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The Geometry of Contact, Separation and Reformation of Continuous Bodies

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THE GEOMETRY OF CONTACT, SEPARATION, AND REFORMATION OF CONTINUOUS BODIES

by Walter Noll

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Introduction

Separation and contact of real objects are everyday phenomena. Real objects may develop cracks, may split, or may shatter to pieces. Fluids may cavitate or mingle. Real objects may stick together, merge, or be pieced together to form composites. The laws that govern these phenomena have been of great interest to engineers for a long time and there has been renewed interest in the discovery and study of these laws in recent years.

The purpose of this paper is to create a mathematical infrastructure which would facilitate the study of the phenomena described above. Good mathematical descriptions of deformations of continuous bodies have been developed during the past 200 years or so, but I have not seen in the literature a satisfactory mathematical description of changes of coherence for such bodies. Here I would like to present such a description. It includes only what I call the "geometry" of changes of coherence. A next step would be a description of "kinematics" of such changes, i.e. of changes of the geometry of coherence with time. Next would come a study of the forces involved in such changes, and finally one would like to develop a general theory of the constitutive laws governing the phenomena governing such changes.

One of the problems with the description of separation and contact of continuous bodies is that the material points on the surfaces of separation or contact disappear or are created. One may ask, therefore, what it is that retains its identity during such processes. The answer, I believe, is the materially ordered set that consists of the subbodies of a given body and the corresponding Boolean algebra. The material points are determined by filters (in the sense of Boolean algebra) of such subbodies. What may change as a result of separation or contact is the set of filters that determine material points.

Changes of coherence of a continuous body can come about not only by separation and contact, but also by processes that can be understood as a separation followed by a simultaneous contact or vice versa. Sliding would be an example. If such separation and simultaneous contact preserve material points, we call the process a "reformation". Such reformations may serve to obtain insight into phenomena such as phase transitions, acceleration waves, and shock waves.

The notation and terminology of [FDS] is used in this paper. In particular, the collection of all subsets of a given set $\mathscr{S}$ is denoted by Sub $\mathscr{S}$. The set of all positive numbers, including zero, is denoted by $\mathbb{P}$. A superscript $^{\times}$ on a set indicates the removal of zero or of the empty set. Let a mapping φ be given. The *domain*, *codomain*, and *range* of φ are denoted by Dom φ, Cod φ, and Rng φ, respectively. For every $A \in \mathrm{Sub\,Dom}\,\varphi$, the *image* of A under φ is denoted by $\varphi_{>}(A) := \{\varphi(x) \mid x \in A\}$. For every $B \in \mathrm{Sub\,Cod}\,\varphi$, the *pre–image* of B under φ is denoted by $\varphi^{<}(B) := \{y \in \mathrm{Dom}\,\varphi\} \mid \varphi(x) \in B\}$. Given $A \in \mathrm{Sub\,Dom}\,\varphi$ and $B \in \mathrm{Sub\,Cod}\,\varphi$ such that $\varphi_{>}(A) \subset B$, the *adjustment* $\varphi|_A^B$ is defined by $\mathrm{Dom}\,\varphi|_A^B := A$, $\mathrm{Cod}\,\varphi|_A^B := B$, and

$$\varphi|_A^B(x) = \varphi(x) \quad \text{for all } x \in A.$$

If φ is invertible, its *inverse* is denoted by $\varphi^{\leftarrow}$. The *identity mapping* of a set $\mathscr{S}$ is denoted by $1_{\mathscr{S}}$.

1. Continuous Bodies

Before giving a precise mathematical definition of a continuous body, one should first specify two classes: (i) a class Fr consisting of subsets of Euclidean spaces, subsets which are candidates for regions occupied by a continuous body when placed in a frame of reference, (ii) a class Tp of mappings which are candidates for the changes of placement of a body in a given frame of reference or from one frame to another.

We take Fr to be the class of fit regions recently introduced in [NV], i.e. we take Fr to be the class of all subsets of Euclidean spaces that are bounded and regularly open and have negligible boundary and finite perimeter. The set of fit regions included in a given Euclidean space $\mathscr{E}$ is denoted by $\mathrm{Fr}(\mathscr{E})$. For all $\mathscr{C},\mathscr{D} \in \mathrm{Fr}(\mathscr{E})$ we then have

$$\mathscr{C} \cap \mathscr{D} \in \mathrm{Fr}(\mathscr{E}) \tag{1.1}$$

$$\mathscr{C} \vee \mathscr{D} := \mathrm{Int\ Clo}\ (\mathscr{C} \cap \mathscr{D}) \in \mathrm{Fr}(\mathscr{E}) \tag{1.2}$$

and

$$\mathscr{C} \quad \mathscr{D} := \mathrm{Int}(\mathscr{C} \backslash \mathscr{D}) \in \mathrm{Fr}(\mathscr{E}). \tag{1.3}$$

We call $\mathscr{C} \vee \mathscr{D}$ the **join** of $\mathscr{C}$ and $\mathscr{D}$ and $\mathscr{C} \quad \mathscr{D}$ the **difference–region** of $\mathscr{C}$ and $\mathscr{D}$. If $\varphi\colon \mathscr{E} \to \mathscr{E}'$ is a C^1–diffeomorphism from the Euclidean space $\mathscr{E}$ to a Euclidean space $\mathscr{E}'$ then $\varphi_{>}(\mathscr{D}) \in \mathrm{Fr}(\mathscr{E}')$ for every $\mathscr{D} \in \mathrm{Fr}(\mathscr{E})$. We may express these facts, roughly, by saying that Fr is stable under intersection, joining, forming of difference–regions, and C^1–diffeomorphisms. If $\mathscr{S}$ is a subset of a given Euclidean space $\mathscr{E}$, we put

$$\mathrm{Fr}(\mathscr{S}) := \mathrm{Sub}\mathscr{S} \cap \mathrm{Fr}(\mathscr{E}) \tag{1.4}$$

We take Tp to be class determined by the following requirements:

(T_1) Every $\lambda \in \mathrm{Tp}$ is an invertible mapping whose domain $\mathrm{Dom}\lambda$ and range $\mathrm{Rng}\lambda$ are subsets of Euclidean spaces denoted by $\mathrm{Dsp}\lambda$ and $\mathrm{Rsp}\lambda$, respectively.

(T_2) We have $\mathrm{Dom}\lambda \in \mathrm{Fr}$ for every $\lambda \in \mathrm{Tp}$.

(T_3) For every $\lambda \in \mathrm{Tp}$, there is a C^2–diffeomorphism $\varphi : \mathrm{Dsp}\lambda \to \mathrm{Rsp}\lambda$ such that $\lambda = \varphi\Big|_{\mathrm{Dom}\lambda}^{\mathrm{Rng}\,\lambda}$.

The following facts are immediate consequences of (T_1) – (T_3):

(T_4) We have $\mathrm{Rng}\lambda \in \mathrm{Fr}$ for every $\lambda \in \mathrm{Tp}$.

(T_5) For all $\lambda, \mu \in \mathrm{Fr}$ with $\mathrm{Rng}\lambda = \mathrm{Dom}\mu$, we have $\mu \circ \lambda \in \mathrm{Tp}$.

(T_6) For every $\lambda \in \mathrm{Tp}$ we have $\lambda^{\leftarrow} \in \mathrm{Tp}$.

(T_7) For every $\lambda \in \mathrm{Tp}$ and $\mathscr{D} \in \mathrm{Fr}(\mathrm{Dsp}\lambda)$ with $\mathscr{D} \subset \mathrm{Dom}\,\lambda$, we have $\lambda\Big|_{\mathscr{D}}^{\lambda_{>}(\mathscr{D})} \in \mathrm{Tp}$.

We call the members of the class Tp **transplacements.**

Remark 1: Strictly speaking, Tp is the class of *morphisms* of a *category* whose *objects* are pairs $(\mathscr{D},\mathscr{E})$, where $\mathscr{E}$ is a Euclidean space and $\mathscr{D} \in \mathrm{Fr}(\mathscr{E})$. ■

Remark 2: For certain purposes, for example when dealing with bodies subject to constraints, one might wish to modify the definitions of Tp or of Fr, or both. ■

Definition 1: *A continuous body* $\mathscr{B}$ *is a non–empty set endowed with structure by the specification of a non–empty class* $\mathrm{Pl}(\mathscr{B})$ *satisfying the following requirements.*

(B_1) Each $\kappa \in \mathrm{Pl}(\mathscr{B})$ is an invertible mapping with $\mathrm{Dom}\,\kappa = \mathscr{B}$ and $\mathrm{Rng}\,\kappa \in \mathrm{Fr}$.

(B_2) For all $\kappa, \gamma \in \mathrm{Pl}(\mathscr{B})$ we have $\kappa \circ \gamma^{\leftarrow} \in \mathrm{Tp}$.

(B_3) For every $\kappa \in \mathrm{Pl}(\mathscr{B})$ and $\lambda \in \mathrm{Tp}$ such that $\mathrm{Rng}\,\kappa = \mathrm{Dom}\,\lambda$, we have $\lambda \circ \kappa \in \mathrm{Pl}(\mathscr{B})$.

We call the members of $\mathrm{Pl}(\mathscr{B})$ the **placements** of $\mathscr{B}$. Given $\kappa \in \mathrm{Pl}(\mathscr{B})$, we call $\mathrm{Rng}\,\kappa$ the **region occupied by $\mathscr{B}$ in the placement** κ; the Euclidean space in which $\mathrm{Rng}\,\kappa$ is a fit region is denoted by $\mathrm{Frm}\ \kappa$ and is called the **frame–space** of κ; the translation space of $\mathrm{Frm}\ \kappa$ is denoted by $\mathrm{Vfr}\ \kappa$ and is called the **frame–vector space** of κ.

Given $\kappa \in \mathrm{Pl}(\mathscr{B})$, it follows from (B_2) and (B_3) that

$$\mathrm{Pl}(\mathscr{B}) = \{\lambda \circ \kappa \mid \lambda \in \mathrm{Tp},\ \mathrm{Dom}\ \lambda = \mathrm{Rng}\ \kappa\}. \tag{1.5}$$

Let a continuous body $\mathscr{B}$ with placement–class $\mathrm{Pl}(\mathscr{B})$ be given. The subsets of $\mathscr{B}$ belonging to

$$\Omega_{\mathscr{B}} := \{\mathscr{P} \in \mathrm{Sub}\ \mathscr{B} \mid \kappa_{>}(\mathscr{P}) \in \mathrm{Fr}\ \text{ for } \textit{some}\ \ \kappa \in \mathrm{Pl}(\mathscr{B})\}$$

are then called the **parts** of $\mathscr{B}$.

It is an immediate consequence of (B_2) and (T_4) that

$$\Omega_{\mathscr{B}} = \{\mathscr{P} \in \mathrm{Sub}\ \mathscr{B} \mid \kappa_{>}(\mathscr{P}) \in \mathrm{Fr} \text{ for } \textit{all}\ \ \kappa \in \mathrm{Pl}(\mathscr{B})\}. \tag{1.6}$$

The non–empty parts of $\mathscr{B}$ are also called **subbodies**, which is justified by the following fact.

Theorem 1: *Every part $\mathscr{P} \in \Omega_{\mathscr{B}}^{\times}$ acquires the natural structure of a continuous body by the specification*

$$\mathrm{Pl}(\mathscr{P}) := \{\kappa\big|_{\mathscr{P}}^{\kappa_{>}(\mathscr{P})} \mid \kappa \in \mathrm{Pl}(\mathscr{B})\} \tag{1.7}$$

for the placement class of $\mathscr{P}$.

Proof: The fact that $\mathrm{Pl}(\mathscr{P})$ satisfies (B_1) follows directly from (1.6). Given $\kappa, \gamma \in \mathrm{Pl}(\mathscr{B})$ we have

$$\gamma\Big|_{\mathscr{P}}^{\gamma_>(\mathscr{P})} \circ \Big(\kappa\Big|_{\mathscr{P}}^{\kappa_>(\mathscr{P})}\Big)^{\leftarrow} = (\gamma \circ \kappa^{\leftarrow})\Big|_{\kappa_>(\mathscr{P})}^{\gamma_>(\mathscr{P})} = (\gamma \circ \kappa^{\leftarrow})\Big|_{\kappa_>(\mathscr{P})}^{(\gamma \circ \kappa^{\leftarrow})_>(\kappa_>(\mathscr{P}))}.$$

Hence, since $\gamma \circ \kappa^{\leftarrow} \in \mathrm{Tp}$ because $\mathrm{Pl}(\mathscr{B})$ satisfies (B_2), it follows from (T_7), with $\mathscr{D} := \kappa_>(\mathscr{P})$ and $\lambda := \gamma \circ \kappa^{\leftarrow}$, that $\mathrm{Pl}(\mathscr{P})$ also satisfies (B_2). Now let $\kappa \in \mathrm{Pl}(\mathscr{P})$ and $\lambda \in \mathrm{Tp}$ be given such that $\mathrm{Rng}(\kappa\Big|_{\mathscr{P}}^{\kappa_>(\mathscr{P})}) = \kappa_>(\mathscr{P}) = \mathrm{Dom}\,\lambda$. By (T_3) we may choose $\varphi : \mathrm{Dsp}\,\lambda \to \mathrm{Rsp}\,\lambda$ such that $\lambda = \varphi\Big|_{\kappa_>(\mathscr{P})}^{\mathrm{Rng}\ \lambda}$. Putting $\bar{\lambda} := \varphi\Big|_{\mathrm{Rng}\ \kappa}^{\varphi_>(\mathrm{Rng}\,\kappa)}$, we have $\bar{\lambda} \in \mathrm{Tp}$ and hence, since $\mathrm{Pl}(\mathscr{B})$ satisfies (B_3), $\bar{\lambda} \circ \kappa \in \mathrm{Pl}(\mathscr{B})$. Therefore, $(\bar{\lambda} \circ \kappa)\Big|_{\mathscr{P}}^{(\bar{\lambda}\circ\kappa)_>(\mathscr{P})} = \lambda \circ (\kappa\Big|_{\mathscr{P}}^{\kappa_>(\mathscr{P})}) \in \mathrm{Pl}(\mathscr{P})$, which shows that $\mathrm{Pl}(\mathscr{P})$ satisfies (B_3). ■

The structure of a continuous body on a given set $\mathscr{B}$ can be specified by the prescription of a single placement of $\mathscr{B}$ as follows.

Proposition 1: *Let $\mathscr{B}$ be a set and let α be an invertible mapping with* $\mathrm{Dom}\,\alpha = \mathscr{B}$ *and* $\mathrm{Rng}\,\alpha \in \mathrm{Fr}$. *Then*

$$\mathrm{Pl}_{\alpha}(\mathscr{B}) := \{\lambda \circ \alpha \mid \lambda \in \mathrm{Tp},\ \mathrm{Dom}\,\lambda = \mathrm{Rng}\,\kappa\} \tag{1.8}$$

endows $\mathscr{B}$ with the structure of a continuous body and we have $\alpha \in \mathrm{Pl}_{\alpha}(\mathscr{B})$.

Proof: Since $\mathrm{Rng}\,\alpha \in \mathrm{Fr}$ we have $1_{\mathrm{Rng}\,\alpha} \in \mathrm{Tp}$ and hence $\alpha = 1_{\mathrm{Rng}\,\alpha} \circ \alpha \in \mathrm{Pl}_{\alpha}(\mathscr{B})$, showing that $\mathrm{Pl}_{\alpha}(\mathscr{B})$ is not empty. The fact that $\mathrm{Pl}_{\alpha}(\mathscr{B})$ satisfies $(\mathrm{B}_1) - (\mathrm{B}_3)$ is an immediate consequence of (T_4), (T_5), and (T_6). ■

2. The Material System of the Parts of a Body, Filters

We assume that a continuous body $\mathscr{B}$ with placement–class $\mathrm{Pl}(\mathscr{B})$ is given. We call the collection $\Omega_{\mathscr{B}}$ of all parts of $\mathscr{B}$, as given by (1.6), the **material system of parts** of $\mathscr{B}$. The following theorem shows that $\Omega_{\mathscr{B}}$ satisfies the axioms for a material system as given, for example, in [N] or [T], (where the term "material universe" rather than "material system" is used).

Theorem 2: *The collection $\Omega_{\mathscr{B}}$, when ordered by inclusion, has the following properties:*

(i) *The intersection of any two parts of $\mathscr{B}$ is a part of $\mathscr{B}$, i.e. for all $\mathscr{P},\mathscr{Q} \in \Omega_{\mathscr{B}}$ $\mathscr{P} \cap \mathscr{Q} \in \Omega_{\mathscr{B}}$.*

(ii) *For any two parts $\mathscr{P},\mathscr{Q} \in \Omega_{\mathscr{B}}$ there is a smallest part of $\mathscr{B}$ that includes both. This smallest member of $\Omega_{\mathscr{B}}$ that includes both $\mathscr{P}$ and $\mathscr{Q}$ is called the* **join** *of $\mathscr{P}$ and $\mathscr{Q}$ and is denoted by $\mathscr{P} \vee \mathscr{Q}$.*

(iii) *For every part of $\mathscr{P} \in \Omega_{\mathscr{B}}$, there is a largest part of $\mathscr{B}$ that is disjoint from $\mathscr{P}$. This largest member of $\Omega_{\mathscr{B}}$ that is disjoint from $\mathscr{P}$ is called the* **exterior of** *$\mathscr{P}$* **relative to** *$\mathscr{B}$ and is denoted by $\mathscr{P}^{\mathrm{b}}$, so that*

$$\mathscr{P} \cap \mathscr{P}^{\mathrm{b}} = \emptyset \tag{2.1}$$

and

$$(\mathscr{P} \cap \mathscr{Q} = \emptyset \quad \Rightarrow \quad \mathscr{Q} \subset \mathscr{P}^{\mathrm{b}}) \qquad \textit{for all } \mathscr{Q} \in \Omega_{\mathscr{B}}. \tag{2.2}$$

For every K e *V\{3) and all 9£* 6 ft^>, *we have*

$$n_y\{9 \text{ n } £) = K\hat{}P) \text{ fl } /c\hat{}.2) \tag{2.3}$$

$$K\hat{}P \text{ V jg}) = *(,?) \text{ V } /c(A) \tag{2.4}$$

$$\kappa_{>}(\mathcal{P}^{b}) = \mathrm{Rng}\,\kappa \quad \kappa(\mathcal{P}) \tag{2.5}$$

PROOF: Let *K* e Pl(^) be given. It follows from (1.6) that («$_>$)$_>$(^^) = {^>(^) I & 6 fi^} is the set Fr(Rng *K)* of all fit regions included in Rng *K* defined according to (1.4). Hence, since *K* is invertible, $n_{>}\Big|^{\mathrm{Fr\,(Rng}\,K)}_{\wedge}$ is an order—isomorphism from ft ~> to Fr(Rng /c). Now, it follows from Theorem 4, Sect. 5 and from (C), (D), (E) of Sect. 4. of [NV] that Fr(Rng *K)* has properties analogous to (i), (ii), and (iii). Hence ft^ has these properties and (2.3)-(2.5) hold. H

It is easily seen that

$$(\hat{}>b)b = y \text{ and } 9 \text{ V } 9^* = 3 \text{ for all } 9\ e \quad \text{ft}\hat{}. \tag{2.6}$$

PiOPOSmoi 1: *The body 3 has exactly one uniform structure that makes all placements uniform homeomorphisms. This uniform structure is determined by any one of the metrics* d *on 3 definedby*

$$d\hat{}ftY) := |*(X) - K(Y)| \quad \text{for all } X,Y\ e\ 3, \tag{2.7}$$

where n€ **Pl($\mathcal{B}$)**.

PROOF: We choose a placement $\kappa \in \mathrm{Pl}(\mathscr{B})$. There is exactly one uniform structure on $\mathscr{B}$ that makes κ uniformly continuous, namely the structure obtained by transporting the uniform structure of $\mathrm{Rng}\,\kappa$ inherited from $\mathrm{Rsp}\,\kappa$ to $\mathscr{B}$ by means of $\kappa^{\leftarrow}$. This uniform structure is the one determined by the metric (2.7). Now let a placement $\gamma \in \mathrm{Pl}(\mathscr{B})$ be given. Showing that γ is a uniform homeomorphism from $\mathscr{B}$ to $\mathrm{Rng}\,\gamma$ then amounts to showing that $\lambda := \gamma \circ \kappa^{\leftarrow}$ is a uniform homeomorphism from $\mathrm{Rng}\,\kappa$ to $\mathrm{Rng}\,\gamma$. Now, it follows from (T_2) that λ has a continuous extension $\bar{\lambda} : \mathrm{Clo}\,\mathrm{Rng}\,\kappa \to \mathrm{Clo}\,\mathrm{Rng}\,\gamma$. Since $\mathrm{Rng}\,\kappa$ is bounded and hence $\mathrm{Clo}\,\mathrm{Rng}\,\kappa$ compact, it follows from the Uniform Continuity Theorem that $\bar{\lambda}$, and hence λ, is uniformly continuous. Interchanging the roles of γ and κ, we see that $\lambda^{\leftarrow} = \kappa \circ \gamma^{\leftarrow}$ is also uniformly continuous, i.e. that λ is a uniform homeomorphism. ∎

Definition 1: *We say that a non-empty subset* $\mathfrak{F}$ *of* $\Omega_{\mathscr{B}}^{\times}$ *if a* **filter** *on* $\Omega_{\mathscr{B}}$ *if*

(F_1) $\mathfrak{F}$ *is stable under pairwise intersection*, i.e., *for all* $\mathscr{P},\mathscr{Q} \in \mathfrak{F}$ *we have* $\mathscr{P} \cap \mathscr{Q} \in \mathfrak{F}$.

(F_2) *Every set in* $\Omega_{\mathscr{B}}$ *that includes a set in* $\mathfrak{F}$ *belongs to* $\mathfrak{F}$, *i.e., for all* $\mathscr{P} \in \mathfrak{F}$ *and all* $\mathscr{R} \subset \Omega_{\mathscr{B}}$ *with* $\mathscr{P} \subset \mathscr{R}$, *we have* $\mathscr{R} \in \mathfrak{F}$.

We say that the filter $\mathfrak{F}$ *is* **fundamental** *if for every entourage* $\mathscr{U}$ *of the uniform space* $\mathscr{B}$, *there is a* $\mathscr{P} \in \mathfrak{F}$ *such that* $\mathscr{P} \times \mathscr{P} \subset \mathscr{U}$. *We say that* $\mathfrak{F}$ *is* **minimally fundamental** *if it is fundamental and if there is no fundamental filter that is strictly included in* $\mathfrak{F}$.

Remark 1: The concept of a filter as defined here coincides with the one used in the theory of Boolean algebras (see Sect. 3 of [S]). Indeed, with the operations of *intersection*, *join*, and *exterior relative to* $\mathscr{B}$ as described in Theorem 2, the material system $\Omega_{\mathscr{B}}$ is just a Boolean algebra. The concept of a filter as used in topology (see Sect. 5 of Ch. I of [B]) is somewhat different. What we call a filter here is a *filter–base* in the sense of topology, and what we call a fundamental filter here would be called the base of a fundamental (or Cauchy–) filter in topology. ∎

Let $\kappa \in \mathrm{Pl}(\mathscr{B})$ be given. Since the uniformity of $\mathrm{Rng}\,\kappa$ is determined by the Euclidean distance in $\mathrm{Frm}\,\kappa$, it follows from Prop. 1 that a given filter Φ is fundamental if and only if, for every $\epsilon \in \mathbb{P}^{\times}$, there is a $\mathscr{P} \in \Phi$ such that

$$\mathrm{diam}(\kappa_{>}(\mathscr{P})) \leq \epsilon. \tag{2.8}$$

For every $X \in \mathscr{B}$, the subset

$$\bar{X} := \{\mathscr{P} \in \Omega_{\mathscr{B}} \,|\, X \in \mathscr{P}\} \tag{2.9}$$

of $\Omega_{\mathscr{B}}$ is easily seen to be a minimally fundamental filter.

Definition 2: *The set* $\bar{\mathscr{B}}$ *of all minimally fundamental filters on* $\Omega_{\mathscr{B}}$ *is called the* **completion** *of* $\mathscr{B}$ *and is denoted by* $\bar{\mathscr{B}}$.

Remark 2: Definition 2 does indeed describe a completion of the uniform space $\mathscr{B}$ in the sense in which the term "completion" is used in the theory of uniform spaces. This is an easy consequence of the theorems on completion stated in Sect. 7 of Ch. II of [B]. ■

It is easily seen that the mapping $(X \mapsto \bar{X}) : \mathscr{B} \to \bar{\mathscr{B}}$ is injective. We identify the range of this mapping with $\mathscr{B}$ itself, i.e., we use the symbol X instead of $\bar{X}$ also for the filter (2.8). The set $\bar{\mathscr{B}}$ has the structure of a separated, *complete* uniform space and the original uniformity of $\mathscr{B}$ coincides with the one that $\mathscr{B}$ inherits from $\bar{\mathscr{B}}$ when $\mathscr{B}$ is regarded as a subset of $\bar{\mathscr{B}}$. Given a subset $\mathscr{S}$ of $\mathscr{B}$, one then must carefully distinguish between the closure $\mathrm{Clo}\,\mathscr{S}$ of $\mathscr{S}$ in $\mathscr{B}$ and the closure of $\mathscr{S}$ when $\mathscr{S}$ is regarded as a subset of $\bar{\mathscr{B}}$. We denote the latter by $\overline{\mathrm{Clo}}\,\mathscr{S}$. An analous distinction must be made between $\mathrm{Bdy}\,\mathscr{S}$ and $\overline{\mathrm{Bdy}}\,\mathscr{S}$. For example, we have

$$\mathrm{Bdy}\,\mathscr{B} = \emptyset\,, \quad \overline{\mathrm{Bdy}}\,\mathscr{B} = \bar{\mathscr{B}}\backslash\mathscr{B} \neq \emptyset. \tag{2.9}$$

The following fact is an immediate consequence of Prop. 1 and the fact that the completion of an open subset of a Euclidean space can be identified with its closure.

Proposition 3: *Every placement* $\kappa \in \mathrm{Pl}(\mathscr{B})$ *has a unique continuous extension*

$$\bar{\kappa} : \bar{\mathscr{B}} \to \mathrm{Clo\,Rng}\,\kappa, \tag{2.10}$$

and this extension is a uniform homeomorphism. For every $\mathscr{S} \in \mathrm{Sub}\,\mathscr{B}$ *we have*

$$\bar{\kappa}_{>}(\overline{\mathrm{Clo}}\,\mathscr{S}) = \mathrm{Clo}\,\kappa_{>}(\mathscr{S}), \qquad \bar{\kappa}_{>}(\overline{\mathrm{Bdy}}\,\mathscr{S}) = \mathrm{Bdy}\,\kappa_{>}(\mathscr{S}). \tag{2.11}$$

The inverse of $\bar{\kappa}$ *is given by*

$$\begin{aligned} \bar{\kappa}^{\leftarrow}(\mathrm{x}) = \{\mathscr{P} \in \Omega_{\mathscr{B}} \mid \kappa_{>}(\mathscr{P}) = \mathrm{Rng}\,\kappa \cap \mathscr{N} \ \text{ for some} \\ \mathscr{N} \in \mathrm{Fr}(\mathrm{Rsp}\,\kappa) \text{ with } \mathrm{x} \in \mathscr{N}\} \end{aligned} \tag{2.12}$$

for all $\mathrm{x} \in \mathrm{Clo\,Rng}\,\kappa$.

We note that (2.12) reduces to

$$\bar{\kappa}^{\leftarrow}(\mathrm{x}) = \{\mathscr{P} \in \Omega_{\mathscr{B}} \mid \mathrm{x} \in \kappa_{>}(\mathscr{P})\} = \overline{\kappa^{\leftarrow}(\mathrm{x})} \cong \kappa^{\leftarrow}(\mathrm{x}) \tag{2.13}$$

when $\mathrm{x} \in \mathrm{Rng}\,\kappa$, as it should.

3. Improper Transplacements and Placements

We now consider a class $\overline{\mathrm{Tp}}$ of mappings which is obtained from Tp by joining certain "improper transplacements".

The class $\overline{\mathrm{Tp}}$ is determined by the following requirements:

($\bar{\mathrm{T}}_1$) Every $\rho \in \overline{\mathrm{Tp}}$ is a continuous invertible mapping whose domain Dom ρ and range Rng ρ are subsets of Euclidean spaces Dsp ρ and Rsp ρ, respectively.

($\bar{\mathrm{T}}_2$) We have Dom $\rho \in \mathrm{Fr}$ for every $\rho \in \overline{\mathrm{Tp}}$.

($\bar{\mathrm{T}}_3$) Every $\rho \in \overline{\mathrm{Tp}}$ has a continuous extension $\bar{\rho}$: Clo Dom $\rho \to$ Clo Rng ρ.

($\bar{\mathrm{T}}_4$) Let $\rho \in \overline{\mathrm{Tp}}$ and $\mathcal{R} \in \mathrm{Fr}\,(\mathrm{Dom}\,\rho)$ be given. Then

$$\bar{\rho}\Big|_{\mathrm{Clo}\,\mathcal{R}} \text{ is injective } \Longrightarrow \rho\Big|_{\mathcal{R}}^{\rho_{>}(\mathcal{R})} \in \mathrm{Tp}.$$

The following facts are easy consequences of ($\bar{\mathrm{T}}_1$) – ($\bar{\mathrm{T}}_4$):

($\bar{\mathrm{T}}_5$) For all $\rho \in \overline{\mathrm{Tp}}$ and $\lambda \in \mathrm{Tp}$ with Rng $\lambda =$ Dom ρ, we have $\rho \circ \lambda \in \overline{\mathrm{Tp}}$.

($\bar{\mathrm{T}}_6$) For every $\rho \in \overline{\mathrm{Tp}}$, Rng ρ is open (but not necessarily regularly open) and ρ is a C^2–diffeomorphism.

Definition 1: *The* **contact set** *of a given* $\rho \in \overline{\mathrm{Tp}}$ *is defined by*

$$\mathrm{Cts}(\rho) := \{x \in \mathrm{Bdy\ Dom}\ \bar{\rho} \mid \rho^{<}(\{\bar{\rho}(x)\}) \text{ is not a singleton}\}. \tag{3.1}$$

Proposition 1: We have

$$\mathrm{Tp} = \{\rho \in \overline{\mathrm{Tp}} \mid \mathrm{Cts}(\rho) = \emptyset\}. \tag{3.2}$$

Proof: Let $\lambda \in \text{Tp}$ be given. It follows from $(T_1)-(T_3)$ that λ has an *invertible* continuous extension $\overline{\lambda}$: Clo Dom $\rho \to$ Clo Rng ρ. In view of (T_7), it follows that λ satisfies $(T_1)-(T_4)$ and that $\text{Cts}(\rho) = \emptyset$.

Now let $\rho \in \text{Tp}$ be given and assume that $\text{Cts}(\rho) = \emptyset$. Then $\overline{\rho}$ is injective. Hence, using (T_4) with $\mathscr{R} =$ Dom ρ, it follows that $\rho \in \text{Tp}$. □

The mappings in Tp\Tp, i.e. the mappings $\rho \in \text{Tp}$ with $\text{Cts}(\rho) \neq \emptyset$, will be called **improper transplacements**. Figure 1 illustrates a situation in which $\text{Cts}(\rho) = \{x,y\}$ is a doubleton.

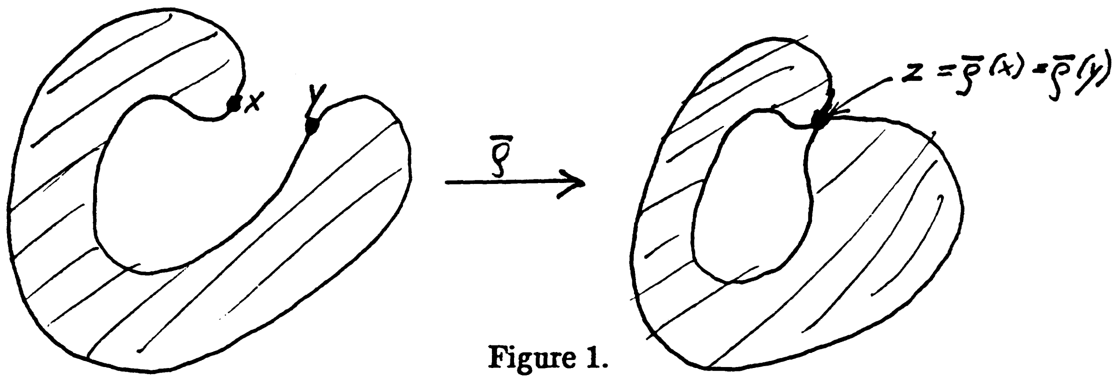

Figure 1.

Remark 1: Let $\rho \in \text{Tp}$ and a point $z \in$ Bdy Rng ρ be given. One can prove that $\overline{\rho}^{<}(\{z\})$ cannot contain more than two points that belong to the *reduced* boundary Rby Dom ρ of the domain of ρ. Figure 1 shows a situation when $\overline{\rho}^{<}(\{z\}) = \{x,y\}$ consists of exactly two such points. It is easy to construct situations in which $\overline{\rho}^{<}(\{z\})$ has more than two points, but then there must be points in $\overline{\rho}^{<}(\{z\})$ at which the boundary of Dom ρ has no tangent in any sense. □

Let ρ be any mapping whose range is an open subset of a Euclidean space. We use the abbreviation

$$\text{Icr } p := \text{Int Clo Rng } p. \tag{3.3}$$

Clearly, Icr p is regularly open and we have

$$\text{Rng } p \text{ C Icr } p. \tag{3.4}$$

If p 6 Tp then Rng p is regularly open by (T4) and hence Icr p = Rng p. The situation illustrated in Fig. 1 shows that one can have Icr p = Rng p even when p e Tp is improper. Figure 2 illustrates a situation in which p € Tp and Icr p t Rng p. The points on the dotted line

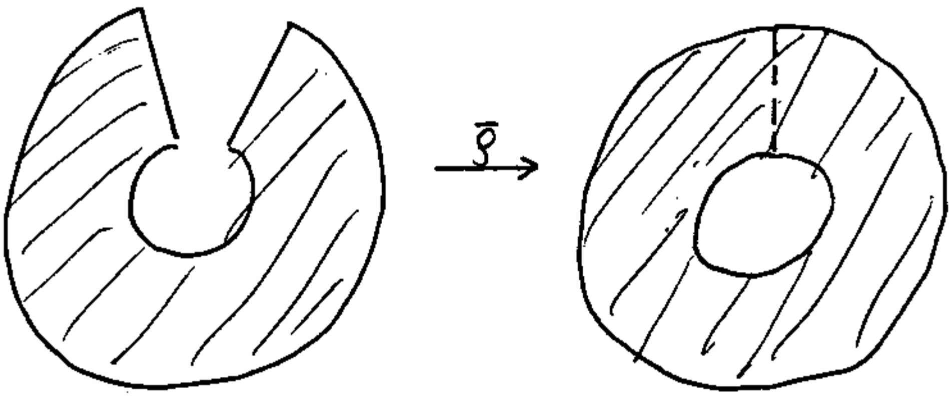

Figure 2.

belong to Icr p but not to Rng p.

Paoposmoi 2: *Let* p € **Tp** *and* A € **Tp** *with* **Icr** p = **Dom** A *be given and put*

$$\rho' := \lambda\Big|_{\text{Rng } p}^{A_{>}(\text{Rngp})} \circ \rho. \tag{3.5}$$

Then $\rho' \in \mathrm{Tp}$,

$$\mathrm{Icr}\,\rho' = \lambda_{>}(\mathrm{Icr}\,\rho) = \mathrm{Rng}\,\lambda, \tag{3.6}$$

and

$$\mathrm{Cts}(\rho) = \mathrm{Cts}(\rho'). \tag{3.7}$$

Proof: It is clear that ρ' satisfies (T_1), (T_2). By (T_2), noting that $\mathrm{Rsp}\,\rho = \mathrm{Dsp}\,\lambda$, we may choose a homeomorphisms $\varphi : \mathrm{Rsp}\,\rho \to \mathrm{Rsp}\,\lambda$ such that

$$\lambda = \varphi\Big|_{\mathrm{Dom}\,\lambda}^{\mathrm{Rng}\,\lambda}, \quad \rho' = \varphi\Big|_{\mathrm{Rng}\,\rho}^{\varphi_{>}(\mathrm{Rng}\,\rho)} \circ \rho. \tag{3.8}$$

Since φ is a homeomorphism, we have

$$\varphi_{>}(\mathrm{Clo}\,\mathrm{Rng}\,\rho) = \mathrm{Clo}\,\varphi_{>}(\mathrm{Rng}\,\rho) = \mathrm{Clo}\,\mathrm{Rng}\,\rho' \tag{3.9}$$

and

$$\varphi_{>}(\mathrm{Icr}\,\rho) = \mathrm{Int}\,\mathrm{Clo}\,(\varphi_{>}(\mathrm{Rng}\,\rho)) = \mathrm{Icr}\,\rho'. \tag{3.10}$$

If $\overline{\rho}$ is the continuous extension of ρ postulated by (T_3), it follows that

$$\overline{\rho}' := \varphi\Big|_{\mathrm{Clo}\,\mathrm{Rng}\,\rho}^{\varphi_{>}(\mathrm{Clo}\,\mathrm{Rng}\,\rho)} \circ \overline{\rho} : \ \mathrm{Clo}\,\mathrm{Dom}\,\rho \to \mathrm{Clo}\,\mathrm{Rng}\,\rho' \tag{3.11}$$

is a continuous extension of ρ' to Clo Dom $\rho =$ Clo Dom ρ' and hence that ρ' satisfies (T_3).

Now let $\mathcal{R} \in$ Fr (Dom ρ) $=$ Fr (Dom ρ') be given such that $\bar{\rho}'|_{\mathrm{Clo}\,\mathcal{R}}$ is injective. Since φ is invertible, it follows from (3.11) that $\bar{\rho}|_{\mathrm{Clo}\,\mathcal{R}}$ is injective. Hence, since ρ satisfies (T_4), we have $\rho\Big|_{\mathcal{R}}^{\rho_>(\mathcal{R})} \in \mathrm{Tp}$. It follows from ($T_4$) that $\rho_>(\mathcal{R}) \in \mathrm{Fr}$ and hence, by (T_7), that $\lambda\Big|_{\rho_>(\mathcal{R})}^{\lambda_>(\rho_>(\mathcal{R}))} \in \mathrm{Tp}$. Using ($T_5$), we conclude from (3.8) that

$$\rho'\Big|_{\mathcal{R}}^{\rho'_>(\mathcal{R})} = \lambda\Big|_{\rho_>(\mathcal{R})}^{\lambda_>(\mathcal{R}))} \circ \rho\Big|_{\mathcal{R}}^{\rho_>(\mathcal{R})} \in \mathrm{Tp},$$

which shows that ρ' satisfies (T_4).

The assertion (3.6) is merely another form of (3.10). The assertion (3.7) follows from (3.11) and the fact that φ is invertible. ∎

Definition 2: *Let a continuous body with placement–class* $\mathrm{Pl}(\mathcal{B})$ *be given. We then define the* **extended placement–class** $\overline{\mathrm{Pl}}(\mathcal{B})$ *by*

$$\overline{\mathrm{Pl}}(\mathcal{B}) := \{\rho \circ \kappa \mid \kappa \in \mathrm{Pl}(\mathcal{B}),\ \rho \in \overline{\mathrm{Tp}},\ \mathrm{Dom}\,\rho = \mathrm{Rng}\,\kappa\}. \tag{3.12}$$

It is an immediate consequence of (T_5) that

$$\overline{\mathrm{Pl}}\,(\mathcal{B}) = \{\rho \circ \kappa \mid \rho \in \overline{\mathrm{Tp}},\ \mathrm{Dom}\,\rho = \mathrm{Rng}\,\kappa\} \tag{3.13}$$

for every $\kappa \in \mathrm{Pl}(\mathcal{B})$. Since $\mathrm{Tp} \subset \overline{\mathrm{Tp}}$, it follows from (1.5) that $\mathrm{Pl}(\mathcal{B}) \subset \overline{\mathrm{Pl}}(\mathcal{B})$. The elements of $\overline{\mathrm{Pl}}(\mathcal{B})\setminus\mathrm{Pl}(\mathcal{B})$ will be called **improper placements** of $\mathcal{B}$.

The following facts are immediate consequences of corresponding facts for improper transplacements:

($\bar{P}_1$) Every $\delta \in \overline{\mathrm{Pl}}(\mathscr{B})$ is a homeomorphism whose domain is $\mathscr{B}$ and whose codomain is an open subset of a Euclidean space $\mathrm{Rsp}\ \delta$.

($\bar{P}_2$) Every $\delta \in \overline{\mathrm{Pl}}(\mathscr{B})$ has a continuous extension

$$\bar{\delta} : \overline{\mathscr{B}} \to \mathrm{Clo\ Rng}\ \delta.$$

($\bar{P}_3$) Let $\delta \in \overline{\mathrm{Pl}}(\mathscr{B})$ and $\mathscr{P} \in \Omega_{\mathscr{B}}$ be given. Then

$$\bar{\delta}|_{\overline{\mathrm{Clo}}(\mathscr{P})} \text{ is injective } \Longrightarrow \delta\Big|_{\mathscr{P}}^{\delta_>(\mathscr{P})} \in \mathrm{Pl}(\mathscr{P}).$$

Definition 3: *The* **contact set** *of a given* $\delta \in \overline{\mathrm{Pl}}(\mathscr{B})$ *is defined by*

$$\mathrm{Cts}(\delta) := \{X \in \overline{\mathscr{B}} \backslash \mathscr{B} \mid \bar{\delta}^<(\{\bar{\delta}(X)\}) \text{ is not a singleton }\}. \tag{3.14}$$

Remark: The condition ($\bar{P}_3$), and the condition ($\bar{T}_4$) from which it is derived have the following significance: If $\mathscr{P}$ is a part of $\mathscr{B}$ which does not "feel" the contacts produced by ρ, then the placement of $\mathscr{P}$ induced by ρ is not improper. ■

The following two Propositions are immediate consequences of Props. 1 and 2 above.

Proposition 3: *We have*

$$\mathrm{Pl}(\mathscr{B}) = \{\delta \in \overline{\mathrm{Pl}}(\mathscr{B}) \mid \mathrm{Cts}(\delta) = \emptyset\}. \tag{3.15}$$

Proposition 4: *Let* $\delta \in \overline{\mathrm{Pl}}(\mathscr{B})$ *and* $\lambda \in \mathrm{Tp}$ *with* $\mathrm{Icr}\ \delta = \mathrm{Dom}\ \lambda$ *be given and put*

$$\delta' := \lambda\Big|_{\mathrm{Rng}\ \delta}^{\lambda_>(\mathrm{Rng}\ \delta)} \circ \delta. \tag{3.16}$$

Then $\delta' \in \overline{\mathrm{PI}}(\mathscr{B})$,

$$\mathrm{Icr}\ \delta' = \lambda_{>}(\mathrm{Icr}\ \delta) = \mathrm{Rng}\ \lambda, \tag{3.17}$$

and

$$\mathrm{Cts}(\delta') = \mathrm{Cts}(\delta). \tag{3.18}$$

Proposition 5: Let $\rho \in \bar{\mathrm{T}}\mathrm{p}$ be given. For every $\mathscr{R} \in \mathrm{Fr}(\mathrm{Dom}\ \rho)$ we have

$$\rho_{>}(\mathscr{R}) = \mathrm{Rng}\ \rho \cap \mathrm{Int\ Clo}\ \rho_{>}(\mathscr{R}) \tag{3.19}$$

Proof: Since ρ is a homeomorphism (see $(\bar{\mathrm{T}}_6)$), it follows that $\rho_{>}$ preserves interiors and relative closures. Since the closure of a subset $\mathscr{S}$ of $\mathrm{Rng}\ \rho$ relative to $\mathrm{Rng}\ \rho$ is given by $\mathrm{Clo}\ \mathscr{S} \cap \mathrm{Rng}\ \rho$, (3.19) follows from the fact that $\mathrm{Int\ Clo}\ \mathscr{R} = \mathscr{R}$ for all $\mathscr{R} \in \mathrm{Fr}(\mathrm{Dom}\ \rho)$. ∎

The following is an immediate consequence of Prop. 5 and (3.13).

Proposition 6: Let $\delta \in \overline{\mathrm{PI}}(\mathscr{B})$ be given. For every $\mathscr{P} \in \Omega_{\mathscr{B}}$ we have

$$\delta_{>}(\mathscr{P}) = \mathrm{Rng}\ \delta \cap \mathrm{Int\ Clo}\ \delta_{>}(\mathscr{P}) \tag{3.21}$$

4. Bodies Obtained by Separation

We assume that a continuous body *3* with placement class *V\{3)* is given. We also assume that a mapping a with the following property is given;

(S) ***There exists K* G Pl(#) *and p* G T̄r\Tr *such that* Icr *p* = Rng *K and***

$$\alpha = \rho^{\leftarrow} \circ \kappa\Big|_{\kappa^{<}(\mathrm{Rng}\ \rho)}^{\mathrm{Rng}\ \rho} . \tag{4.1}$$

The following result states, roughly, that the placement *K* for which (S) holds can be chosen arbitrarily.

PEOPOSITIOI 1: ***If a given mapping a satisfies the condition* (S), *then for every n* G Pl(^) *there is a p* G T̄r\Tr *such that* Icr *p* = Rng *K and* (4.1) *holds.***

PROOF: Since *a* satisfies (S), we may choose 7GPl(v#)and aGT̄r\Tr such that Icr *a* = Rng 7 and

$$\alpha = \sigma^{\leftarrow} \circ \gamma\Big|_{7^{<}(\mathrm{Rngp})}^{\mathrm{R}\wedge\mathrm{g}\ a} . \tag{4.2}$$

Now let *K* G Pl(^) be given. By (B_2) of Def. 1 of Sect. 1, we have

$$\text{A} := \kappa \circ 7'' \text{ G Tp}, \quad \text{Dom A} = \text{Rng 7} = \text{Icr a}, \tag{4.3}$$

and hence, by Prop. 2 of Sect. 3,

$$\rho := \lambda\Big|^{\mathrm{Rng}\ \rho}_{\kappa^{<}(\mathrm{Rng}\ \rho)} \circ \sigma \in \mathrm{Tp}. \tag{4.4}$$

and Icr $\rho = \mathrm{Rng}\ \lambda$. Since $\mathrm{Rng}\ \kappa = \mathrm{Rng}\ \lambda$ by (4.3), it follows that Icr $\rho = \mathrm{Rng}\ \kappa$. On the other hand, it follows from (4.4) that

$$\rho^{\leftarrow} \circ \kappa\Big|^{\mathrm{Rng}\ \rho}_{\kappa^{<}(\mathrm{Rng}\ \rho)} = \sigma^{\leftarrow} \circ \lambda^{\leftarrow}\Big|^{\mathrm{Rng}\ \sigma}_{\lambda_{>}(\mathrm{Rng}\ \sigma)} \circ \kappa\Big|^{\mathrm{Rng}\ \rho}_{\kappa^{<}(\mathrm{Rng}\ \rho)}.$$

Since $\lambda^{\leftarrow} = \gamma \circ \kappa^{\leftarrow}$ and $\mathrm{Rng}\ \rho = \lambda_{>}(\mathrm{Rng}\ \sigma)$, we conclude that

$$\rho^{\leftarrow} \circ \kappa\Big|^{\mathrm{Rng}\ \rho}_{\kappa^{<}(\mathrm{Rng}\ \sigma)} = \sigma^{\leftarrow} \circ \gamma\Big|^{\mathrm{Rng}\ \sigma}_{\gamma^{<}(\mathrm{Rng}\ \sigma)} = \alpha\,,$$

i.e. that (4.1) is valid. □

Proposition 2: *The set*

$$\mathscr{B}' := \mathrm{Dom}\ \alpha \tag{4.5}$$

is a open subset of $\mathscr{B}$ *and we have*

$$\mathrm{Clo}\ \mathscr{B}' = \mathscr{B}. \tag{4.6}$$

Proof: Choose κ and ρ such that (S) is valid. By (4.5) and (4.1) we then have $\mathscr{B}' = \kappa^{<}(\mathrm{Rng}\ \rho)$. Since $\mathrm{Rng}\ \rho$ is open by $(\overline{\mathrm{T}}_7)$ and since κ is continuous, it follows that $\mathscr{B}'$ is open in $\mathscr{B}$. Since $\mathrm{Icr}\,\rho$ is dense in $\mathrm{Rng}\,\rho$ and since $\kappa^{\leftarrow}$ is a homeomorphism from $\mathrm{Icr}\,\rho$ to $\mathscr{B}$, it follows that $\mathscr{B}' = (\kappa^{\leftarrow})_{>}(\mathrm{Rng}\,\rho)$ is dense in $\mathscr{B}$, i.e. that (4.6) holds. ■

Since $\mathrm{Rng}\,\alpha = \mathrm{Dom}\,\rho \in \mathrm{Fr}$ by $(\overline{\mathrm{T}}_2)$ when κ and ρ are chosen according to (S), it follows from Prop. 1 of Sect. 1 that

$$\mathrm{Pl}_{\alpha}(\mathscr{B}') := \{\lambda \circ \alpha \mid \lambda \in \mathrm{Tp},\ \mathrm{Dom}\,\lambda = \mathrm{Rng}\,\alpha\} \tag{4.7}$$

endows $\mathscr{B}'$ with the structure of a continuous body. We say that the continuous body $\mathscr{B}'$ obtained in this way is obtained from $\mathscr{B}$ by **separation via α**.

Pitfall: It may happen that $\mathscr{B} = \mathscr{B}'$ and yet $\alpha \notin \mathrm{Pl}(\mathscr{B})$, so that $\mathrm{Pl}_{\alpha}(\mathscr{B}) \neq \mathrm{Pl}(\mathscr{B})$. In fact, this is the case if and only if ρ in (S) can be chosen such that $\mathrm{Icr}\,\rho = \mathrm{Rng}\,\rho$. An example of an improper transplacement ρ with this property is indicated in Fig. 1 of Sect. 3. In such a situation, one must consider two *distinct* continuous bodies having the *same* underlying set of material points. ■

5. Bodies Obtained by Contact

Again, we assume that a continuous body $\mathscr{B}$ with placement class $\mathrm{Pl}(\mathscr{B})$ is given. We also assume that an improper placement $\delta \in \overline{\mathrm{Pl}}(\mathscr{B})\backslash \mathrm{Pl}(\mathscr{B})$ (as defined in Sect. 3) is given such that $\mathrm{Icr}\,\delta \in \mathrm{Fr}$. For every $x \in \mathrm{Icr}\,\delta$, we put

$$\Phi_x := \{\mathscr{P} \in \Omega_{\mathscr{B}} \mid x \in \mathrm{Int}\,\mathrm{Clo}(\delta_>(\mathscr{P}))\}. \tag{5.1}$$

Proposition 1: *For each* $x \in \mathrm{Icr}\,\delta$, *the collection* Φ_x *is a filter. If* $x \in \mathrm{Rng}\,\delta$ *we have*

$$\Phi_x = \overline{\delta^{\leftarrow}(x)} = \{\mathscr{P} \in \Omega_{\mathscr{B}} \mid \delta^{\leftarrow}(x) \in \mathscr{P}\} \tag{5.2}$$

and hence Φ_x *is a fundamental filter.*

Proof: Let $x \in \mathit{I}\mathrm{cr}\,\delta$ be given. The fact that Φ_x is a filter is an immediate consequence of the definition (5.1).

Now assume that $x \in \mathrm{Rng}\,\delta$. By Prop. 6 of Sect. 3 and (5.1) we then have, for all $\mathscr{P} \in \Omega_{\mathscr{B}}$,

$$\mathscr{P} \in \Phi_x \iff x \in \delta_>(\mathscr{P}).$$

Since δ is invertible, it follows from (2.9) that $\Phi_x = \overline{\delta^{\leftarrow}(x)}$, which is a fundamental filter. ∎

Remark: It may happen that Φx is a fundamental filter for a given $x \in \mathrm{Icr}\,\delta$ even though $x \notin \mathrm{Rng}\,\delta$. An example may be read off from Fig. 1, when we put $\delta = \rho \circ \kappa$ for a given $\kappa \in \mathrm{Pl}(\mathscr{B})$.

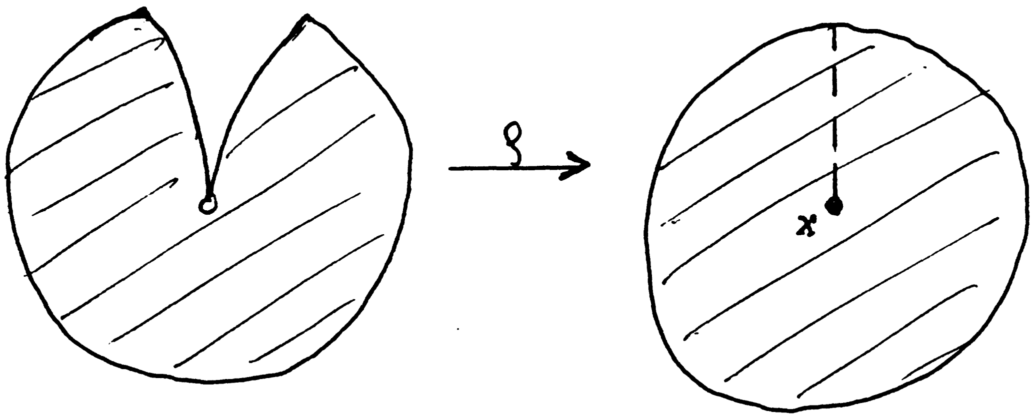

Figure 1.

However, one can show that Φx is not fundamental when $x \in \bar{\delta}_{>}(\mathrm{Cts}(\phi))$. ■

We now put

$$\hat{\mathscr{B}} := \{\Phi_x \mid x \in \mathrm{Icr}\,\delta\}. \tag{5.3}$$

Using the identification $X \mapsto \bar{X}$ of material points in $\mathscr{B}$ with their corresponding filters (2.9), we may regard $\mathscr{B}$ as a subset of $\hat{\mathscr{B}}$.

It is easily seen that the mapping

$$(x \mapsto \Phi_x) : \mathrm{Icr}\,\delta \to \hat{\mathscr{B}} \tag{5.4}$$

given by (5.1) is injective and hence invertible. We denote its inverse by

$$\hat{\delta} : \hat{\mathscr{B}} \to \mathrm{Icr}\,\delta\,, \tag{5.5}$$

and we put

$$\mathrm{Pl}_{\hat{\delta}}(\hat{\mathscr{B}}) := \{A \circ \hat{6} \mid A \; 6 \; Tp, \mathrm{Dom}\, A = \mathrm{Icr}\, p). \tag{5.6}$$

Since Icr *p* 6 Fr by assumption, it follows from Prop. 1 of Sect. 1 that (5.6) endows $\hat{3}$ with the structure of a continuous body. We say that the continuous body $\hat{3}$ obtained in this way is obtained from *3* by contact via *6.*

Pitfall: It may happen that *6,6'* 6 *¥1(3)* lead to the *same* set $\hat{3}$ of filters and yet *V\jf^3)* ≠ PI^ *fi,{3).* An example can be read off from Figure 2., in which *K* 6 Pl(*#) is given and *p* := *6* o /c~, *p'* := *6'* o*~. Therefore, contacts as well as separations can lead to distinct continuous bodies having the same underlying set of material points.

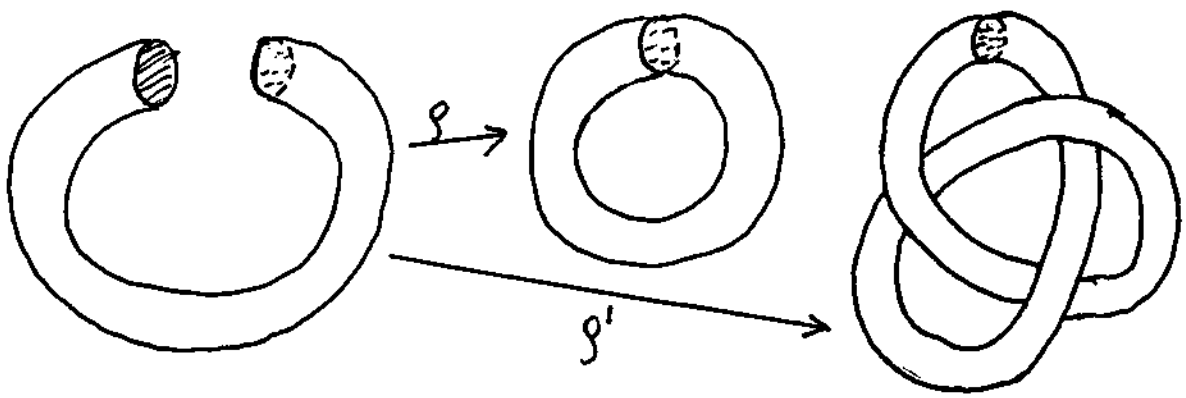

Figure 2.

It may even happen that $\hat{3} = 3$ and yet Pl(^) ≠ $\mathrm{Pl}_{\hat{\delta}}(\mathscr{B})$, which is illustrated by Figure 1 of Sect. 3 when *p* = *6* o *tt* for a given *K* 6 Pl(^). a

6. Restorations and Reformations

We assume again that a continuous body $\mathscr{B}$ with placement class $\mathrm{Pl}(\mathscr{B})$ is given.

Theorem 1: *Let the continuous body* $\mathscr{B}'$, *with placement class* $\mathrm{Pl}_{\alpha}(\mathscr{B}')$, *be obtained from* $\mathscr{B}$ *by separation via a mapping* α *satisfying the condition* (S) *of* Sect. 4. *Then* $\mathscr{B}$ *can be restored from* $\mathscr{B}'$ *by contact via an improper placement* δ *of* $\mathscr{B}'$. *In fact, if* κ *and* ρ *are chosen as in condition* (S), *we may take* δ *to be* $\delta := \kappa\Big|_{\mathscr{B}'}^{\mathrm{Rng}\,\rho}$.

Proof: Let $\hat{\mathscr{B}}'$ be the body obtained from $\mathscr{B}'$ by contact via $\delta := \kappa\Big|_{\mathscr{B}'}^{\mathrm{Rng}\,\rho}$. It follows from Prop. 1 of Sect. 5 that the mapping $\hat{\delta}$ described by (5.4) and (5.5) satisfies $\hat{\delta}\Big|_{\mathscr{B}'}^{\mathrm{Rng}\,\delta} = \delta$ when $\mathscr{B}'$ is considered as a subset of $\hat{\mathscr{B}}'$. Since $\mathrm{Rng}\,\delta = \mathrm{Rng}\,\rho$, it follows that $\hat{\delta}$ agrees with κ on $\mathscr{B}'$ and that $\kappa^{\leftarrow} \circ \hat{\delta} : \hat{\mathscr{B}}' \to \mathscr{B}$ is an invertible mapping that can be used to identify $\hat{\mathscr{B}}'$ with $\mathscr{B}$. ∎

Theorem 2: *Let the continuous body* $\hat{\mathscr{B}}$, *with placement class* $\mathrm{Pl}_{\hat{\delta}}(\mathscr{B})$, *be obtained from* $\mathscr{B}$ *by contact via a given improper placement* δ *of* $\mathscr{B}$. *Then* $\mathscr{B}$ *can be restored from* $\hat{\mathscr{B}}$ *by separation via a mapping* α *satisfying the condition* (S) *of* Sect. 4 *relative to* $\hat{\mathscr{B}}$. *In fact, we may take* α *to be any placement of* $\mathscr{B}$ *(regarded as a subset of* $\hat{\mathscr{B}}$*).*

Proof: As in the previous proof, we have $\hat{\delta}\Big|_{\mathscr{B}}^{\mathrm{Rng}\,\delta} = \delta$. Put $\rho := \delta \circ \alpha^{\leftarrow}$. Then

$$\alpha = \rho^{\leftarrow} \circ \delta = \rho^{\leftarrow} \circ \hat{\delta}\Big|_{\mathscr{B}}^{\mathrm{Rng}\,\delta},$$

which shows that the condition (S) is satisfied for $\hat{\mathscr{B}}$ because (4.1) holds when κ there is replaced by $\hat{\delta} \in \mathrm{Pl}_{\hat{\delta}}(\hat{\mathscr{B}})$. Observing (4.7) with $\mathscr{B}'$ replaced by $\mathscr{B}$, we see that $\mathrm{Pl}_{\alpha}(\mathscr{B}) = \mathrm{Pl}(\mathscr{B})$, i.e. that $\mathscr{B}$, as obtained by separation from $\hat{\mathscr{B}}$ via α, retains the original placement class. ∎

If $\mathscr{B}'$ is obtained from $\mathscr{B}$ by separation, then $\mathscr{B}'$ may be a proper subset of $\mathscr{B}$, i.e. some fo the material points of $\mathscr{B}$ may "disappear". If $\hat{\mathscr{B}}$ is obtained from $\mathscr{B}$ by contact, then $\mathscr{B}$ may be a proper subset of $\hat{\mathscr{B}}$, i.e. some of the material points of $\hat{\mathscr{B}}$ have been "created" (by means of filters). However, the system $\Omega_{\mathscr{B}}$ of parts of $\mathscr{B}$ *does* maintain its identity after separation and contact in the sense described by he following result:

Theorem 3: *Assume that the continuous bodies $\mathscr{B}'$ and $\hat{\mathscr{B}}$ are obtained from $\mathscr{B}$ by separation and contact, respectively. Then the mappings*

$$\mathscr{P} \mapsto (\mathscr{P} \cap \mathscr{B}') : \Omega_{\mathscr{B}} \to \Omega_{\mathscr{B}'} \tag{6.1}$$

and

$$(\mathscr{P} \mapsto \mathrm{Int}\,\mathrm{Clo}_{\hat{\mathscr{B}}}\, \mathscr{P}) : \Omega_{\mathscr{B}} \to \Omega_{\hat{\mathscr{B}}} \tag{6.2}$$

are order–isomorphisms with respect to inclusion and hence material–system–isomorphisms.

Many familiar alterations involving continuous bodies can be described mathematically as the result of a separation followed by a simultaneous contact. An example is sliding, as illustrated in Figure 1.

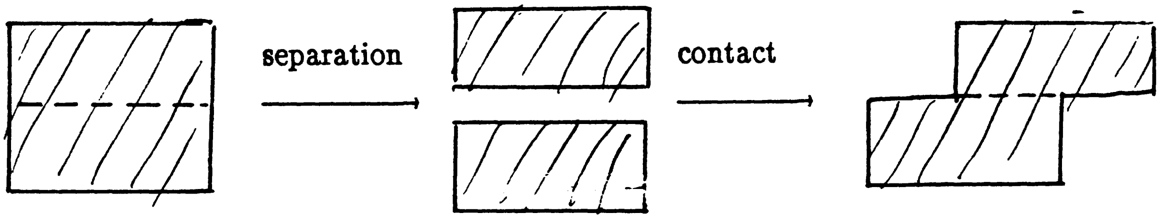

Figure 1.

If a separation followed by a contact leads to a body having the *same* set of material points as the original body, but perhaps a different placement class, we say that the resulting body is obtained form the original one by **reformation.**

To describe reformations more explicitly, we consider a class Rf of mappings obtained from the class $\bar{\mathrm{T}}\mathrm{p}$ as follows:

Let $\rho,\rho' \in \bar{\mathrm{T}}\mathrm{p}$ be given such that

(i) Dom $\rho =$ Dom ρ',

(ii) Icr ρ, Icr $\rho' \in$ Fr,

(iii) the mapping

$$\beta := \rho' \circ \rho'^{\leftarrow} : \mathrm{Rng}\,\rho \to \mathrm{Rng}\,\rho' \tag{6.3}$$

has an *invertible* continuous extension

$$\bar{\beta} : \mathrm{Clo\,Rng}\,\rho \to \mathrm{Clo\,Rng}\,\rho'. \tag{6.4}$$

If the conditions (i) and (ii) are satisfied, it is easily seen that $\bar{\beta}_{>}(\mathrm{Icr}\ \rho) = \mathrm{Icr}\ \rho'$ and that

$$\sigma := \bar{\beta}\Big|_{\mathrm{Icr}\ \rho}^{\mathrm{Icr}\ \rho'} \tag{6.5}$$

is a homeomorphism from $\mathrm{Icr}\ \rho$ to $\mathrm{Icr}\ \rho'$. We denote the class of all mappings σ obtained in the manner described by (6.3) – (6.5) by Rf and call its members **reformings**. It is clear that $\mathrm{Tp} \subset \mathrm{Tp} \subset \mathrm{Rf}$.

Theorem 4: *Assume that a placement class* $\mathrm{Pl}'(\mathscr{B})$ *is obtained from the original placement class* $\mathrm{Pl}(\mathscr{B})$ *by a reformation, i.e. a separation followed by a contact that preserves the set* $\mathscr{B}$. *Then, for every* $\kappa \in \mathrm{Pl}(\mathscr{B})$, *there is a* $\sigma \in \mathrm{Rf}$ *such that* $\mathrm{Dom}\ \sigma = \mathrm{Rng}\ \kappa$ and

$$\mathrm{Pl}'(\mathscr{B}) = \{\lambda \circ \sigma \circ \kappa \mid \lambda \in \mathrm{Tp},\ \mathrm{Dom}\ \lambda = \mathrm{Rng}\ \sigma\}. \tag{6.6}$$

Remark 1: Even though a reforming is a homeomorphism, it need only be "piecewise" of class C^2. Figure 2 illustrates a situation where the gradient of a reforming has a jump–discontinuity along a plane surface.

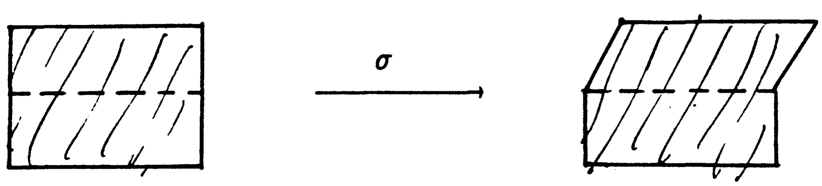

Figure 2.

Even if a reforming *is* a C^2—diffeomorphism, it need not belong to $\tilde{T}p$. An example can be read off from Fig. 2 of Sect. 5 when *a* is obtained from *p* and *p'* as described by (6.3)-(6.5). D

Reformings can serve to describe the underlying geometry of acceleration waves, shock waves, phase boundaries, etc.

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