A REMARK ON THE REGULARITY OF SOLUTIONS OF MAXWELL'S EQUATIONS ON LIPSCHITZ DOMAINS

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Abstract

Let $\vec{u}$ be a vector field on a bounded Lipschitz domain in R^3, and let $\vec{u}$ together with its divergence and curl be square integrable. If either the normal or the tangential component of $\vec{u}$ is square integrable over the boundary, then $\vec{u}$ belongs to the Sobolev space $H^{1/2}$ on the domain. This result gives a simple explanation for known results on the compact embedding of the space of solutions of Maxwell's equations on Lipschitz domains into L^2.

Let ft C R^3 be a bounded simply connected domain with connected Lipschitz boundary F. This means that F can be represented locally as the graph of a Lipschitz function. For properties of Lipschitz domains, see [7], [3], [2]. In particular, F has the strict cone property.

We consider real vector fields $\vec{u}$ on *ft* satisfying in the distributional sense

$$\vec{u} \; 6 \; L^2(tt); \qquad \operatorname{div} \vec{u} \; e \; L^2(Q); \qquad \operatorname{curl} \vec{u} \; e \; L^2\{Q) \; . \tag{1}$$

We denote the inner product in $L^2(il)$ by $(\cdot,\cdot)$.

It is well known that functions $\vec{u}$ satisfying (1) have boundary values $\vec{n}$ x *it* and $\vec{n} \cdot \vec{u}$ in the Sobolev space $_JH^{r}\sim^{1}/^{2}(F)$ defined in the distributional sense by the natural extension of the Green formulas

$$(\operatorname{curl} \vec{u}, \vec{v}) - (\vec{w}, \operatorname{curl} \vec{t};) \;=\; <\vec{n} \times \text{fl}, \vec{v}> \tag{2}$$

$$(\operatorname{div} \vec{u}, (p) + (\vec{w}, \operatorname{grad} \Phi) \;=\; <\vec{n} \cdot \vec{w}, tp> \tag{3}$$

for all $\vec{v}, ipe \; H^1(\Omega)$.

Here $\vec{n}$ denotes the exterior normal vector which exists almost everywhere on F, and $< \cdot, \cdot >$ is the natural duality in $H^{-1/2}(T) \times H^{1/2}(T)$ extending the $L^2(T)$ inner product.

It is known that for smooth domains (e.g., $\Gamma \in C^{1,1}$), each one of the two boundary conditions

$$\vec{n} \times \vec{u} \in H^{1/2}(\Gamma) \qquad \text{or} \qquad \vec{n} \cdot \vec{u} \in H^{1/2}(\Gamma) \tag{4}$$

implies $\vec{u} \in H^1(\Omega)$, see [2] and, for the case of homogeneous boundary conditions, [6], where one finds also a counterexample for a nonsmooth domain. Such counterexamples are derived from nonsmooth weak solutions $v \in H^1(\Omega)$ of the Neumann problem ($\partial_n := \vec{n} \cdot \operatorname{grad}$ denotes the normal derivative)

$$\Delta v = g \in L^2(\Omega)\,; \qquad \partial_n v = 0 \quad \text{on } \Gamma \tag{5}$$

If $\vec{u} = \operatorname{grad} v$, then $\vec{u}$ satisfies (1) and $\vec{n} \cdot \vec{u} = 0$ on Γ, and $\vec{u} \in H^s(\Omega)$ if and only if $v \in H^{1+s}(\Omega)$. For smooth or convex domains, one knows that $v \in H^2(\Omega)$. If Ω has a nonconvex edge of opening angle $\alpha\pi$, $\alpha > 1$, then, in general, the solution v of (5) is not in $H^{1+s}(\Omega)$ for $s = 1/\alpha$, hence $\vec{u} \notin H^s(\Omega)$. This upper bound s for the smoothness of $\vec{u}$ can be arbitrary close to $1/2$.

Regularity theorems for (1), (4) have applications in the numerical approximation of the Stokes problem [2] and in the analysis of initial-boundary value problems for Maxwell's equations [6]. The compact embedding into $L^2(\Omega)$ of the space of solutions of the time-harmonic Maxwell equations is needed for the principle of limiting absorption. This compact embedding result was shown by Weck [10] for a class of piecewise smooth domains and by Weber [9] and Picard [8] for general Lipschitz domains. In these proofs, no regularity result for the solution $\vec{u}$ was used or obtained. See Leis' book [6] for a discussion.

In this note, we use the result by Dahlberg, Jerison, and Kenig [4], [5] on the $H^{3/2}$ regularity for solutions of the Dirichlet and Neumann problems with L^2 data in potential theory (see Lemma 1 below). Together with arguments similar to those described by Girault and Raviart [2], this yields $\vec{u} \in H^{1/2}(\Omega)$ (Theorem 2). The compact embedding in L^2 is an obvious consequence of this regularity. If instead of Lemma 1, one uses only the more elementary tools from [1], one obtains $H^{3/2-\epsilon}$ regularity for solutions of the Dirichlet and Neumann problems in potential theory and, consequently $\vec{u} \in H^{1/2-\epsilon}(\Omega)$ for any $\epsilon > 0$. This kind of regularity is also known for the case of an open manifold Γ (screen problem). It suffices, of course, for the compact embedding result.

The proof of the following result can be found in [4].

Lemma 1. *(Dahlberg-Jerison-Kenig) Let $v \in H^1(\Omega)$ satisfy $\Delta v = 0$ in Ω. Then the two conditions*

$$\text{(i)} \quad v \restriction_\Gamma \in H^1(\Gamma) \qquad \textit{and} \qquad \text{(ii)} \quad \partial_n v \restriction_\Gamma \in L^2(\Gamma)$$

are equivalent. They imply $v \in H^{3/2}(\Omega)$.

Remarks.

a.) The first assertion in the Lemma goes back to Nečas [7].

b.) There are accompanying norm estimates, viz.

There exist constants C_1, C_2, C_3, independent of v such that

$$C_1\|\partial_n v\|_{L^2(\Gamma)} \leq \|\vec{n}\times \operatorname{grad} v\|_{L^2(\Gamma)} \leq C_2\|\partial_n v\|_{L^2(\Gamma)},$$

$$\|v\|_{H^{3/2}(\Omega)} \leq C_3\|v\restriction_\Gamma\|_{H^1(\Gamma)}.$$

c.) The boundary values are attained in a stronger sense than the distributional sense (2), (3), namely pointwise almost everywhere in the sense of nontangential maximal functions in $L^2(\Gamma)$.

Theorem 2. *Let $\vec{u}$ satisfy the conditions* (1) *in Ω and either*

$$\vec{n}\times\vec{u}\in L^2(\Gamma) \tag{6}$$

or

$$\vec{n}\cdot\vec{u}\in L^2(\Gamma)\,. \tag{7}$$

Then $\vec{u}\in H^{1/2}(\Omega)$.
If (1) *is satisfied, then the two conditions* (6) *and* (7) *are equivalent.*

Proof. The proof follows the lines of [2]. It is presented in detail to make sure that it is valid for Lipschitz domains.
Let $\vec{f}:=\operatorname{curl}\vec{u}\in L^2(\Gamma)$. Then $\operatorname{div}\vec{f}=0$ in Ω.
According to [2, Ch. I, Thm 3.4] there exists $\vec{w}\in H^1(\Omega)$ with

$$\operatorname{curl}\vec{w}=\vec{f}\,,\quad \operatorname{div}\vec{w}=0\quad \text{in } \Omega. \tag{8}$$

The construction of $\vec{w}$ is as follows:
Choose a ball $\mathcal{O}$ containing $\overline{\Omega}$ in its interior and solve in $\mathcal{O}\setminus\overline{\Omega}$ the Neumann problem: $\chi\in H^1(\mathcal{O}\setminus\overline{\Omega})$ with

$$\Delta\chi=0 \text{ in } \mathcal{O}\setminus\overline{\Omega}\,;\ \partial_n\chi=\vec{n}\cdot\vec{f} \text{ on } \Gamma\,;\ \partial_n\chi=0 \text{ on } \partial\mathcal{O}\,. \tag{9}$$

Note that $\vec{n}\cdot\vec{f}\in H^{-1/2}(\Gamma)$ satisfies the solvability condition $<\vec{n}\cdot\vec{f},\,1>=0$ because $\operatorname{div}\vec{f}=0$ in Ω.
Define $\vec{f_0}:=\vec{f}$ in Ω, $\vec{f_0}:=\operatorname{grad}\chi$ in $\mathcal{O}\setminus\overline{\Omega}$, $\vec{f_0}:=0$ in $\mathbf{R}^3\setminus\overline{\mathcal{O}}$. Then $\vec{f_0}\in L^2(\mathbf{R}^3)$ has compact support and satisfies $\operatorname{div}\vec{f_0}=0$ in $\mathbf{R}^3$. Therefore $\vec{f_0}=\operatorname{curl}\vec{w}$ for some $\vec{w}\in H^1(\mathbf{R}^3)$ with $\operatorname{div}\vec{w}=0$ in $\mathbf{R}^3$. One obtains $\vec{w}$ for example by convolution of $\vec{f_0}$ with a fundamental solution of the Laplace operator in $\mathbf{R}^3$ and taking the curl.

Thus (8) is satisfied. The function $\vec{z} := \vec{u} - \vec{w}$ satisfies

$$\vec{z} \in L^2(\Omega) \quad \text{and} \quad \operatorname{curl} \vec{z} = 0 \quad \text{in} \quad \Omega. \tag{10}$$

Since Ω is simply connected, there exists $v \in H^1(\Omega)$ with

$$\vec{z} = \operatorname{grad} v . \tag{11}$$

Then v satisfies

$$\Delta v = \operatorname{div} \vec{u} \in L^2(\Omega). \tag{12}$$

We can apply Lemma 1 to v, because by subtraction of a suitable function in $H^2(\Omega)$, we obtain a homogeneous Laplace equation from (12).

Now, since $\vec{w}|_\Gamma \in H^{1/2}(\Gamma)$, condition (i) in the Lemma is equivalent to

$$\vec{n} \times \operatorname{grad} v = \vec{n} \times \vec{z} = \vec{n} \times \vec{u} - \vec{n} \times \vec{w} \in L^2(\Gamma)$$

and hence to (6), and condition (ii) is equivalent to

$$\vec{n} \cdot \operatorname{grad} v = \vec{n} \cdot \vec{z} = \vec{n} \cdot \vec{u} - \vec{n} \cdot \vec{w} \in L^2(\Gamma)$$

and hence to (7). Therefore the Lemma implies that (6) and (7) are equivalent. Also, $v \in H^{3/2}(\Omega)$ is equivalent to $\operatorname{grad} v \in H^{1/2}(\Omega)$, hence to

$$\vec{u} = \vec{z} + \vec{w} = \operatorname{grad} v + \vec{w} \in H^{1/2}(\Omega) .$$

∎

Remark. The accompanying norm estimates are:
There exist constants C_1, C_2, C_3, independent of $\vec{u}$ such that

$$\begin{aligned}
\|\vec{n} \times \vec{u}\|_{L^2(\Gamma)} &\leq C_1 \left(\|\vec{u}\|_{L^2(\Omega)} + \|\operatorname{div} \vec{u}\|_{L^2(\Omega)} + \|\operatorname{curl} \vec{u}\|_{L^2(\Omega)} + \|\vec{n} \cdot \vec{u}\|_{L^2(\Gamma)} \right) \\
\|\vec{n} \cdot \vec{u}\|_{L^2(\Gamma)} &\leq C_2 \left(\|\vec{u}\|_{L^2(\Omega)} + \|\operatorname{div} \vec{u}\|_{L^2(\Omega)} + \|\operatorname{curl} \vec{u}\|_{L^2(\Omega)} + \|\vec{n} \times \vec{u}\|_{L^2(\Gamma)} \right) \\
\|\vec{u}\|_{H^{1/2}(\Omega)} &\leq C_3 \left(\|\vec{u}\|_{L^2(\Omega)} + \|\operatorname{div} \vec{u}\|_{L^2(\Omega)} + \|\operatorname{curl} \vec{u}\|_{L^2(\Omega)} + \|\vec{n} \times \vec{u}\|_{L^2(\Gamma)} \right) .
\end{aligned}$$

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