact Riemann surface and g_0 is the Riemannian metric induced from a hermitian metric on M, then a solution pair* (77, u) *of (3.8)-(3.9) can be used to get a cosmic string solution triplet* (<7,<£,A) *of the coupled Einstein-BogomoVnyi equations (2.8+) so that $g = e^v go$, \(f>\$^2$ = e^u, the strings are located at* J>i,-..,p_m, *and the corresponding winding numbers are* ni,...,n_m.

Proof. Let h_0 be the hermitian metric on the Riemann surface M so that g_0 is induced from h_0. Define $h = e^v ho$ and $g = e^v go$. Then g is induced from the new hermitian metric h over M. For Pi,...,p_m and ni,...,n_m given in the theorem, introduce the divisor

$$D = \sum_{\ell=1}^{m} n_\ell \cdot p_\ell$$

on M. Since M is non-compact, it is standard that D is the divisor of global meromorphic function $\tilde{4}$> on M. The fact that *rn* > 0 *(£* = l,...,m) implies <$\tilde{£}$ is actually holomorphic. We can construct a line bundle $\tilde{L}$ from D. Denote by {e_a : U —> $\tilde{L}$} (U_a C M) a set of local holomorphic frames of $\tilde{L}$ and by

$\tilde{g}_a b$: £/« n l/i -> GL(1, C) = C - {0}

the corresponding transition functions: e& = $\wedge_a$6$\tilde{e}_a$. The triviality of $\tilde{L}$ says that $<\tilde{f}>$ gives rise to a global holomorphic section of $\tilde{L}$ —• Af. For simplicity, such a section is still denoted by ^. Of course (<?,u) = (e^po^tx) solves (3.3) and (3.6). We now define a global hermitian metric $\tilde{h}$ for $\tilde{L}$ by setting

*«,.«.) = i£p. (3.10)

where <£ = $\wedge_a \tilde{e}_o$. By virtue of *fo* = *g~j<j>$_a$*, $\tilde{h}$ is well-defined.

Let $\tilde{V}$^ be the unique metric connection of $\tilde{L}$ associated with /i, $\tilde{A}$ being the connection vector which is of type (1,0):

$\tilde{A}$ = (A$\tilde{i}$ — L$\tilde{4}_2$)d2:

$$= \tilde{A}_1 dx^1 + \tilde{A}_2 dx^2 - i(\tilde{A}_2 dx^1 - \tilde{A}_1 dx^2).$$

On t/o, we have *gjk* = n ^ ^ and

$$(\tilde{\nabla}_{\tilde{A}} \tilde{\phi})^{(0,1)}_a = 0. \quad (3.11)$$

Moreover, since $\tilde{\nabla}_{\tilde{A}}$ is canonical, using (3.3), (3.10), and $\Delta \ln|\tilde{\phi}_a| = 2\pi\sum_{\ell=1}^{m} n_\ell \delta_{p_\ell}$, we see [9] that the curvature 2-form $\tilde{F} = \mathrm{d}\tilde{A}$ satisfies

$$\begin{aligned}\tilde{F}_{12} &= \partial_1 \tilde{A}_2 - \partial_2 \tilde{A}_1 = -\tfrac{1}{2}\Omega_{U_a}\Delta \ln \tilde{h}(\tilde{\mathrm{e}}_a, \tilde{\mathrm{e}}_a) \\ &= -\tfrac{1}{2}\Omega_{U_a}(e^u - 1) = -\tfrac{1}{2}\Omega_{U_a}(\tilde{h}(\tilde{\phi},\tilde{\phi}) - 1),\end{aligned} \tag{3.12}$$

where, and in the sequel, $\Delta = \partial_1^2 + \partial_2^2$. Our goal is to show that (2.8+) can be recovered from (3.11)–(3.12).

Let $L = M \times \mathbf{C}$ be the trivial line bundle equipped with the standard hermitian structure. We shall find our solution triplet (g, ϕ, A) of (2.8+) from $(g, \tilde{\phi}, \tilde{A})$ so that the complex scalar ϕ is a cross-section of $L \to M$, the vector field A is represented as a real-valued connection 1-form: $(\nabla_A \phi)_j = D_j\phi = \partial_j \phi - \mathrm{i}A_j\phi$ (in local coordinates), and F_{jk} is recognized as the curvature 2-form of $L \to M$ determined by A: $F = \mathrm{d}A$. For this purpose, we recall that there is an isomorphism $f : L \to \tilde{L}$. Denote by $\{\mathrm{e}_a : U_a \to L\}$ a set of local frames of L and $\{g_{ab}\} \subset \mathbf{C} - \{0\}$ the corresponding set of transition functions. If $f(\mathrm{e}_a) = f_a \tilde{\mathrm{e}}_a$, then $f_b = g_{ab} f_a \tilde{g}_{ab}^{-1}$. Thus by setting

$$\tau(\mathrm{e}_a) = \frac{f(\mathrm{e}_a)|\mathrm{e}_a|}{|f_a|\tilde{h}^{1/2}(\tilde{\mathrm{e}}_a, \tilde{\mathrm{e}}_a)},$$

we obtain from f an isomorphism $\tau : L \to \tilde{L}$ satisfying

$$\tau(\mathrm{e}_a) = \tau_a \tilde{\mathrm{e}}_a, \qquad |\tau_a| = \frac{|\mathrm{e}_a|}{\tilde{h}^{1/2}(\tilde{\mathrm{e}}_a, \tilde{\mathrm{e}}_a)}.$$

As a consequence, we get a 1-1 map τ_* so that if ϕ and A are a cross-section and a connection 1-form of $L \to M$, then

$$\begin{aligned}\tilde{\phi} &= \tau_*(\phi) = \tau_a \phi_a \tilde{\mathrm{e}}_a, \\ \tilde{A} &= \tau_*(A) = \mathrm{i}\tau_a \mathrm{d}\tau_a^{-1} + A\end{aligned}$$

are those for the line bundle $\tilde{L} \to M$. Thus there holds

$$\tilde{\nabla}_{\tau_*(A)}\tau_*(\phi) = \tau_a(\mathrm{d}\phi_a - \mathrm{i}A\phi_a)\tilde{\mathrm{e}}_a. \tag{3.13}$$

Besides, (3.10) implies the relation

$$\tilde{h}(\tau_*(\phi), \tau_*(\psi)) = \phi\psi^*. \tag{3.14}$$

We are now ready to construct a solution of (2.8+). Let $\tilde{A}$ be the complex-valued 1-form which yields the unique metric connection of $\tilde{L} \to M$ equipped with the hermitian structure $\tilde{h}$. Choose a 1-form A on M so that $\tau_*(A) = \tilde{A}$. Assume that ∇_A is the connection of $L \to M$ induced from A. Using (3.13)–(3.14) we see that for any two cross-sections (or complex-valued functions) ϕ, ψ of $L \to M$,

$$\begin{aligned} \mathrm{d}(\phi\psi^*) &= d\tilde{h}(\tau_*(\phi), \tau_*(\psi)) \\ &= \tilde{h}(\tilde{\nabla}_{\tau_*(A)}\tau_*(\phi), \tau_*(\psi)) + \tilde{h}(\tau_*(\phi), \tilde{\nabla}_{\tau_*(A)}\tau_*(\psi)) \\ &= (\nabla_A\phi)\psi^* + \phi(\nabla_A\psi)^*, \end{aligned}$$

which implies that ∇_A is the metric connection of $L \to M$. In particular, A must be a real-valued 1-form.

With $\tau_*(\phi) = \tilde{\phi}$ and $\tau_*(A) = \tilde{A}$ where $\tilde{\phi}$ and $\tilde{A}$ satisfy (3.11)–(3.12), we have after a straightforward calculation that, in the chart $(U_a, (x^j))$,

$$\begin{aligned} \tfrac{1}{2}(D_1\phi + \mathrm{i}D_2\phi) &= (\nabla_A\phi)_a^{(0,1)} \\ &= \tau_a^{-1}(\tilde{\nabla}_{\tilde{A}}\tilde{\phi})_a^{(0,1)} = 0, \\ F_{12} &= \mathrm{Re}(\mathrm{d}\tilde{A})_a \\ &= \tilde{F}_{12} = -\tfrac{1}{2}\Omega_{U_a}(|\phi|^2 - 1). \end{aligned}$$

Hence the first two equations in (2.8+) are recovered. This fact also says that the current vector J_k satisfies (3.4). However, from the definitions of $\tilde{\phi}$ and ϕ, we have $|\phi|^2 = e^u$. Consequently the third equations in (2.8+) follows immediately from (3.5) and the theorem is proved. □

4 Existence of Cosmic Strings

In this section we consider the existence of a solution to the system (3.8)–(3.9) where the non-compact Riemannian 2-manifold (M, g_0) is to be specified. Of course, the most convenient assumption is that (M, g_0) =the Euclidean plane $\mathbf{R}^2$. Thus, in (3.8)–(3.9), $\Delta_{g_0} = \Delta$, $K_0 = 0$, and the unknown functions η, u satisfy the equations

$$\begin{aligned} \Delta\eta &= 2\pi G(e^\eta[e^u - 1] - \Delta e^u), \\ \Delta u &= e^\eta(e^u - 1) + 4\pi\sum_{\ell=1}^{m} n_\ell\delta_{p_\ell}, \end{aligned} \qquad x \in \mathbf{R}^2. \tag{4.1}$$

At this moment, we are not able to prove a general existence theorem for (4.1). However, when the points $p_1, ..., p_m$ coincide, we can obtain the following result.

Theorem 4.1. *Assume that $p_1 = \cdots = p_m = p$ and that $N = n_1 + \cdots + n_m$ satisfies*

$$2\pi NG < 1. \tag{4.2}$$

For any constant α verifying

$$\alpha \geq \ln\left(1 + \frac{1}{2\pi G}\right), \tag{4.3}$$

the equations (4.1) have a solution pair $(\eta^{(\alpha)}, u^{(\alpha)})$ so that it is radially symmetric about the point p, $u^{(\alpha)} < 0$ in $\mathbf{R}^2$, and

$$\max_{x\in\mathbf{R}^2}\{u^{(\alpha)}(x)\} = -\alpha.$$

Moreover, as functions of the radial variable $r = |x - p|$, $u^{(\alpha)}$ is strictly concave and there hold the asymptotic decay estimates

$$e^{\eta^{(\alpha)}} = o(r^{-4}), \quad e^{u^{(\alpha)}} = o(r^{-2N-\frac{2}{\pi G}(1-2\pi NG)}) \quad \textit{for large } r. \tag{4.4}$$

This theorem will be established in several steps.

First we recall a useful reduction for (4.1) made in [5] . Put

$$u_0(x) = 2\sum_{\ell=1}^{m} n_\ell \ln|x - p_\ell|$$

and $u = u_0 + v$. Then (4.1) becomes

$$\begin{aligned} \Delta\eta &= 2\pi G(e^{\eta}[e^{u_0+v} - 1] - \Delta e^{u_0+v}), \\ \Delta v &= e^{\eta}(e^{u_0+v} - 1), \end{aligned} \qquad x \in \mathbf{R}^2. \tag{4.5}$$

Insert the second equation in (4.5) into the first one, we have

$$\Delta(\eta - 2\pi Gv + 2\pi Ge^{u_0+v}) = 0.$$

Thus it is reasonable to impose the relation

$$\eta = 2\pi G(v - e^{u_0+v}) + a, \tag{4.6}$$

where a is an arbitrary constant. As a consequence, (4.5) is reduced to

$$\Delta v = a_0 e^{2\pi G(v - e^{u_0+v})}(e^{u_0+v} - 1), \qquad x \in \mathbf{R}^2 \tag{4.7}$$

with $a_0 = e^\circ > 0$. We shall study the special case of (4.7) when $p_x = \bullet\bullet\bullet = p_m = p$. Without loss of generality, we assume that p is the origin. Then $u_0 = 2N$ In r (r = \x\) and a radially symmetric solution of (4.7) must satisfy

$$v_{rr} + \frac{1}{r} v_r = a_o e \wedge^{G(v'} \wedge^{v} \backslash e \wedge^{v} - 1), \qquad r > 0. \tag{4.8}$$

We hope to obtain a solution of (4.7) from a solution t; of (4.8). However, by virtue of the well-known removable singularity theorem (see [19]), we easily see that, v can be extended to the full R^2 to get a smooth solution of (4.7) if and only if

$$\wedge_{r \to o} \frac{l}{\ln r} = 0 . \tag{4.9}$$

In fact, assume that $v(r)$ is a solution of (4.8)-(4.9). Then, for any 6, &i > 0, there holds

$$]\mathrm{imiV}{**}>_{r \to 0} = \mathrm{lim}_{r \to 0}\, e \wedge \wedge \wedge) = 0.$$

As a consequence, we can view the right-hand side of (4.8) as an L^2(ft) function, where ft is a small neighborhood of the origin. Therefore the *I?* theory of elliptic equations says that there is a $W^{2j2}(Ct)$ function w so that Aw =the right-hand side of (4.8) in ft. ^From the embedding $W^{2,2}$(ft) -* C*($\bar{ft}$) (0 < a < 1), we see that w € $C^a(\bar{£l})$. However, since / = v — tx; is harmonic in ft — {0} and satisfies (4.9), we see that / is smooth and harmonic in the entire ft. Therefore v is C^a in ft. Finally, the C^a theory and a bootstrap argument imply that t; is smooth.

Thus, from now on, the equation (4.9) will serve as our boundary condition at r = 0 for a solution of (4.8).

To work on (4.8), it may be more convenient to replace v by the old variable $u = u_0 + v$ = 2JVlnr + v. Thus (4.8) is equivalent to

$$u_{rr} + \frac{1}{r} u_r = a_o r^{-4 \wedge G} e^{2 \wedge u - cW})(e^{w} - 1), \qquad r > 0, \tag{4.10}$$

and (4.9) takes the form

$$\lim_{r \to 0} \frac{\wedge}{\ln r} = 2N. \tag{4.11}$$

The statement in Theorem 4.1 says that we are looking for solutions of (4.10) satisfying

$$\mathrm{Um}_{r \to \infty} \mathrm{ti}(r) = -oo. \tag{4.12}$$

Those solutions must enjoy the following simple property.

Lemma 4.2. *Ifu{r) verifies (4.10)-(4.12), then u(r) < 0 for all r > 0.*

Proof. The boundary condition (4.11) implies in particular that $\lim_{r\to 0} u(r) = -\infty$. Therefore we can get an $r_0 > 0$ such that

$$u(r_0) = \max_{r>0}\{u(r)\},$$

$u_{rr}(r_0) \le 0$, and $u_r(r_0) = 0$. Using these facts in (4.10) we find $u(r_0) \le 0$. However, $u(r_0) \ne 0$ since otherwise we would have by using $u_r(r_0) = 0$ the conclusion $u(r) \equiv 0$ (the uniqueness theorem for the initial value problems of ordinary differential equations), which is false. Thus $u(r) \le u(r_0) < 0$ for all $r > 0$. □

Consequently, for a solution of (4.10)–(4.12), there are constants $r_0 > 0$ and $\alpha > 0$ so that

$$u(r_0) = -\alpha, \qquad u_r(r_0) = 0. \tag{4.13}$$

In the following, we shall find suitable $r_0, \alpha > 0$ so that the solution of (4.10) subject to the initial condition (4.13) will satisfy the boundary constraints (4.11)–(4.12). Such a goal will be achieved by a two-side shooting argument.

It is useful to employ the change of variable

$$t = \ln r, \qquad t_0 = \ln r_0.$$

Then we are led to the initial value problem

$$\begin{aligned} u'' &= a_0 e^{2(1-2\pi NG)t} e^{2\pi G(u-e^u)}(e^u - 1), \quad -\infty < t < \infty, \\ u(t_0) &= -\alpha, \quad u'(t_0) = 0, \end{aligned} \tag{4.14}$$

where $u' = \mathrm{d}u/\mathrm{d}t$ and $u(t)$ denotes the dependence of u on the new variable t which should not be confused with the notation $u(r)$.

We have the following basic result concerning (4.14).

Lemma 4.3. *For given $\alpha > 0$, (4.14) has a unique global solution $u(t)$. This solution satisfies $u(t) < 0$ and*

$$\lim_{t\to-\infty} u(t) = -\infty, \qquad \lim_{t\to\infty} u(t) = -\infty. \tag{4.15}$$

Moreover, both

$$\lim_{t\to-\infty} u'(t) = \beta_- \quad \textit{and} \quad \lim_{t\to\infty} u'(t) = -\beta_+ \tag{4.16}$$

are finite numbers and

$$\beta_+ > \max\left\{0, \frac{1-2\pi NG}{\pi G}\right\}. \tag{4.17}$$

Proof. For a local solution of (4.14), we have

$$u'(t) = a_0 \int_{t_0}^{t} e^{2(1-2\pi NG)s} e^{2\pi G(u(s)-e^{u(s)})}(e^{u(s)} - 1)\,\mathrm{d}s \tag{4.18}$$

in the interval of existence. We claim that, for all t where $u(t)$ exists, we have $u(t) < 0$. Otherwise, if there is a $\bar{t}$ verifying $u(\bar{t}) \geq 0$, we may first assume $\bar{t}$ to be such that $\bar{t} \geq t_0$ and

$$\bar{t} = \inf\{t \geq t_0 \,|\, u(t) \text{ exists and } u(t) \geq 0\}.$$

The fact that $\alpha < 0$ implies $\bar{t} > t_0$ and $u(\bar{t}) = 0$. Of course $u(t) < 0$, $t \in [t_0, \bar{t})$. Inserting this result into (4.18) we get $u'(t) < 0$, $t \in (t_0, \bar{t}]$. In particular, $u(\bar{t}) < 0$, a contradiction. Similarly, the assumption $\bar{t} < t_0$ will also lead to a contradiction.

Using the property $u(t) < 0$ and (4.18), we see that $|u'(t)|$ cannot blow up in finite time. Therefore the existence holds globally in $(-\infty, \infty)$ for a solution of (4.14).

On the other hand, applying $u < 0$ in (4.14) we have $u''(t) < 0$, $t \in (-\infty, \infty)$. So $u'(t)$ is strictly decreasing. In particular, either $u'(t) \to \infty$ or $u'(t) \to$ a finite positive number as $t \to -\infty$ since $u'(t_0) = 0$. Thus we always have $u(t) \to -\infty$ as $t \to -\infty$. Similarly, since $u'(t) \to -\infty$ or $u'(t) \to$a finite negative number as $t \to \infty$, we must have $u(t) \to -\infty$ as $t \to \infty$. Hence the boundary behavior (4.15) is established.

To prove the first result in (4.16), we assume otherwise that

$$\begin{aligned} \lim_{t\to-\infty} u'(t) &= -a_0 \int_{-\infty}^{t_0} e^{2(1-2\pi NG)s} e^{2\pi G(u(s)-e^{u(s)})}(e^{u(s)} - 1)\,\mathrm{d}s \\ &= \infty. \end{aligned} \tag{4.19}$$

Therefore the L'Hôpital rule implies

$$\lim_{t\to-\infty} \frac{u(t)}{t} = \lim_{t\to-\infty} u'(t) = \infty. \tag{4.20}$$

On the other hand, the integral in (4.19) has the bound

$$\left| \int_{-\infty}^{t_0} \right| \leq C_1 + C_2 \int_{-\infty}^{t_1} e^{\left[2(1-2\pi NG)+2\pi G\frac{u(s)}{s}\right]s}\,\mathrm{d}s, \tag{4.21}$$

where $t_1 \leq \min\{t_0, -1\}$ and $C_1, C_2 > 0$ are constants depending on t_1. ¿From (4.20) we can find a $t_2 \leq t_1$ so that

$$2(1 - 2\pi NG) + 2\pi G\frac{u(s)}{s} \geq 1, \qquad s \leq t_2.$$

Thus the right-hand side of (4.21) is finite, which contradicts (4.19).

Finally, we show the validity of the second result in (4.16). As observed earlier, we have $u'(t) \to -\infty$ or $u'(t) \to$a finite negative number, $-\beta_+$, as $t \to \infty$. However, assuming the former possibility will lead to

$$\lim_{t\to\infty} \frac{u(t)}{t} = -\infty.$$

Then using the same argument as that for the case $t \to -\infty$, we reach the contradiction $|\lim_{t\to\infty} u'(t)| < \infty$. Hence $u'(t) \to -\beta_+$ as $t \to \infty$ and $u'(t) > -\beta_+$. In particular,

$$u(t) > -\beta_+ t + C, \qquad t \geq t_0,$$

where C is a constant. But the convergence of the integral (4.18) as $t \to \infty$ and (4.15) imply the convergence of

$$\int_{t_0}^{\infty} e^{2(1-2\pi NG)s+2\pi Gu(s)}\,\mathrm{d}s.$$

Consequently,

$$e^C \int_{t_0}^{\infty} e^{(2[1-2\pi NG]-2\pi G\beta_+)s}\,\mathrm{d}s \;<\; \int_{t_0}^{\infty} e^{2(1-2\pi NG)s+2\pi Gu(s)}\,\mathrm{d}s$$
$$< \infty.$$

Therefore

$$\beta_+ > \frac{1-2\pi NG}{\pi G}.$$

□

We now denote the dependence of the solution u of (4.14) on the initial data t_0, α by $u = u(t; t_0, \alpha)$. Using Lemma 4.3 and (4.20), the boundary condition (4.11) reads

$$\beta_-(t_0, \alpha) = \lim_{t\to-\infty} u'(t; t_0, \alpha) = 2N. \tag{4.22}$$

Recall that $\beta_-(t_0, \alpha)$ can be expressed by the formula

$$\beta_-(t_0, \alpha) = -a_0 \int_{-\infty}^{t_0} e^{2(1-2\pi NG)s} e^{2\pi G(u(s;t_0,\alpha)-e^{u(s;t_0,\alpha)})}(e^{u(s;t_0,\alpha)} - 1)\,\mathrm{d}s \tag{4.23}$$

(see (4.18)). Assume that (4.2) is fulfilled. Since $u < 0$, the integral in (4.23) is uniformly convergent with respect to the variables t_0, α. Hence $\beta_-(t_0, \alpha)$ is continuous. We shall show that, for suitable $t_0 \in \mathbf{R}$ and $\alpha > 0$, (4.22) can be verified.

Lemma 4.4. *Suppose that the condition (4.2) holds. Then for any α satisfying (4.3), there is a $t_0 = t_0(\alpha)$ such that the unique solution $u(t; t_0, \alpha)$ of (4.14) fulfills (4.22).*

Proof. From (4.14) we have

$$u'' \geq -a_0 e^{2(1-2\pi NG)t+2\pi Gu}.$$

Set $w = 2(1 - 2\pi NG)t + 2\pi Gu$. Then

$$w'' \geq -a_0 e^{w}. \tag{4.24}$$

However, since $u' > 0$ for $t < t_0$, we have in view of (4.2) that $w' > 0$, $t \in (-\infty, t_0]$. Multiplying (4.24) by w' and integrating on the interval (t, t_0), we arrive at

$$4(1-2\pi NG)^2 - (w'(t))^2 \geq 2a_0 (e^{w(t)} - e^{2(1-2\pi NG)t_0 - 2\pi G\alpha}), \quad t < t_0.$$

Thus

$$0 < 2\pi Gu'(t;t_0,\alpha) < \sqrt{4(1-2\pi NG)^2 + 2a_0 e^{2(1-2\pi NG)t_0-2\pi G\alpha}} - 2(1-2\pi NG)$$
$$\equiv 2\pi G\sigma, \quad t < t_0, \tag{4.25}$$

which implies the useful inequality

$$-\alpha > u(t;t_0,\alpha) > -\alpha - \sigma(t_0 - t), \qquad t < t_0. \tag{4.26}$$

Again, from (4.14), we have by using $u < 0$ that

$$u'' \leq a_1 e^{2(1-2\pi NG)t} e^{2\pi Gu}(e^{u} - 1), \tag{4.27}$$

where $a_1 = a_0 e^{-2\pi G}$.

Consider the function $f(u) = e^{2\pi Gu}(e^u - 1)$. It is easily seen that $f(u)$ is decreasing for $u \in (-\infty, -\ln(1 + 1/2\pi G)]$. Thus the condition (4.3) and (4.26) imply that

$$-\ln\left(1 + \frac{1}{2\pi G}\right) > u(t;t_0,\alpha) > -\alpha - \sigma(t_0 - t), \qquad t < t_0. \tag{4.28}$$

Using (4.28) in (4.27) gives

$$u'' \leq a_1 e^{2(1-2\pi NG)t} e^{-2\pi G(\alpha+\sigma(t_0-t))}(e^{-(\alpha+\sigma(t_0-t))} - 1), \qquad t < t_0. \tag{4.29}$$

Integrating (4.29) over $(-\infty, t_0)$, we find

$$B(t_0,\alpha) > a_1 e^{2(1-2\pi NG)t_0 - 2\pi G\alpha}\left(\frac{1}{2(1-2\pi NG)+2\pi G\sigma} - \frac{e^{-\alpha}}{2(1-2\pi NG)+2\pi G\sigma+\sigma}\right).$$

However, since the condition (4.3) can be rewritten as

$$e^{-\alpha} \leq \frac{2\pi G}{2\pi G + 1},$$

we obtain from the above

$$\begin{aligned}\beta_-(t_0,\alpha) &\geq a_1 e^{2(1-2\pi NG)t_0-2\pi G\alpha}\frac{1}{(2[1-2\pi NG]+2\pi G\sigma)(2\pi G+1)} \\ &= \frac{a_1 h(t_0,\alpha)}{(2\pi G+1)\sqrt{4(1-2\pi NG)^2+2a_0 h(t_0,\alpha)}},\end{aligned} \tag{4.30}$$

where

$$h(t_0,\alpha) = e^{2(1-2\pi NG)t_0-2\pi G\alpha}.$$

¿From (4.30) we see that for given α satisfying (4.3), we can find $t_0 = t_0'$ so that $\beta_-(t_0',\alpha) > 2N$. On the other hand using (4.25), we see that for fixed α, there is some $t_0 = t_0''$ to make $\beta_-(t_0'',\alpha) < 2N$. Consequently the continuity of $\beta_-(t_0,\alpha)$ implies the existence of a point $t_0 = t_0(\alpha)$ between t_0' and t_0'' so that $\beta_-(t_0,\alpha) = 2N$. Hence the lemma follows. □

We now improve the lower bound for β_+ given in (4.17).

Lemma 4.5. *Let $u(t)$ be a solution of (4.14) produced in Lemma 4.3 so that (4.22) is fulfilled. Then the constant β_+ in (4.16) satisfies*

$$\beta_+ > 2N + \frac{2}{\pi G}(1-2\pi NG). \tag{4.31}$$

Proof. Multiplying (4.14) by u' and integrating over $(-\infty,\infty)$, we obtain formally

$$\begin{aligned}\beta_+^2 - 4N^2 &= -\frac{a_0}{\pi G}\int_{-\infty}^{\infty} e^{2(1-2\pi NG)t}\left[e^{2\pi G(u-e^u)}\right]' dt \\ &= -\frac{a_0}{\pi G}\left[e^{2(1-2\pi NG)t}e^{2\pi G(u(t)-e^{u(t)})}\right]_{-\infty}^{\infty} \\ &\quad +\frac{2a_0}{\pi G}(1-2\pi NG)\int_{-\infty}^{\infty} e^{2(1-2\pi NG)t}e^{2\pi G(u-e^u)}\, dt.\end{aligned} \tag{4.32}$$

Obviously (4.17) implies

$$\begin{aligned}\lim_{t\to\infty} e^{2(1-2\pi NG)t}e^{2\pi G(u(t)-e^{u(t)})} &\leq \lim_{t\to\infty} e^{2t[(1-2\pi NG)+\pi G\frac{u(t)}{t}]} \\ &= 0.\end{aligned}$$

Thus the first term on the right-hand side of (4.32) vanishes whereas the second term may be rewritten by using (4.18) and Lemma 4.3 in the form

$$\frac{2}{\pi G}(1-2\pi NG)\Big\{(\beta_+ + 2N) + a_0\int_{-\infty}^{\infty} e^{2(1-2\pi NG)t}e^{2\pi G(u-e^u)+u}\, dt\Big\}.$$

Inserting the above into (4.32) we have

$$\beta_+^2 - \frac{2}{\pi G}(1-2\pi NG)\beta_+ - 2N\Big(\frac{2}{\pi G}[1-2\pi NG]+2N\Big) > 0,$$

namely,

$$(\beta_+ + 2N)\Big(\beta_+ - \Big[\frac{2}{\pi G}(1-2\pi NG)+2N\Big]\Big) > 0.$$

Therefore the desired lower bound is found. □

Replacing t by the original variable $r = e^t$ in (4.14) and using Lemmas 4.3–4.5 and (4.6), we see that Theorem 4.1 is established.

Thus using Theorems 3.1 and 4.1, we obtain a family of string solutions of the Einstein–Bogomol'nyi equations (2.8+). In fact for our problem discussed here, we can actually construct a solution triplet (g, ϕ, A) of (2.8+) directly from a solution pair (η, u) of (4.1) produced in Theorem 4.1 by setting

$$\begin{aligned} g_{jk}(z) &= e^{\eta(z)}\delta_{jk}, \\ \phi(z) &= e^{\frac{1}{2}u(z)+\mathrm{i}N\arg(z)}, \\ A_1(z) &= -\operatorname{Re}\{2\mathrm{i}\partial^*\ln\phi(z)\}, \\ A_2(z) &= -\operatorname{Im}\{2\mathrm{i}\partial^*\ln\phi(z)\}, \end{aligned} \tag{4.33}$$

where

$$z = (x) = x^1 + \mathrm{i}x^2, \qquad \partial = (\partial_1 - \mathrm{i}\partial_2)/2.$$

Let us now compute several relevant physical quantities.

First of all, the total magnetic flux is, in view of (2.8+), (4.6), (4.18), (4.16), and Lemma 4.5,

$$\begin{aligned} \Phi &= \frac{1}{4\pi}\int_M \{\epsilon^{jk}F_{jk}\}\sqrt{g}\,\mathrm{d}^2x \\ &= -\frac{1}{4\pi}\int_{\mathbb{R}^2} e^{\eta}(e^u-1)\,\mathrm{d}^2x \\ &= -\frac{a_0}{2}\int_0^\infty r e^{2\pi G(v(r)-e^{u(r)})}(e^{u(r)}-1)\,\mathrm{d}r \\ &= -\frac{a_0}{2}\int_{-\infty}^\infty e^{2(1-2\pi NG)t+2\pi G(u(t)-e^{u(t)})}(e^{u(t)}-1)\,\mathrm{d}t \\ &= \frac{1}{2}(\beta_- + \beta_+) = N + \frac{1}{2}\beta_+. \end{aligned}$$

Next, using (4.4), (4.31), (4.16), and (2.8+), we have

$$
\begin{aligned}
g &= \mathrm{o}(r^{-4}), \\
|\phi|^2 &= \mathrm{o}(r^{-2N-\frac{2}{\pi G}(1-2\pi NG)}), \\
|D_j\phi|^2 &= \mathrm{o}(r^{-2N-\frac{2}{\pi G}(1-2\pi NG)-2}), \\
|F_{jk}| &= \mathrm{o}(r^{-4})
\end{aligned}
\qquad \text{for large } r. \tag{4.34}
$$

Furthermore, from (3.4), we can calculate

$$\nabla_j(\epsilon^{jk}J_k) = \frac{1}{4}\epsilon^{jk}F_{jk} + \frac{1}{2}\,\mathrm{Im}\{\mathrm{div}_g(\epsilon^{jk}\phi^*[D_k\phi])\}.$$

Therefore using the decay estimates (4.34) we find

$$\int_M \nabla_j(\epsilon^{jk}J_k)\sqrt{g}\,\mathrm{d}^2x = \pi\Phi.$$

Inserting the above result into (2.4) and (2.8+) we obtain the total energy (3.2) (the energy per unit length of the strings) for a solution of (2.8+):

$$E = \pi\Phi + 8\pi^2 G\Phi = \pi(8\pi G+1)\Big(N + \frac{1}{2}\beta_+\Big).$$

In summary, we can state

Theorem 4.6. *Suppose that N is a positive integer satisfying (4.2). Then the Einstein–Bogomol'nyi equations (2.8+) have a family of finite-energy distinct cosmic string solutions $(g^{(\alpha)}, \phi^{(\alpha)}, A^{(\alpha)})$ labelled by the parameter α in the range (4.3) so that the 2-manifold M, on which the strings reside, is $\mathbf{R}^2$, that the solutions all realize an arbitrarily prescribed string location $p \in \mathbf{R}^2$ and are radially symmetric about the point p, and that the winding number of the string is N. Moreover, there hold the decay estimates (4.34) for the solutions and*

$$\max_{x\in\mathbf{R}^2} |\phi^{(\alpha)}(x)|^2 = e^{-\alpha}.$$

Furthermore there is a constant $\beta(\alpha) > 1/\pi G$ such that the flux and the energy of the solution $(g^{(\alpha)}, \phi^{(\alpha)}, A^{(\alpha)})$ are given by

$$\Phi^{(\alpha)} = \beta(\alpha), \qquad E^{(\alpha)} = \pi(8\pi G+1)\beta(\alpha).$$

Note. From (3.7), (4.5), (4.6), and (4.34), the decay estimate for the Gaussian curvature K can also be obtained. For example, if $N \geq 1$, then $K = \mathrm{O}(1)$ at infinity.

5 The Case When $N = 0$

In such a situation (4.1) take the form

$$\begin{aligned} \Delta\eta &= 2\pi G(e^{\eta}[e^{u}-1]-\Delta e^{u}), \\ \Delta u &= e^{\eta}(e^{u}-1), \end{aligned} \qquad x \in \mathbf{R}^2. \tag{5.1}$$

Thus we can use the ansatz

$$\eta = 2\pi G(u - e^{u}) + a, \qquad a \in \mathbf{R}$$

as in Section 4 to reduce (5.1) into

$$\Delta u = a_0 e^{2\pi G(u-e^{u})}(e^{u}-1), \qquad a_0 = e^{a} > 0. \tag{5.2}$$

As in our previous paper [20], we can use the method of Chen and Li [4] to prove the symmetry of all global solutions of (5.2) under the hypothesis

$$\int_{\mathbf{R}^2} \exp 2\pi G u < \infty \tag{5.3}$$

This gives the following result which we state without proof.

Theorem 5.1. *Let u be a global solution of (5.2) satisfying the finiteness condition (5.3). Then u is radially symmetric and strictly decreasing about some point $p \in \mathbf{R}^2$. Moreover, $u(p) < 0$ and u is asymptotic to $-\beta \log r$ as $r = |x|$ tends to infinity, for some $\beta > 2$*

Therefore, it is natural to look for solutions of (5.2) which are radially symmetric about an arbitrarily prescribed point $p \in \mathbf{R}^2$. Without loss of generality, we can choose p to be the origin. In this special limit, (5.2) becomes

$$u_{rr} + \frac{1}{r}u_r = a_0 e^{2\pi G(u-e^{u})}(e^{u}-1), \qquad r > 0. \tag{5.4}$$

In order to obtain a solution of (5.2) from a solution of (5.4), it is standard to impose the boundary condition

$$u(0) = -\alpha, \qquad u_r(0) = 0. \tag{5.5}$$

It is well-known that, for given $\alpha \in \mathbf{R}$, (5.4)–(5.5) allow a unique local solution (see Berestycki, Lions, and Peletier [2]).

Lemma 5,2. *For any a € R, (5.4)"(5.5) have a unique global solution in* r > 0. *Furthermore*

$$\lim u(r) = oo, \quad \lim ru_r(r) = oo \quad t/a<0 \tag{5.6}$$

anrf

$$\text{Um}_{r\to\infty} u(r) = -oo, \quad \text{Um}_{r\to\infty} ru_r(r) = -p+ \quad if\, a > 0, \tag{5.7}$$

where the constant /?+ satisfies

$$\beta_+ > \wedge \quad . \tag{5.8}$$

Proof. Locally we have the representation

$$ru_r(r) = ao \int_0^r pe^{2\wedge u \wedge '' '' c^{u(p)}}(e^{''\wedge'} - l)dp. \tag{5-9}$$

Therefore, if u(0) = —a > 0, then *u(r)* > 0 for all r > 0 where u(r) exists. Suppose there is an ro > 0 so that *u(r)* exists in $[0,r_0)$ and *u(r)* —> oo as r —• r_0. However, since

$$\lim_{r\to r_0} e^{2\pi G(u(r)-e^{u(r)})}(e^{u(r)}-1) = 0,$$

we see that $ru_r(r)$ is bounded in [0,ro). This contradicts the assumption that *u(r)* —• oo as r —» r_0. Hence the solution is global.

An easy implication of (5.9) is that $u_r(r)$ > 0 in (0, oo). In particular *u(r)* is an increasing function. Thus either *u(r)* —>a finite positive number or (5.6) is true. It is obvious that the former possibility cannot occur in view of (5.9). Thus (5.6) holds.

If ti(0) = —a < 0, then u(r) < 0 for all r > 0 where *u(r)* exists. Hence the solution must be globally defined for all r > 0. Using (5.9) again we see that *u(r)* is decreasing. Thus we may show in a similar way that the first limit in (5.7) holds. As a consequence, $ru_r(r)$ —• —oo or $ru_r(r)$ —* —/?+ (a finite negative number) as r —• oo. We first exclude the former possibility.

In fact, if there is an $r_0 > 0$ so that

$$ru_r(r) <-\wedge\frac{2}{Q} \qquad \text{for } r \geq r_0,$$

then

$$tz(r) < u(r_0) - \frac{2}{\,}\ln\frac{r}{_0}, \qquad r \geq r_0.$$

Therefore

$$\int_{r_0}^{\infty} pe^{2\pi Gu(r)}\, dp < C \int_{r_0}^{\infty} /T^3\, d/> < oo$$

and the integral on the right-hand side of (5.9) is convergent when $r = \infty$. This contradicts the assumption made earlier. Hence the second limit in (5.7) holds as well and

$$ru_r(r) > -\beta_+, \qquad r > 0,$$

which says that

$$u(r) > -\beta_+ \ln r + u(1) \qquad \text{for } r \geq 1, \tag{5.10}$$

where $\beta_+ > 0$ is to be determined. However, since (5.10) implies

$$\int_1^\infty r^{(1-2\pi G\beta_+)}\,\mathrm{d}r < C\int_1^\infty re^{2\pi Gu(r)}\,\mathrm{d}r < \infty,$$

we find the condition

$$\beta_+ > \frac{1}{\pi G}. \tag{5.11}$$

Finally, multiplying (5.4) by $r^2 u_r(r)$, integrating by parts over $(0,\infty)$, and using (5.9), (5.11), we arrive at

$$\begin{aligned}\beta_+^2 &= -\frac{a_0}{\pi G}\Big[r^2 e^{2\pi G(u(r)-e^{u(r)})}\Big]_0^\infty + \frac{2a_0}{\pi G}\int_0^\infty re^{2\pi G(u(r)-e^{u(r)})}\,\mathrm{d}r \\ &= \frac{2}{\pi G}\beta_+ + \frac{2a_0}{\pi G}\int_0^\infty re^{2\pi G(u(r)-e^{u(r)})+u(r)}\,\mathrm{d}r.\end{aligned}$$

Therefore (5.8) follows. □

The inequality (5.8) can be viewed as a special case of (4.31) for $N = 0$. As in Section 4, we can use the solutions produced in Lemma 5.2 to construct a family of nontrivial 0-string solutions of (2.8+):

Theorem 5.3. *For any $\alpha > 0$ and $p \in \mathbf{R}^2$, the Einstein–Bogomol'nyi equations (2.8+) have a finite energy solution triplet $(g^{(\alpha)}, \phi^{(\alpha)}, A^{(\alpha)})$ over $\mathbf{R}^2$ so that there hold the decay estimates (4.34) with $N = 0$ and the solution is radially symmetric about p. Moreover*

$$\max_{x\in\mathbf{R}^2}|\phi^{(\alpha)}(x)|^2 = |\phi(p)|^2 = e^{-\alpha},$$

and $|\phi^{(\alpha)}|^2$ is strictly monotone decreasing with respect to the variable $r = |x-p|$.

6 The Matter-Gauge Sector

In the last two sections, we have obtained a family of solution triplet (g,ϕ,A) of the Einstein–Bogomol'nyi equations (2.8+) so that (M,g) is conformally equivalent to $\mathbf{R}^2$ and

$$g_{jk} = e^\eta \delta_{jk} = \mathrm{o}(r^{-4}) \quad \text{for large } r = |x|. \tag{6.1}$$

In view of (6.1), when the gravity sector is assumed to be known and only the string solutions of the Bogomol'nyi system (2.6) (namely the matter-gauge sector) are considered, we expect to find some new results. Such solutions describe magnetic strings or topological defects in a superconductor in a cosmological scale and are of independent interest. For greater generality, we assume in this section that $-\eta$ blows up at infinity like $\ln r$ and

$$e^{\eta} = o(r^{-3\kappa}), \qquad 1 < \kappa \leq 2 \tag{6.2}$$

for large $r = |x|$. Here κ is a constant. Note that no radial symmetry is imposed on η.

From (3.3) we see that the existence of a string solution of (2.6) (we shall concentrate on (2.6+)) for which the strings are located at $p_1, ..., p_m \in \mathbf{R}^2$ with the corresponding winding numbers $n_1, ..., n_m$ is equivalent to the solvability of the equation

$$\Delta u = e^{\eta}(e^u - 1) + 4\pi \sum_{\ell=1}^{m} n_\ell \delta_{p_\ell}. \tag{6.3}$$

In the case $e^{\eta} \equiv 1$, an existence and uniqueness theorem has been established in Jaffe and Taubes [12]. In our case (6.2), we shall adapt the method of McOwen [18] for the study of conformal deformation equations as for the electroweak vortices [21]. We proceed as follows.

Let u_0 be defined by

$$u_0 = -\sum_{\ell=1}^{m} n_\ell \ln(\sigma + |x - p_\ell|^{-2}), \qquad \sigma > 0.$$

Then

$$\Delta u_0 = 4\pi \sum_{\ell=1}^{m} n_\ell \delta_{p_\ell} - f_\sigma$$

with

$$f_\sigma(x) = 4\sigma \sum_{\ell=1}^{m} n_\ell (1 + \sigma|x - p_\ell|^2)^{-2}. \tag{6.4}$$

The change of variable $v = u - u_0$ then reduces (6.3) to the form

$$\Delta v = e^{\eta + u_0 + v} - e^{\eta} + f_\sigma. \tag{6.5}$$

Note that $e^{u_0} \geq 0$ is smooth. Choose $v_0 \in C^\infty(\mathbf{R}^2)$ to verify

$$v_0(x) = -\ln r, \qquad r = |x| \geq 1.$$

An integration by parts gives

$$-\int_{\mathbf{R}^2} \Delta v_0 \, \mathrm{d}x = -\int_{|x|\leq 1} \Delta v_0 \, \mathrm{d}x = 2\pi.$$

Define now $w = v - \alpha v_0$. Then (6.5) becomes

$$\Delta w = e^{\eta + u_0 + \alpha v_0 + w} - F, \tag{6.6}$$

where

$$F = e^{\eta} + \alpha \Delta v_0 - f_\sigma.$$

Since $\int f_\sigma \to 0$ as $\sigma \to \infty$ (see (6.4)), we can fix α and σ to make

$$0 < \alpha < \frac{1}{2\pi}\left(\int_{\mathbf{R}^2} e^{\eta}\,\mathrm{d}x - \int_{\mathbf{R}^2} f_\sigma\,\mathrm{d}x \right). \tag{6.7}$$

Construct the functionals

$$\begin{aligned} I(w) &= \int_{\mathbf{R}^2} \left(\frac{1}{2}|\nabla w|^2 - Fw \right) \mathrm{d}x, \\ J(w) &= \int_{\mathbf{R}^2} e^{\eta + u_0 + \alpha v_0 + w}\,\mathrm{d}x. \end{aligned}$$

In order to have I, J well-defined, we need to consider a suitable weighted Sobolev space: Choose $h_0 \in C^\infty(\mathbf{R}^2)$ so that $h_0 > 0$ and

$$h_0(x) = r^{-2\kappa} \qquad \text{for } r = |x| \geq 1.$$

Define the measure $\mathrm{d}\mu = h_0 \mathrm{d}x$ and set $L^q(\mathrm{d}\mu) = L^q(\mathbf{R}^2, \mathrm{d}\mu)$ $(q \geq 1)$. Denote by $\mathcal{H}$ the Hilbert space of L^2_{loc} functions for which

$$\|w\|^2_{\mathcal{H}} = \|\nabla w\|^2_{L^2(\mathrm{d}x)} + \|w\|^2_{L^2(\mathrm{d}\mu)} < \infty.$$

Then $\mathbf{R} \subset \mathcal{H}$ and the closed subspace $\mathbf{R}^\perp = \tilde{\mathcal{H}}$ in $\mathcal{H}$ is given by

$$\tilde{\mathcal{H}} = \left\{ w \in \mathcal{H} \,\Big|\, \int_{\mathbf{R}^2} w\,\mathrm{d}\mu = 0 \right\}.$$

The following important results are cited from [18].

Lemma 6.1. *There are constants $C, \beta > 0$ so that for any $w \in \tilde{\mathcal{H}}$,*

$$\int_{\mathbf{R}^2} e^{|w|}\,d\mu \leq C \exp\left[\beta \|\nabla w\|^2_{L^2(dx)} \right].$$

Lemma 6.2. *The Poincaré inequality is valid in $\tilde{\mathcal{H}}$:*

$$\|w\|^2_{L^2(d\mu)} \leq C \|\nabla w\|^2_{L^2(dx)}, \qquad w \in \tilde{\mathcal{H}}.$$

Lemma 6.3. *The injection $\mathcal{H} \to L^2(d\mu)$ is a compact embedding.*

In view of the above results, it is straightforward to show that I is weakly lower semi-continuous while J is weakly continuous in $\mathcal{H}$.

Consider the optimization problem

$$\min_{w\in\mathcal{S}}\{I(w)\}, \tag{6.8}$$

where

$$\mathcal{S}=\{w\in\mathcal{H}\,|\,J(w)=C_0\},\qquad C_0=\int_{\mathbf{R}^2}F\,\mathrm{d}x.$$

The condition (6.7) implies that $C_0>0$ and $\mathcal{S}\neq\emptyset$.

Lemma 6.4. *The problem (6.8) has a solution.*

Proof. For any $w\in\mathcal{H}$, we have a unique decomposition $w=\bar{w}+\tilde{w}$ with $\bar{w}\in\mathbf{R}$ and $\tilde{w}\in\tilde{\mathcal{H}}$. Thus for $w\in\mathcal{S}$ there holds

$$\bar{w}=\ln C_0-\ln\left[\int_{\mathbf{R}^2}e^{\eta+u_0+\alpha v_0+\tilde{w}}\,\mathrm{d}x\right]. \tag{6.9}$$

As a consequence,

$$\begin{aligned} I(w) &= \frac{1}{2}\|\nabla\tilde{w}\|^2_{L^2(\mathrm{d}x)}-\int_{\mathbf{R}^2}F\tilde{w}\,\mathrm{d}x-C_0\bar{w}\\ &= \frac{1}{2}\|\nabla\tilde{w}\|^2_{L^2(\mathrm{d}x)}-\int_{\mathbf{R}^2}F\tilde{w}\,\mathrm{d}x-C_0\ln C_0 \\ &\quad +C_0\ln\left[\int_{\mathbf{R}^2}e^{\eta+u_0+\alpha v_0+\tilde{w}}\,\mathrm{d}x\right]. \end{aligned} \tag{6.10}$$

However, by $h_0^{-1}\geq\varepsilon_0>0$ and Jensen's inequality,

$$\begin{aligned} \int_{\mathbf{R}^2}e^{\eta+u_0+\alpha v_0+\tilde{w}}\,\mathrm{d}x &\geq \varepsilon_0\int_{\mathbf{R}^2}e^{\eta+u_0+\alpha v_0+\tilde{w}}\,\mathrm{d}\mu\\ &\geq \varepsilon_0 C\exp\left[\int_{\mathbf{R}^2}(\eta+u_0+\alpha v_0)\,\mathrm{d}\mu\Big/\int_{\mathbf{R}^2}\mathrm{d}\mu\right]\\ &\geq C_1, \end{aligned} \tag{6.11}$$

where C_1 is a constant. Inserting (6.11) into (6.10) and using the Schwarz inequality and Lemma 6.2, we arrive at

$$I(w)\geq C_2\|\nabla\tilde{w}\|^2_{L^2(\mathrm{d}x)}-C_3, \tag{6.12}$$

where $C_2, C_3>0$ are constants independent of $w\in\mathcal{S}$. In particular, I is bounded from below on $\mathcal{S}$.

Let $\{w_n\}$ be a minimizing sequence of (6.8). Then (6.12) and Lemma 6.2 say that $\{\bar{w}_n\}$ is bounded in $\tilde{H}$. Using (6.9) and Lemma 6.1 we conclude that $\{\bar{w}_n\}$ is bounded in R as well. Hence $\{w_n\}$ has a subsequence which approaches some point w G H weakly. Such w is a solution of (6.8). a

Lemma 6.5. *Let w £7i be a solution of (6.8). Then w is a smooth solution of (6.6).*

Proof. By the Lagrange multiplier rule, there is a number A € R so that for any $\in \mathcal{H}$

$$j_{R2} (Vw \cdot V£ - FO\, dx = A\, J_{R2}\, c«+*+"*+\wedge\, dx. \qquad (6.13)$$

In (6.13), set **f** = 1. Then we find A = - 1 . Hence w is a weak solution of (6.6). The standard elliptic regularity theory then implies w is smooth. E

We now discuss the asymptotic behavior of the solution of (6.6). As in [18], for 6 G R and s € N, define $W^{*}_{\delta,s}$ to be the closure of the set of compactly supported C^{∞} functions over R^2 in the norm

$$\|w\|^2_{\delta,s} = \sum_{|\gamma|\le s} \|(1+|x|)^{\delta+|\gamma|} D^\gamma w\|^2_{L^2(dx)}.$$

The following useful results can be found in [18].

Lemma 6.6. *If s > 1 and 6 > —1, then W?$_6$ functions are continuous and vanishing at infinity.*

Lemma 6.7. *For —1 < 6 < 0, the Laplace operator* A *maps Wfj onto W£$_6$+$_2$* **and** *its range is*

$$\Delta(W^2_{2,\delta}) = \{h \in W^2_{0,\delta+2} \mid \int h\,dx = 0\}.$$

Lemma 6.8. *If(eH and* **A£ = 0,** *then £ = const.*

Lemma 6.9. *The solution w obtained in Lemma 6.5 tends to a constant at infinity.*

Proof. Denote the right-hand side of (6.6) by h_x. Then h_x G $L(dx)$ and

$$\int_{R^2} h_1\, dx = 0.$$

Using (6.2) it is seen that h_x € W£$_{+2}$ for -1 < 6 < 0. Thus by Lemma 6.7, there is a wi G W£$_5$ so that Awi = hi. Lemma 6.6 says Wi vanishes at infinity. Consequently Wi € $L^2(dfi)$. Moreover, since $Vw\backslash$ G $WQ^2_{>6+1}$ and 6 > —1, so Vt^ G

$L^2(dx)$. Therefore $w_1 \in \mathcal{H}$. Thus $w - w_1 \in \mathcal{H}$ and $\Delta(w - w_1) = 0$. By Lemma 6.8, $w - w_1$ =const. □

Let us now return from the solution w of (6.6) produced above to the original variable u:

$$u = u_0 + \alpha v_0 + w.$$

Thus we have the following sharp decay rate

$$e^u = O(r^{-\alpha}),$$

where α satisfies (6.7). Of course the function u here again gives rise to a solution pair (ϕ, A) of (2.6) so that

$$\begin{aligned} \Phi &= -\frac{1}{4\pi}\int_{\mathbb{R}^2} e^{\eta}(e^u - 1)\,dx \\ &= \frac{\alpha}{2} + \frac{1}{4\pi}\int_{\mathbb{R}^2} f_\sigma\,dx. \end{aligned} \tag{6.14}$$

In summary we have

Theorem 6.10. *For any $p_1, ..., p_m \in \mathbb{R}^2$, $n_1, ..., n_m \in \mathbb{N}$, and α, σ satisfying (6.7), the Bogomol'nyi equations (2.6) have a multi-string solution (ϕ, A) so that the strings are located exactly at $p_1, ..., p_m$ with respective winding numbers $n_1, ..., n_m$, the matter field ϕ obeys the sharp decay estimate $|\phi|^2 = O(r^{-\alpha})$ for large $r = |x|$, and the magnetic flux has the representation (6.14).*

7 Concluding Remarks

In this paper we have established the existence and behavior of a continuous family of finite energy solutions of the Einstein–Bogomol'nyi equations which are automatically the solutions of the coupled Einstein-matter-gauge equations. These solutions are all cylindrically symmetric and represent cosmic strings living in a conformally flat space. In particular, we conclude that a string distribution cannot uniquely determine a solution configuration of the Einstein-matter-gauge system. Furthermore we make the following remarks.

(i) The condition (4.2) presents a (sufficient) bound to the total number of strings superimposed at a point. Whether such a bound can be further improved remains open. However, since in our normalization, the gravitational constant G is typically of order 10^{-40}, (4.2) is not too severe a restriction.

(ii) Our solutions do not yield an asymptotically Euclidean spacetime metric as is generally expected for the gravitational effect. In fact this is a special feature of the Einstein–Bogomol'nyi system (2.8+) as was already observed in [5]. To see this, we rewrite (4.6) in the form

$$\sqrt{g} = e^{\eta} = a_0\Big(\Pi_{\ell=1}^{m}|x - p_\ell|^{n_\ell}\Big)^{-4\pi G}|\phi|^{4\pi G}e^{-2\pi G|\phi|^2},$$

where (g, ϕ, A) is a solution triplet of (2.8+) so that ϕ is related to u by the formula $\ln|\phi|^2 = u$. It can be shown that, if (g, ϕ, A) is a finite energy solution of (2.8+), then $|\phi|^2 \leq 1$ everywhere. Such a result may be obtained by an adaptation of the argument in the proof of Lemma 2.1 in [23] (although, here, the metric g is no longer asymptotically Euclidean). Thus we see that the metric g is necessarily asymptotically zero. As a consequence, it yields considerable flexibility for the behavior of field configurations in the category of finite energy . Due to this reason, non-uniqueness occurs.

(iii) Theorem 5.3 presents a family of 0-string solutions so that the gravitational metric enjoys essentially the same asymptotic properties as those in N-string solutions. This result suggests that strings may not be responsible for the cosmological phenomena occurring far away from local regions.

(iv) Our solutions are all in the sector $|\phi|^2 = 0$ at infinity. It is not known whether the Einstein–Bogomol'nyi system also allows nontrivial solutions in the sector $|\phi|^2 \neq 0$ at infinity.

(v) To show that the multi-string solutions of the matter-gauge Bogomol'nyi system (2.6) have finite energy, some extra assumptions on the decay rate of the first derivatives of the prescribed metric $g_{jk} = e^{\eta}\delta_{jk}$ in addition to (6.2) may have to be made.

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