On the Weak Lower Semicontinuity of Energies with Polyconvex Integrands

Wilfrid Gangbo
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

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Abstract: Let $f : \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times N} \longrightarrow [0,\infty)$ be a Borel measurable function such that $f(x,u,\cdot)$ is polyconvex in the last variable ξ for almost every $x \in \Omega$ and for every $u \in \mathbf{R}^N$. It is shown that if f is continuous and $F(u) := \int_\Omega f(x,u(x),\nabla u(x))dx,\ \ u \in W^{1,N}(\Omega,\mathbf{R}^N)$, then F is weakly lower semicontinuous in $W^{1,p}$, $p > N-1$, in the sense that $F(u) \leq \liminf_{n\to\infty} F(u_\nu)$ for $u_\nu, u \in W^{1,N}(\Omega,\mathbf{R}^N)$ such that $u_\nu \rightharpoonup u$ in $W^{1,p}$. On the contrary if f is only a Carathéodory function then in general F is not weakly lower semicontinuous in $W^{1,p}$ for $N > p > N-1$. Precisely, it is shown that if $F(u) := \int_K |det(\nabla u(x))|dx$ where K is a compact set, then F is weakly lower semicontinuous in $W^{1,p}$, $N > p > N-1$ if and only if $meas(\partial K) = 0$.

Contents

*Carnegie Mellon University, Department of Mathematics, Pittsburgh PA 15213-3890, USA.

1 Introduction.

Let $N \geq 2$ be an integer number, $\Omega \subset \mathbf{R}^N$ be an open bounded set and let $f : \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times N} \longrightarrow [0,\infty)$ be a Borel measurable function. We set

$$F(u) := \int_\Omega f(x, u(x), \nabla u(x))dx, \quad u \in W^{1,p}(\Omega, \mathbf{R}^N) := W^{1,p}.$$

If one uses the direct method of the calculus of variations to obtain existence of minimum for F, one needs to show that F is weakly lower semicontinuous in $W^{1,p}$. Since Morrey's works ([Mo1], [Mo2]) and later Acerbi-Fusco ([AF]), Marcellini ([Ma2]) and others, it is well known that if $1 \leq p < \infty$ and if

$$0 \leq f(x,u,\xi) \leq a + b|\xi|^p, \quad \forall (x,u,\xi) \in \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times N} \tag{1}$$

then F is weakly lower semicontinuous in $W^{1,p}$ if only if f is quasiconvex with respect to the last variable ξ. We recall that f is said to be quasiconvex if it verifies the following Jensen's inequality

$$\frac{1}{\Omega}\int_\Omega f(x_0, u_0, \xi + \nabla u(x))dx \geq f(x_0, u_0, \xi_0)$$

for almost every $x_0 \in \Omega$, for every $(u_0, \xi_0) \in \mathbf{R}^N \times \mathbf{R}^{N\times N}$ and for every $u \in W_0^{1,\infty}(\Omega, \mathbf{R}^N)$. As it is very hard to check whether or not a given function is quasiconvex, following Morrey's work Ball introduced the notion of polyconvex function which is a sufficient condition for quasiconvexity. A function $f(x_0, u_0, \cdot)$ is said to be polyconvex if $f(x_0, u_0, \cdot)$ is a convex function of all minors of the matrix ξ. (See Definitions 1.2 and see [Da] for more details about polyconvexity and quasiconvexity). We recall that quasiconvexity does not imply weakly lower semicontinuous of F in $W^{1,p}$ if the growth condition (1) fails. An example due to Tartar, is given in [BM]. Recently Dacorogna and Marcellini in [DM] proved that if $f(x,u,\xi) \equiv f(\xi) \geq 0$ is polyconvex, with no particular growth condition, then

$$F(u) \leq \liminf_{n\to\infty} F(u_\nu),$$

for $u_\nu, u \in W^{1,N}(\Omega, \mathbf{R}^N)$ such that $u_\nu \rightharpoonup u$ in $W^{1,p}$ provided $p > N-1$. Actually in [DM] the most interesting case to study is $N - 1 < p < N$. The case $p \geq N$ is easily proved. Indeed, using the convexity of f in the last variable ξ we can approximate f by a non decreasing sequence of smooth functions f_j such that

$$0 \leq f_j(x,u,\xi) \leq C_j(x,u)(1 + |\xi|^N),$$

(see Lemma 2.3). Then we apply Proposition 1.3 to $f_j(x,u,\xi)$ for j fixed and when j goes to infinity we obtain

$$F(u) \leq \liminf_{n\to\infty} F(u_\nu).$$

In the case where $p < N-1$ we cannot expect weak lower semicontinuity of F. In fact, Maly proved in [Mal] that for each $p < N-1$ there is a sequence

$$u_\nu, u \in W^{1,N}, \;\; u_\nu \rightharpoonup u \;\text{ in }\; W^{1,p}$$

verifying

$$\liminf_{n\to\infty} \int_\Omega |det(\nabla u_n(x))| dx < \int_\Omega |det(\nabla u(x))| dx, \quad u(x) \equiv x.$$

In this paper we study the general case where f is a non negative function polyconvex in the last variable and

$$F(u) := \int_\Omega f(x, u(x), \nabla u(x)) dx.$$

We prove that if f is continuous in $\Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times N}$ then F is weakly lower semicontinuous in $W^{1,p}$, $N-1 < p < N$ in the sense that

$$F(u) \leq \liminf_{n\to\infty} F(u_\nu),$$

for $u_\nu, u \in W^{1,N}(\Omega, \mathbf{R}^N)$ and $u_\nu \rightharpoonup u$ in $W^{1,p}$. This generalizes [DM] result to include the case where f depends on (x, u). Continuity is a necessary condition. Indeed if f is not continuous but is simply Carathéodory function then in general F is not weakly lower semicontinuous on $W^{1,p}$, $N-1 < p < N$. To illustrate this, we show that if K is a compact subset of Ω and $N-1 < p < N$ then

$$meas(\partial K) \neq 0$$

if and only if

$$\liminf_{n\to\infty} \int_K |det(\nabla u_n(x))| dx < \int_K |det(\nabla u(x))| dx,$$

for a suitable $u_\nu, u \in W^{1,N}(\Omega, \mathbf{R}^N)$ such that $u_\nu \rightharpoonup u$ in $W^{1,p}$.

We give some definitions relevant for this work.

Definition 1.1 *Let $N, M \geq 1$ be two integer numbers, $\Omega \subset \mathbf{R}^M$ an open set. A function $f : \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times M} \longrightarrow \mathbf{R}$ is said to be a* **Carathéodory function** *if $f(\cdot, u, \psi)$ is measurable for every $(u, \psi) \in \mathbf{R}^N \times \mathbf{R}^{N\times M}$ and $f(x, \cdot, \cdot)$ is continuous for almost every $x \in \Omega$.*

Definition 1.2 *(See* [Da] *).*
Let $f : \mathbf{R}^{N\times M} \longrightarrow \mathbf{R}$ be a Borel measurable function defined on the set of the $N \times M$ real matrices.

- *f is said to be* **convex** *if $f(\lambda\xi + (1-\lambda)\eta) \leq \lambda f(\xi) + (1-\lambda) f(\eta)$ for every $\xi, \eta \in \mathbf{R}^{N\times M}$ and every $\lambda \in (0,1)$.*

• *f is said to be* **polyconvex** *if there exists a function* $h : \mathbf{R}^{\tau(N,M)} \longrightarrow \mathbf{R}$ *convex such that* $f(\xi) = h\big(T(\xi)\big)$ *for every* $\xi \in \mathbf{R}^{N\times M}$ *, where* $\tau(N,M) = \sum_{1\leq s\leq \min(N,M)} \binom{M}{s}\binom{N}{s}$, $T(\xi) = (adj_1\xi, \cdots, adj_{\min(N,M)}\xi)$ *and* $adj_s\xi$ *stands for the matrix of all* $s\times s$ *minors of* ξ.

When $N = M = 2$ *then* $T(\xi) = (\xi, det(\xi))$.

• *f is said to be* **quasiconvex** *if* $\frac{1}{|\Omega|}\int_\Omega f\big(\xi + \nabla\phi\big) \geq f(\xi)$ *for every* $\xi \in \mathbf{R}^{N\times M}$, *for every* $\Omega \subset \mathbf{R}^N$ *open bounded set and for every* $\phi \in W_0^{1,\infty}\big(\Omega\big)^M$, *(it is equivalent to assume the previous inequality for one fixed open bounded* $\Omega \subset \mathbf{R}^N$ *).*

For completeness we state some well known results.

Proposition 1.3 *Let* $N, M \geq 2$ *be two integer numbers,* $\Omega \subset \mathbf{R}^N$ *an open bounded set and* $f : \Omega \times \mathbf{R}^M \times \mathbf{R}^{N\times M} \longrightarrow \mathbf{R}$ *a continuous function such that* $f(x,u,\cdot)$ *is quasiconvex for each* $(x,u) \in \Omega \times \mathbf{R}^M$. *Assume further more that* f *satisfies for* $1 \leq q < p < \infty$

$$-\alpha(|u|^q + |\xi|^q) - \gamma(x) \leq f(x,u,\xi) \leq \alpha(|u|^p + |\xi|^p) + \gamma(x)$$

where $\alpha > 0,\ \ \gamma \in L^1(\Omega)$

$$|f(x,u,\xi) - f(x,v,\eta)| \leq \beta(1 + |u|^{p-1} + |v|^{p-1} + |\xi|^{p-1} + |\eta|^{p-1}) \times (|u-v| + |\xi - \eta|)$$

where $\beta > 0$ *and*

$$|f(x,u,\xi) - f(y,u,\xi)| \leq \nu(|x-y|)(1 + |u|^p + |\xi|^p),$$

where ν *is a continuous increasing function with* $\nu(0) = 0$. *Let*

$$F(u) := \int_\Omega f(x,u(x),T(\nabla u(x)))dx,\ \ u \in W^{1,p}(\Omega,\mathbf{R}^M).$$

Then F *is weakly lower semicontinuous in* $W^{1,p}$.

Proof: For the proof we refer the reader to Theorem 2.4 in [Da].

Lemma 1.4 *Let* $\phi : \mathbf{R} \longrightarrow \mathbf{R}$ *be a bounded lipschitz function. Let* $\Omega \subset \mathbf{R}^N$ *be an open bounded set and* $\psi \in C_0^\infty(\Omega,\mathbf{R}^\tau)$. *If* $p > N-1$, *if* $u_\nu, u \in W^{1,N}(\Omega,\mathbf{R}^N)$ *and if* $u_\nu \rightharpoonup u$ *in* $W^{1,p}$ *then*

$$\lim_{\nu\to\infty}\int_\Omega \phi'(u_\nu^1)\cdots\phi'(u_\nu^N) < \psi; T(\nabla u_\nu) > dx = \int_\Omega \phi'(u^1)\cdots\phi'(u^N) < \psi; T(\nabla u) > dx.$$

Moreover the results stands for $p = N-1,\ \ N = 2$. *Here* $<\ ;\ >$ *is the scalar product in* $\mathbf{R}^\tau$ *and* $\tau \sum_{1\leq s\leq N}\binom{N}{s}^2$.

Proof: Lemma 1.4 is obtained as a slight modification of the proof of Lemma 1 in [DM].

2 The case of continuous integrands.

Let us first state the main result of this section.

Theorem 2.1 *Let $N \geq 2$ be an integer number, $\Omega \subset \mathbf{R}^N$ an open bounded set, $\tau(N) = \sum_{1 \leq s \leq N} \binom{N}{s}^2$. Let $f : \Omega \times \mathbf{R}^N \times \mathbf{R}^{\tau(N)} \longrightarrow [0, \infty)$ be a continuous function such that $f(x, u, \cdot)$ is convex for each $(x, u) \in \Omega \times \mathbf{R}^N$. Let*

$$F(u) := \int_\Omega f(x, u(x), T(\nabla u(x)))dx, \quad u \in W^{1,p}(\Omega, \mathbf{R}^N) := W^{1,p}.$$

Then,

$$F(u) \leq \liminf_{\nu \to \infty} F(u_\nu), \tag{2}$$

if $u_\nu, u \in W^{1,N}(\Omega, \mathbf{R}^N)$ and $u_\nu \rightharpoonup u$ in $W^{1,p}$, $p > N - 1$. Moreover , if $N = 2$ the result is true even if $p = N - 1 = 1$.

We recall that $T(\nabla u)$ stands for the matrix of all minors of ∇u.

Remark 2.2
1.- The case $p \geq N$ is well known. Indeed, as f is convex in the last variable ξ, using Lemma 2.3 we can approximate f by a non decreasing sequence of smooth functions f_j such that

$$0 \leq f_j(x, u, \xi) \leq C_j(x, u)(1 + |\xi|^N).$$

Then we apply Proposition 1.3 to $f_j(x, u, \xi)$ for j fixed and when j goes to infinity we obtain

$$F(u) \leq \liminf_{n \to \infty} F(u_\nu).$$

In the general case (2) would be obvious if we knew that $T(\nabla u_\nu) \rightharpoonup T(\nabla u)$ L^1. However this is not necessarily true. In [DM] there is an example where $u_\nu \rightharpoonup u$ in $W^{1,p}$, $N > p > N - 1$ and $T(\nabla u_\nu) \not\rightharpoonup T(\nabla u) L^1$ (cf also [BM]). For instance if $N = 2$, $\Omega = (0,1)^2, 1 < p \leq 2$,

$$u_\nu \equiv \nu^{\frac{1}{p}-1}(1 - y)^\nu (\sin \nu x, \cos \nu x) \rightharpoonup (0, 0) \quad in \;\; W^{1,p}.$$

Then $det(\nabla u_\nu) = -\nu^{\frac{2}{p}}(1 - y)^{2\nu - 1}$ is not bounded in L^1 if $p < 2$, and so it does not converge in L^1 weak to $det(\nabla u) = 0$, even if $p = 2$.

2.- The assumption that $u_\nu, u \in W^{1,N}$ is important. It can be useful to extend the definition of $F(u)$ to functions $u \in W^{1,p}, p < N$ (cf [Mal]). Also Theorem 2.1 is false if one omits this assumption (cf [BM]).

3.- If $1 \leq p < N-1$, and $N \geq 3$, then F is not necessarily weakly lower semicontinuous (cf [Mal]). But if $p = N-1, N \geq 3$, the question to know whether or not F is weakly lower semicontinuous is still open. However Malý proved in [Mal] that if $u, u_\nu \in W^{1,N-1}$ are sense preserving diffeomorphisms such that $u_\nu \rightharpoonup u$ in $W^{1,N-1}$, then $F(u) \leq \liminf_{\nu \to \infty} F(u_\nu)$.

4.- The basic idea to prove Theorem 2.1 is the following: in the first step, we write f as a sum of functions in form $a(u)g(x, T(\nabla u))$, with $g(x, \cdot) \geq 0$ being convex. This can be done using Weierstrass's Approximation Theorem (see Lemma 2.4). In the second step, changing variables we write $a(u)g(x, T(\nabla u))$ on form $h = h(x, T(\nabla v))$. Then, following the idea of Dacorogna and Marcellini in [Da-Ma] who studied integrands of the form $h = h(T(\nabla v))$, we conclude the theorem.

Lemma 2.3 (**De Giorgi's Lemma.**)
Let $N, \tau \geq 1$ be two integer numbers, $\Omega \subset \mathbf{R}^N$ be an open bounded set, and $g : \Omega \times \mathbf{R}^\tau \longrightarrow [0, \infty)$ be a continuous function such that $g(x, \cdot)$ is convex for each $x \in \Omega$. There exists a non decreseasing sequence of functions (g_k) of class C^∞ such that:
i) $g_k \geq -1$;
ii) g_k converge uniformly to g in every compact subset of $\Omega \times \mathbf{R}^\tau$;
iii) g_k is convex in the variable T;
iv) $g_k(x, T) = 0$ if $dist(x, \partial\Omega) \leq \frac{1}{k}$;
v) $D_T g_k(x, T)$ is uniformly bounded in k in every compact subset of $\Omega \times \mathbf{R}^\tau$ by a constant which does not depend on x, where $D_T g_k = \left(\frac{\partial}{\partial T_1} g_k, \cdots, \frac{\partial}{\partial T_\tau} g_k\right)$.

Proof: For the proof we refer the reader to [Ma2].

Lemma 2.4 *(* **Weierstrass's Approximation Theorem** *)*
Let $f : [0,1] \longrightarrow \mathbf{R}$ be acontinuous function. Then,

$$f(u) = \lim_{n \to \infty} \sum_{0 \leq k \leq n} \binom{n}{k} f\left(\frac{k}{n}\right) u^k (1-u)^{n-k}$$

and the convergence is uniform.

Proof: For the proof we refer the reader to [Kl].

Remark 2.5
1.- If f is non negative, then the sequence $g_n(u) = \sum_{0 \leq k \leq n} \binom{n}{k} f(\frac{k}{n}) u^k (1-u)^{n-k}$ is non decreasing.
2.- One can deduce from Weierstrass's Approximation Theorem that for $h > 0$. a real

number, N an integer number and $f : [-h,h]^N \longrightarrow [0,\infty)$ a continuous function, there exists a sequence $(f_n)_n$ of non negative functions of class $C^\infty(\mathbf{R}^N)$ such that

$$f = \lim_{n\to\infty} \sum_{k=0}^{n} f_k$$
$$f_i(u_1,\cdots,u_N) = \phi_i^1(u_1)\cdots\phi_i^N(u_N) \;\; i = 1,2,\cdots.$$
$$\phi_i^j \in C^\infty(\mathbf{R}),\;\; j = 1,\cdots,N \;\; i = 1,2,\cdots.$$

Proof of Theorem 2.1 We give the proof of Theorem 2.1 only in the case where $N > p > N-1$ since the case $p \geq N$ is well known (see Remark 2.2). In the first step of the proof, we truncate the functions $(u_\nu)_\nu$ and u to get a new sequence which is uniformly bounded in L^∞. Then we write f as a sum of functions of the form $a(u)g(x,T(\nabla u))$, where $g(x,\cdot) \geq 0$ are convex. In the second step we study the particular case where f has the form $a(u)g(x,T(\nabla u))$. In the last step we study the general case where f satisfies the hypotheses of Theorem 2.1.

First step.
a) Fix $h > 0$ and $\delta(h) << 1$. Truncate u and u_ν by considering $\phi(u)$, where ϕ is given by

$$\phi(u) = \prod_{i=1}^{N} \psi(u^i), \quad \phi'(u) = \prod_{i=1}^{N} \psi'(u^i) \quad \text{with} \quad \psi'(t) = \frac{d\psi}{dt}(t), \tag{3}$$

and $\psi \in C^\infty(\mathbf{R},\mathbf{R})$ is defined in the following way

$$\psi(t) = \begin{cases} -h & \text{si } t < -h - \delta(h) \\ t & \text{if } |t| \leq h \\ h & \text{if } t > h + \delta(h), \end{cases}$$

$0 \leq \psi'(t) \leq 1$ for every $t \in \mathbf{R}$ and $\psi'(t) = 0$ if and only if $|t| \geq h + \delta(h)$.

b) By Lemma 2.4 and Remark 2.5 we get

$$f(x,u,T) = \lim_{n\to\infty} \sum_{j=0}^{n} a_j(u) f_j(x,T) \quad \forall\,(x,u,T) \in \Omega \times [-2h,2h]^N \times \mathbf{R}^\tau, \tag{4}$$

with

$$f_j \in C(\Omega \times \mathbf{R}^\tau, \mathbf{R}), \;\; f_j \geq 0,$$

$$a_j(u) = \prod_{i=1}^{N} a_j^i(u^i), \;\; u \in [-2h,2h]^N, \;\; j = 1,2,\cdots, \qquad i = 1,\cdots,N$$

and

$$a_j^i \in C^\infty(\mathbf{R},\mathbf{R}), \;\; a_j^i \geq 0, j = 1,2,\cdots, \qquad i = 1,\cdots,N.$$

Using explicitly Weierstrass's Approximation formula it is easy to see that $f_j(x,\xi)$ is in the form $b_j f(x, \frac{k_j}{n_j}, \xi)$. Therefore $f_j(x,\cdot)$ is convex in $\mathbf{R}^\tau$ for every $j = 1,2,\cdots$.

Second step. We show that for all j

$$\liminf_{\nu\to\infty} \int_\Omega a_j(u_\nu)\phi'(u_\nu) f_j(x, T(\nabla(u_\nu))) \geq \int_\Omega a_j(u)\phi'(u) f_j(x, T(\nabla(u))). \tag{5}$$

- If $\liminf_{\nu\to\infty} \int_\Omega a_j(u_\nu)\phi'(u_\nu) f_j(x, T(\nabla(u_\nu))) = \infty$ then (5) is trivial.

- Assume that $\liminf_{\nu\to\infty} \int_\Omega a_j(u_\nu)\phi'(u_\nu) f_j(x, T(\nabla(u_\nu))) < \infty$. By Lemma 2.3 we can approximate f_j by a sequence of non decreasing functions $(f_j^k)_k$ which converge uniformly to f_j in every compact set of $\Omega \times \mathbf{R}^\tau$ and such that $(f_j^k)_k$ verifies all properties given in Lemma 2.3. Without loss of generality we may assume that f_j verifies

$$f_j \in C^\infty(\Omega \times \mathbf{R}^\tau, [0,\infty)),$$

$$f_j(x,\cdot) \text{ is convex},$$

$$f_j(x,\cdot) \equiv 0 \text{ if } dist(x, \partial\Omega) \leq \frac{1}{k} \text{ for a suitable k },$$

$$D_T f_j(x,T) \text{ is uniformly bounded in every compact set of } \Omega \times \mathbf{R}^\tau \text{ independly on } T. \tag{6}$$

We can also assume that $u \in C^\infty(\Omega, \mathbf{R}^N)$. If this wasn't the case then it would suffice to replace u by $u_\epsilon \in C^\infty(\Omega, \mathbf{R}^N)$ such that $||u_\epsilon - u||_{W^{1,N}} \leq \epsilon$ following the proof with the suitable modifications.The convexity of f_j implies that

$$\begin{aligned}
& \liminf_{\nu\to\infty} \int_\Omega a_j(u_\nu)\phi'(u_\nu) f_j(x, T(\nabla(u_\nu))) \\
\geq\ & \liminf_{\nu\to\infty} \int_\Omega a_j(u_\nu)\phi'(u_\nu) f_j(x, T(\nabla(u))) \\
+\ & \liminf_{\nu\to\infty} \int_\Omega a_j(u_\nu)\phi'(u_\nu) < D_T f_j(x, T(\nabla(u))); T(\nabla(u_\nu)) - T(\nabla(u)) > \\
\geq\ & \int_\Omega a_j(u)\phi'(u) f_j(x, T(\nabla(u))) \\
+\ & \liminf_{\nu\to\infty} \int_\Omega a_j(u_\nu)\phi'(u_\nu) < D_T f_j(x, T(\nabla(u))); T(\nabla(u_\nu)) - T(\nabla(u)) >,
\end{aligned}$$

where we used Fatou's Lemma and the fact that

$$a_j(u_\nu)\phi'(u_\nu) \longrightarrow a_j(u)\phi'(u) \ \ a.e.$$

For $T \in \mathbf{R}^\tau$, we set $T = (\bar{T}, t)$, $t \in \mathbf{R}$. For fixed $x \in \Omega$, let $D_{\bar{T}} f_j(x,\cdot)$ denote the matrix of the partial derivatives of $f_j(x,\cdot)$ with respect to the $\tau - 1$ first variables in $\mathbf{R}^\tau$. Let H be the functional defined on $\Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times N}$ by

$$H(x,v,\xi)) = a_j(v)\phi'(v) < D_{\bar{T}} f_j(x, T(\nabla(u))); \bar{T}(\xi) - \bar{T}(\nabla(u)) > .$$

It is easy to see that H and $-H$ are quasiconvex in the last variable. Using the fact that $u \in C^\infty(\Omega, \mathbf{R}^N)$, (6) and the fact that $|\phi'(u_\nu)| \leq 1$, we get that H and $-H$ verify the assumptions of Proposition 1.3. We deduce that

$$\liminf_{\nu\to\infty} \int_\Omega a_j(u_\nu)\phi'(u_\nu) < D_{\bar{T}} f_j(x, T(\nabla(u))); \bar{T}(\nabla(u_\nu)) - \bar{T}(\nabla(u)) >= 0. \tag{7}$$

On the other hand setting

$$v^i_\nu = A^i_j(\psi(u^i_\nu)),\ v^i = A^i_j(\psi(u^i)),\ \text{ where } A^i_j(t) = \int_{-h-\delta(h)}^t a^i_j \circ \psi^{-1}(s)ds,$$

then we obtain

$$v^i_\nu \longrightarrow v^i \ \ a.e.,$$
$$a^i_j(u^i_\nu)\psi'(u^i_\nu) \longrightarrow a^i_j(u^i)\psi'(u^i) \ \ a.e.$$

and

$$\frac{\partial}{\partial t} f_j((x, T\nabla(u)) \in C^\infty_0(\Omega).$$

By Lemma 1.4 we obtain

$$\begin{aligned}
&\liminf_{\nu\to\infty} \quad \int_\Omega a_j(u_\nu)\phi'(u_\nu)\frac{\partial}{\partial t} f_j(x, T(\nabla(u)))\big(det(\nabla(u_\nu)) - det(\nabla(u))\big) \\
&= \liminf_{\nu\to\infty} \quad \Big(\int_\Omega \frac{\partial}{\partial t} f_j(x, T\nabla(u))\big(det(\nabla(v_\nu)) - det(\nabla(v))\big) \\
&- \quad \int_\Omega \Big(a_j(u_\nu)\phi'(u_\nu) - a_j(u)\phi'(u)\Big)\frac{\partial}{\partial t} f_j(x, T\nabla(u))det(\nabla(u))\Big) = 0
\end{aligned}$$

which together with (7), yields (5).

Third step.
Let $n \in \mathbf{N}$ be fixed. By (4), (5) and the definition of ψ we deduce that

$$\begin{aligned}
& \liminf_{\nu\to\infty} \int_\Omega f(x, u_\nu, T(\nabla u_\nu)) \\
\geq \quad & \liminf_{\nu\to\infty} \int_\Omega \Big(\sum_{j=0}^n a_j(u_\nu)\phi'(u_\nu) f_j(x, T(\nabla(u_\nu)))\Big) \\
\geq \quad & \sum_{j=0}^n \liminf_{\nu\to\infty} \int_\Omega a_j(u_\nu)\phi'(u_\nu) f_j(x, T(\nabla(u_\nu))) \\
\geq \quad & \sum_{j=0}^n \int_\Omega a_j(u)\phi'(u) f_j(x, T(\nabla(u))).
\end{aligned}$$

When n goes to infinity, Lebesgue's Monotone Theorem and (4) give

$$\liminf_{n\to\infty} \int_\Omega f(x, u_\nu, T(\nabla u_\nu)) \geq \int_\Omega \phi'(u) f(x, u, T(\nabla(u))).$$

Letting h go to infinity in the previous inequality we obtain (2). ∎

3 The case of Carathéodory integrands.

We state the main result of this section.

Theorem 3.1 *Let $N \geq 2$ be an integer number, $N-1 < p < N$, $\Omega \subset \mathbf{R}^N$ and open bounded set, and $K \subset \Omega$ a compact set. The two following assertions are equivalent:*

$$meas(\partial K) \neq 0, \tag{8}$$
$$\liminf_{n\to\infty} \int_K |det(\nabla u_n(x))| dx < \int_K |det(\nabla u(x))| dx \tag{9}$$

for a suitable $u_\nu, u \in W^{1,N}(\Omega, \mathbf{R}^N)$ such that $u_\nu \rightharpoonup u$ in $W^{1,p}$.

Before proving Theorem 3.1 we begin with some remarks.

Remark 3.2
Let us recall that if $F(u) = \int_K |det(\nabla u(x))| dx$ and K is a compact set then, for $p \geq N$, F is weakly lower semicontinuous on $W^{1,p}$ even if $meas(\partial K) \neq 0$ (see Proposition 1.3). For $p < N-1$ then F is not weakly lower semicontinuous on $W^{1,p}$ even if $meas(\partial K) = 0$ (see [Mal]).

We state and prove the following lemma that will be used to prove that (8) implies (9).

Lemma 3.3
Let $N, r \geq 2$ be two integer numbers, $\Omega \subset \mathbf{R}^N$ an open bounded set and $K \subset \Omega$ a compact set such that $meas(\partial K) > 0$. Let $p < N$ be a real number. Then there exists a sequence $u_k \in W^{1,N}(\Omega, \mathbf{R}^N)$ such that

$$
\begin{aligned}
i) &\quad u_k \rightharpoonup u = id \text{ in } W^{1,p}(\Omega, \mathbf{R}^N) \text{ with } id(x) := x,\\
ii) &\quad |det(\nabla u_k(x))| \leq 1 \quad \text{on } K,\\
iii) &\quad meas\{x \in \partial K : \ det(\nabla u_k(x)) \neq 0\} < \frac{1}{2^k}.
\end{aligned}
$$

Proof: we divide the proof into five steps. We assume without loss of generality that $\Omega = (0,1)^N$.
First step. We construct the sequence u_k. Let $k \in \mathbf{N}$ be fixed. Using Vitali's Covering Theorem we find two sequences $(x_i^k)_i \subset \partial K$, $(\beta_i^k)_i \subset (0, \frac{1}{2^k})$ such that

$$\partial K \subset \tilde{N}_k \bigcup \Big(\bigcup_{i=1}^{i=\infty} B(x_i^k, \beta_i^k) \Big),$$
$$B(x_i^k, \beta_i^k) \cap B(x_j^k, \beta_j^k) = \emptyset \quad \text{for } i \neq j, \quad i,j = 1, \cdots, \infty,$$

$$meas(\tilde{N}_k) \leq \frac{meas(\partial K)}{2^{k+1}}, \tag{10}$$

$$meas\Big(\bigcup_{i=1}^{i=\infty} B(x_i^k,\beta_i^k) - meas(\partial K)\Big) \leq \frac{meas(\partial K)}{2^{k+1}}, \tag{11}$$

$$B(x_i^k,\beta_i^k) \subset \Omega \quad \text{for } i=1,\cdots,\infty,$$

where $B(x,\beta)$ stands for the open ball in $\mathbf{R}^N$ with center x and radius β and $\tilde{N}_k$ is an open set. Since K is a compact set we have

$$\partial K \subset \tilde{N}_k \bigcup \Big(\bigcup_{i=1}^{i=T(k)} B(x_i^k,\beta_i^k)\Big), \tag{12}$$

where $T(k)$ is a constant depending on k. Now we want to change the centers x_i^k by other centers which belong to the complementary of K. Using (10), (11), (12) and the fact that $x_i^k \in \partial K$, we deduce that there exist an open set N_k and two sequences $a_i^k \in B(x_i^k,\beta_i^k)\setminus K$, $\;0 < \epsilon_i^k < \beta_i^k$, such that

$$\partial K \subset N_k \bigcup \Big(\bigcup_{i=1}^{i=T(k)} B(a_i^k,\epsilon_i^k)\Big), \tag{13}$$

$$B(a_i^k,\epsilon_i^k) \subset B(x_j^k,\beta_j^k) \;\; i=1,\cdots,T(k),$$

$$meas(N_k) \leq \frac{meas(\partial K)}{2^k} \tag{14}$$

$$meas\Big(\bigcup_{i=1}^{i=T(k)} B(a_i^k,\epsilon_i^k) - meas(\partial K)\Big) \leq \frac{meas(\partial K)}{2^k}. \tag{15}$$

Since $\Omega \setminus K$ is an open set and $a_i^k \in B(x_i^k,\beta_i^k)\setminus K$, there exists $\delta_i^k > 0$ such that

$$\delta_i^k < \Big(\frac{1}{T(k)(2^k\cdot\epsilon_i^k)^p}\Big)^{\frac{1}{N-p}} \qquad i=1,\cdots,T(k) \tag{16}$$

and

$$B(a_i^k,\delta_i^k) \subset \Omega\setminus K \quad i=1,\cdots,T(k). \tag{17}$$

We define

$$u_k(x) = \begin{cases} a_i^k + \frac{\epsilon_i^k}{\delta_i^k}(x-a_i^k),\; x \in B(a_i^k,\delta_i^k) \\ a_i^k + \frac{\epsilon_i^k}{|x-a_i^k|}(x-a_i^k),\; x \in B(a_i^k,\epsilon_i^k)\setminus B(a_i^k,\delta_i^k) \\ x,\; x \in \Omega\setminus(\cup_{i=1}^{i=T(k)} B(a_i^k,\epsilon_i^k)) \end{cases}.$$

It is easy to see that u_k is a diffeomorphism from $B(a_i^k,\delta_i^k)$ into $B(a_i^k,\epsilon_i^k)$ and u_k maps $B(a_i^k,\epsilon_i^k)\setminus B(a_i^k,\delta_i^k)$ into $\partial B(a_i^k,\epsilon_i^k)$.

Second step. In this step we show that $u_k \in W^{1,\infty}\Big(\Omega, \mathbf{R}^N\Big)$. As

$$u_k \in C^1\Big(\bar{B}(a_i^k, \delta_i^k), \mathbf{R}^N\Big),$$

$$u_k \in C^1\Big(\bar{B}(a_i^k, \epsilon_i^k) \setminus B(a_i^k, \delta_i^k), \mathbf{R}^N\Big)$$

and

$$u_k \text{ is continuous on } \bar{B}(a_i^k, \epsilon_i^k),$$

we have

$$u_k \in W^{1,\infty}\Big(B(a_i^k, \epsilon_i^k), \mathbf{R}^N\Big) \tag{18}$$

and since

$$u_k(x) = x \quad \text{on} \quad \partial B(a_i^k, \epsilon_i^k) \tag{19}$$

we conclude that

$$u_k \in C^0\Big(\Omega, \mathbf{R}^N\Big). \tag{20}$$

Using the definition of u_k on $\Omega \setminus (\bigcup_{i=1}^{i=T(k)} B(a_i^k, \epsilon_i^k))$ it is obvious that

$$u_k \in W^{1,\infty}(\Omega \setminus (\bigcup_{i=1}^{i=T(k)} \bar{B}(a_i^k, \epsilon_i^k)), \tag{21}$$

which together with (18) and (20) yields

$$u_k \in W^{1,\infty}\Big(\Omega, \mathbf{R}^N\Big). \tag{22}$$

Third step. We show that, up to a subsequence, $u_k \rightharpoonup u = id$ in $W^{1,p}(\Omega, \mathbf{R}^N)$. Using the definition of u_k on Ω we obtain

$$|u_k(x) - x| \leq \frac{1}{2^k} \text{ for every } x \in \Omega, \tag{23}$$

and

$$(\Sigma), \qquad \nabla u_k(x) = \begin{cases} \frac{\epsilon_i^k}{\delta_i^k} I_N & x \in B(a_i^k, \delta_i^k) \\ \frac{\epsilon_i^k}{|x-a_i^k|}\Big(I_N - \frac{(x-a_i^k)\otimes(x-a_i^k)}{|x-a_i^k|^2}\Big) & x \in B(a_i^k, \epsilon_i^k) \setminus B(a_i^k, \delta_i^k) \\ I_N & x \in \Omega \setminus (\bigcup_{i=1}^{i=T(k)} B(a_i^k, \epsilon_i^k)) \end{cases}$$

where I_N is the identity matrix in $\mathbf{R}^{N\times N}$. For $a, b \in \mathbf{R}^N$, $a \otimes b$ denote the $N \times N$ matrix with component $a_i b_j$ and $|a| = \sqrt{a_1^2 + \cdots + a_N^2}$. Cleary, there exits a constant $C = C(N)$ such that

$$|\nabla u_k(x)| \leq \begin{cases} C\frac{\epsilon_i^k}{\delta_i^k} & x \in B(a_i^k, \delta_i^k) \\ C\frac{\epsilon_i^k}{|x-a_i^k|} & x \in B(a_i^k, \epsilon_i^k) \setminus B(a_i^k, \delta_i^k) \\ C & x \in \Omega \setminus (\bigcup_{i=1}^{i=T(k)} B(a_i^k, \epsilon_i^k)) \end{cases} .$$

Thus by (15) and (16) we have

$$\int_{\Omega} |\nabla u_k(x)|^p dx \leq C^p\Big(1+\sum_{i=1}^{i=T(k)}\Big(\int_{B(a_i^k,\epsilon_i^k)}(\frac{\epsilon_i^k}{|x-a_i^k|})^p dx+\int_{B(a_i^k,\delta_i^k)}(\frac{\epsilon_i^k}{\delta_i^k})^p dx\Big)\Big)$$
$$\leq w_N C^p\Big(1+(\sum_{i=1}^{i=T(k)} N(\frac{(\epsilon_i^k)^N}{N-p})+\frac{1}{2^k}\Big),$$

where w_N = measB(O,1). Recalling that B(af,cf) does not intersect £(a£,e*) for $i \neq j$ and $B(a_i^k$, cf) C ft = (0, l)^W we conclude that

$$\int_{\Omega} |\nabla u_k(x)|^p dx \leq w_N C^p\Big(1+\frac{N}{w_N(N-p)})+\frac{1}{2^k}\Big). \tag{24}$$

Therefore (ujt)* is bounded in W$^{ltJ>}$ and by (23) we deduce that, up to a subsequence,

$$u_k \wedge u = id \quad in \ W^{1,p}(\Omega, \mathbf{R}^N).$$

Fourth step. We show that *\det(Vuk(x))* ≤ 1 a.e. on A'. Indeed (Ē) implies that

$$\text{dc*(Vu*(*)) = 1 a.e. } x \in ft \setminus (\bigcup^{:=T(fc)} £(a?,e*)). \tag{25}$$

We know that $u_k \in C^l[B\{a^k{}_i,e^k{}_i) \setminus B(q^k,6^k_i),R^N)$ **and**

$$|u*(«) - of| = e_i^k \quad Vx \in 5(«f, ef) \setminus J5(af, 5,^{fc}).$$

As u* is the identity on $dB(a\wedge$ cf) we obtain

$$u_k(B(ale^k) \setminus B(a^k,6^k)) = a£(a,^{fc},ef).$$

Therefore u_k is not invertible at any point $x \in B(q^k,e_i^k) \setminus B(a_i^k,6^k_i)$. We conclude that

$$det(Vu_k(x)) = 0 \text{ a.e. } x \ 6 \ £(o?_f«?) \setminus B(a?,\wedge^{fc}), \tag{26}$$

•which, together with (17) and (25) implies that

$$0 \leq \text{<fet}(Vu_t(x)) \leq 1 \text{ a.e. } x \in K. \tag{27}$$

Fifth step. We claim that *meas{x € dK : det(Vuk(x))* ^ 0} ≤ meaAaA". By (13). (17), (25) and (26) we have

$$\{x \in \partial K : \ det(Vu_k(x)) \ ? \ 0\} \ C \ N_k \tag{28}$$

and the result follows now from (14). ∎

Proof of Theorem 3.1.
We prove that (8) implies (9). Assume that $meas(\partial K) \neq 0$. By Lemma 3.3 there exists a sequence $u_k \in W^{1,N}(\Omega, \mathbf{R}^N)$ such that:

$$\begin{aligned} &i) \quad u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega, \mathbf{R}^N), \; u(x) := x, \\ &ii) \quad |det(\nabla u_k(x))| \leq 1 \quad \text{a.e on } K, \qquad (29) \\ &iii) \quad \{x \in \partial K : \; det(\nabla u_k(x)) \neq 0\} < \frac{1}{2^k}. \qquad (30) \end{aligned}$$

(29) and (30) imply that

$$\begin{aligned} \int_K |det(\nabla u_k(x))| dx &= \int_{\partial K} |det(\nabla u_k(x))| dx + \int_{K \setminus \partial K} |det(\nabla u_k(x))| dx \\ &\leq \frac{meas(\partial K)}{2^k} + meas(K \setminus \partial K) \end{aligned}$$

and so

$$\begin{aligned} \liminf_{k \to \infty} \int_K |det(\nabla u_k(x))| dx &\leq meas(K \setminus \partial K) \\ &< meas(K) = \int_K |det(\nabla u(x))| dx, \end{aligned}$$

and we conclude (9).

In order to prove that (9) implies (8), we assume that $meas(\partial K) = 0$. It is easy to construct a sequence $a_n \in C^0(\Omega, \mathbf{R}^N)$ such that (see [Ga])

$$a_n(x) \to 1_K(x) \quad \text{a.e } x \in \Omega, \qquad (31)$$

$$0 \leq a_n(x) \leq a_{n+1}(x) \leq 1_K(x) \quad \text{a.e } x \in \Omega. \qquad (32)$$

Let $u_k, u \in W^{1,N}(\Omega, \mathbf{R}^N)$ be such that $u_k \rightharpoonup u \;\; W^{1,p}(\Omega, \mathbf{R}^N)$. Theorem 2.1 implies that

$$\begin{aligned} \int_\Omega a_n(x) |det(\nabla u(x))| dx &\leq \liminf_{k \to \infty} \int_\Omega a_n(x) |det(\nabla u_k(x))| dx \\ &\leq \liminf_{k \to \infty} \int_K |det(\nabla u_k(x))| dx, \end{aligned}$$

for each fixed n. Using (31), (32) and Fatou's Lemma we conclude that .

$$\int_K |det(\nabla u(x))| dx \leq \liminf_{k \to \infty} \int_K |det(\nabla u_k(x))| dx.$$

∎

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