An Equivalence Theorem for String Solutions of the Einstein-Matter-Gauge Equations

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Abstract

String-like static solutions of the Einstein-matter-gauge equations have interesting implications in cosmology. It has been shown recently that, at a critical coupling phase, this system of equations allows a reduction into a coupled Einstein–Bogomol'nyi system. In this paper, we prove that, in the important case where the underlying two-dimensional Riemannian manifold is either compact or asymptotically Euclidean, the two systems are actually equivalent. Moreover, we show that the standard assumption that the strings reside in a conformally Euclidean surface will give us a metric which fails to be asymptotically Euclidean. In particular, in the radially symmetric case, we establish under the finite energy condition the boundary behavior of the metric. These results may indicate that a string solution will inevitably lead to nonflatness of the space at infinity even on the cross section.

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1 Introduction

It is well-known that an important role played by the string-like solutions of the Einstein-matter-gauge (EMG) equations is that these solutions may provide an uneven distribution of magnetic excitation and matter, and hence, are seeds for galaxy formation in the early phase transition stages of the universe [8],[20]. However, the EMG equations are difficult to solve exactly and only plausible argument and numerical simulations about their properties are available in the literature. The new light came when Linet [9],[10] and Comtet and Gibbons [4] showed that, at a critical phase called the Bogomol'nyi coupling, the second-order EMG equations allow a reduction into a mixed second- and first-order Einstein–Bogomol'nyi (EB) equations.

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Namely, the solutions of the EB equations also solve the EMG equations. Furthermore, Comtet and Gibbons [4] proved that the EB system can be put into a scalar elliptic equation in $\mathbf{R}^2$ with an interesting superimposed critical nonlinearity. In [17] Spruck and the author have succeeded in using a special shooting method [16] to rigorously construct for the first time a continuous family of string solutions of finite energies. An interesting question arises: Are the solutions of EMG equations always the solutions of the EB equations? The purpose of the present paper is to give an affirmative answer to the above question in two physically important cases. More precisely, we shall show that the EMG and EB systems are actually equivalent if the underlying two-dimensional Riemannian manifold in which the strings reside is either compact or asymptotically Euclidean. Such equivalence problem has a rich history in classical field theories. When the gauge group G is nonabelian, the answer is in general negative [2],[19],[15],[13]. On the other hand, when $G = U(1)$, the work of Taubes [18] (see also [7]) established the equivalence of the full equations of motion of the abelian Higgs model and the reduced first-order Bogomol'nyi system in $\mathbf{R}^2$. However, when $\mathbf{R}^2$ is replaced by a curved two-manifold M which is usually assumed to be compact [11],[1],[5] or asymptotically Euclidean [21], the equivalence is still an open question. The main difficulty is due to the lack of a suitable Pohazaev type identity for finite energy solutions on M.

In this paper we show the equivalence of the full EMG equations and the EB equations on M using the method of [18],[7]. The presence of the Einstein equations and the form of the string solutions determine the structure of the energy-momentum tensor of the theory (in the matter-gauge sector) which in turn gives us something like a pointwise version of the Pohazaev identity. This feature plays a key role in our proof of the equivalence.

We then study the behavior of the solutions of the EB system at infinity with the standard assumption of the underlying Riemannian manifold being conformally $\mathbf{R}^2$. It is proved that in this case a finite-energy[1] string solution cannot yield an asymptotically Euclidean metric. Thus the equivalence theorem of the paper says that strings cannot live in a surface that is both conformally and asymptotically Euclidean.

An outline of the contents of the paper is as follows. In Section 2 we state our problem and the equivalence theorem. In Section 3 we present some preliminary results. In Section 4 we finish the proof of the theorem. In Section 5, we study the boundary behavior of a finite-energy solution of the EB system. In particular, we will

[1]By this we always mean finite energy per unit length of a string as usual.

consider a solution with radial symmetry. We show that a solution of this type leads to a metric vanishing at infinity in a suitable sense.

2 The Equivalence Theorem

Let $\eta_{\mu\nu}$ be the metric tensor of a four-dimensional Minkowskian spacetime, $R_{\mu\nu}$ the Ricci tensor, and R the scalar curvature. Then the Einstein tensor takes the form

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R.$$

The standard $U(1)$ matter-gauge Lagrangian in the Bogomol'nyi coupling is defined in the expression

$$\mathcal{L} = \frac{1}{4}\eta^{\mu\mu'}\eta^{\nu\nu'}F_{\mu\nu}F_{\mu'\nu'} + \frac{1}{2}\eta^{\mu\nu}(D_\mu\phi)(D_\nu\phi)^* + \frac{1}{8}(|\phi|^2 - 1)^2,$$

where ϕ is a complex scalar matter field, $D_\mu\phi = \partial_\mu - \mathrm{i}A_\mu\phi$ is the gauge-covariant derivative, A_μ is a gauge vector field, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field.

The Euler–Lagrange equations of the action

$$\frac{1}{2}\int R\sqrt{-\eta}\,\mathrm{d}x + \int \mathcal{L}\sqrt{-\eta}\,\mathrm{d}x$$

are the coupled EMG equations

$$\left\{\begin{aligned} G_{\mu\nu} &= T_{\mu\nu}, \\ \frac{1}{\sqrt{-\eta}}D_\mu(\eta^{\mu\nu}\sqrt{-\eta}[D_\nu\phi]) &= \frac{1}{2}(|\phi|^2 - 1)\phi, \\ \frac{1}{\sqrt{-\eta}}\partial_{\mu'}(\eta^{\mu\nu}\eta^{\mu'\nu'}\sqrt{-\eta}F_{\nu\nu'}) &= \frac{\mathrm{i}}{2}\eta^{\mu\nu}(\phi[D_\nu\phi]^* - \phi^*[D_\nu\phi]), \end{aligned}\right.$$

where

$$T_{\mu\nu} = \eta^{\mu'\nu'}F_{\mu\mu'}F_{\nu\nu'} + \frac{1}{2}(D_\mu\phi[D_\nu\phi]^* + [D_\mu\phi]^*D_\nu\phi) - \eta_{\mu\nu}\mathcal{L}$$

is the energy-momentum tensor of the matter-gauge sector.

We assume from now on the string ansatz that

$$\begin{aligned} \mathrm{d}s^2 &= \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \\ &= -\mathrm{d}t^2 + \mathrm{d}z^2 + g_{jk}\mathrm{d}x^j\mathrm{d}x^k, \qquad j,k = 1,2, \end{aligned}$$

where $g = (g_{jk})$ is the metric tensor of a two-dimensional Riemannian manifold M, and that A_μ, ϕ depend only on the coordinates on M and

$$A_\mu = (0, 0, A_1, A_2).$$

Then $T_{\mu\nu}$ verifies

$$\begin{aligned} T_{tt} &= \mathcal{E}_{\mathrm{MG}}, \quad T_{zz} = -\mathcal{E}_{\mathrm{MG}}, \quad T_{tz} = T_{tj} = T_{zj} = 0, \\ T_{jk} &= g^{j'k'} F_{jj'} F_{kk'} + \frac{1}{2}(D_j\phi[D_k\phi]^* + [D_j\phi]^* D_k\phi) - g_{jk}\mathcal{E}_{\mathrm{MG}}, \end{aligned}$$

where

$$\mathcal{E}_{\mathrm{MG}} = \frac{1}{4} g^{jj'} g^{kk'} F_{jk} F_{j'k'} + \frac{1}{2} g^{jk} (D_j\phi)(D_k\phi)^* + \frac{1}{8}(|\phi|^2 - 1)^2$$

is the energy density of the matter-gauge sector. Besides, if we use K_g to denote the Gaussian curvature of the two-manifold (M, g), the Einstein tensor is simplified to

$$\begin{aligned} G_{tt} &= -G_{zz} = K_g, \\ G_{\mu\nu} &= 0 \quad \text{for other values of } \mu, \nu. \end{aligned}$$

As a consequence, the string version of the EMG equations become

$$\left\{ \begin{aligned} K_g &= \mathcal{E}_{\mathrm{MG}}, \quad T_{jk} = 0, \quad j, k = 1, 2, \\ \frac{1}{\sqrt{g}} D_j (g^{jk} \sqrt{g} [D_k\phi]) &= \frac{1}{2}(|\phi|^2 - 1)\phi, \\ \frac{1}{\sqrt{g}} \partial_{j'} (g^{jk} g^{j'k'} \sqrt{g} F_{kk'}) &= \frac{\mathrm{i}}{2} g^{jk} (\phi [D_k\phi]^* - \phi^* [D_k\phi]), \end{aligned} \right. \qquad x \in M. \tag{2.1}$$

The unknown is the metric-matter-gauge triplet (g, ϕ, A). Denote by ∇_j the covariant derivative with respect to the metric g and J_k the current vector

$$J_k = \frac{1}{2} A_k - \frac{\mathrm{i}}{4}(\phi^* [D_k\phi] - \phi [D_k\phi]^*).$$

Then, in terms of the skew-symmetric Levi–Civita tensor ϵ_{jk} with $\epsilon_{12} = \sqrt{g}$, we have

$$\begin{aligned} \mathcal{E}_{\mathrm{MG}} = \; & \frac{1}{4} g^{jj'} g^{kk'} \Big(F_{jk} \pm \frac{1}{2}\epsilon_{jk}(|\phi|^2 - 1) \Big) \Big(F_{j'k'} \pm \frac{1}{2}\epsilon_{j'k'}(|\phi|^2 - 1) \Big) \\ & + \frac{1}{4} g^{jk} (D_j\phi \pm \mathrm{i}\epsilon_j^{j'} D_{j'}\phi)(D_k\phi \pm \mathrm{i}\epsilon_k^{k'} D_{k'}\phi)^* \\ & \pm \nabla_j (\epsilon^{jk} J_k). \end{aligned} \tag{2.2}$$

This decomposition enables Comtet and Gibbons [4] to show that (2.1) can be reduced to the following EB system

$$\begin{cases} K_g \mp \nabla_j(\epsilon^{jk}J_k) &= 0, \\ D_j\phi \pm i\epsilon_j^k D_k\phi &= 0, \qquad x \in M. \\ F_{jk} \pm \frac{1}{2}\epsilon_{jk}(|\phi|^2 - 1) &= 0, \end{cases} \tag{2.3}$$

Namely, any solution of (2.3) is a solution of (2.1). We shall prove the converse: Finite energy solutions of (2.1) also verify (2.3) provided that (M, g) is either compact or asymptotically Euclidean [14],[3],[12].

DEFINITION. The Riemannian manifold (M, g) is said asymptotically Euclidean if there is a compact set $K \subset M$ such that $M - K$ is the disjoint union of a finite number of subsets $M_1, ..., M_m$, called the ends of M, each diffeomorphic to the complement of a contractable compact set in $\mathbf{R}^2$. Under this diffeomorphism the metric tensor of $M_\ell \subset M$ takes the form

$$g_{jk} = \delta_{jk} + h_{jk}$$

in the standard coordinates (x^j) on $\mathbf{R}^2$ and

$$h_{jk}(x) \to 0 \quad \text{as } r = |x| \to \infty.$$

It is often convenient to identify the end $M_\ell \subset M$ $(\ell = 1, ..., m)$ with the corresponding set in $\mathbf{R}^2$.

We now state

Theorem 2.1. *In the category of finite-energy field configuration triplets (g, ϕ, A) for which (M, g) defines either a compact or asymptotically Euclidean Riemannian manifold, the systems (2.1) and (2.3) are equivalent.*

The proof of the theorem will be carried out in the following two sections.

3 Preliminary Results

In this section we use the notation of Section 2. Assume that (g, ϕ, A) is a finite-energy solution of the EMG equations (2.1) so that the Riemannian manifold (M, g) is either compact or asymptotically Euclidean.

Lemma 3.1. *On M, either $|\phi| \equiv 1$ or $|\phi| < 1$.*

Proof. Let Δ denote the (negative) Laplace–Beltrami operator

$$\Delta u = \frac{1}{\sqrt{g}}\partial_j(\sqrt{g}g^{jk}\partial_k u), \qquad u \in C^2(M)$$

induced from g. Then (2.1) gives us

$$\begin{aligned}\Delta(|\phi|^2 - 1) &= 2g^{jk}(D_j\phi)(D_k\phi)^* + |\phi|^2(|\phi|^2 - 1)\\ &\geq |\phi|^2(|\phi|^2 - 1), \qquad x \in M.\end{aligned} \tag{3.1}$$

If M is compact, then the maximum principle and the above elliptic inequality imply that $|\phi(x)|^2 \leq 1$ for all $x \in M$.

We next assume that (M, g) is asymptotically Euclidean. In order to simplify the notation, let us prove the lemma assuming that M has only one end, say M_0, which is identified with the complement of a contractable compact set in $\mathbf{R}^2$. Hence, if $r_0 > 0$ is sufficiently large, then

$$\mathbf{R}^2 - B_{r_0} \subset M_0,$$

where

$$B_{r_0} = \{x \in \mathbf{R}^2 \mid |x| < r_0\}.$$

Let $M - M_0 = K$. Set

$$\Omega_r = K \cup (B_r \cap M_0), \qquad r \geq r_0.$$

Define a function $\eta \in C_0^\infty(\mathbf{R})$ with the properties

$$\eta(s) = \begin{cases} 1, & |s| \leq r_0,\\ 0, & |s| \geq 2r_0,\end{cases}$$

and $0 \leq \eta \leq 1$. Since

$$M = \Omega_{r_0} \cup (\mathbf{R}^2 - B_{r_0}),$$

we can define a function $\eta_r \in C_0^\infty(M)$ ($r \geq r_0$) by putting

$$\eta_r(x) = \begin{cases} 1, & x \in \Omega_{r_0},\\ \eta\Big(r_0\Big[1 + \dfrac{|x| - r_0}{r}\Big]\Big), & x \in \mathbf{R}^2 - B_{r_0}.\end{cases}$$

Let $\psi \in W_0^{1,2}(\Omega)$ where Ω is a bounded open subset of M. Then we have

$$\int_M \sqrt{g}\mathrm{d}x\Big\{g^{jk}(D_j\psi)(D_k\phi)^* + g^{jk}(D_j\psi)^*(D_k\phi) + \frac{1}{2}(|\phi|^2 - 1)(\phi^*\psi + \phi\psi^*)\Big\} = 0. \tag{3.2}$$

For $r \geq r_0$ define $\psi_r \in W_0^{1,2}(\Omega_{2r})$ by

$$\psi_r(x) = \eta_r(x)(|\phi(x)| - 1)^+ \frac{\phi(x)}{|\phi(x)|},$$

where $b^+ = \max\{0, b\}$. Let

$$\Omega_r^+ = \{x \in \Omega_r \mid |\phi(x)| > 1\}.$$

Define $f = \phi/|\phi|$ on M^+. Then $ff^* = 1$ and on Ω_{2r}^+

$$D_j\psi_r = (\partial_j\eta_r)(|\phi| - 1)f + ([\partial_j|\phi|]f + [|\phi| - 1]D_jf)\eta_r.$$

Replacing ψ in (3.2) by ψ_r, we have

$$\int_{\Omega_{2r}^+} \sqrt{g}\mathrm{d}x\{2(g^{jk}\partial_j|\phi|\partial_k|\phi|)\eta_r + 2(|\phi| - 1)g^{jk}\partial_j\eta_r\partial_k|\phi|$$
$$+2(|\phi| - 1)|\phi|g^{jk}(D_jf)(D_kf)^*\eta_r + (|\phi| - 1)^2(|\phi| + 1)|\phi|\eta_r\} = 0. \tag{3.3}$$

From $(|\phi| - 1) \leq (|\phi|^2 - 1)$ (on Ω_{2r}^+) and the Schwarz inequality, we obtain

$$\left|\int_{\Omega_{2r}^+} (|\phi| - 1)g^{jk}\partial_j\eta_r\partial_k|\phi| \sqrt{g}\mathrm{d}x\right|$$
$$\leq \left(\int_M (|\phi|^2 - 1)^2 \sqrt{g}\mathrm{d}x\right)^{1/2} \left(\int_{\Omega_{2r}^+} (g^{jk}\partial_j\eta_r\partial_k|\phi|)^2 \sqrt{g}\mathrm{d}x\right)^{1/2}. \tag{3.4}$$

However, using the simple inequalities

$$|\partial_j|\phi|| \leq |D_j\phi|, \qquad |\nabla\eta_r| \leq C\frac{r_0}{r},$$

where $C > 0$ is a constant independent of $r \geq r_0$, we get

$$\int_{\Omega_{2r}^+} (g^{jk}\partial_j\eta_r\partial_b|\phi|)^2 \sqrt{g}\mathrm{d}x \leq C_1\frac{r_0^2}{r^2} \int_M g^{jk}(D_j\phi)(D_k\phi)^* \sqrt{g}\mathrm{d}x. \tag{3.5}$$

Inserting (3.4) into (3.3) and using (3.5) we obtain

$$\int_{\Omega_{2r}^+} \sqrt{g}\mathrm{d}x\, \eta_r\left\{g^{jk}\partial_j|\phi|\partial_k|\phi| + (|\phi| - 1)|\phi|g^{jk}(D_jf)(D_kf)^* + \frac{1}{2}(|\phi| - 1)^2(|\phi| + 1)|\phi|\right\}$$
$$\leq \sqrt{2C_1}\,\frac{r_0}{r} \int_M \mathcal{E}_{\mathrm{MG}} \sqrt{g}\mathrm{d}x.$$

Letting $r \to \infty$ we find $\mathrm{vol}(M^+) = 0$. Hence the bound $|\phi| \leq 1$ again follows.

Finally, applying the strong maximum principle (or the Hopf theorem, see [6]) to (3.1) in view of $|\phi| \leq 1$, we see that either $|\phi| \equiv 1$ or $|\phi| < 1$ on M. □

Lemma 3.2. *On M, either $\epsilon^{jk}F_{jk} + (|\phi|^2 - 1) \equiv 0$ or $\epsilon^{jk}F_{jk} + (|\phi|^2 - 1) < 0$.*

Proof. The last equation in (2.1) can be rewritten as

$$\partial_j(\epsilon^{j'k'}F_{j'k'}) + \mathrm{i}\epsilon_{jj'}g^{j'k'}(\phi[D_{k'}\phi]^* - \phi^*[D_{k'}\phi]) = 0. \tag{3.6}$$

Therefore

$$\frac{1}{\sqrt{g}}\partial_j(\sqrt{g}g^{jk}\partial_k[\epsilon^{j'k'}F_{j'k'}]) + \frac{\mathrm{i}}{\sqrt{g}}\partial_j(\sqrt{g}\epsilon^{jk}[\phi(D_k\phi)^* - \phi^*(D_k\phi)]) = 0.$$

Thus

$$\Delta(\epsilon^{jk}F_{jk}) - |\phi|^2(\epsilon^{jk}F_{jk}) + 2\mathrm{i}\epsilon^{jk}(D_j\phi)(D_k\phi)^* = 0. \tag{3.7}$$

From (3.1) and (3.7) we obtain

$$\begin{aligned}\Delta(\epsilon^{jk}F_{jk} + [|\phi|^2 - 1]) &= |\phi|^2(\epsilon^{jk}F_{jk} + [|\phi|^2 - 1]) \\ &\quad + g^{jk}(D_j\phi + \mathrm{i}\epsilon_j^{j'}D_{j'}\phi)(D_k\phi + \mathrm{i}\epsilon_k^{k'}D_{k'}\phi)^* \\ &\geq |\phi|^2(\epsilon^{jk}F_{jk} + [|\phi|^2 - 1]).\end{aligned} \tag{3.8}$$

Hence if M is compact, we can use the maximum principle in (3.8) to conclude that $\epsilon^{jk}F_{jk} + (|\phi|^2 - 1) \leq 0$.

Suppose now (M, g) is asymptotically Euclidean. Use the notation in the proof of Lemma 3.1. Then from (3.8), for any $u \in W_0^{1,2}(\Omega)$, we have

$$\begin{aligned}\int_\Omega &\sqrt{g}\mathrm{d}x\{g^{jk}\partial_j u\partial_k(\epsilon^{j'k'}F_{j'k'} + [|\phi|^2 - 1]) + u|\phi|^2(\epsilon^{jk}F_{jk} + [|\phi|^2 - 1]) \\ &+ ug^{jk}(D_j\phi + \mathrm{i}\epsilon_j^{j'}D_{j'}\phi)(D_k\phi + \mathrm{i}\epsilon_k^{k'}\phi)^*\} \\ &= 0.\end{aligned} \tag{3.9}$$

Since (M, g) is asymptotically Euclidean, we have by virtue of Lemma 3.1 and (3.6) the estimate

$$\begin{aligned}&\left|\int_{\Omega_{2r}^+} \sqrt{g}\mathrm{d}x\{g^{jk}(\partial_j\eta_r)(\epsilon^{j'k'}F_{j'k'} + [|\phi|^2 - 1])\partial_k(\epsilon^{j''k''}F_{j''k''} + [|\phi|^2 - 1])\}\right| \\ &\leq C\frac{r_0}{r}\Big(\int_M \sqrt{g}\mathrm{d}x\{g^{jj'}g^{kk'}F_{jk}F_{j'k'} + (|\phi|^2 - 1)^2\}\Big)^{1/2} \times \\ &\qquad \times\Big(\int_M \sqrt{g}\mathrm{d}x\{g^{jk}(D_j\phi)(D_k\phi)^*\}\Big)^{1/2} \\ &\leq C_1\frac{r_0}{r}\int_M \mathcal{E}_{\mathrm{MG}}\sqrt{g}\mathrm{d}x,\end{aligned} \tag{3.10}$$

where

$$\Omega_{2r}^{+} = \{x \in \Omega_{2r} \mid \epsilon^{jk}F_{jk} + (|\phi|^2 - 1) > 0\}.$$

Introduce as before the trial function $u_r \in W_0^{1,2}(\Omega_{2r})$ defined by

$$u_r = \eta_r(\epsilon^{jk}F_{jk} + [|\phi|^2 - 1])^{+} \equiv \eta_r U^{+}.$$

Then, replacing u in (3.9) by u_r and using (3.10), we have

$$\int_{\Omega_{2r}^{+}} \sqrt{g}\mathrm{d}x\, \eta_r\{g^{jk}\partial_j U \partial_k U + |\phi|^2 U^2 + U g^{jk}(D_j\phi + \mathrm{i}\epsilon_j^{j'} D_{j'}\phi)(D_k\phi + \mathrm{i}\epsilon_k^{k'} D_{k'}\phi)^*\}$$

$$\leq C_1 \frac{r_0}{r}\int_M \mathcal{E}_{\mathrm{MG}}\sqrt{g}\mathrm{d}x, \qquad r \geq r_0. \tag{3.11}$$

However, each term on the left-hand side of (3.11) is non-negative, thus letting $r \to \infty$ we find $M^{+} = \emptyset$. Consequently there again holds as in the compact case $U = \epsilon^{jk}F_{jk} + (|\phi|^2 - 1) \leq 0$.

By virtue of the strong maximum principle and the inequality (3.8), we see that either $U \equiv 0$ or $U < 0$ on M. □

Lemma 3.3. *On M, either $\epsilon^{jk}F_{jk} - (|\phi|^2 - 1) \equiv 0$ or $\epsilon^{jk}F_{jk} - (|\phi|^2 - 1) > 0$.*

Proof. From (3.1), (3.7), and a straightforward calculation, we get the following dual version of (3.8):

$$\begin{aligned} \Delta(\epsilon^{jk}F_{jk} - [|\phi|^2 - 1]) &= |\phi|^2(\epsilon^{jk}F_{jk} - [|\phi|^2 - 1]) \\ &\quad - g^{jk}(D_j\phi - \mathrm{i}\epsilon_j^{j'} D_{j'}\phi)(D_k\phi - \mathrm{i}\epsilon_k^{k'} D_{k'}\phi)^* \\ &\leq |\phi|^2(\epsilon^{jk}F_{jk} - [|\phi|^2 - 1]). \end{aligned} \tag{3.12}$$

Thus we can duplicate the proof of Lemma 3.2 in (3.12) to obtain the desired conclusions. □

4 Proof of Equivalence

Let (g, ϕ, A) be a finite-energy solution of the EMG equations (2.1) discussed in Section 3. We shall show that it also solves the EB equations (2.3).

In fact, a lengthy calculation gives us

$$(\epsilon^{jk}F_{jk}+[|\phi|^2-1])(\epsilon^{j'k'}F_{j'k'}-[|\phi|^2-1])=4g^{jk}T_{jk}\equiv 0 \quad \text{in } M.$$

Therefore for any fixed $p\in M$, either

$$(\epsilon^{jk}F_{jk}+[|\phi|^2-1])_p=0 \tag{4.1}$$

or

$$(\epsilon^{jk}F_{jk}-[|\phi|^2-1])_p=0. \tag{4.2}$$

If (4.1) holds, then Lemma 3.2 says that it must be true for all $p\in M$; while, if (4.2) holds, then Lemma 3.3 says that it must be true for all $p\in M$. Hence we conclude that either (4.1) or (4.2) holds for all $p\in M$. Now we recall (3.8) and (3.12). It is seen that, correspondingly, either

$$D_j\phi+\mathrm{i}\epsilon_j^k D_k\phi\equiv 0$$

or

$$D_j\phi-\mathrm{i}\epsilon_j^k D_k\phi\equiv 0.$$

As a consequence, the Bogomol'nyi sector in the EB equations (2.3) is recovered. From this fact and the expression of $\mathcal{E}_{\mathrm{MG}}$ given in (2.2), we immediately see that the first equation in (2.3) also holds.

The proof of the theorem is complete.

5 Nonexistence of Conformally and Asymptotically Euclidean Metric

In [17], we have constructed a class of cosmic string solutions of (2.3) (or (2.1)) so that the string metric defines a surface which curls up at infinity with the Gaussian curvature assuming a nontrivial value there. A natural question then arises: Is there a solution realizing a string metric which yields an asymptotically Euclidean surface? The answer to this question is negative under a further standard assumption of the surface being globally conformally Euclidean. Namely, we shall show that the EMG equations (2.1) do not allow a finite-energy cosmic string solution triplet (g,ϕ,A) so that (M,g) is both conformally and asymptotically Euclidean. This fact seems to be striking. The result is stated as follows.

Theorem 5.1. *Let (g, ϕ, A) be a finite-energy cosmic string solution of (2.3) so that $(M, g) = (\mathbf{R}^2, e^{\zeta}\delta_{jk})$. Besides, assume that there hold the usual uniformity properties at infinity:*

$$\partial_j\zeta(x) \to 0 \quad \text{and} \quad \partial_j\partial_k\zeta(x) \to 0 \quad \text{for } j,k = 1,2 \quad \text{as } |x| \to \infty. \tag{5.1}$$

Then g cannot be asymptotically Euclidean. More precisely, one or both of the following cases must occur:

$$\limsup_{|x|\to\infty} \zeta(x) = \infty, \tag{5.2}$$

$$\liminf_{|x|\to\infty} \zeta(x) = -\infty. \tag{5.3}$$

Proof. Suppose that the strings are located at $p_1, ..., p_m \in \mathbf{R}^2$ with local winding numbers $n_1 \geq 1, ..., n_m \geq 1$ and $N = n_1 + \cdots + n_m > 0$ be the total string number. Then it is well-known that $u = \ln|\phi|^2$ and ζ satisfy the elliptic equations

$$\begin{cases} \Delta u &= e^{\zeta}(e^u - 1) + 4\pi\sum_{\ell=1}^{m} n_\ell\delta_{p_\ell}, \\ \Delta\zeta &= \frac{1}{4}(e^{\zeta}[e^u - 1] - \Delta e^u), \end{cases} \quad x \in \mathbf{R}^2. \tag{5.4}$$

Assume otherwise that ζ does not verify (5.2) and (5.3). Then the range of ζ lies in a compact interval of $\mathbf{R}$. Using this fact, the condition (5.1), and a specialization of the proofs in [21], we can show that $u = 0$ at infinity (or equivalently $|\phi|^2 = 1$ at infinity).

Set

$$u_0(x) = 2\sum_{\ell=1}^{m} n_\ell \ln|x - p_\ell|$$

and $u = u_0 + v$. Then (5.4) become

$$\begin{cases} \Delta v &= e^{\zeta}(e^{u_0+v} - 1), \\ \Delta\zeta &= \frac{1}{4}(e^{\zeta}[e^{u_0+v} - 1] - \Delta e^{u_0+v}), \end{cases} \quad x \in \mathbf{R}^2. \tag{5.5}$$

Inserting the first equation of (5.5) into the second one, we arrive at

$$\Delta\left(\zeta - \frac{1}{4}v + \frac{1}{4}e^{u_0+v}\right) = 0 \quad \text{in } \mathbf{R}^2.$$

Put

$$V = \zeta - \frac{1}{4}v + \frac{1}{4}e^{u_0+v}.$$

Then V is harmonic in the entire $\mathbf{R}^2$ and

$$\lim_{|x|\to\infty}\frac{V(x)}{\ln|x|} = \lim_{|x|\to\infty}\frac{\zeta-\frac{1}{4}u+\frac{1}{2}\sum_{\ell=1}^m n_\ell\ln|x-p_\ell|+\frac{1}{4}e^u}{\ln|x|}$$

$$= \frac{N}{2} > 0.$$

In particular, $V(x)\to\infty$ as $|x|\to\infty$. However, by the maximum principle, there does not exist such a harmonic function on $\mathbf{R}^2$. □

We next show that the property (5.3) is most likely to happen in reality.

Theorem 5.2. *Let (g,ϕ,A) be a finite-energy radially symmetric solution of (2.3) so that the $N>0$ strings are all superimposed at the origin and (ζ,u) is a solution of (5.4) depending only on the variable $r=|x|, x\in\mathbf{R}^2$. Then there is a sequence $\{r_n\}$ $(n=1,2,...)$ so that $r_n\to\infty$ as $n\to\infty$ and*

$$\lim_{n\to\infty}\zeta(r_n)=-\infty. \tag{5.6}$$

Proof. First we note that the proof of Lemma 3.1 can be used to show that $u=\ln|\phi|^2\le 0$ (namely $|\phi|\le 1$) everywhere. From (5.4) and $\sum_{\ell=1}^m n_\ell\delta_{p_\ell}=N\delta(x)$, we see that $u(r)$ is a solution of the equation

$$u_{rr}+\frac{1}{r}u_r=e^\zeta(e^u-1), \qquad r>0 \tag{5.7}$$

satisfying the boundary condition

$$\lim_{r\to 0} ru_r(r)=\lim_{r\to 0}\frac{u(r)}{\ln r}=2N. \tag{5.8}$$

Using (5.7)–(5.8), we obtain

$$ru_r(r)=2N+\int_0^r \rho e^{\zeta(\rho)}(e^{u(\rho)}-1)\,d\rho, \qquad r>0. \tag{5.9}$$

We have two possibilities to study.

Case 1. There is an $r_0>0$ so that $u_r(r_0)=0$.

It is easily seen that such an r_0 is unique. In fact since $ru_r(r)-2N$ is nonincreasing by (5.9), if there is an $r_1>r_0$ to make $u_r(r_1)=0$, then

$$\int_{r_0}^{r_1}\rho e^{\zeta(\rho)}(e^{u(\rho)}-1)\,d\rho=0.$$

As a consequence, $u(r)=0$ for $r\in(r_0,r_1)$. The uniqueness theorem of the initial value problem of ordinary differential equations says that $u(r)=0$ for all $r>0$, contradicting (5.8).

Thus $ru_r(r) < 0$ for $r > r_0$. However, $ru_r(r)$ is monotone decreasing (see (5.9)), therefore setting $r_1 > r_0$ and $\varepsilon = -r_1 u_r(r_1) > 0$, we have

$$ru_r(r) < -\varepsilon \qquad r > r_1;$$

or

$$u(r) < u(r_1) - \varepsilon \ln \frac{r}{r_1}, \qquad r > r_1.$$

In particular, $u(r) \to -\infty$ as $r \to \infty$.

On the other hand, the finite energy condition implies the convergence of the integral

$$\int_0^\infty r e^{\zeta(r)} (e^{u(\rho)} - 1)^2 \, dr.$$

Thus the conclusion of the theorem follows.

Case 2. For any $r > 0$, we have $u_r(r) > 0$.

Hence there is a constant $a \leq 0$ so that

$$\lim_{r \to \infty} u(r) = a. \tag{5.10}$$

Suppose otherwise that

$$\inf_{r>0} \zeta(r) \; > -\infty. \tag{5.11}$$

Since (ζ, u) is a solution pair of the equations (5.4), we see that the argument in the proof of Theorem 5.1 applies. In other words, (5.10)–(5.11) will imply that

$$V = \zeta - \frac{1}{4} u + \frac{N}{2} \ln |x| + \frac{1}{4} e^u$$

is harmonic in the entire $\mathbf{R}^2$ but $V \to \infty$ as $|x| \to \infty$. This is false. So (5.6) must hold for some sequence $\{r_n\}$ as stated in the theorem. □

Remark 5.1. Combining Theorems 2.1 and 5.1, we conclude that an asymptotically Euclidean metric might only be obtained from a cosmic string solution on a Riemannian manifold which is not globally conformally Euclidean.

Remark 5.2. In [17], Spruck and the author have constructed a family of radially symmetric string solutions of the EB system (2.3) so that the surface has a nonvanishing curvature at infinity (hence non-asymptotically Euclidean) and is characterized by the property

$$\lim_{r \to \infty} \zeta(r) = -\infty. \tag{5.12}$$

Theorem 5.2 says that, in the category of radially symmetric solutions, our solutions are probably the only finite-energy solutions that one can expect to get. In other words, the boundary condition (5.12) is "almost" the best possible condition one may impose at infinity. This fact should have interesting implications in physics.

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