NAMT
92-030

Nonlocal Superconductivity

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Research Report No. 92-NA-030

July 1992

Sponsors

U.S. Army Research Office
Research Triangle Park
NC 27709

National Science Foundation
1800 G Street, N.W.
Washington, DC 20550

[7] M. Chipot and D. Kinderlehrer. *Equilibrium Configurations of Crystals.* Technical Report 326, Institute for Mathematics and its Applications, 1987.

[8] Q. Du, M.D. Gunzburger, and J.S. Peterson. Analysis and approximation of the Ginzburg-Landau model of superconductivity, *SIAM Reviews*, 34:54–81, 1991.

[9] A.C. Eringen. Memory-dependent nonlocal electromagnetic elastic solids and superconductivity, *J. Math. Phys.* 32:787–796, 1991.

[10] I. Fonseca. Variational methods for elastic crystals. *Arch. Rat. Mech. Anal.*, 97:189–220, 1987.

[11] R.D. James and D. Kinderlehrer. Frustration in ferromagnetic materials, to appear in *Continuum Mechanics and Thermodynamics,* 1990.

[12] A.B. Pippard. *Proc. Roy. Soc. Lond., Ser. A*, 216:547, 1953.

[13] R.C. Rogers. A nonlocal model for the exchange energy in ferromagnetic materials. *J. of Integral Equations and Applications*, 3:85-127, 1991.

[14] L. Tartar. Compensated compactness and applications to partial differential equations. In R.J. Knops, editor, *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium.* Pitman, 1979.

[15] L. Tartar. The compensated compactness method applied to systems of conservation laws. In John M. Ball, editor, *Systems of Nonlinear Partial Differential Equations*, NATO ASI, C. Reidel Pub. Co., 1983.

[16] J.D. van der Waals. The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density, (in Dutch) *Verhandel. Konink. Akad. Weten. Amsterdam*, Vol. 8, 1893. (Translation by J.S. Rowlinson. *J. of Statistical Physics,* 20:197-244, 1979.)

[17] J.W. Walter, Jr. *On a Nonlocal Model of Fluid Phases.* Ph.D. thesis, Univ. of Minnesota, 1985.

[18] L.C. Young. *Calculus of Variations and Optimal Control Theory.* Chelsea, 1969.

2. Regions in the interior are expected to have lower concentrations of carriers (smaller values of ψ). In fact, interior regions would be nonsuperconducting ($\psi = 0$) if $\omega(\mathbf{x})/\gamma > 1$.

3. We expect to be able to formulate problems such that for certain values of γ and $\mathbf{H}$, we will be able to compute solutions such that for $\mathbf{x}$ in interior regions of the body we will have

$$0 < \psi(\mathbf{x}) < \sqrt{\frac{2}{3}\left(1 - \frac{\omega(\mathbf{x})}{\gamma}\right)}.$$

To us this would represent a Type-II superconducting material exhibiting an oscillatory vortex structure: Corollary 4 (and the construction in the proof of Lemma 3) indicates that at such points the minimizing sequence for the energy can be constructed by oscillating between the nonsuperconducting phase ($\psi = 0$) and a superconducting phase with concentration of charge carriers $\psi = \sqrt{\frac{2}{3}\left(1 - \frac{\omega(\mathbf{x})}{\gamma}\right)}$.

Note that one of the defects of our theory (and of the use of Young-measures in general) is that our results give no indication of the geometric structure of the vortices. However, we have retained important information about the average charge carrier concentration and the states that are averaged to get this quantity. In many applications this information may be sufficient to make this a useful model.

We are currently working on numerical calculations for these problems.

References

[1] S.S. Antman and R.C. Rogers. Steady-state problems of nonlinear electro-magneto-thermo-elasticity. *Arch. Rat. Mech. Anal.*, 95:279–323, 1986.

[2] J.M. Ball. A version of the fundamental theorem of Young measures. *PDE's and Continuum Models of Phase Transitions,* Lecture Notes in Physics, 344, M. Rascle, D. Serre, and M. Slemrod, eds. Springer, pp. 207-215.

[3] J.M. Ball and R.D. James. Fine phase mixtures as minimizers of energy. *Arch. Rat. Mech. Anal.*, 100:13–52, 1987.

[4] J. Bardeen. Theory of Superconductivity, *Encyclopedia of Physics*, 15, Springer, 1956.

[5] D. Brandon and R.C. Rogers. Nonlocal regularization of L.C. Young's tacking problem. *Applied Math. and Optimization*, 25:387–301, 1992.

[6] D. Brandon and R. Rogers. The Coercivity Paradox and Nonlocal Ferromagnetism. *Continuum Mechanics and Thermodynamics*, 4:1–21, 1992.

Proof. The existence of a sequence satisfying (4.38) is guaranteed by Lemma 3. Thus, we need only verify that this sequence is an infimizing sequence for $\mathcal{E}$. Suppose not, then there exists $(\tilde{\psi}, \tilde{\mathbf{A}}) \in L^4(\Omega; \mathbb{R}^+) \times H^1(\Omega; \mathbb{R}^3)$ such that $\mathcal{E}(\tilde{\psi}, \tilde{\mathbf{A}}) < \lim \mathcal{E}(\psi_n, \mathbf{A}_n)$. However, using (4.6) and the fact that $G^* \leq G$, we get

$$\begin{aligned} \mathcal{E}_R(\tilde{\psi}, \tilde{\mathbf{A}}) &\leq \mathcal{E}(\tilde{\psi}, \tilde{\mathbf{A}}) \\ &< \lim_{n\to\infty} \mathcal{E}(\psi_n, \mathbf{A}_n) \\ &= \mathcal{E}_R(\bar{\psi}, \bar{\mathbf{A}}). \end{aligned}$$

This contradicts the fact that $(\bar{\psi}, \bar{\mathbf{A}})$ is a minimizer of $\mathcal{E}_R$. □

Using the direct method of the calculus of variations described above and some other standard techniques, one can prove the following existence theorem for the relaxed problem.

Theorem 7 *Let* $\mathbf{H} \in L^2(\Omega; \mathbb{R}^3)$ *be given. Then there exists a solution* $(\psi, \mathbf{A}) \in L^4(\Omega; \mathbb{R}^+) \times H^1(\Omega; \mathbb{R}^3)$ *of the relaxed nonlocal minimization problem.*

To finish this section, we state the Euler-Lagrange necessary conditions for the relaxed nonlocal minimization problem.

Theorem 8 *Suppose* $\mathbf{H} \in L^2(\Omega; \mathbb{R}^3)$, *and in addition* $\mathbf{H} \times \mathbf{n}$ *(where n is the unit outward normal to* $\partial\Omega$*) is well defined in the sense of trace on the boundary of* Ω. *Then every solution* $(\psi, \mathbf{A}) \in L^4(\Omega; \mathbb{R}^+) \times H^1(\Omega; \mathbb{R}^3)$ *of the relaxed nonlocal minimization problem satisfies*

$$\text{curl curl } \mathbf{A}(\mathbf{x}) = \text{curl } \mathbf{H}(\mathbf{x}) + \psi(\mathbf{x}) \int_\Omega \mathbf{K}_1(\mathbf{x} - \mathbf{y}) \mathbf{A}(\mathbf{y}) \psi(\mathbf{y})\, d\mathbf{y}, \tag{4.40}$$

$$G^*_\psi(\psi(\mathbf{x}), \mathbf{x}) = \frac{2}{\gamma} \int_\Omega K_2(\mathbf{x} - \mathbf{y}) \psi(\mathbf{y})\, d\mathbf{y} - 2\mathbf{A}^*(\mathbf{x}) \int_\Omega \mathbf{K}_1(\mathbf{x} - \mathbf{y}) \mathbf{A}(\mathbf{y}) \psi(\mathbf{y})\, d\mathbf{y}. \tag{4.41}$$

at almost every $\mathbf{x} \in \Omega$. *Furthermore, at almost every boundary point* $\mathbf{x} \in \partial\Omega$ *we have*

$$\text{curl } \mathbf{A}(\mathbf{x}) \times \mathbf{n}(\mathbf{x}) = \mathbf{H}(\mathbf{x}) \times \mathbf{n}(\mathbf{x}). \tag{4.42}$$

Once again, the proof of this is standard.

5 Comments

Our results suggest that for various values of the parameter γ and the applied field $\mathbf{H}$, we can expect minimizers of the modified nonlocal energy to have the following characteristics.

1. Because of the dependence of the local lower-order term G on $\mathbf{x}$, we expect Superconducting charge carriers to be concentrated at the boundaries of the material.

1. *At almost every point* x € *tl at which* $(x) € [0, J§ (1 — ^^Elj]t&e *Young-measure fix of the sequence %}>$_n$ is a convex combination of two Dirac masses centered at* 0 *and* $\sqrt{\frac{2}{3}\left(1-\frac{\omega(x)}{\gamma}\right)}$

2. *At almost every point* x G £2 at tu/icfc $(x) ≥ A/| (l — £ ^) the *Young-measure fi$_x$ of the sequence ip$_n$ is a single Dirac mass.*

We now can prove that a minimizing sequence for £ (or at least a weakly convergent subsequence) yields a minimizer of the relaxed energy *£R.*

Theorem 5 *Suppose there is a sequence (tp$_{ny}$A$_n$)* € I^4(fi;IR$^+$) x #*($}; IR^3) *such that*

(V>$_n$, A$_n$) - (^ Ā) *in* I^4(n;IR$^+$) x JJ^IR^3), (4.33)

and

lim £(V>$_n$, A$_n$) ≤ £(V>', A^1) *for all* M>*, A^1) e I^4(f2;IR$^+$) x H^OjlR^8). (4.34)

TAen (V̄», Ā) mmtimtże^ ffce *relaxed nonlocal energy; i.e.,*

£*(& Ā) ≤ ^(V*^{1}, A') /or a« (V>», A^1) € L^4(fi;IR$^+$) x ^(njIR^3). (4.35)

Proof. Suppose not. Then there exists (V̄>, Ā) € L^4(f2;IR$^+$) x Jf^a(fi;IR^8) such that

£ J I (^ Ā) < £ H (^ Ā) . (4.36)

Now, by Lemma 3, there exists a sequence *rp̄$_n$* € *L^A(£l]* IR$^+$) such that

$$\begin{aligned}\lim_{n\to\infty} \mathcal{E}(\&, \bar{\mathbf{A}}) &= \mathcal{E}_R(\wedge, \bar{\mathbf{A}}) \\ &< \mathcal{E}_R(\bar{\psi}, \bar{\mathbf{A}}) \\ &\le \lim \mathcal{E}_R(\psi_n, \mathbf{A}_n)\end{aligned}$$

(where we have used Lemma 2 for the final inequality). This lead to a contradiction of (4.34). D

We now state the most important result of the section which says that minimizing sequences for £ can be constructed from classical minimizers for *£R.*

Theorem 6 *Suppose* (V̄^, Ā) *minimizes the relaxed nonlocal energy; i.e.,*

£$_R$$,A)¯< £«(V>$^($, A") *for all* (^,A^1) € X^4(fi;lR$^+$) x *H\Sl;TB?).* (4.37)

Then there is a sequence (i>$_n$,A$_n$) € I^4(ft;]R$^+$) x If^fylR^3) *such that*

(V-n, A$_n$) - *$,* Ā) *in* X^4(ri;lR$^+$) x H^IR^3), (4.38)

and

lim £(Y>n, A$_n$) ≤ 5(V>$^($, A«) /or aH (V>«, A^1) e I^4(n;IR$^+$) x ^(tylR^3). (4.39)

We can now deduce the following properties of ψ_n. Since $\psi \mapsto G_n^*(\psi, \mathbf{x})$ is affine (on each cube on which ψ_n is not constant), it follows that

$$\int_\Omega G_n^*(\psi_n(\mathbf{x}), \mathbf{x})\, d\mathbf{x} = \int_\Omega G_n^*(\bar{\psi}_n(\mathbf{x}), \mathbf{x})\, d\mathbf{x}. \tag{4.27}$$

Furthermore, since ψ_n is bounded in $L^4(\Omega; \mathbb{R}^+)$ we have

$$\int_\Omega [G_n^*(\bar{\psi}_n(\mathbf{x}), \mathbf{x}) - G^*(\bar{\psi}_n(\mathbf{x}), \mathbf{x})]\, d\mathbf{x} \to 0. \tag{4.28}$$

Finally, using our choice of $\bar{\mathbf{x}}$, we get

$$G^*(\psi_n(\mathbf{x}), \mathbf{x}) = G(\psi_n(\mathbf{x}), \mathbf{x}) \text{ for all } \mathbf{x} \in \Omega. \tag{4.29}$$

Putting these results together, we get (4.6).

We now need to show that $\psi_n \rightharpoonup \bar{\psi}$ in $L^4(\Omega : \mathbb{R}^+)$. Let K be such that

$$\|\psi_n - \bar{\psi}\|_{L^4(\Omega;\mathbb{R}^+)} \leq K. \tag{4.30}$$

Let $\phi \in L^{4/3}(\Omega; \mathbb{R})$ be given. For any $\epsilon > 0$, we can choose N sufficiently large such that there exists a simple function ϕ_ϵ, constant on each cube of the grid $\mathcal{G}_N$ such that

$$\|\phi - \phi_\epsilon\|_{L^{4/3}(\Omega;\mathbb{R})} < \frac{\epsilon}{K}. \tag{4.31}$$

Since our grids are nested, ϕ_ϵ is constant on the cubes of $\mathcal{G}_n$ for any $n > N$. Thus, for $n > N$ we have

$$\int_\Omega \psi_n \phi_\epsilon\, d\mathbf{x} = \int_\Omega \bar{\psi}_n \phi_\epsilon\, d\mathbf{x} = \int_\Omega \bar{\psi} \phi_\epsilon\, d\mathbf{x}. \tag{4.32}$$

Thus, for n sufficiently large, we can use Hölder's inequality to get

$$\begin{aligned} \left| \int_\Omega (\psi_n - \bar{\psi})\phi\, d\mathbf{x} \right| &= \left| \int_\Omega (\psi_n - \bar{\psi})(\phi - \phi_\epsilon)\, d\mathbf{x} \right| \\ &\leq \|\psi - \psi_n\|_{L^4(\Omega;\mathbb{R})} \|\phi - \phi_\epsilon\|_{L^{4/3}(\Omega;\mathbb{R})} < \epsilon. \end{aligned}$$

This completes the proof. □

We have chosen not to use any of the tools of Young-measures in this paper. (One of the main reasons for this being that the natural space for ψ is L^4 (where Young-measures are rather complicated) rather than L^∞.) However, the following result on the Young-measure of the weakly convergent sequence constructed above follows naturally from the construction. We state it without proof.

Corollary 4 *Let the hypotheses of Lemma 3 hold and in addition suppose* $\bar{\psi} \in L^\infty(\Omega; \mathbb{R}^+)$. *Then the Young-measure of the sequence* ψ_n *constructed in Lemma 3 has the following properties.*

Let Ω be covered with a nested sequence of grids $\mathcal{G}_n$ of cubes with side lengths 2^{-n}. We approximate $\bar{\psi}$ by the sequence of simple functions $\bar{\psi}_n$ that take the average value of $\bar{\psi}$ within each cube completely contained in Ω; i.e., for any cube $C \subset \Omega$ in the grid $\mathcal{G}_n$ we let

$$\bar{\psi}_n(\mathbf{x}) := 8^n \int_C \bar{\psi}(\mathbf{y})\, d\mathbf{y}, \quad \mathbf{x} \in C. \tag{4.17}$$

It follows from standard results of analysis that

$$\bar{\psi}_n \rightarrow \psi \text{ (strongly) in } L^4(\Omega; \mathbb{R}^+), \tag{4.18}$$

and hence

$$\lim_{n\rightarrow\infty} \mathcal{E}_L^*(\bar{\psi}_n, \bar{\mathbf{A}}) = \mathcal{E}_L^*(\bar{\psi}, \bar{\mathbf{A}}). \tag{4.19}$$

Now, for each cube C in $\mathcal{G}_n$, pick $\tilde{\mathbf{x}} \in C$ such that

$$\omega(\tilde{\mathbf{x}}) = \min_{\mathbf{x}\in C} \omega(\mathbf{x}). \tag{4.20}$$

Now define

$$G_n^*(\psi, \mathbf{x}) := G^*(\psi, \tilde{\mathbf{x}}) \text{ for } \mathbf{x} \in C. \tag{4.21}$$

Note that since ω is uniformly Lipschitz and $\bar{\psi}_n$ is bounded in $L^4(\Omega; \mathbb{R}^+)$ we can show

$$\int_\Omega [G_n^*(\bar{\psi}_n, \mathbf{x}) - G^*(\bar{\psi}_n, \mathbf{x})]\, d\mathbf{x} \rightarrow 0 \tag{4.22}$$

We now construct ψ_n from $\bar{\psi}_n$ in the following way. Let C be any cube in which

$$\bar{\psi}_n(\mathbf{x}) \in \left(0, \sqrt{\frac{2}{3}\left(1 - \frac{\omega(\tilde{\mathbf{x}})}{\gamma}\right)}\right) \quad \text{for all } \mathbf{x} \in C. \tag{4.23}$$

(Here $\tilde{\mathbf{x}} \in C$ is the point picked in the construction of G_n^*.) Thus there is a $\theta_C \in (0,1)$ such that

$$\bar{\psi}_n(\mathbf{x}) = \theta_C \sqrt{\frac{2}{3}\left(1 - \frac{\omega(\tilde{\mathbf{x}})}{\gamma}\right)} \text{ for } \mathbf{x} \in C. \tag{4.24}$$

Pick any measurable set $S_{\theta_C} \subset C$ with

$$|S_{\theta_C}| = 8^{-n}\theta_C \tag{4.25}$$

and let

$$\psi_n(\mathbf{x}) := \begin{cases} \sqrt{\frac{2}{3}\left(1 - \frac{\omega(\tilde{\mathbf{x}})}{\gamma}\right)}, & \mathbf{x} \in S_{\theta_C}, \\ 0, & \mathbf{x} \in C \backslash S_{\theta_C}. \end{cases} \tag{4.26}$$

By making this modification in every cube in which (4.23) is satisfied and letting $\psi_n = \bar{\psi}_n$ in all other cubes of $\mathcal{G}_n$ we construct ψ_n.

Lemma 3 *For any* $(\bar{\psi}, \bar{\mathbf{A}}) \in L^4(\Omega; \mathbb{R}^+) \times H^1(\Omega; \mathbb{R}^3)$ *there exists a sequence* $\psi_n \in L^4(\Omega; \mathbb{R}^+)$ *such that*

$$\psi_n \rightharpoonup \bar{\psi} \text{ in } L^4(\Omega; \mathbb{R}^+), \tag{4.5}$$

and

$$\lim_{n\to\infty} \mathcal{E}(\psi_n, \bar{\mathbf{A}}) = \mathcal{E}_R(\bar{\psi}, \bar{\mathbf{A}}). \tag{4.6}$$

Proof. We first note that we can write

$$\begin{aligned} \mathcal{E} &= \mathcal{E}_L + \mathcal{E}_N, & (4.7)\\ \mathcal{E}_R &= \mathcal{E}_L^* + \mathcal{E}_N, & (4.8)\\ & & (4.9) \end{aligned}$$

where

$$\mathcal{E}_L(\psi, \mathbf{A}) := \int_\Omega G(\psi(\mathbf{x}), \mathbf{x})\, d\mathbf{x} \tag{4.10}$$

$$\mathcal{E}_L^*(\psi, \mathbf{A}) := \int_\Omega G^*(\psi(\mathbf{x}), \mathbf{x})\, d\mathbf{x} \tag{4.11}$$

$$\begin{aligned} \mathcal{E}_N(\psi, \mathbf{A}) := & \int_\Omega |\text{curl } \mathbf{A} - \mathbf{H}|^2\, dx - \frac{1}{\gamma}\int_\Omega\int_\Omega K_2(\mathbf{x}-\mathbf{y})\psi(\mathbf{x})\psi(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y} \quad (4.12)\\ & + \int_\Omega\int_\Omega \mathbf{A}^*(\mathbf{x})\mathbf{K}_1(\mathbf{x}-\mathbf{y})\mathbf{A}(\mathbf{y})\psi(\mathbf{x})\psi(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y}. \end{aligned}$$

Recall that the integral operators

$$L^4(\Omega; \mathbb{R}) \ni \psi \mapsto \int_\Omega K_2(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\, d\mathbf{y} \in L^{4/3}(\Omega; \mathbb{R}) \tag{4.13}$$

and

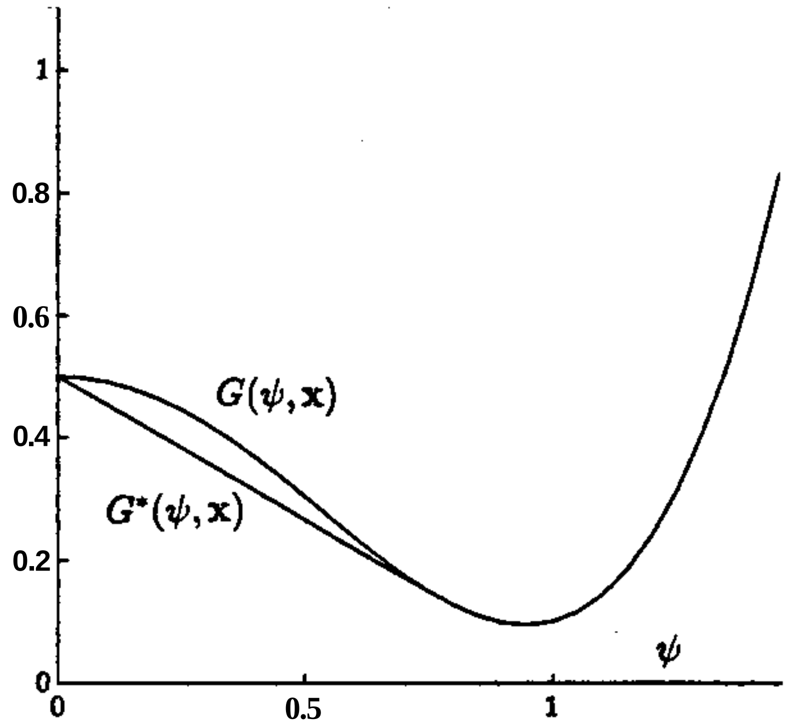
$$L^4(\Omega; \mathbb{R}) \ni \psi \mapsto \int_\Omega \bar{\mathbf{A}}^*(\mathbf{x})\mathbf{K}_1(\mathbf{x}-\mathbf{y})\bar{\mathbf{A}}(\mathbf{y})\psi(\mathbf{y})\, d\mathbf{y} \in L^{4/3}(\Omega; \mathbb{R}) \tag{4.14}$$

are assumed to be compact. We use this, the fact that compact operators map weakly convergent sequences in strongly convergent sequences, and the fact that the integral of the product of weakly and strongly convergent sequences converges weakly to get

$$\mathcal{E}_N(\psi_n, \bar{\mathbf{A}}) \to \mathcal{E}_N(\bar{\psi}, \bar{\mathbf{A}}) \tag{4.15}$$

for *any* weakly convergent sequence satisfying (4.5). Thus, to show (4.6), we need only construct a sequence satisfying (4.5) and

$$\lim_{n\to\infty} \mathcal{E}_L(\psi_n, \bar{\mathbf{A}}) = \mathcal{E}_L^*(\bar{\psi}, \bar{\mathbf{A}}). \tag{4.16}$$



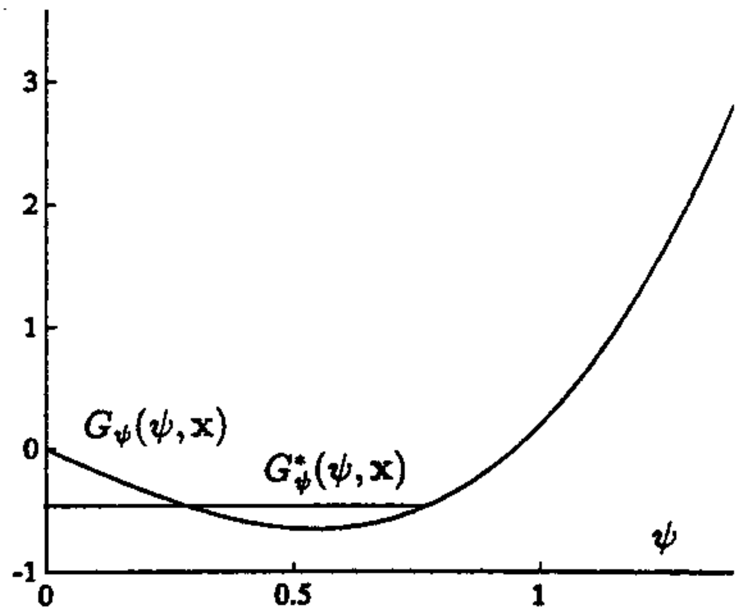


Figure 3: The relaxed energy density G'' and its derivative.

- The minimizers of the relaxed energy $\mathcal{E}_R$ are classical, and hence we can try to compute them using classical techniques of the calculus of variations (Euler-Lagrange equations, etc.).
- There is an easy way to construct minimizing sequences for $\mathcal{E}$ (or to deduce the structure of the Young-measure for the minimizing sequence) from a minimizer of the relaxed energy $\mathcal{E}_R$.

The relaxed problem is obtained by replacing the nonconvex energy density G with its convexification. This relaxed or convexified energy density is given by

$$G^*(\psi, \mathbf{x}) := \begin{cases} -\left[\frac{2}{3}\left(1 - \frac{\omega(\mathbf{x})}{\gamma}\right)\right]^{3/2}\psi + \frac{1}{2} & 0 \le \psi \le \sqrt{\frac{2}{3}\left(1 - \frac{\omega(\mathbf{x})}{\gamma}\right)} \\ \frac{1}{2}\psi^4 - \left(1 - \frac{\omega(\mathbf{x})}{\gamma}\right)\psi^2 + \frac{1}{2} & \sqrt{\frac{2}{3}\left(1 - \frac{\omega(\mathbf{x})}{\gamma}\right)} < \psi \end{cases} \tag{4.1}$$

Note that G^* is convex (though not strictly convex) and continuously differentiable in its first variable (cf. Figure 3). (Recall that in this section $1 - \frac{\omega}{\gamma} > 0$.)

The relaxed nonlocal minimization problem. *Let* $\mathbf{H} \in L^2(\Omega; \mathbb{R}^3)$ *be given. Find a pair* $(\psi, \mathbf{A}) \in L^4(\Omega; \mathbb{R}^+) \times H^1(\Omega; \mathbb{R}^3)$ *such that the relaxed energy functional*

$$\begin{aligned} \mathcal{E}_R(\psi, \mathbf{A}) &= \int_\Omega \left\{G^*(\psi(\mathbf{x}), \mathbf{x}) + |\mathrm{curl}\, \mathbf{A} - \mathbf{H}|^2\right\} d\mathbf{x} \\ &\quad - \frac{1}{\gamma}\int_\Omega\int_\Omega K_2(\mathbf{x} - \mathbf{y})\psi(\mathbf{x})\psi(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y} \\ &\quad + \int_\Omega\int_\Omega \mathbf{A}^*(\mathbf{x})\mathbf{K}_1(\mathbf{x} - \mathbf{y})\mathbf{A}(\mathbf{y})\psi(\mathbf{x})\psi(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y} \end{aligned} \tag{4.2}$$

is minimized.

We begin our study of the relationship of the relaxed problem to the modified nonlocal problem with the following lemma.

Lemma 2 *Suppose there is a sequence* $(\psi_n, \mathbf{A}_n) \in L^4(\Omega; \mathbb{R}^+) \times H^1(\Omega; \mathbb{R}^3)$ *such that*

$$(\psi_n, \mathbf{A}_n) \rightharpoonup (\bar{\psi}, \bar{\mathbf{A}}) \text{ in } L^4(\Omega; \mathbb{R}^+) \times H^1(\Omega; \mathbb{R}^3). \tag{4.3}$$

Then

$$\mathcal{E}_R(\bar{\psi}, \bar{\mathbf{A}}) \le \liminf_{n \to \infty} \mathcal{E}(\psi_n, \mathbf{A}_n) \tag{4.4}$$

The proof of this is standard in the calculus of variations. It uses the weak lower semicontinuity of all the terms in the two energies except for the nonconvex G, together with the fact that G^* is the convexification of G.

The next lemma is actually a construction of a weakly convergent sequence such that the limit of the modified energy $\mathcal{E}$ is the relaxed energy $\mathcal{E}_R$ of the weak limit.

2. Upper and lower bounds on the energy of the sequence are then used to obtain *a priori* estimates on the sequence in some Banach space.

3. A weakly convergent subsequence is then extracted from the minimizing sequence (by the Banach-Alaoglu theorem).

4. Finally, one shows that the weak limit of the subsequence is actually a minimizer for the problem.

Unfortunately, the final step usually uses the convexity of the highest order terms of the energy density in a crucial way. Because G, a leading order term in our energy, is nonconvex, it is not in general true that the weak limit of a minimizing sequence is itself a minimizer. When this occurs, we say that the minimum energy is *not attained.* One way of addressing nonattainment is to accept the minimizing sequences themselves (rather than their limits) as solutions of the minimization problem. However, while this makes sense theoretically, it is obviously a rather cumbersome process to try to compute minimizing sequences. For any given body Ω and applied field $\mathbf{H}$ we would have to compute not just a single function, the minimizer, but an entire sequence of functions that minimize the energy. In addition, when computing sequences, the classical tools used to compute minimizers (Euler-Lagrange equations and other necessary conditions) no longer seem to be useful.

Fortunately, for the modified nonlocal problem, there is a relatively easy way to determine the character of minimizing sequences. In the next section we introduce a *relaxed* version of the nonconvex problem stated above. We will show that this problem has a classical minimizer. Furthermore, we will show that using this classical minimizer of the relaxed problem, we can construct a minimizing sequence of the nonlocal energy $\mathcal{E}$ or, alternatively, deduce the structure of the Young-measure of the minimizing sequence directly. (Young-measures are probability measures describing the oscillations of weakly convergent sequences. Since in this paper we are able to describe minimizing sequences using classical variational techniques, we do not give a detailed description of Young-measures here. For a more complete theoretical discussion one could consult [2, 14, 15, 18]. Examples of applications of Young-measures in a setting similar to the one in this paper include [3, 5, 7, 11, 13].)

4 The relaxed nonlocal problem

Because of the special structure of the modified Bardeen energy $\mathcal{E}$, we will be able to deduce the following facts about the relationship between $\mathcal{E}$ and the *relaxed nonlocal energy* $\mathcal{E}_R$ (to be described below).

- The weak limit of any minimizing sequence for the modified Bardeen energy $\mathcal{E}$ will be a minimizer for a relaxed energy $\mathcal{E}_R$.

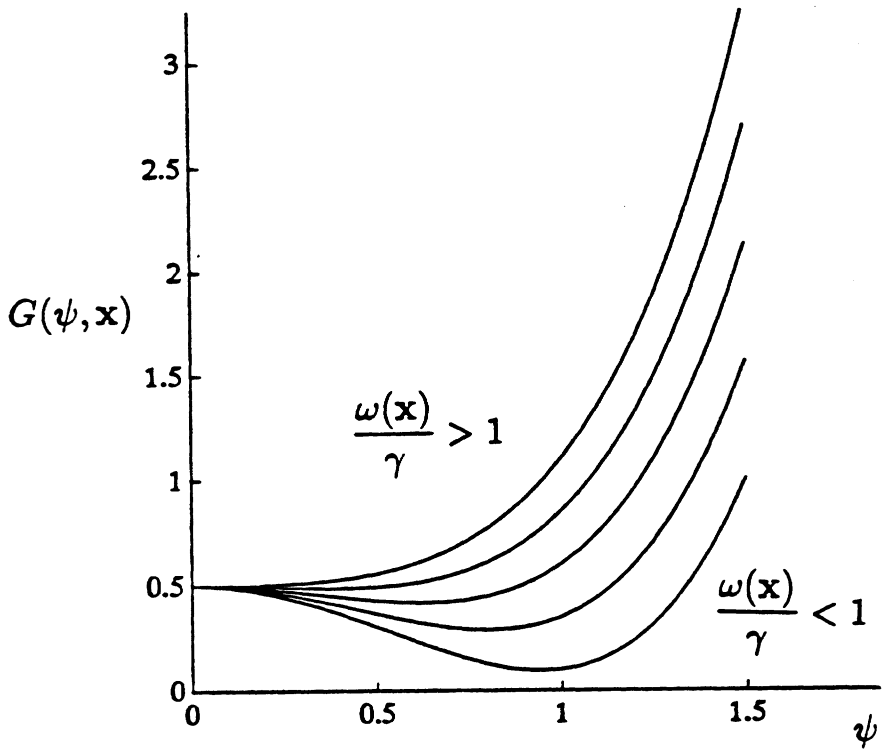


Figure 1: The local energy density G.

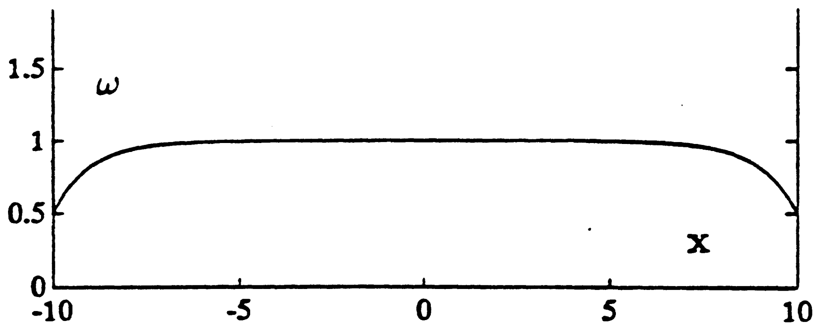


Figure 2: Weighting funciton ω when Ω is an infinite plate of thickness $2L$ and $K_2(\mathbf{x}-\mathbf{y}) \sim e^{-\zeta|\mathbf{x}-\mathbf{y}|}/|\mathbf{x}-\mathbf{y}|$.

The first term on the right is often absorbed into other terms in the total energy.

It is interesting to note that our justification of nonlocal energy densities by referring to the derivation of the gradient term is the opposite of the procedure used by Van der Waals [16], who modeled density phase transitions using a nonlocal term and then approximated the nonlocal term using a gradient term.

One should also note that the success of higher order "viscosity" models has been in applications to relatively stable, nonoscillatory phenomena. The analysis of phenomena such as turbulent flow of fluids and hysteresis subloops in ferromagnetism using higher order models is difficult at best, and it is for such applications that nonlocal models seem to be most promising.

One analytic "cost" we pay for dispensing with the higher order term is that now the behavior of the energy is more heavily influenced by the local lower-order term

$$\int_\Omega G(\psi(\mathbf{x}),\mathbf{x})\, d\mathbf{x},$$

where

$$\begin{aligned} G(\psi,\mathbf{x}) &:= \frac{1}{2}(|\psi|^2-1)^2 + \frac{\omega(\mathbf{x})}{\gamma}|\psi|^2 \\ &= \frac{1}{2}\left[|\psi|^2 - \left(1-\frac{\omega(\mathbf{x})}{\gamma}\right)\right]^2 + \frac{\omega(\mathbf{x})}{\gamma} - \frac{1}{2}\left(\frac{\omega(\mathbf{x})}{\gamma}\right)^2. \end{aligned} \tag{3.13}$$

The energy density G(^,x) is a perturbation of the nonconvex local energy density term (|V>|^2 ~~ I)^2 found in the Ginzburg-Landau energy by an additional local penalty for superconducting charge carriers ^^^l^l^2. (The additional penalty is the remnant of the nonlocal penalty for variations (cf. 3.12).) Note that the entire energy density is nonconvex when 1 — a>(x)/7 > 0. Furthermore, in this case it is minimized at *if)* = $\sqrt{1 - o>(x)/_7}$ (cf. Figure 1).

In order to understand *G* more fully, we need to examine the function tt>(x) = /$_n$i^2(x — y) *dy.* At present, our hypotheses simply imply that its values lie in the interval [0,1]. In practice, as we indicated earlier, we think of *K^* as satisfying

$$\#_2(x-y):=*(|x-y|), \tag{3.14}$$

where *k* : (0, oo) —» IR* is a smooth decreasing function, concentrated at the origin. In this case, if the body is large compared to the region of concentration of fc, then a> is close to one in the interior of the body and significantly less than one at the boundary (cf. Figure 2). Since *G* is minimized at y1 — a>(x)/7, this leads us to the conclusion that the local energy density *G* encourages a greater concentration of superconducting charge carriers at the boundary of the body.

One might now try to use the "direct method^{7}' of the calculus of variations to get a classical existence result. This method is described as follows.

1. One first considers an minimizing sequence for the energy (such a sequence always exists since every energy has an infimum).

whose distance from $\mathbf{x}_j$ is h), and let n_j be the number of nearest neighbors to $\mathbf{x}_j$. In order to penalized rapid transitions in u, we define an energy

$$E(u; G_M) := \frac{C}{M} \sum_{j=1}^{M} \frac{1}{n_j} \sum_{m \in N_j} |u(\mathbf{x}_j) - u(\mathbf{x}_m)|^2, \tag{3.6}$$

where $C > 0$ is a constant. A routine calculation shows that as the step size goes to zero we have

$$E(u; G_M) \to \bar{C} \int_\Omega |\nabla u(\mathbf{x})|^2 \, d\mathbf{x}. \tag{3.7}$$

Note that one of the key assumptions in the derivation of (3.7) is that as the grid size becomes smaller, one continues to penalize only differences between nearest neighbor grid points. Another obvious possibility for penalization is to consider differences between the values of u based on *distance* between grid points. In elasticity, the argument for such a penalty has been made based on the effects of capillarity and viscosity [16, 17]; and in ferromagnetism a similar argument can be based on exchange forces [13, 6]. One can argue that in superconductors the existence of Cooper pairs and the linkage between these pairs makes a distance-based penalty for the phase parameter even more compelling than in elasticity or ferromagnetism.

Suppose we wish to use such a method of penalization. For simplicity, let us take $k(|x_i - x_j|)$ to be the weighting function for the penalties, where $k : (0, \infty) \to \mathbb{R}^+$ is a smooth function such that k is nonincreasing (so that distant points are penalized no more than nearby points) and

$$\int_0^\infty k(r) r^2 dr < \infty. \tag{3.8}$$

We consider an energy of the form

$$\hat{E}(u; G_M) := \frac{C}{M^2} \sum_{j=1}^{M} \sum_{m=1}^{M} k(|x_j - x_m|) |u(\mathbf{x}_j) - u(\mathbf{x}_m)|^2. \tag{3.9}$$

As the step size goes to zero we have

$$\hat{E}(u; G_M) \to \hat{C} \int_\Omega \int_\Omega k(|\mathbf{x} - \mathbf{y}|) |u(\mathbf{x}) - u(\mathbf{y})|^2 \, d\mathbf{x} \, d\mathbf{y}. \tag{3.10}$$

If we let

$$\beta(\mathbf{x}) := \int_\Omega k(|\mathbf{x} - \mathbf{y}|) \, d\mathbf{y} \tag{3.11}$$

we get

$$\begin{aligned} \hat{C} \int_\Omega \int_\Omega \; & k(|\mathbf{x} - \mathbf{y}|) |u(\mathbf{x}) - u(\mathbf{y})|^2 \, d\mathbf{x} \, d\mathbf{y} \\ & = 2\hat{C} \left[\int_\Omega \beta(\mathbf{x}) u^2(\mathbf{x}) \, d\mathbf{x} - \int_\Omega \int_\Omega k(|\mathbf{x} - \mathbf{y}|) u(\mathbf{x}) u(\mathbf{y}) \, d\mathbf{x} \, d\mathbf{y} \right]. \end{aligned} \tag{3.12}$$

Here u represents a typical variable whose phase change is being penalized. Such a replacement has the "advantage" of allowing for both discontinuities and oscillations. In Bardeen's model the higher order terms in ψ and $\mathbf{A}$ remain. We suggest the following modification of Bardeen's model which will allow us to consider a much weaker space for the order parameter ψ.

Modified nonlocal minimization problem. *Let* $\mathbf{H} \in L^2(\Omega; \mathbb{R}^3)$ *be given. Find a pair* $(\psi, \mathbf{A}) \in L^4(\Omega; \mathbb{R}^+) \times H^1(\Omega; \mathbb{R}^3)$ *such that the energy functional*

$$\begin{aligned} \mathcal{E}(\psi, \mathbf{A}) &= \int_\Omega \left\{ \frac{1}{2}(|\psi|^2 - 1)^2 + \frac{\omega(\mathbf{x})}{\gamma}|\psi|^2 + |\text{curl}\, \mathbf{A} - \mathbf{H}|^2 \right\} dx \\ &-\frac{1}{\gamma} \int_\Omega \int_\Omega K_2(\mathbf{x} - \mathbf{y})\psi(\mathbf{x})\psi(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y} \\ &+ \int_\Omega \int_\Omega \mathbf{A}^*(\mathbf{x})\mathbf{K}_1(\mathbf{x} - \mathbf{y})\mathbf{A}(\mathbf{y})\psi(\mathbf{x})\psi(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y} \end{aligned} \tag{3.1}$$

is minimized.

Here $\gamma > 0$ is the appropriate nondimensionalization constant,

$$\omega(\mathbf{x}) := \int_\Omega K_2(\mathbf{x} - \mathbf{y})\, d\mathbf{y}, \tag{3.2}$$

and we assume that $K_2 : \mathbb{R}^3 \to \mathbb{R}^+$ is even and satisfies

$$\int_{\mathbb{R}^3} K_2(\mathbf{x} - \mathbf{y})\, d\mathbf{y} = 1, \tag{3.3}$$

and that the operator

$$L^4(\Omega; \mathbb{R}) \ni \psi \mapsto \int_\Omega K_2(\mathbf{x} - \mathbf{y})\psi(\mathbf{y})\, d\mathbf{y} \in L^{4/3}(\Omega; \mathbb{R}) \tag{3.4}$$

is compact. The type of kernel we have in mind is one of the form

$$K_2(\mathbf{x} - \mathbf{y}) \sim \frac{e^{-\zeta|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|}, \tag{3.5}$$

which satisfies the compactness condition.

In order to motivate the replacement of a gradient term with a nonlocal term as a phase change penalty, let us examine a typical justification for the use of the gradient term. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Suppose $u : \Omega \to \mathbb{R}$ is some physical quantity whose phase transitions we wish to penalize. Let $G_M := \{\mathbf{x}_m\}_{m=1}^M$ be a uniform rectangular grid on Ω with step size h. For each $j = 1, \ldots, M$ let N_j be the set of indices of the grid points that are nearest neighbors to the grid point $\mathbf{x}_j$ (i.e., those

2.2 Bardeen's nonlocal theory

Bardeen [4, p. 326] proposed a nonlocal generalization of the Ginzburg-Landau theory designed to incorporate Pippard's theory. Bardeen considered a one-dimensional model; we consider a multidimensional generalization of Bardeen's model below. Neither Bardeen's model or our generalization of it is gauge invariant. Indeed, to the authors' knowledge the problem of whether a gauge invariant nonlocal theory can be formulated remains open and the question of whether such a theory is desirable remains problematic. Thus, in the following, we assume that the superconducting charge density ψ is real and nonnegative.

The Bardeen nonlocal minimization problem. *Let* $\mathbf{H} \in L^2(\Omega;\mathbb{R}^3)$ *be given. Find a pair* $(\psi,\mathbf{A}) \in H^1(\Omega;\mathbb{R}^+) \times H^1(\Omega;\mathbb{R}^3)$ *such that the energy functional*

$$\begin{aligned}\bar{\mathcal{E}}(\psi,\mathbf{A}) &= \int_\Omega \left\{\frac{1}{2}(|\psi|^2-1)^2 + \frac{1}{\kappa^2}|\nabla\psi|^2 + |\text{curl } \mathbf{A} - \mathbf{H}|^2\right\} dx \\ &\quad + \int_\Omega\int_\Omega \mathbf{A}^*(\mathbf{x})\mathbf{K}_1(\mathbf{x}-\mathbf{y})\mathbf{A}(\mathbf{y})\psi(\mathbf{x})\psi(\mathbf{y}) d\mathbf{x}\, d\mathbf{y}.\end{aligned} \tag{2.8}$$

is minimized.

Here, $\mathbf{A}^*$ indicates the transpose of the vector $\mathbf{A}$, and $\mathbb{R}^+ := [0,\infty)$.

In the rest of this paper we assume $\mathbf{K}_1 : \mathbb{R}^3 \to$ Psym (where Psym denotes the space of positive definite symmetric 3×3 real matrices) is even and satisfies

$$\det\left(\int_{\mathbb{R}^3} \mathbf{K}_1(\mathbf{x}-\mathbf{y})\, d\mathbf{x}\right) = 1. \tag{2.9}$$

In addition, we require that for any fixed $\mathbf{A} \in H^1(\Omega;\mathbb{R}^3)$, the operator

$$L^4(\Omega;\mathbb{R}^+) \ni \psi \mapsto \int_\Omega \mathbf{A}^*(\mathbf{x})\mathbf{K}_1(\mathbf{x}-\mathbf{y})\mathbf{A}(\mathbf{y})\psi(\mathbf{y})\, d\mathbf{y} \in L^{4/3}(\Omega;\mathbb{R}) \tag{2.10}$$

is compact. While Bardeen did not make this compactness assumption explicitly, the Pippard model employs a kernel of the form

$$\mathbf{K}_1(\mathbf{x}-\mathbf{y}) := C\frac{e^{-|\mathbf{x}-\mathbf{y}|/\xi}}{|\mathbf{x}-\mathbf{y}|^4}(\mathbf{x}-\mathbf{y})\otimes(\mathbf{x}-\mathbf{y}) \tag{2.11}$$

where C and ξ are positive constants and $\otimes$ indicates the dyadic tensor product. With this kernel, the operator described in (2.10) is compact (cf. Theorem 5.3 in [1]).

3 Modified Bardeen model

The use of nonlocal energy densities in Bardeen's model has some interesting similarities to nonlocal models in ferromagnetism [6, 13] and phase transitions [17]. However, in [6, 13, 17], a nonlocal energy density was used to replace a higher order term; i.e.,

$$\int_\Omega\int_\Omega k(\mathbf{x}-\mathbf{y})u(\mathbf{x})u(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y} \quad \text{replaces} \quad \int_\Omega |\nabla u(\mathbf{x})|^2\, d\mathbf{x}.$$

is minimized.

Here, $\Omega \subset \mathbb{R}^3$ is a bounded domain occupied by the superconductor, $\mathbf{A}$ is the *magnetic vector potential* which is related to the *magnetic field* $\mathbf{h}$ generated by the superconductor through the identity

$$\mathbf{h} = \text{curl } \mathbf{A}, \tag{2.2}$$

$\mathbf{H}$ is the *applied magnetic field*, ψ is the complex-valued *order parameter*, and κ is the *Ginzburg-Landau parameter*, a material constant. Sometimes the problem is supplemented with other boundary conditions.

In the Ginzburg-Landau theory, ψ is regarded as an "averaged wave-function of the superconducting electrons". It can be normalized so that $|\psi|^2$ is proportional to the density of superconducting charge-carriers, and the phase of ψ is related to the current in the superconductor.

The Euler-Lagrange equations for (2.1) are known as the **Ginzburg-Landau equations:**

$$\left|\frac{-i}{\kappa}\nabla - \mathbf{A}\right|^2 \psi + \psi(|\psi|^2 - 1) = 0 \text{ in } \Omega \tag{2.3}$$

$$\text{curl curl } \mathbf{A} = \text{curl } \mathbf{H} - |\psi|^2\mathbf{A} \text{ in } \Omega. \tag{2.4}$$

The current $\mathbf{j}$ in the superconductor is given by

$$\mathbf{j} = \frac{c}{4\pi}\text{curl } \mathbf{h} = \frac{c}{4\pi}\text{curl curl } \mathbf{A}, \tag{2.5}$$

where c is the speed of light. Thus, we can rewrite (2.4) as follows

$$\frac{4\pi}{c}\mathbf{j} = \text{curl } \mathbf{H} - |\psi|^2\mathbf{A}. \tag{2.6}$$

Using standard techniques (cf. e.g., [8, §3.2]) one can prove the following existence result for the Ginzburg-Landau minimization problem.

Theorem 1 *There exists a solution of the Ginzburg-Landau minimization problem; i.e., there exists* $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^3)$ *such that*

$$\bar{\mathcal{E}}(\psi, \mathbf{A}) \leq \bar{\mathcal{E}}(\psi', \mathbf{A}') \quad \textit{for all } (\psi', \mathbf{A}') \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^3). \tag{2.7}$$

Unfortunately, solutions of the Ginzburg-Landau equations have proved to be very hard to compute, particularly in regimes in which one observes highly oscillatory vortex phenomena.

characteristics. In Section 4, we use relaxation techniques to prove an existence theorem for the modified model and to describe some of the behavior of solutions. Finally in Section 5, we make a number of concluding remarks.

Notation. We use nonboldface Greek and Latin letters to denote scalars (both real and complex). Boldface letters are used to denote both vectors and tensors. The space $\boldsymbol{L^p}(\boldsymbol{\Omega}, \mathrm{IR}^n)$ is the set of (measureable) functions

$$\mathbf{f : n -» m^n}$$

such that

$$\|\mathbf{f}\|_{L^p(\Omega;\mathbb{R}^n)} := \quad |f(x)|^p \text{<facj} \quad < 00.$$

Here | • | indicates the Euclidean norm in IR^n. The space $H^1(\mathrm{fl},\mathrm{IR}^n)$ is the set of (measureable) functions f as above such that

$$\|\mathbf{f}\|_{H^1(\Omega;\mathbb{R}^n)} := \left\{\|\mathbf{f}\|^2_{L^2(\Omega;\mathbb{R}^n)} + \|\nabla\mathbf{f}\|^2_{L^2(\Omega;\mathbb{R}^n)}\right\}^{1/2} < \infty.$$

Here Vf indicates the gradient of f (the tensor of first partials of the components of f).

2 Classical Models of Superconductivity

In this section we give a brief description of two classical models of superconductors that can be compared directly with the model to be proposed in Section 3. Our theory is basically a modification of the Ginzburg-Landau theory (described in § 2.1), which models superconductors in terms of a local variational problem. In our model, we replace higher-order terms in Ginzburg-Landau with terms that are nonlocal in space. Spatially nonlocal terms have been used in mathematical models of superconductors for some time. Notably, the theory of Pippard [12] introduced a nonlocal constitutive law for the current inside the superconductor and Eringen [9] studies a very general memory-dependent nonlocal model. However, instead of describing these nonlocal theories, we examine in § 3.2 a nonlocal model of Bardeen, which is in variational form and is therefore easier to compare to the model we introduce below.

2.1 The Ginzburg-Landau theory

The Ginzburg-Landau theory of superconductivity is based on the following variational problem.

The Ginzburg-Landau minimization problem. *Let* **H** € L^2(£2;IR^3) *be given. Find* ***a pair (if>,*** **A)** **€** ***H^l(Sl\ <D)*** **x** **if^1(f2;IR^3)** ***such that the energy functional***

$$E\{1>, \mathbf{A}) = /_n \tfrac{1}{2} i (M^2 - I)^2 + | (\wedge V - \mathbf{A}) \text{ vf} + \text{lend } A - H|^2 | \text{dcc} \qquad (2.1)$$

Nonlocal Superconductivity

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July 27, 1992

1 Introduction

Superconducting materials exhibit a variety of highly oscillatory behavior. In particular, classical type-II superconductors display quantized vortices in certain regimes, and new high-T_c materials are characterized by finely layered structures. Recently, new techniques of analysis such as relaxation and the use of Young-measures have been used successfully to study other multi-dimensional, highly oscillatory phenomena. Examples of such applications include fine-phase structures in elastic crystals (cf. e.g. [3, 7, 10]) and magnetic domains in ferromagnetic materials (cf. e.g. [6, 11]). Precursors of the techniques used in these problems, under various names such as *generalized curves* and *chattering controls*, have a long history of use in problems governed by ordinary differential equations. It seems reasonable that the mathematical tools developed for the phase transition applications cited above would be well suited to describe the physical behavior of superconductors. Unfortunately, classical mathematical models of superconductors (in particular, Ginzburg-Landau) are not amenable to the application of these techniques. The purpose of this work is to describe a modified model for superconductors to which these new techniques can be applied and to do some of the basic analysis for this model.

The rest of this paper is organized as follows. In Section 2, we give a brief description of some classical theories of superconductivity that particularly influenced our model. In Section 3, we introduce the modified model and describe some of its basic mathematical

*The research of D. Brandon was partially supported by the National Science foundation through grant # DMS-9296011 and by the Army Research Office and the National Science Foundation through the Center for Nonlinear Analysis.

†The research of R.C. Rogers was partially supported by the National Science foundation through grant # DMS-9204304 and by the Army Research Office and the National Science Foundation through the Center for Nonlinear Analysis.

NONLOCAL SUPERCONDUCTIVITY

Deborah Brandon and Robert C. Rogers

Abstract

Classical Type-II superconductors display quantized vortices in certain regimes, and certain high-Tc materials exhibit finely layered structures. These highly oscillatory phenomena suggest that mathematical models involving Young-measures may be a plausible choice. In this paper we introduced a new model for superconductors which allows for "measure-valued" solutions. The new model is a modification of Bardeen's nonlocal theory, and is in the form of a variational problem involving minimization of a nonlocal energy.

The results obtained for this model seem to suggest that superconducting charge carriers will concentrate at the boundaries of the material, whereas in the interior of the material, there will be a lower concentration of carriers. In addition, (under certain conditions) a minimizing sequence for the energy can be constructed by oscillating between the nonsuperconducting phase and a superconducting phase similarly to Type-II superconductors exhibiting an oscillatory vortex structure.